Kodaira dimension of moduli of special cubic fourfolds

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This is joint work with Sho Tanimoto.

1 Main Theorem

Definition 1 *A (labelled) special cubic fourfold is a smooth cubic fourfold x, together with a rank two saturated lattice*

$$K = \langle h^2, T \rangle \subset H^{2,2}(X) \cap H^4(X, \mathbb{Z})$$

where h is the hyperplane class.

The discriminant *of X is the determinant of the lattice K.*

Let $\mathcal{C}_D$ denote the special cubic fourfolds with a labelling of discriminant D, and let $\mathcal{C}$ the coarse moduli space of all cubic fourfolds.

Theorem 1 (Hassett) *$\mathcal{C}_D$ is an irreducible algebraic divisor in $\mathcal{C}$, non-empty iff $D > 6$ and $D \equiv 0, 2 \pmod 6$.*

Examples:

1. If T is a plane then $X \in \mathcal{C}_8$ and K_8 is

	h^2	T
h^2	3	1
T	1	3

2. $D = 6n$ then

$$K_D \simeq \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 0 \\ T & 0 & 2n \end{array}$$

3. $D = 6n + 2$ then

$$K_D \simeq \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 1 \\ T & 1 & 2n+1 \end{array}$$

For cubic fourfolds $H^4(X, \mathbb{Z}) \simeq (+1)^{\oplus 21} \oplus (-1)^{\oplus 2}$ and

$$K_D^{\perp}(-1) \simeq \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & \epsilon \\ 0 & \epsilon & 2n \end{pmatrix} \oplus U \oplus E_8(-1)^{\oplus 2}$$

where

$$\epsilon = \begin{cases} 0 \text{ if } D \equiv 0 \pmod 6 \\ 1 \text{ if } D \equiv 2 \pmod 6 \end{cases}.$$

Theorem 2 (Tanimoto-VA '15) *1. If $D = 6n + 2$*

(a) *If $n > 18, n \notin \{20, 21, 23\}$ then $\mathcal{C}_D$ is general type.*

(b) *If $n = 14, 18, 20, 21, 25$ then $\kappa(\mathcal{C}_D) \geq 0$.*

(c) *If $n \leq 7$ then $\kappa(\mathcal{C}_D) < 0$ (Nuer, Hassett).*

2. If $D = 6n$

(a) *If $n > 18, n \notin \{20, 22, 23, 27, 32, 33\}$ then $\mathcal{C}_D$ is general type.*

(b) *If $n = 17, 23, 27, 33$ then $\kappa(\mathcal{C}_D) \geq 0$.*

(c) *If $n \leq 6$ then $\kappa(\mathcal{C}_D) < 0$ (Nuer, Hassett).*

2 Arithmetic Motivation

Let $X/\mathbb{Q}$ be a K3 surface. Skorobogatov/Zarhin '08 showed that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite where

$$\mathrm{Br}_0(X) := \mathrm{im}(\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(X)).$$

Note: $\mathrm{Br}(\bar{X}) = (\mathbb{Q}/\mathbb{Z})^{22-\rho(\bar{X})}$, so "most" elements of the transcendental Brauer group do not descend to the ground field.

Question 1 (Uniform boundedness for Brauer groups) *Let $X/\mathbb{Q}$ be a K3 surface with $\mathrm{NS}(\bar{X}) \simeq L$ where L is a fixed lattice. Is there a constant $c(L)$ such that $|\mathrm{Br}(X)/\mathrm{Br}_0(X)| < c(L)$?*

In joint work with McKinnie-Sawon-Tanimoto from 2014, we found the following picture: Let $\mathcal{K}_{2d}$ denote the coarse moduli space of K3 surfaces. Let $\mathcal{K}_{2d}(\langle\alpha\rangle)$ denote the moduli space of pairs of X and a subgroup

$$0 \neq \langle\alpha\rangle \subset \mathrm{Br}(X)[p], \quad p \nmid d$$

where p is prime. (This description works when $\mathrm{Pic}(X)$ is cyclic.) This has three irreducible components:

- $\mathcal{K}_{2dp^2}$
- $\mathcal{C}_{2p^2}$ if $d = 1, p \equiv 1 \pmod 3$
- mystery component

Sanity check: If the Brauer groups are to be controlled then the spaces should be of general type for $p \gg 0$.

Previous work in this direction is due to Kondo (for $\mathcal{K}_{2p^2}$ and $p \equiv 2 \pmod 3$) and Gritsenko-Hulek-Sankaran (for $\mathcal{K}_d$ for $d \gg 0$).

Question: (Charles) Fixing the ground field, are there just finitely many lattices that arise for the Picard group of a K3 surface?

3 Orthogonal modular varieties

Let L be an integral lattice of signature $(2, m)$ with $m \geq 9$, e.g., $L = K_D^{\perp}(-1)$. Consider the (local) period domain

$$\mathcal{D}_L = \{x \in \mathbb{P}(L \otimes \mathbb{C}) : (x, x) = 0, (x, \bar{x}) > 0\}^+,$$

which is a component of $\mathcal{D}_L \cup \mathcal{D}'_L$. Let $O^+(L) \subset O(L)$ denote the subgroup of the integral orthogonal group preserving $\mathcal{D}_L$. We have an exact sequence

$$1 \to \tilde{O}(L) \to O(L) \to O(D(L)) \to 1$$

with kernel the *stable orthogonal group*. Here $D(L)$ is the discriminant group. We set

$$\tilde{O}^+(L) = \tilde{O}(L) \cap O^+(L).$$

For $\Gamma \subset O^+(L)$ of finite index let

$$\mathcal{F}_L(\Gamma) = \Gamma \backslash \mathcal{D}_L,$$

be a *modular variety of orthogonal type*. It is quasi-projective (Baily-Borel) of dimension m.
Idea:

$$\mathcal{C}_D \hookrightarrow \Gamma \backslash \mathcal{D}_{K_D^\perp(-1)}$$

for a suitable Γ. Modular forms for Γ are roughly differential forms for $\mathcal{F}_L(\Gamma)$.

Theorem 3 (Gritsenko-Hulek-Sankaran '07) *'low weight cusp form trick' The variety $\mathcal{F}_L(\Gamma)$ is of general type if there exists a non-zero $F_a \in S_a(\Gamma, \chi)$ for $\chi : \Gamma \to \mathbb{C}^*$ either the trivial or the determinant character, of weight $a < m$ that vanishes along the ramification divisor of the projection*

$$\mathcal{D}_L \to \mathcal{F}_L(\Gamma).$$

Idea: $F_a^k M_{(m-a)k}(\Gamma) \subset \Gamma(\bar{Y}, \omega_{\bar{Y}}^{\otimes k})$ where $\bar{Y}$ is a desingularization of a compactification of $\mathcal{F}_L(\Gamma)$.

How do we produce an F_a?

Theorem 4 (Borcherds) *For $L_{2,26} := U^{\oplus 2} \oplus E_8(-1)^{\oplus 3}$ there exists $0 \neq \Phi_{12} \in M_{12}(O^+(L_{2,26}), \det)$.*

We leverage this form.

Theorem 5 (Borcherds-Katzarkov-Pantev-Shepherd Barron; GHS) *Let $L \hookrightarrow L_{2,26}$ be a primitive embedding giving $\mathcal{D}_L \hookrightarrow \mathcal{D}_{L_{2,26}}$. Let*

$$R_{-2}(L) = \{r \in L_{2,26} : r^2 = -2, (r, L) = 0\}$$

and $N(L) = \#R_{-2}(L)/2$. Then

$$\Phi|_L = \left. \frac{\Phi_{12}(Z)}{\prod_{r \in R_{-2}(L)/\pm 1} (Z, r)} \right|_{\mathcal{D}_L} \in M_{12+N(L)}(\tilde{O}^+(L), \det)$$

and if $N_L > 0$ then $\Phi|_L$ is a cusp form.

We need $L := K_D^{\perp}(-1) \hookrightarrow L_{2,26}$ such that $0 < N_L < 7$. This is OK if $D \equiv 2 \pmod 6$. If $D \equiv 0 \pmod 6$ then

$$\mathcal{C}_D \hookrightarrow \Gamma_D^+ \backslash \mathcal{D}_{K_D^{\perp}(-1)}$$

but $\tilde{O}^+(L) \subset \Gamma_D^+$ as an index two subgroup, and one has to show that the quasi-pullback $\Phi|_L$ is modular for the larger group Γ_D^+

To get embeddings $K_D^{\perp}(-1) \hookrightarrow L_{2,26}$, set

$$B = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & \epsilon \\ 0 & \epsilon & 2n \end{pmatrix},$$

and consider embeddings $B \hookrightarrow U \oplus E_8(-1)$. This requires delicate analysis of theta-series and mass formulas, excluding elements where N_L is too large. For $n \geq 150,000$ this all works. A computer search finds embeddings for $n \leq 4000$. The intermediate cases required some clever analysis.