Integral Tate conjecture for cubic fourfolds

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Let $\mathbb{F}$ be a finite field of characteristic p and ℓ a prime distinct from p. Let $\overline{\mathbb{F}}$ be an algebraic closure with $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$.

For X smooth and projective over $\mathbb{F}$, let

$$c_\ell^i \colon \mathrm{CH}^i(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}^{2i}_{\text{ét}}(X, \mathbb{Z}_\ell(i))$$

be the cycle class map. Passing to algebraic closures we get maps:

$$\bar{c}_\ell^i \colon \mathrm{CH}^i(\overline{X}) \otimes \mathbb{Z}_\ell \to \bigcup_{H \subset G \text{ open}} \mathrm{H}^{2i}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(i))^H.$$

Remark 1 *If $X/\mathbb{C}$ smooth projective then we get maps*

$$c^i : \mathrm{CH}^i(X) \to \mathrm{Hdg}^{2i}(X, \mathbb{Z}).$$

Facts:

1. $i = 1$: The Kummer sequence gives

   $$\mathrm{coker}(c_\ell^1) \hookrightarrow T_\ell \mathrm{Br}(X),$$

   where the right-hand-side has no torsion. Then the Tate conjecture implies the integral version.

2. If $i = \dim(X)$ then the Lang-Weil estimates give a zero-cycle of degree 1.

3. If $i = \dim(X) - 1$ then Schoen shows $\bar{c}_\ell^{\dim(X)-1}$ is surjective *provided* the Tate conjecture holds for divisors on surfaces.

We turn to $i=2$: Here we have

$$\mathrm{coker}(c_\ell^2) = \mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/\text{max. div. subgroup}$$

here

$$\mathrm{H}^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = \varinjlim \mathrm{H}^3_{\mathrm{nr}}(X, \mu_{\ell^n}^{\otimes 2}).$$

which is conjecturally finite.

Theorem 1 (Parimala-Suresh) *Let S be a smooth projective surface over $\mathbb{F}$ and $X \to S$ a conic bundle. Then $\mathrm{H}^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ for all ℓ.*

Theorem 2 (Charles, P-) *Let $X \subset \mathbb{P}^5_{\mathbb{F}}$ be a smooth cubic with $p \geq 5$. Then $\bar{c}_\ell^2$ is surjective.*

It remains open whether $\mathrm{H}^3_{\mathrm{nr}}(X)$ is nonzero.

1. Voisin established the integral Hodge conjecture in this case, over $\mathbb{C}$, via normal functions of Zucker.

2. method for Theorem 2: algebraic normal functions

3. (F. Charles): the Tate conjecture holds for X.

Idea: Given $X/\mathbb{F}$ we consider the variety of lines F. Lift to characteristic zero, to get $\mathcal{X}$ and $\mathcal{F}$, where the latter is holomorphic symplectic of type $K3^{[2]}$. Under some conditions, F, being a reduction of $\mathcal{F}$, satisfies the Tate conjecture. Using Beauville-Donagi results on the incidence correspondence, we deduce the conjecture for X.

Proof: (1) Take $\overline{X} \subset \mathbb{P}^5_{\overline{\mathbb{F}}}$ and $\alpha \in \mathrm{H}^4_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(2))^H$. Take a Lefschetz pencil

$$\begin{array}{ccccc} \overline{X} & \leftarrow & \mathrm{Bl}_S\overline{X} & \supset & Y \\ & & \downarrow & & \downarrow \\ & & \mathbb{P}^1 & \supset & U \end{array}$$

where U is the smooth locus of the pencil and S is a cubic surface.

Lemma 1

i. If $\alpha|Y = 0$ then $\alpha = 0$.

ii. One can assume up to replacing α by $\alpha + b[L \times \mathbb{P}^1]$ that $\alpha|Y_t = 0$ for all $t \in U$.

Consider the incidence correspondence

$$\begin{array}{ccccc} & & \{(L,x), x \in L\} & & \\ & & || & & \\ & & V & & \\ & \stackrel{p}{\swarrow} & & \stackrel{q}{\searrow} & \\ F & & & & Y \\ & \stackrel{\psi}{\searrow} & & \stackrel{\pi}{\swarrow} & \\ & & U & & \end{array}$$

where F_t is the Fano variety of lines of Y_t. The Leray spectral sequence gives

$$\mathrm{H}^1(U, \mathbb{R}^3\pi_*\mathbb{Z}_\ell(2)) \to \mathrm{H}^4_{\text{ét}}(Y, \mathbb{Z}_\ell(2)) \to \mathrm{H}^0(U, \mathbb{R}^4\pi_*\mathbb{Z}_\ell(2)).$$

As $\alpha|Y_t = 0$ for all $t \in U$, the element α comes from $\beta \in \mathrm{H}^1(U, \mathbb{R}^3\pi_*\mathbb{Z}_\ell(2))$, where this first term maps to

$$\mathrm{H}^1(U, \mathbb{R}^1\psi_*\mathbb{Z}_\ell(1)).$$

Let γ be the image of β. Set $\mathcal{J} = \mathrm{Pic}^0 F/U$. Then the Kummer sequence gives

$$\mathcal{J}(U) \otimes \mathbb{Z}_\ell \xrightarrow{\eta} \mathrm{H}^1(U, \mathbb{R}^1\psi_*\mathbb{Z}_\ell(1)) \to T_\ell \mathrm{H}^1(U, \mathcal{J})$$

where the last term has no torsion. If $\alpha = \bar{c}_\ell^2 z$ for $z \in \mathrm{CH}^2(\overline{X}) \otimes \mathbb{Z}_\ell$ then $z' = p_* q^* z \in \mathcal{J}(U) \otimes \mathbb{Z}_\ell$.

Fact: If α is algebraic, $\eta(z')$ coincides with the image of α along the map $\mathrm{H}^1(U, \mathbb{R}^3\pi_*\mathbb{Z}_\ell) \to \mathrm{H}^1(U, \mathbb{R}^1\phi_*\mathbb{Z}_\ell(1))$.

Next, for α a cohomology class as before, the ordinary Tate conjecture implies there exists $N > 0$ such that $N\alpha$ is algebraic. Thus $N\gamma \in \mathrm{Im}(\eta)$ and $\gamma \in \mathrm{Im}(\eta)$.

Definition 1 *An element $z' \in \mathcal{J}(U) \otimes \mathbb{Z}_\ell$ such that $\eta(z') = \gamma$ is a normal function associated with α.*

(2) (Markushevich-Tikhomirov-Druel) Consider the moduli space $\mathcal{M} \to U$ of semistable torsion-free rank-two sheaves on Y with $c_2 = 0, c_1 = 2[L]$, defined in this generality by Langer. Thus we have

$$\begin{array}{c} \mathcal{M} \\ \downarrow \\ \mathcal{J} \\ \downarrow \\ U \end{array}$$

such that for any $t \in U$, $\mathcal{M}_t \to \mathcal{J}_t$ induces a birational map from at least one component of $\mathcal{M}_t$ to $\mathcal{J}_t$. By (1) we get $z' \in \mathcal{J}(U) \otimes \mathbb{Z}_\ell$. It is enough to assume $z' \in \mathcal{J}(U)$, i.e., a section of $\mathcal{J} \to U$.

Let $K = \overline{\mathbb{F}}(U)$ and regard $z'_K \in \mathcal{J}(K)$ which yields $y \in \mathcal{M}(K)$. Realize $\mathcal{M}_K = \mathrm{Quot}/\mathrm{GL}$ and let $C \subset \mathrm{Quot}$ denote the unique closed orbit above y, i.e., a projective homogeneous space for the action of GL. Using a result of Springer gives the rational point, i.e., $C(K) \neq \emptyset$, whence a rank two sheaf $\mathcal{F}$ on Y, because the cohomological dimension of $\overline{\mathbb{F}}(t)$ is 1 . The class

$$c_2(\mathcal{F}) - 2[L \times \mathbb{P}^1]$$

is the desired cycle of class $\alpha|Y$.

Counterexamples

1) Atiyah-Hirzebruch, Totaro, P-, Yagita: There exists $\alpha \in \bigcup_H \mathrm{H}^4(\bar{X}, \mathbb{Z}_\ell(2))^H$ such that α is not algebraic but $\ell\alpha = 0$. Even modulo torsion it is possible to get examples. The method involves analyzing cycles on classifying spaces via approximations by quotients of linear spaces by algebraic groups.

2) Over $\mathbb{C}$: Kollár: $X_d \subset \mathbb{P}^4_d$ hypersurface with d sufficiently divisible (by ℓ). Then any curve class is proportional to ℓ, thus the class of 1 is not represented by algebraic cycles. This should not be possible over $\overline{\mathbb{F}}$ by Schoen's Theorem, however.

3) The map

$$\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to \mathrm{CH}^2(\overline{X})^G \otimes \mathbb{Z}_\ell$$

need not be surjective.