Good reduction of K3 Surfaces

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This is joint work with Y. Matsumoto.

1 Introduction

Setup: Let $\mathcal{O}_K$ be a local Henselian DVR of characteristic 0, with perfect residue field k and fraction field K. Let p denote the characteristic of k.

Examples: $\mathbb{C}[[t]]$ and the ring of integers in a p-adic field.

Let $G_K = \mathrm{Gal}(\bar{K}/K)$ and $G_k = \mathrm{Gal}(\bar{k}/k)$ be the absolute Galois groups. Note that G_K acts on the integral closure of $\mathcal{O}_K$; reducing modulo a nonzero prime ideal we get an action on $\bar{k}$, so we have a map $G_K \to G_k$. This map is surjective and fits into the exact sequence

$$1 \to I_K \to G_K \to G_k \to 1,$$

whose kernel is called the *inertia* group.

Let X be smooth and proper over K, so we have a representation

$$\rho_m : G_K \to \mathrm{Aut}(\mathrm{H}^m_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell))$$

with $\ell \neq p$. It is called *unramified* if the inertia acts trivially, i.e., $\rho_m(I_K) = \{\mathrm{id}\}$.

The following essentially follows from proper base change, with some inputs from Matsumoto's thesis in order to extend it to algebraic spaces:

Theorem 1 *Let X be smooth and proper over K. Suppose X has good reduction, i.e., there exists a smooth proper algebraic space*

$$\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$$

with generic fiber $\mathcal{X}_\eta \simeq X$. Then ρ_m is unramified for each m.

What about the converse?

- Serre-Tate and Néron-Ogg-Shafarevich: If X is an abelian variety and ρ_1 is unramified then X has good reduction.
- It fails for curves of genus $g \geq 2$, e.g., when the reduction is of compact type.
- T. Oda: If X is a curve of genus $g \geq 2$ over K then X has good reduction if and only if

$$\rho_1 : G_K \to \mathrm{Out}(\pi_1(X_{\bar{K}})_\ell)$$

  is unramified, i.e., if $\rho_1(I_K)$ is trivial.

Are there analogues for K3 surfaces, expressed in terms of ρ_2?

2 Kulikov models

We will refer to Assumption $(*)$:

Assumption 1 *A K3 surface X over K satisfies $(*)$ if there exists a finite extension L/K and a flat proper morphism*

$$\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_L)$$

where $\mathcal{X}$ is a regular algebraic space with generic fiber $\mathcal{X}_\eta \simeq X_L$ and special fiber $\mathcal{X}_0$ a strict normal crossings divisor, and $\omega_{\mathcal{X}/\mathcal{O}_L} = \mathcal{O}_{\mathcal{X}}$.

When Assumption (*) holds, we call $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_L)$ as above a *Kulikov model* for X.

It is known that Assumption (*) holds for $\mathcal{O}_K = \mathbb{C}[[t]]$, by work of Kulikov and Pinkham-Persson. For mixed characteristic, it should follow from resolution of singularities and toroidalization, and then applying Kawamata's version of MMP in mixed characteristic.

Proposition 1 (Maulik $+\varepsilon$) *Given a K3 surface X over K, if there exists a very ample $\mathcal{L}$ on X with $\mathcal{L}^2 + 4 < p$ then Assumption $(*)$ holds.*

Theorem 2 (Kulikov, Pinkham-Persson; Nakkajima) *Let X be a K3 surface over K satisfying Assumption $(*)$. Let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_L)$ be a Kulikov model of X. Then the geometric special fiber $(\mathcal{X}_0)_{\bar{k}}$ is one of the following:*

(I) smooth

(II) a chain of elliptic surfaces

(III) a configuration of rational surfaces

Morover, we can compute the weight filtration on $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ from the configuration of the components of the special fiber via the Steenbrink-Rapoport-Zink spectral sequence. Since the weight-monodromy conjecture is known in this case, the weight filtration can be computed from the I_K-action. This way, the I_K-action can detect the three Types of special fibers above. The filtration is trivial precisely in the Type I case.

Theorem 3 (Matsumoto) *If X is a K3 surface over K satisfying $(*)$ with ρ_2 unramified then X has potential good reduction, i.e., there exists L/K such that X_L has good reduction.*

Can we do better?

Theorem 4 (Matusmoto-L.) *Let X be a K3 surface over K satisfying $(*)$ with the G_K-action on $\mathrm{H}^2_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unramified. Then there exists an unramified extension L/K such that X_L has good reduction (as an algebraic space) . Moreover, there exists a model of X (as a projective scheme) $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ such that the central fiber is a K3 surface with rational double point singularities.*

We have the following applications similar to the case of abelian varieties:

1. 'independence of ℓ': if X is a K3 surface satisfying $(*)$, then the G_K-action on $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unramified for one ℓ if and only if it is so for all ℓ.

2. If X is a K3 surface having potential good reduction and $p \geq 23$, then good reduction can be achieved after a finite and *tame* extension.

We also have results relating the reduction behavior of a polarized K3 surface over K and its associated Kuga-Satake abelian variety.

Theorem 5 (Matsumoto-L.) *For all $p \geq 5$, there exist smooth quartics $X \subset \mathbb{P}^3_{\mathbb{Q}_p}$ such that*

1. *The G_K action on $\mathrm{H}^2_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unramified,*

2. *X has good reduction over $\mathbb{Q}_{p^2}$, and*

3. *X does not have good reduction over $\mathbb{Q}_p$.*

Sketch of Proof of Theorem 4: Let X satisfy the assumptions of the Theorem. Then there exists an L/K finite and (without loss of generality) Galois with $G = \mathrm{Gal}(L/K)$, such that $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_L)$ is a smooth model.

Idea: The group G acts on the generic fiber. Try to extend this to $\mathcal{X}$ and then, $\mathcal{X}/G$ should be the desired model.

Choose $\mathcal{L}$ ample on X stable under the G-action. This gives an action on $\mathrm{H}^0(X_L, \mathcal{L}^{\otimes n})$. We claim that after a sequence of flops there exists $\mathcal{X}$ such that $\mathcal{L}|_{\mathcal{X}_0}$ is big and nef. Thus we obtain

$$\begin{array}{ccc} \mathcal{X} & \rightarrow & \mathcal{X}' := \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathrm{H}^0(\mathcal{X}_L, \mathcal{L}^{\otimes n})\right) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(\mathcal{O}_L) & \end{array}$$

Then, $\mathcal{X}'$ is a projective scheme with generic fiber X_L, whose central fiber is a K3 surface with rational double point singularities, and the G-action on X_L extends to $\mathcal{X}'$.

Let I_G be the inertia subgroup of G. We first claim that the I_G-action extends to $\mathcal{X}$ - otherwise, we can use the exceptional locus of $\mathcal{X} \to \mathcal{X}'$ to cook up a class in $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ upon which the I_K-action is non-trivial, contradicting that the Galois action is unramified.

Since the I_K-action is trivial on $\mathrm{H}^2_{\text{ét}}((\mathcal{X}_0)_{\bar{k}}, \mathbb{Q}_\ell)$, also the I_K-action on $(\mathcal{X}_0)_{\bar{k}}$ itself is *trivial*.

We are done once we show that the special fiber of $\mathcal{X}/I_G$ is $(\mathcal{X}_0)/I_G = \mathcal{X}_0$. There are some technical issues when the order of the group divides the residue characteristic. Wild ramification issues may be surmounted by using the fact that K3 surfaces admit no non-trivial vectors fields. This yields the desired smooth model after a finite and unramified extension.

Finally, $\mathcal{X}/G$ is a model of X, whose central fiber is a K3 surface with rational double point singularities.