Derived Torelli and applications

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This is joint with Max Lieblich.

1 Statement of result

Let k be field and X a smooth projective scheme over k. Let $D(X)$ denote the bounded derived category of coherent sheaves. Let $K(X)_{\mathbb{Q}}$ denote the Grothendieck group of $D(X)$ tensored with $\mathbb{Q}$. The Riemann-Roch theorem gives an identification

$$K(X)_{\mathbb{Q}} \simeq A^*(X)_{\mathbb{Q}}$$

which gives a filtration by on $K(X)_{\mathbb{Q}}$ by codimension.

Thus each X yields a pair (D, F) where D is a triangulated category and F is a filtration on $K(D)_{\mathbb{Q}}$. Given two such pairs (D, F) and (D', F'), an equivalence is an equivalence of triangulated categories $\sigma : D \xrightarrow{\sim} D'$ such that $K(\sigma)\colon K(D)_{\mathbb{Q}} \xrightarrow{\sim} K(D')_{\mathbb{Q}}$ respects the filtrations.

Example. Let X be a K3 surface over $k = \bar{k}$. Then

$$F^2 = A^2(X)_{\mathbb{Q}}, \quad F^1 = \mathrm{NS}(X)_{\mathbb{Q}} \oplus A^2(X)_{\mathbb{Q}}, \quad F^0 = A^*(X)_{\mathbb{Q}}.$$

Theorem 1 *Let X be a K3 surface over $k = \bar{k}$ and Y/k a smooth projective scheme such that $(D(X), F_X) \simeq (D(Y), F_Y)$. Then $X \simeq Y$.*

This is a well-known statement in characteristic zero; but in positive characteristics we need a mechanism that plays the role of the Hodge structure.

[Question: Does this yield a new proof of the Torelli Theorem? Answer: No, we have to lift to characteristic zero and use Torelli there to obtain our result.]

2 Fourier-Mukai transforms

Let X, Y be smooth and projective over k and $P \in D(X \times Y)$. Consider the Fourier-Mukai transform

$$\begin{array}{rcl} \Phi^P : D(X) & \to & D(Y) \\ K & \mapsto & Rp_{2*}(Lp_1{}^*K \otimes^{\mathbb{L}} P) \end{array}$$

Theorem 2 (Orlov) *If $F : D(X) \xrightarrow{\sim} D(Y)$ is an equivalence of triangulated categories then $F = \Phi^P$ for some $P \in D(X \times Y)$.*

Write $A^*(X)_{num} = A^*(X)_{\mathbb{Q}}/\sim_{num}$,

$$\beta(P) = \mathrm{ch}(P)\sqrt{\mathrm{Td}_{X\times Y}} \in A^*(X \times Y)_{num},$$

and consider

$$\Phi^P_{A^*} : A^*(X)_{num} \to A^*(Y)_{num}$$

using the same formula as above. Instead of the identification coming from Riemann–Roch, we will use the isomorphism

$$\sqrt{\mathrm{td_X}} \cdot \mathrm{ch}(-) : K(X)_{\mathbb{Q}} \xrightarrow{\sim} A^*(X)_{\mathbb{Q}}.$$

For a surface X we have a pairing on $A^*(X)_{num}$

$$\langle (a, b, c), (a', b', c') \rangle = bb' - ac' - a'c$$

and Φ^P is compatible with this pairing.

Corollary 1 $F^1_{A^*(X)_{num}} = (F^2_{A^*})^{\perp}$ *and Φ^P preserves the codimension filtration on $A^*(X)_{num}$ if and only if $\Phi^P(F^2) = F^2$.*

Alternate formulation:

Theorem 3 *Let X and Y be K3 surfaces over $k = \bar{k}$. Let $P \in D(X \times Y)$ be an object such that*

$$\Phi^P : D(X) \xrightarrow{\sim} D(Y)$$

and

$$\Phi^P_{A^*} : A^*(X)_{num} \to A^*(Y)_{num}$$

preserves the codimension filtration. Then $X \simeq Y$.

3 Moduli spaces of sheaves

Let X/k be a K3 surface and $E \in D(X)$ and consider the Mukai vector

$$v(E) = (\mathrm{rk}(E), c_1(E), \mathrm{rk}(E) + \frac{c_1(E)^2}{2} + c_2(E)) \in A^*(X)_{num}.$$

Let h be a polarization on X and $\mathcal{M}_h(v)$ the stack of Gieseker semistable sheaves on X with Mukai vector v.

Theorem 4 (Mukai) *For suitable v, every semistable sheaf is stable, and $\mathcal{M}_h(v)$ is a $\mathbb{G}_m$-gerbe over $M_h(v)$ and $X \times M_h(v)$ carrises a universal family:*

$$\begin{array}{ccc} & X \times M_h(v) & \\ \swarrow & & \searrow \\ X & & M_h(v) \end{array}$$

Theorem 5 *Let X, Y be K3 surfaces over $k = \bar{k}$ with $D(X) \simeq D(Y)$. Then after composing $D(X) \xrightarrow{\sim} D(M_h(v))$ then you can arrange for*

$$D(Y) \simeq D(X) \simeq D(M_h(v))$$

to respect filtrations, whence $Y \simeq M_h(v)$.

4 Idea of proof

Given X, Y, Φ_P respecting filtrations:

1. We can an arrange that $\Phi^P(1,0,0) = (1,0,0)$ and that $\Phi^P(\text{ample cone}) = \pm(\text{ample cone})$.

2. Consider the deformation functor

$$\mathrm{Def}_X : \mathrm{Art}_W \to \mathrm{Set}$$

   where the former is the category of Artinian local W-algebras with residue field k. Here $W = W(k)$ is the Witt vectors.

Proposition 1 *There is an isomorphism of deformation functors*

$$\delta : \mathrm{Def}_X \to \mathrm{Def}_Y$$

such that for each $L \in \mathrm{Pic}(X)$

$$\delta(\mathrm{Def}_{(X,L)}) = \mathrm{Def}_{(Y,\Phi(L))}.$$

Idea: Let $\mathcal{D}(X)$ denote the stack of perfect complexes on X which are simple and universally gluable. Simple means $\mathrm{Aut}(E) = \mathbb{G}_m$ and universally gluable means $\mathrm{Ext}^i(E, E) = 0$ for $i < 0$. Without the latter condition, you will not get a stack structure. This was worked out by Lieblich previously.

We can think of $P \in D(X \times Y)$ as

$$\begin{array}{rcl} Y & \to & \mathcal{D}_X \\ y & \mapsto & P_y \end{array}$$

The fact that P is an FM equivalence means the image lands in a special open set.

Picture: $Y \to \mathcal{D}_X$ is an open immersion.

$$\begin{array}{rcl} Y & \hookleftarrow & Y \\ {\scriptstyle P_A}\downarrow & & \downarrow {\scriptstyle P} \\ \mathcal{D}_{X_A} & \hookleftarrow & \mathcal{D}_X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \hookleftarrow & \mathrm{Spec}(k) \end{array}$$

induces

$$\begin{array}{rcl} \delta : \mathrm{Def}_X(A) & \to & \mathrm{Def}_Y(A) \\ X_A & \mapsto & Y_A \end{array}$$

In fact, δ identifies Def_X and Def_Y but P deforms to $X_A \times_A \delta(X_A)$.

3 Choose L ample and lift X, Y, P to characteristic 0.

5 Realizations

Consider a Weil cohomology theory

$$\mathrm{H}^* : (\text{sm. proj. var.}/k)^{op} \to \text{gr. v. spaces}/K$$

with X even dimensional. Consider

$$\tilde{\mathrm{H}}(X) = \bigoplus_{i=-\delta}^{\delta} \mathrm{H}^{d+2i}(X)(i)$$

and assume it is pure. For example, when X is a surface take

$$\mathrm{H}^0(X)(-1) \oplus \mathrm{H}^2(X) \oplus \mathrm{H}^4(X)(1).$$

The transform $\Phi^P : D(X) \to D(Y)$ yields $\Phi^P_{\tilde{\mathrm{H}}} : \tilde{\mathrm{H}}(X) \xrightarrow{\sim} \tilde{\mathrm{H}}(Y)$ with the same formula.

Corollary 2 (Huybrechts) *Let X and Y be K3 surfaces over $\mathbb{F}_q$ with $D(X) \simeq D(Y)$. Then $\#X(\mathbb{F}_q) = \#Y(\mathbb{F}_q)$.*

Two more results

1. If X is a K3 surface over $k = \bar{k}$ then X has only finitely many FM-partners.

2. If X is supersingular and $D(X) \simeq D(Y)$ then $X \simeq Y$.

Katrina Honigs obtained analogous statements for abelian varieties over finite fields.