# Stability conditions on threefolds

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## 1 Classical setting: slope stability for sheaves

Let $X$ be smooth projective of dimension $n$ with ample divisor $H$.

**Definition 1** *For $E$ coherent on $X$ we define*

$$\mu_n(E) = \begin{cases} \frac{H^{n-1}c_1(E)}{H^n r(E)} & \text{if } r(E) \neq 0 \\ \infty & \text{if } r(E) = 0 \end{cases}$$

*We say $E$ is $\mu$-*semistable *if $\mu(F) \leq \mu(E)$ for all $0 \neq F \subset E$.*

This forces purity if $E$ is not torsion; torsion sheaves are automatically semistable.

**Theorem 1 (Bogomolov)** *Assume the basefield is $\mathbb{C}$. If $E$ is $\mu$-semistable and $r(E) \neq 0$ then*

$$\Delta(E) = H^{n-2}(c_1(E)^2 - 2r(E)\mathrm{ch}_2(E)) \geq 0.$$

**Idea of proof:**

- Restriction theorems reduce to the case where $X$ is a surface.
- After pulling back to a cover and twisting by line bundles, we reduce to the case $c_1(E) = 0$ and $E$ is locally free. Here we just need to show $\mathrm{ch}_2(E) \leq 0$.
- **Euler characteristic:**

$$\chi(E) = r(E)\chi(\mathcal{O}_X) + \mathrm{ch}_2(E) \leq h^0(E) + h^2(E).$$

- Replace $E$ by $S^m E$ (symmetric power), or $E \otimes \cdots \otimes E$, or (in characteristic $p$) the Frobenius pull-back $F_p^* E$.
- At least the first two remain semistable. The terms $h^0(E), h^2(E)$ can be bounded or controlled whereas $\mathrm{ch}_2(E)$ grows quickly. Frobenius does not always respect semistability but we can work around this.

# 2 Tilt stability

This is a precursor to Bridgeland stability and agrees with it in the case of surfaces.

Let $D^b(X)$ denote the bounded derived category of coherent sheaves. Write

$$\begin{aligned}\mathrm{coh}^{H,0}(X) &= \{(C^{-1} \xrightarrow{f} C^0) \in D^b(X) : \\ &\quad \text{for all } \mathrm{coker}(f) \twoheadrightarrow Q, \mu_H(Q) > 0 \\ &\quad \text{for all } F \hookrightarrow \ker(F)\mu_H(F) \leq 0\}.\end{aligned}$$

This is 'the heart of a tilted $T$-structure'; it is an abelian subcategory of $D^b(X)$. Note that $H^{n-1}c_1 \geq 0$ on $\mathrm{coh}^{H,0}(X)$, so we can define a notion of stability.

**Definition 2** *For $C \in \mathrm{coh}^{H,0}(X)$ we define*

$$\nu_{H,0}(C) = \begin{cases} \infty & \text{if } H^{n-1}c_1 = 0 \\ \frac{H^{n-2}\mathrm{ch}_2 - H^n r}{H^{n-1}c_1} & \text{otherwise.} \end{cases}$$

*We say $C$ is $\nu$-semistable if for all $0 \neq D \subset C$ we have $\nu(D) \leq \nu(C)$.*

**Remark 1 (Bridgeland, Arcara-Bertram)** *The Bogomolov Theorem implies tilt-stability is well-defined: If $H^{n-1}c_1 = 0$ then $H^{n-2}\mathrm{ch}_2 - H^n r \geq 0$.*

Bad news: we lack tools for analyzing complexes that we have for bundles and sheaves. Good news: we can vary the stability conditions using the parameters in the construction.

Formally replace $H$ by $\alpha H$ where $\alpha \in \mathbb{R}$, $\alpha > 0$ and ch by $\mathrm{ch}^{\beta H} := e^{-\beta H}\mathrm{ch}$, $\beta \in \mathbb{R}$. Let $\nu_{\alpha,\beta}$ denote the corresponding slope and $\mathrm{coh}^{\alpha,\beta}(X)$ the corresponding subcategory of $D^b(X)$.

**Theorem 2 (Bayer-M-Toda)**

*a. If $C$ is $\nu_{\alpha,\beta}$ semistable then*

$$\overline{\Delta}_H(C) = (H^{n-1}c_1^\beta(C))^2 - 2H^n(r(C)H^{n-2}\mathrm{ch}_2^\beta(C)) \geq 0$$

*b. $\nu$-stability depends 'continuously' on $\alpha, \beta$*

'Continuity' means there is a well-defined wall and chamber structure for semistable object; stability is an open property.

**Idea of proof:** Fix $\beta$ to be rational. Let $\alpha \to \infty$. In this case stability reduces to slope stability. Here the Theorem is true because it holds for sheaves.

Assume for the moment the continuity: Reduce $\alpha$ and observe behavior as we cross each wall. Do induction on $H^{n-1}c_1^\beta$.

How do we *get* the continuity? Use induction to construct the wall and chamber structure.

Now introduce Bridgeland central charge

$$\overline{Z} = (\omega^{n-2}\mathrm{ch}_2^B - \omega^n r) + \sqrt{-1}\omega^{n-1}c_1^\beta$$

with $\omega = \alpha H$ and $B = \beta H$. Then results of Bridgeland imply we can deform in $B$.

To complete the proof of Theorem: prove (b), followed by (a) when $\beta \notin \mathbb{Q}$.

**Conjecture 1 (BMT, Bertram, Stellari)** *Let* $\dim(X) = 3$*,* $\omega = \alpha H$*, and* $B = \beta H$*. If $C$ is $\nu$-semistable then*

$$\overline{\Delta}_\omega(C) + \overline{\nabla}_{\omega,B}(C) \geq 0$$

*where*

$$\overline{\nabla}_{\omega,B}(C) = 12(\omega\mathrm{ch}_2^\beta)^2 - 18(\omega^2 c_1^\beta)\mathrm{ch}_3^\beta.$$

This implies Reider-type theorems; interesting also when applied to ideal sheaves of curves. Here we obtain weak versions of Castelnuovo's inequalities for genera of space curves.

**Remark 2 (Bogomolov; Kobayashi-Lübke; Simpson)** *If $C$ $\nu$-semistable and $\overline{\Delta}(C) = 0$, then the conjecture holds. To see this, reduce to the case where $C$ is a vector bundle, slope semistable with $\overline{\Delta}_H = 0$. Then the conjecture becomes an equality.*

**Examples:**

- $\mathbb{P}^3$ by [BMT,M], using quiver representations
- $Q \subset \mathbb{P}^4$ quadric [Schmidt]
- abelian threefolds [Maciocia-Piyaratne, BMS], as well as étale quotients like Calabi Yau's

# 3 Bridgeland stability

Repeat the construction for $\alpha, \beta, a, b \in \mathbb{R}$ with $\alpha > 0$ and $a > \frac{1}{6}\alpha^2 + \frac{1}{2}|b|\alpha$. Then $\mathcal{A}_{\alpha,\beta} \subset D^b(X)$ is the new heart.

$$Z^{a,b}_{\alpha,\beta} = (-\mathrm{ch}^\beta_3 + bH\mathrm{ch}^\beta_2 + aH^2c^\beta_1) + \sqrt{-1}(H\mathrm{ch}^\beta_2 - \frac{1}{2}\alpha^2 H^3 r).$$

Then

1. get a family of Bridgeland stability conditions
2. these conditions depend continuously on parameters
3. same inequality holds as in conjecture for semistable objects

Let $\mathrm{Stab}_H X$ be the space of Bridgeland stability conditions which contains

$$\tilde{\wp} = \mathrm{GL}(2, \mathbb{R})\{\sigma^{a,b}_{\alpha,\beta}\}$$

where each $\sigma$ is given by a pair $(Z, \mathcal{A})$.

**Theorem 3** *$\tilde{\wp}$ is a universal cover of*

$$\{Z \in \mathrm{Hom}(\mathbb{Q}^4, \mathbb{C}) : \ker(Z) \cap \mathcal{C} \subset \mathbb{P}^3_{\mathbb{R}} \textit{ is empty}\}$$

*where* $\mathcal{C} = [x^3, x^2y, \frac{1}{2}xy^2, \frac{1}{6}y^3]$.

*If $X$ is an abelian threefold then $\tilde{\wp}$ is a connected component of* $\mathrm{Stab}_H X$.