Sheaves of CATEGORIES

Plan | 1) Definition
2) Examples
3) Applications

1) X - CW complex with stratification S

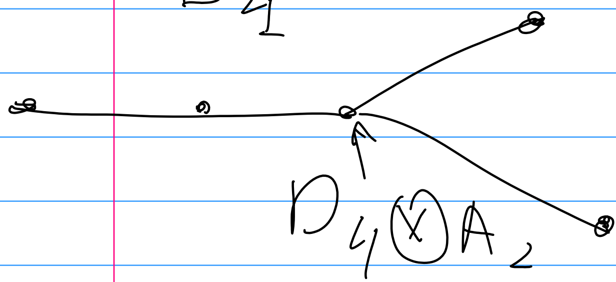
Definition (PSC) $\mathcal{F}$ is an Abelian sub CATEGORY of $D^b_{constr}(X, S)$.

Granov → Kontsevich → Construct stable objects (17KK)

Get the 17N filtration KNPS

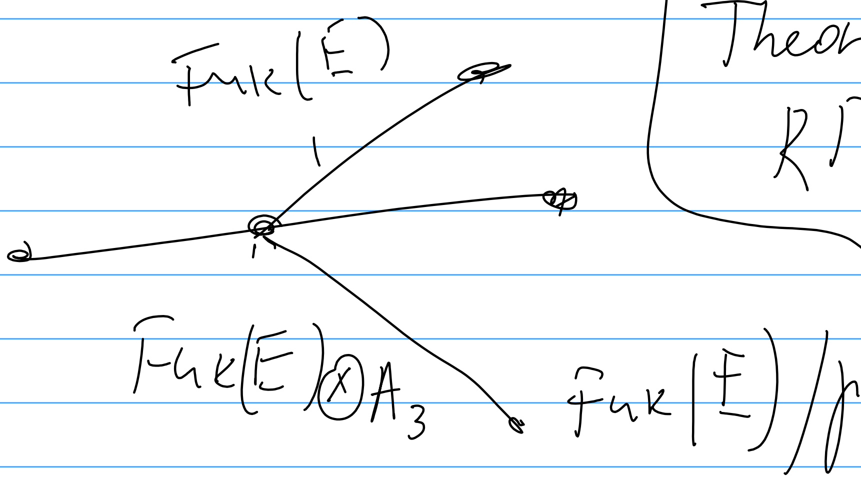
Examples

1) D_4



$$R\Gamma(\ ,\ \mathcal{F}) = A_2 \otimes A_2 \otimes A_2 = D^b(\mathbb{P}^1(a,b,c)) = D^b(E/\mathbb{Z}_3)$$

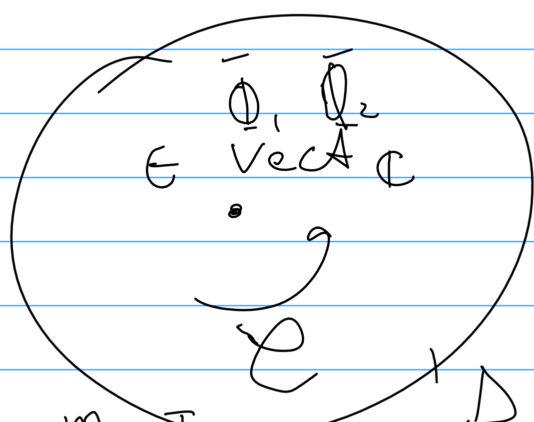
2)



Theorem (Kontsevich)

$$R\Gamma(\ ,\ \mathcal{F}) = D^b(\mathbb{P}^2)$$

3) Bogomolov (PSC)



$$Perv(\Delta, \mathcal{C}) = \{m, n, \Phi_1, \Phi_2\}$$

$\Delta \quad T = \mathcal{C} - m \cdot n$

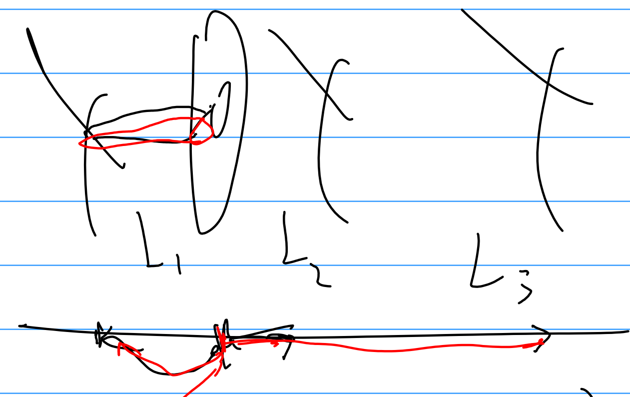
$$\Phi_1 \underset{n}{\overset{m}{\rightleftarrows}} \Phi_2$$

HMS

$$D^b(\mathbb{P}^2) \longleftrightarrow (\mathbb{C}^*)^2,\ W = x+y+\frac{1}{xy}$$

$\|$

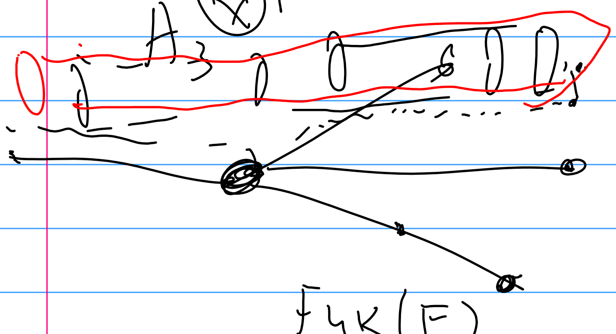
$\langle E_1, E_2, E_3 \rangle$



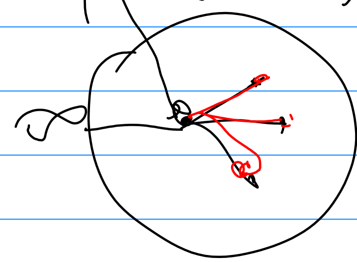
$\cong \langle L_1, L_2, L_3 \rangle$

$\|$ $\cong$

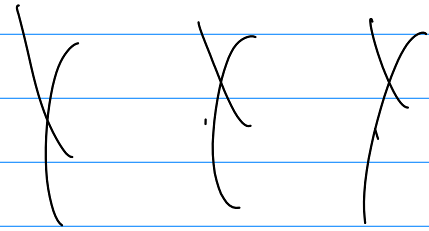
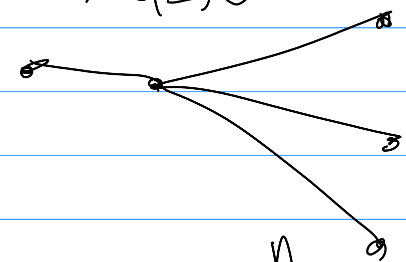
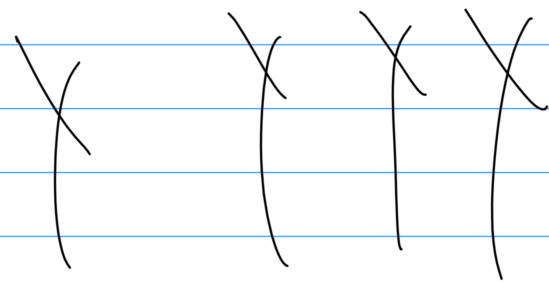
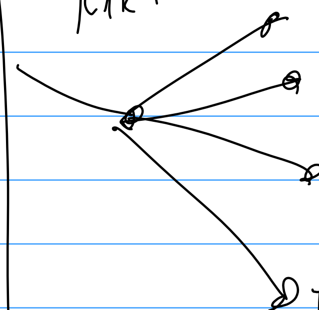
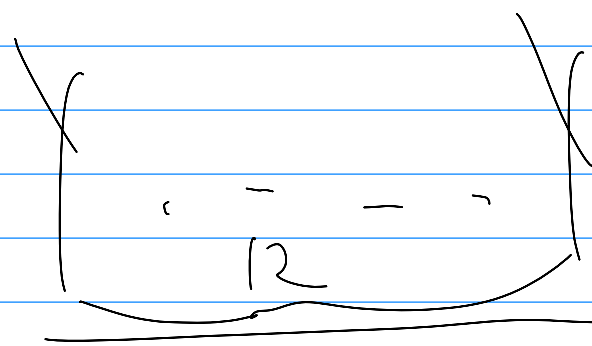
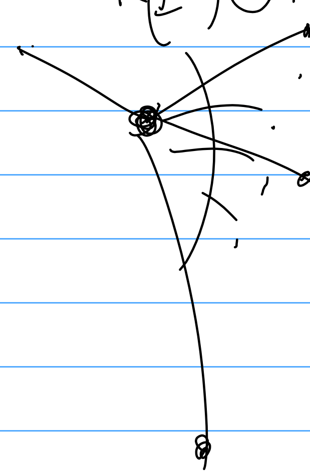
$A_3 \otimes \mathrm{Fuk}(E)$



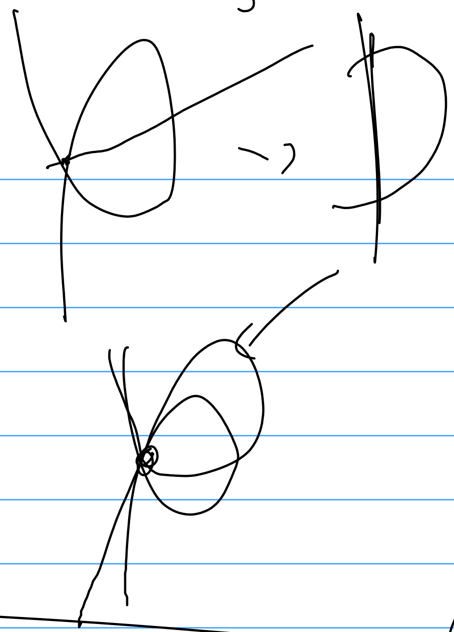
$\mathrm{Fuk}(E)/\emptyset$

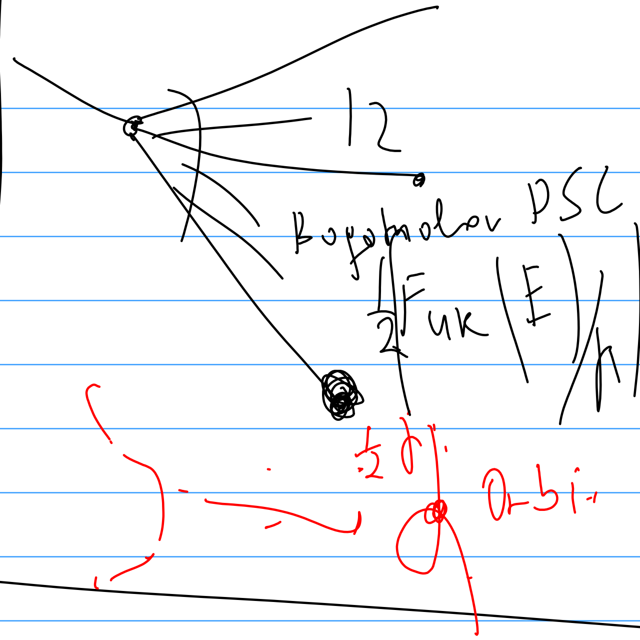


Geometry on PSC

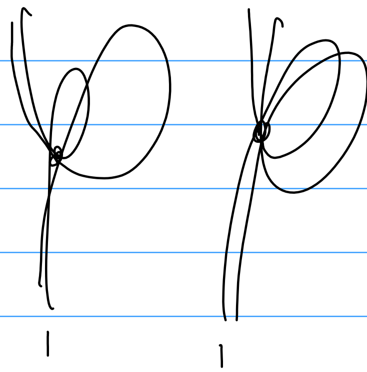
B side	A side	PSC
$\mathbb{P}^2$		$\mathrm{Fuk}(E)\otimes A_3$  $\mathrm{Fuk}(E)/p$
$\mathbb{P}^2_p$		$\mathrm{Fuk}(E)\otimes A_4$  $\mathrm{Fuk}(E)/p$
$\mathbb{P}^2_{p_1,\ p_j}$		$\mathrm{Fuk}(E)\otimes A_{12}$  $\mathrm{Fuk}(E)/p$

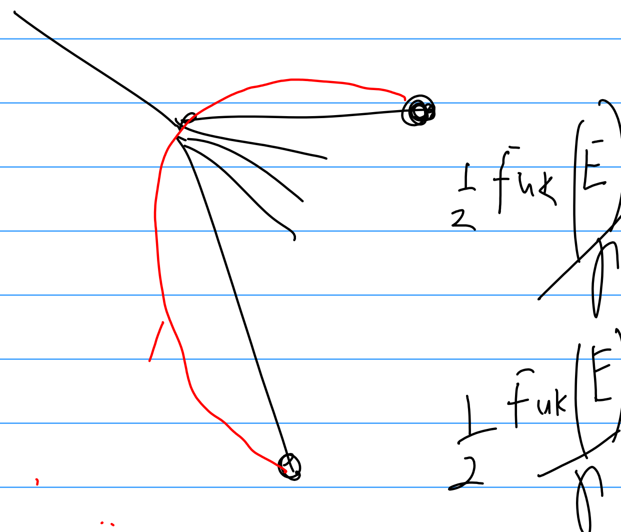
rational blow down





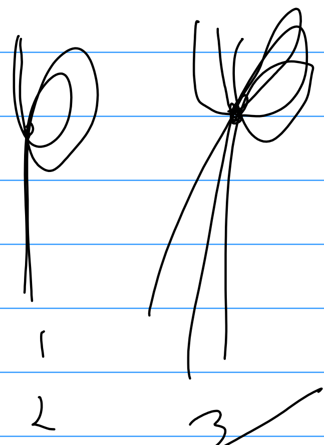
Enriques Surface



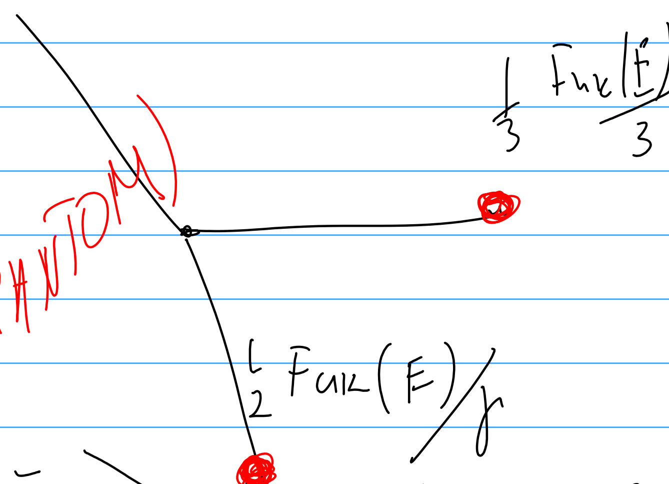


CATEGORICAL NORMAL function

Dolgachev surface



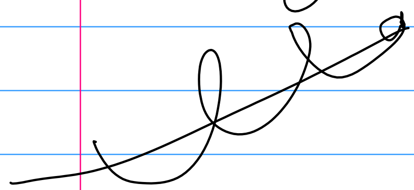
(PHANTOM)



$D^b(\mathrm{Dolg}) = \langle \alpha, E_1, \ldots, E_{12} \rangle$ $K^0(\alpha) = 0$

Nevalinna theory, CATEGORICAL NORMAL Functions and height.

Nevalinna theory



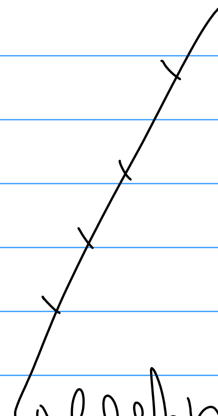
defect

phantom

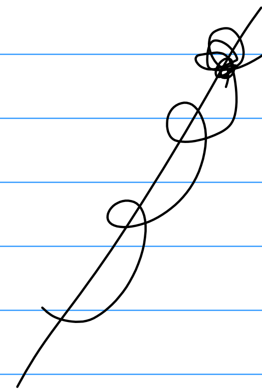
$\langle \mathcal{B}, E, E_e \rangle$

$\langle \alpha, E, \ldots, E_k \rangle$

$K^0(\alpha) = 0$



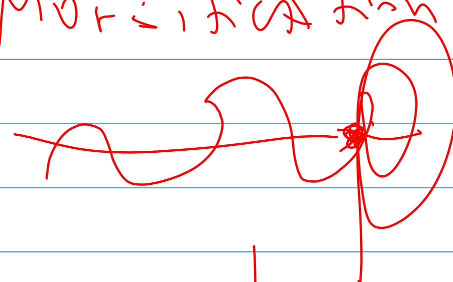
Algebraic LG models

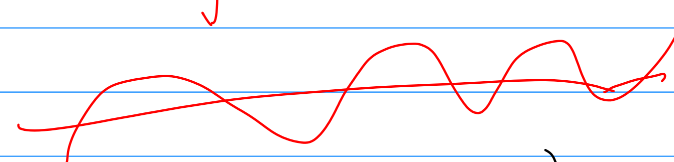


Symplectic LG models

$f : Y \to \mathbb{C}$ symplectic.

Morification



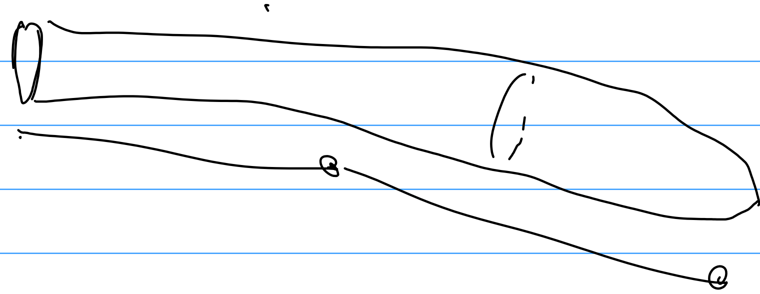


Theorem (Kontsevich)
Phantom cannot be morsified ALGEBRAICALLY

Normal Functions	CATEGORICAL NORMAL FUNCTIONS

$Y(X_t)$

admissible



height

$(D \supset B)$

Kuznetsov Normal Cohomology

$$NHH^*(B) \to HH^*(D) \to HH^*(B^\perp)$$

height (D/B)

exceptional coll.

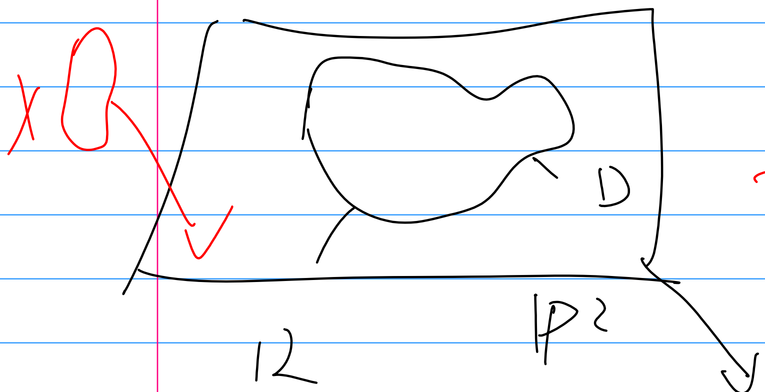
Theorem $\langle E_1, \dots, E_n, \alpha \rangle$

B

categorical normal function

$$NHH^*(B) \neq 0 \iff \alpha \text{ is a phantom}$$

Conic Bundles

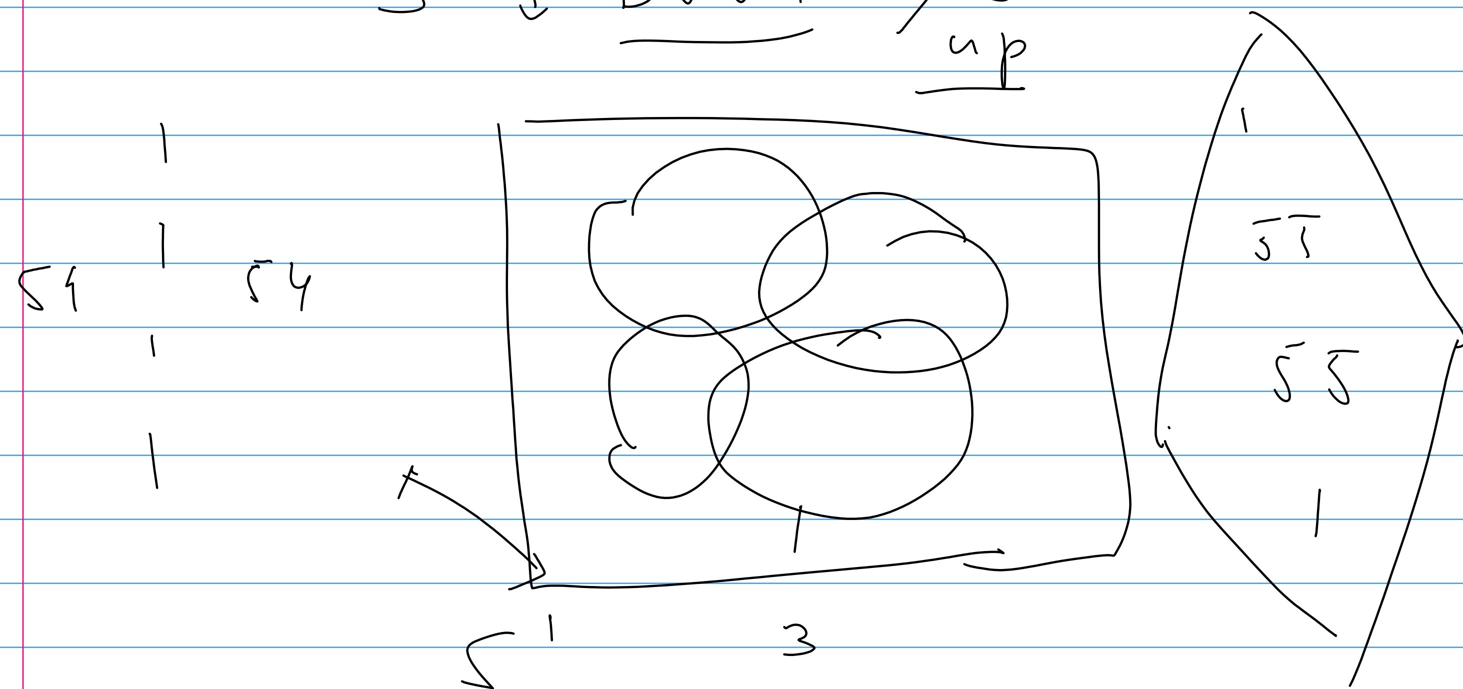


$$|K_S + 4D| \neq 0$$

Theorem (Sarkisov) X not rational

transitions

$S^2 \downarrow$ Down $\nearrow S^2$ up

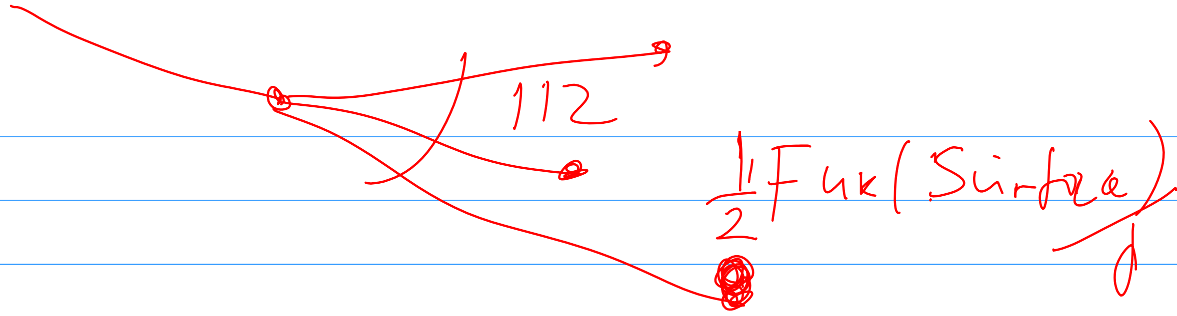


$$|K_{S'} + D'| \neq 0$$

Hassett Tschinkel

$D^b(X') =$

$= \langle E_1, \dots, E_{112}, \alpha \rangle$ ⟷ Nontrivial element in Br.



Theorem (Kuznetsov)
X conic bundle

$$D^b(X) = \langle \text{Hasset-Tschinkel root stack}, E, E_{1/2} \rangle$$

Conjecture 1 X a threefold.

$$D^b(X) = \langle \mathcal{A}, E_1, \ldots, E_n \rangle$$

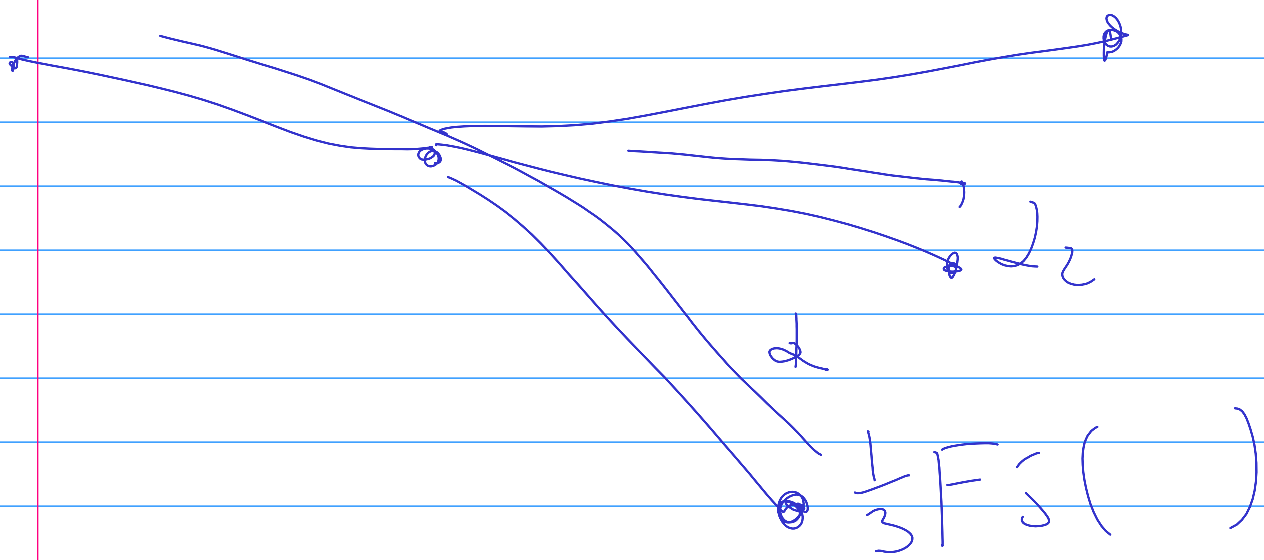
$$K_0(\mathcal{A}) = 0$$

$\Rightarrow$ X is stably nonrational.

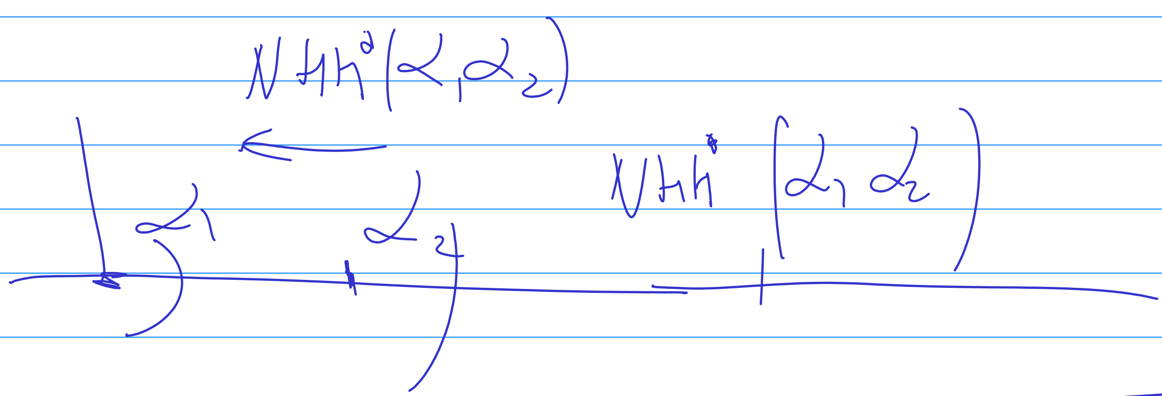
Conjecture 2 X a threefold

$$D^b(X) = \langle \mathcal{A}, \ldots \rangle$$

$\Rightarrow$ X is stably nonrational.



$$\alpha_1 \subset \alpha_2 \subset \alpha_3$$



Recall G generator

G - generates D

via sums

summands

shifts

triangles

$t(G) = \{\text{min number of tilings to generate } D\}$

$O_{spec} = \{ t(G_1), \quad , \quad , t(G_2) \}$

$O_{spec}(D^b(\mathbb{P}^1)) = \{1, 2\}$

$O_{spec}\, D^b(\mathbb{P}^2) = \{2, \ldots \}$

$\alpha \in D^b(\mathbb{P}^2)$ $\qquad$ $\underline{\mathrm{NHH}^{\cdot}(\alpha, D^b(\mathbb{P}^2)) = 2}$

$\mathrm{GAP}\, O_{spec}(\quad) =$

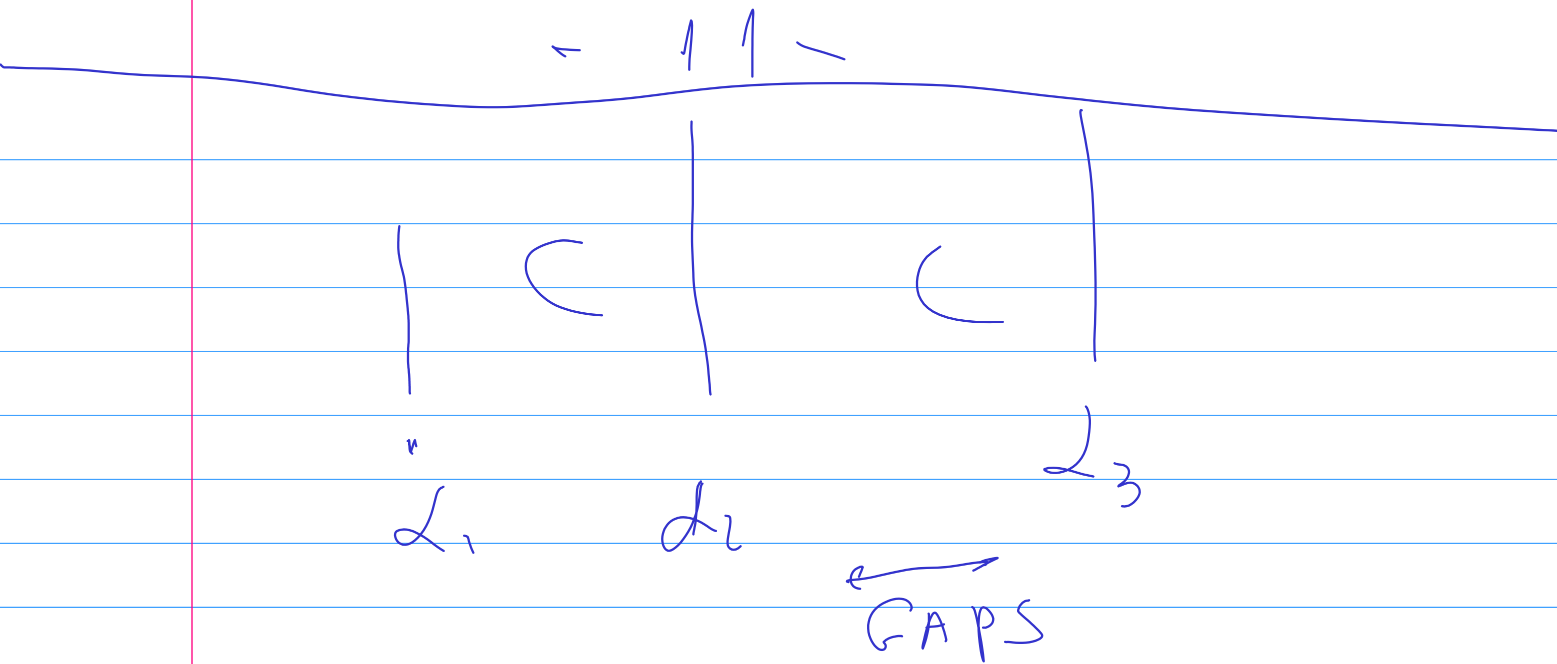
$\{ t(G_1), \ldots, t(G_n), \ldots, t(G_{n+1}) \}$

Conjecture 3 X rational

$\dim_{\mathbb{C}} X = n$

$\Rightarrow \mathrm{GAP}\, O_{spec}(D^b(X)) =$

$\underline{n-2}$



Exceptional heights $\longleftrightarrow$ nonexceptional GAPS

Defects!

Conjecture 4 (Gromov)

GAP Spec restricts

λ_1 λ_2 λ_3

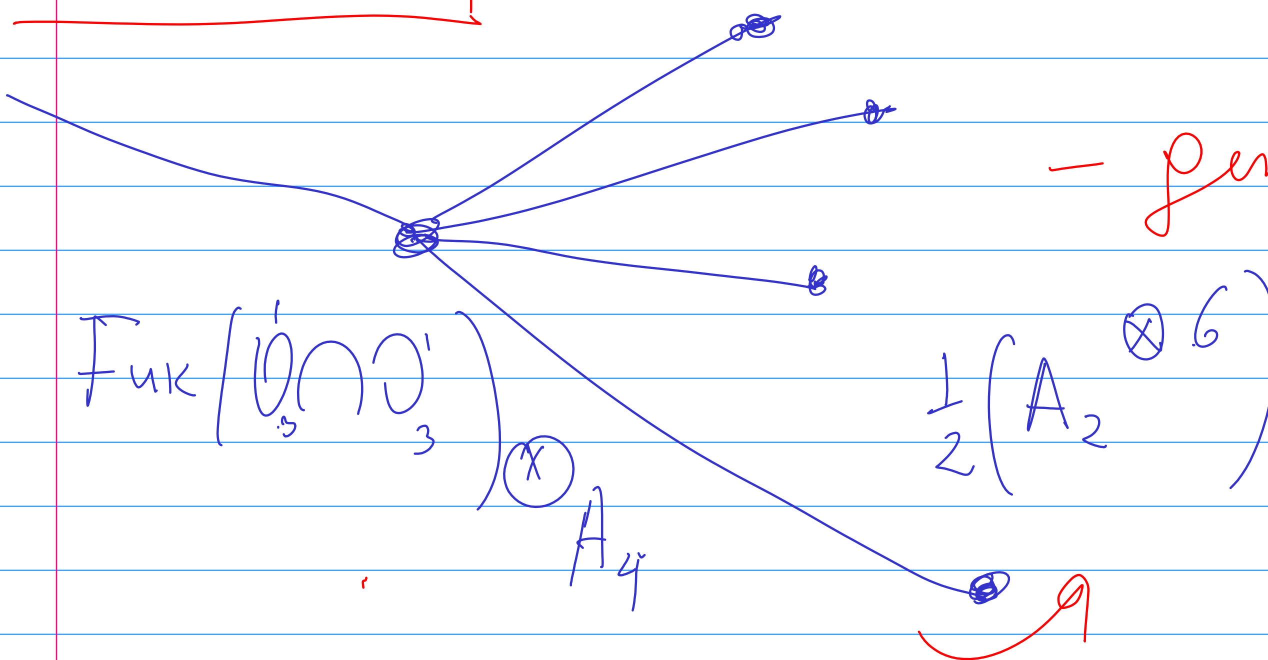
dynamical spectrum.

Conjecture |

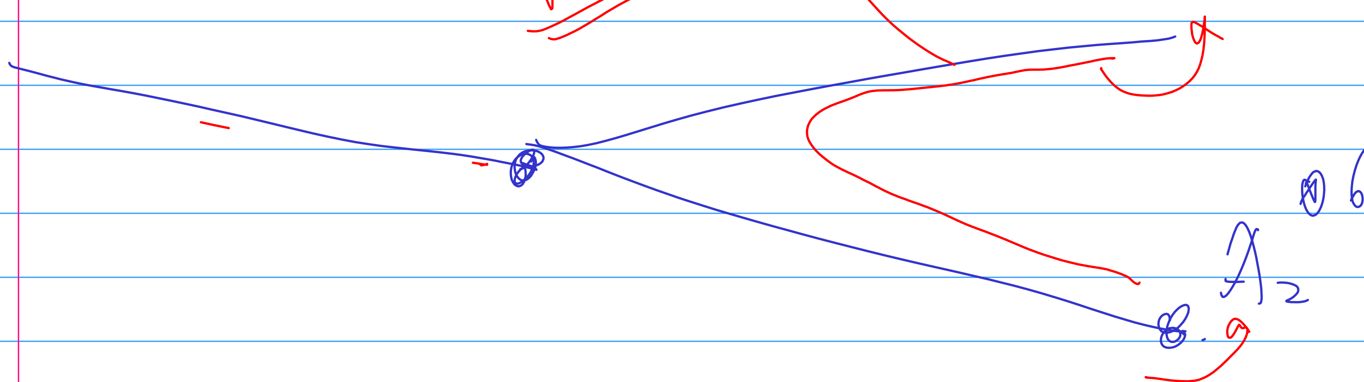
$$\mathrm{Min}\left(\mathrm{Heights}(M^F)\right) = \mathrm{GAP}\, \mathcal{O}_{\mathrm{spec}}(D^b)$$

$\alpha_1 C \Omega_K$

4d - Cubics



NEW CATEGORICAL NORMAL FUNCTION including $\mathbb{P}^2$



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$$\underset{\substack{\| \\ \text{GAP}}}{\text{NMH}}^{\alpha}(1) = 3 \longrightarrow \underset{\substack{\| \\ \text{GAP}}}{\text{NMH}}^{\alpha}(2) = 2$$