THE HASSE PRINCIPLE FOR 9–NODAL CUBIC 3–FOLDS

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1 Introduction

It has been known since the 19th century that if X is a cubic threefold over a field K of characteristic zero and $X \otimes \overline{K}$ has just d isolated singularities, then $d \leq 10$. Coray [C] has shown that if d is prime to 3, then $X(K)$ is non–empty, while Colliot–Thélène and Salberger [CT–Sal] have shown that if $d = 3$ and K is a number field then X satisfies the Hasse principle. More recently Coray *et al.* [C–L–SB–SD] have proved the Hasse principle if $d = 6$. This paper fills the gap, as far as singular cubic threefolds are concerned, by showing that if $d = 9$, then (Theorem 4.5 below) the only obstruction to the Hasse principle is the Brauer–Manin obstruction described in [CT–San2]. We do this by descent. More precisely, we show first that, provided that X has points everywhere locally, universal torsors over the smooth locus X^0 of X exist, that they are K–birational to cones over certain singular cubic 7–folds and that all of them satisfy the Hasse principle.

We also prove (Theorem 7.5) that on 10–nodal cubic threefolds the only obstruction to weak approximation is the Brauer–Manin one, and prove a partial such result (Proposition 5.2) in the 9–nodal case.

As mentioned, the proofs depend on the consideration of various torsors, under tori. What often makes various such torsors computable is that the base variety is given by particularly simple (for example, linear) equations inside some torus embedding, or rather an equivariant compactification of some torsor under a torus. 9–nodal cubic threefolds follow this pattern: they turn out to be hyperplane sections of some Galois twist of the *Perazzo cubic 4–fold P*, defined in $\mathbb{P}^5$ by $x_1 x_2 x_3 = y_1 y_2 y_3$. However, 10–nodal cubic 3–folds X are different. They are not, apparently, usefully embeddable in a toric variety, and yet universal torsors over X^0 are simple; they are birational to cones over the Grassmannian $G(2, 6)$.

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2 Universal torsors over the smooth locus of a Perazzo cubic

The object that renders 9–nodal cubic threefolds tractable is the Perazzo cubic fourfold P [S–R]. This is given by the equation $x_1 x_2 x_3 = y_1 y_2 y_3$. It contains nine 3–planes L_{ij}, given by $x_i = y_j = 0$. The complement $P^0 = P - \cup L_{ij}$ is a torsor under a trivial 4-dimensional torus T, so P is, by abuse of language, a torus embedding. This makes it easy to compute universal torsors over it and those of its twists that are also torus embeddings (i.e., those that have K–points); we shall do this explicitly in the next section.

Note that the nine 3–planes L_{ij} are conjugate under the wreath product $\Gamma = S_3 \wr S_2$; this is the subgroup of the symmetric group S_6 generated by $S_3 \times S_3$ and the involution ι of $S_3 \times S_3$ that switches the two factors. The first factor S_3 permutes the x_i, the second permutes the y_i and ι switches x_i with y_i. The singular locus $\mathrm{Sing}(P)$ consists of nine lines l_{pq}, where l_{pq} is given by $x_i = y_j = 0$ for $i \neq p$, $j \neq q$. The six points $w_1 = (1, 0, \ldots, 0), \ldots, w_6 = (0, \ldots, 0, 1)$ are Γ-conjugate; note that under the subgroup $S_3 \times S_3$ of Γ of index two they fall into two orbits. Because the complement T of the 3–planes is a 4–dimensional torus and the embedding $T \hookrightarrow P$ is T–invariant, the class group $\mathrm{Pic}\, P^0 = \mathrm{Cl}(P \otimes \overline{K})$ is torsion–free of rank $9 - \dim S = 5$; it is generated by the classes L_{ij}, subject to the relations of the form

$$0 = (x_1/y_2) = L_{11} + L_{13} - L_{22} - L_{32}.$$

Lemma 2.1 *The automorphism group scheme $G = \mathrm{Aut}(P, \mathcal{O}(1))$ over $\mathbb{Q}$ is a split extension of Γ by T.*

PROOF: The six points w_i are distinguished as those points where two or more of the l_{pq} meet. So the connected component $\mathrm{Aut}^0(P, \mathcal{O}(1))$ preserves each of them, and so preserves the 3–planes L_{ij}. Now the result is obvious, with T being the complement in P of $\cup L_{ij}$.   □

More generally, we define a *Perazzo cubic fourfold* to be a cubic 4–fold Y that is a Gal_K–twist of P. Its smooth locus will be denoted by Y^0.

Proposition 2.2 *Every Perazzo cubic Y satisfies the Hasse principle.*

PROOF: Quadratic base extensions are harmless, so that we can assume that the 6 distinguished points fall into two Galois orbits of three points each. Since it is known [CT–Sal] that cubics with 3 conjugate nodes satisfy the Hasse principle, we are done.   □

Definition 2.3 *A double–three is a configuration of six 2–planes L_i, M_j in $\mathbb{P}^8$, where $i, j = 1, 2, 3$, such that $\cup L_i$ and $\cup M_j$ each span $\mathbb{P}^8$ and each intersection $L_i \cap (\cup M_j)$ and $M_j \cap (\cup L_i)$ consists of three non–collinear points.*

Note that any double–three has a unique decomposition into the union of two threes, where the 2–planes in each three are mutually disjoint.

Proposition 2.4 *Suppose that Y is a Perazzo cubic with a K–point.*

(1) There exist universal torsors over Y^0.

(2) Every such universal torsor is K–birational to $\mathbb{A}^9_K$.

(3) The corresponding rational map $\mathbb{A}^9_K - \to Y$ factors through the standard projection $\mathbb{A}^9 - \to \mathbb{P}^8_K$.

(4) The rational map $\mathbb{P}^8_K - \to Y \hookrightarrow \mathbb{P}^5$ is given by a linear system of cubics passing doubly through a double–three.

PROOF: We know that Y^0 is a torus embedding $S_1 \hookrightarrow Y^0$, where S_1 is the complement of nine 3–planes in Y and a torsor under the torus $S = \mathrm{Aut}^0(Y)$, and that $\overline{K}[Y^0]^* = \overline{K}^*$. Hence universal torsors over Y^0 exist and can be constructed according to the procedure described in [CT–San1]. That is, there is an exact sequence

$$1 \to S_0 \to M \to S \to 1$$

of tori, where $\hat{S}_0 \cong \mathrm{Pic}(Y^0)$ as Gal_K–modules and $\hat{M}$ is the free module spanned by the classes of the 3–planes. The coboundary map $S(K) \to H^1(K, S_0)$ is surjective and the universal torsors over Y^0 form a torsor under this H^1, so any universal torsor $\mathcal{T} \to Y^0$ has the property that $\mathcal{T}|_S \to S$ is the pull–back of $\alpha : M \to S$ via the translation ϕ_x by $-x$, for some $x \in S(K)$.

It therefore remains to describe the map $M \to S$. We first do this in the untwisted case, where $Y = P$ and $S = T$.

The group $\mathrm{Pic}(P^0)$ is generated by the planes L_{ij} subject to the relations $\sum_q L_{iq} - \sum_p L_{pj} = (x_i/y_j) \sim 0$. Introduce nine new variables z_{ij}; then there is a morphism $\pi : \mathbb{A}^9_K = \mathrm{Spec}\, K[\{z_{ij}\}] \to P$ defined by $x_i = \prod_q z_{iq}$ and $y_j = \prod_p z_{pj}$. Identify M with the open subset of $\mathbb{A}^9$ given by $\prod z_{ij} \neq 0$; then π restricts to α.

Consider the variables z_{ij} as the entries of a 3×3 matrix. Then there are six triples of variables z_{ab}, z_{cd}, z_{ef} such that no two of any triple lie in the same row or the same column. For each such triple, let $L_{ab,cd,ef}$ be the 2–plane defined by the vanishing of the other variables. Then every cubic in the linear system defining $\mathbb{P}^8 - \to Y$ is double along each $L_{ab,cd,ef}$; it is clear that these 2–planes form a double–three, with $L_{11,22,33}$, $L_{23,12,31}$ and $L_{21,32,13}$ forming one three.

For the general case, note that S is the twist of T by some cocycle (in fact, homomorphism) $\mathrm{Gal}_K \to \Gamma$. The action of Γ on T lifts to a π–equivariant linear action on $\mathbb{A}^9$; taking the twists gives the results. $\qquad\square$

3 Geometry

In this section we investigate the basic geometry of 9–nodal cubic threefolds. Some of this material can also be found in [C–T–Z].

We let K denote a perfect field of characteristic zero, $\overline{K}$ an algebraic closure of K and X a cubic threefold over K whose singular locus $\mathrm{Sing}\,X$ consists of exactly nine $\overline{K}$-points. We also assume that every Galois–conjugate subset of $\mathrm{Sing}\,X$ has at least 2 members, for else X is K-rational and there is nothing more to be said. The smooth locus of X will be denoted by X^0.

Lemma 3.1 *(1) Every singularity of X is an ordinary node.*
(2) X contains just nine 2–planes.
(3) The 2–planes in X and the points of $\mathrm{Sing}\,X$ form a $(9,4)$-configuration G which is the 1–skeleton of the cell decomposition of the 2–torus formed as the product of two triangles.
(4) The symmetry group of G is Γ.

PROOF: For (1), we can assume that $K = \mathbb{C}$.

Since cubic surfaces have at most 4 isolated singularities, X is not a cone and so, by I, Theorem 1.7 of [Z], the dual variety $X^\vee$ of X is a hypersurface. Moreover,

$$\deg X^\vee = 3.2^3 - \sum_{v \in \mathrm{Sing}(X)} m(v),$$

where the class $m(v)$ of the singularity (X, v) has the property that $m(v) \geq 2$ and $m(v) = 2$ if and only if v is an ordinary node. Precisely, $m(v) = \mu(v) + \mu'(v)$, where μ is the Milnor number of an isolated hypersurface singularity (X, v) and μ' is the Milnor number of a general hyperplane section [T]. If (X, v) is defined locally analytically (or formally) in $\mathbb{C}^n$ by $f(x_1, \ldots, x_n) = 0$, then

$$\mu(v) = \dim_\mathbb{C} \mathcal{O}_{\mathbb{C}^n}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

If $\mu(v') \geq 2$ for some v then $m(v) \geq 4$ for at least two points v, so that

$$\deg X^\vee \leq 24 - 7 \times 2 - 2 \times 4 = 2.$$

But $\deg X^\vee \geq 3$, by the biduality theorem, so that $\mu(v') = 1$ for all v; this means that every v' is simple, of type A_1, and v is simple of type A.

Say that X has n_r points of type A_r, so that $\sum n_r = 9$. Note that $m(A_r) = r + 1$, so that

$$3 \leq 24 - \sum_{r \geq 1} n_r(r + 1) = 6 - \sum_{r \geq 2}(r - 1)n_r.$$

This gives $n_r = 0$ for all $r \geq 5$ and $n_2 + 2n_3 + 3n_4 \leq 3$, so that $n_1 \geq 6$.

Choose an A_1-point P. Projecting from P identifies $\mathrm{Bl}_P X$ with $\mathrm{Bl}_C \mathbb{P}^3$, where C is a reduced curve of bidegree $(3, 3)$ on a smooth quadric with $n_1 - 1$ points of type A_1 and n_r of type n_r for all $r \geq 2$. Suppose that $\widetilde{C} = \sqcup \widetilde{C}_j \to C$ is the normalization; then

$$\sum \chi(\mathcal{O}_{\widetilde{C}_j}) = \chi(\mathcal{O}_C) + (n_1 - 1).1 + n_2.1 + n_3.2 + n_4.2 = 5 + n_3 + n_4.$$

So, if $n_3+n_4 > 0$, then C is the transverse union of six lines, and then $n_3 = n_4 = 0$, contradiction. Therefore $n_3 = n_4 = 0$, so that $C = \sum_1^5 C_j$ where C_5 is a conic and the other components are lines. Then also $n_2 = 0$ and (1) is proved.

Remark: If there is a tenth singular point, then it is known classically that X is a form of the Segre cubic.

For (2), note that the description of the curve C in the proof of (1) shows that there are exactly four 2–planes on X through each node. Each plane in X contains 4 nodes of X, so that there are just 9 planes in X.

(3): From the description of C it also follows that through each node of X there are 4 lines in X that pass through a further node and every such line is the intersection of a unique pair of 2–planes in X. Each of the 4 nodes in a 2–plane L lies on two lines of the form $L \cap M$, since each line in C meets two others. This proves (3).

(4): it is clear that the symmetry group of G is isomorphic to $S_3 \sim S_2$. $\square$

Remark: We can also view G as a graph whose vertices are the planes in X and whose edges are the pairs of planes that meet in a point.

Proposition 3.2 *Any 9–nodal cubic 3–fold X over K is K–isomorphic to a hyperplane section of some Perazzo 4–fold.*

PROOF: Assume first $K = \overline{K}$. By 3.1 there are nine 2–planes in X; denote them by L_{ij}, where $i,j \in \mathbb{Z}/3$ and L_{ij} meets $L_{i\pm1,j}$ and $L_{i,j\pm1}$, each in a line. So for all i and j, $\sum_i L_{ij}$ and $\sum_j L_{ij}$ are hyperplane sections. Say $\sum_j L_{ij} = (x_i = 0)$ and $\sum_i L_{ij} = (y_j = 0)$. Then $\sum_{ij} L_{ij}$ is the complete intersection $\prod x_i = \prod y_j = 0$, so that X is a member of the pencil $|3H - \sum_{ij} L_{ij}|$. Hence the equation of X is $\alpha \prod x_i = \beta \prod y_j$, as required.

Next, we check (still assuming $K = \overline{K}$) that if X, X' are isomorphic sections of P, then there is an automorphism of P taking X to X'.

There exists $\sigma \in PGL_6$ such that $\sigma(X) = X'$. Put $P' = \sigma(P)$; then there is a hyperplane H such that $P.H = P'.H$, and it is enough to find $\tau \in PGL_6$ such that $\tau(P) = P'$ and $\tau|_H = 1$. Put $G = \{\tau \in PGL_6 | \tau|_H = 1\}$; then G acts transitively on $\mathbb{P}^5 - H$.

Both P and P' contain nine 3–planes M_{ij}, M'_{ij}, respectively; we can take them to be ordered so that $M_{ij}.H = M'_{ij}.H$ for all i,j. Suppose that P, resp. P', is given by $F = 0$, resp. $F' = 0$, where $F = x_1x_2x_3 - y_1y_2y_3$; then we can take M_{ij} to be given by $x_i = y_j = 0$. Put $v = (1,0,0,0,0,0)$, so that $v = \cap_{i\neq1}L_{ij}$. So, after applying a suitable $\tau \in G$, we have $\cap_{i\neq1}L_{ij} = \cap_{i\neq1}L'_{ij}$. Since $L_{ij}.H = L'_{ij}.H$, we now have $L_{ij} = L'_{ij}$ for all $i \neq 1$. So $F' \in \cap_{i\neq1}(x_i, y_j) = (x_2x_3, y_1y_2y_3)$; the equality of these two ideals is verified by noting that both define subschemes of codimension 2 and degree 6 and that obviously $(x_2x_3, y_1y_2y_3) \subseteq \cap_{i\neq1}(x_i, y_j)$. Hence $F' = lx_2x_3 + \alpha y_1y_2y_3$, where $\alpha \in K$ and l is linear. Say that H is defined by $m = 0$; it is then easy to see, since X is irreducible, that, after rescaling,

$m = l - x_1$ and $F' = (x_1 + m)x_2x_3 - y_1y_2y_3$. Then $\tau \in GL_6$ given by

$$x_1 \mapsto x_1 + m, x_i \mapsto x_i \quad \text{for} \quad i \neq 1, y_j \mapsto y_j$$

has the required effect.

Now drop the assumption that $K = \overline{K}$. There is an embedding $\phi : X_{\overline{K}} \hookrightarrow P_{\overline{K}}$ with the property that for all $\sigma \in \mathrm{Gal}_K$, there exists $\psi_\sigma \in \mathrm{Aut}\, P(\overline{K})$ such that $\phi^\sigma = \psi_\sigma \circ \phi$. Then $\psi_{\sigma\tau}\phi = (\psi_\sigma)^\tau \psi_\tau \phi$ for $\sigma, \tau \in \mathrm{Gal}_K$.

Write $\omega = \psi_{\sigma\tau}^{-1}(\psi_\sigma)^\tau \psi_\tau$. So $\omega\phi = \phi$. That is, ω is an automorphism of $P_{\overline{K}}$ that acts trivially on the hyperplane H cutting out $X_{\overline{K}}$.

We want to show that any such ω is the identity. For this, we can assume once more that $K = \overline{K}$ and that $\omega \neq 1$. After moving H by an element of S we can suppose also that $(1, \ldots, 1) \in H$. Then $\omega \in \Gamma_0$, where $\Gamma_0 \subseteq \mathrm{Aut}\, P$ is a copy of Γ that splits the surjection $\mathrm{Aut}\, P \to \Gamma$. Since ω is trivial on H, it is conjugate to $(s, 1) \in S_3 \times S_3 \subseteq \Gamma_0$, where s is a transposition; we can take $s = (12)$. Then H is given by $x_1 = x_2$, which contradicts the fact that $P.H$ has isolated singularities. So $\omega = 1$ and $(\psi_\sigma) \in Z^1(\mathrm{Gal}_K, \mathrm{Aut}\, P(\overline{K}))$. It is now easy to see that X embeds into the twist of P by (ψ_σ). □

Now suppose that X is a hyperplane section of Y and that $\mathrm{Sing}\, X = X \cap \mathrm{Sing}\, Y$. So X is a 9–nodal cubic threefold. Let $\widetilde{X} \to X$ the blow–up of the nodes. It is then easy to see that the Betti numbers of $\widetilde{X}$ are determined by $e(\widetilde{X}) = -6 + 9.4 = 30$ and $b_3 = 0$, so that $\mathrm{rank}\, \mathrm{Cl}(X \otimes \overline{K}) = 5$.

Lemma 3.3 *(1) The natural map* $\mathrm{Cl}(Y_{\overline{K}}) \to \mathrm{Cl}(X_{\overline{K}})$ *is a* Gal_K–*isomorphism.*

(2) $\mathrm{Cl}(X_{\overline{K}})$ *is generated by the classes* L_{ij} *subject to the relations* $R_i - C_j = 0$, *where* $R_i = \sum_i L_{ij}$ *and* $C_j = \sum_j L_{ij}$.

PROOF: We check first that the 2–planes on X generate $\mathrm{Cl}(X)$.

Fix a node P on X; then, as before, $\mathrm{Bl}_P X$ is identified with $\mathrm{Bl}_C \mathbb{P}^3$ and we see that $\mathrm{Cl}(\mathrm{Bl}_P X)$ is generated by $L_1, \ldots, L_4, Q, H$, where L_i is the strict transform of a plane that projects to a line C_i in C, Q the strict transform of a quadric cone projecting to the conic C_5 in C and H is the pull–back of the hyperplane class on $\mathbb{P}^3$.

Let H_1 be the hyperplane class on X. Then $H_1 - E \sim H$ in $\mathrm{Cl}(\mathrm{Bl}_P X)$ and there are 2–planes L', L'' on X such that $H_1 \sim L_1 + L_3 + L'$ and $H_1 \sim Q + L''$ in $\mathrm{Cl}(X)$. Since $\mathrm{Cl}(X) \cong \mathrm{Cl}(\mathrm{Bl}_P X)/\mathbb{Z}.[E]$, we get relations $H \sim H_1 \sim L_1 + L_3 + L'$ and $Q \sim H_1 - L''$ in $\mathrm{Cl}(X)$, and the planes on X do generate $\mathrm{Cl}(X)$. The relations $R_i = C_j$ follow from the observation that $R_i \sim H_1 \sim C_j$.

In Y, the 3–planes form the boundary $Y - U$, as described above, and so generate $\mathrm{Cl}(Y)$. Hence $\mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is surjective. Since both groups have rank 5, it is enough to prove that $\mathrm{Cl}(X)$ is torsion–free. Since X has isolated hypersurface singularities and is 3–dimensional, $\pi_1(X^0) \to \pi_1(X)$ is an isomorphism. Since $\pi_1(X) = 1$, we are done. □

Lemma 3.4 *The image W of Gal_K on $\mathrm{Cl}(X_{\overline{K}})$ equals its image in Γ.*

PROOF: This is a consequence of the fact, which has already been remarked, that the nine 2–planes in X generate $\mathrm{Cl}(X_{\overline{K}})$. □

4 The Hasse principle

Now suppose that K is a number field and that X is a 9–nodal cubic threefold over K. We know that X is a hyperplane section of some Perazzo cubic 4-fold Y over K.

Lemma 4.1 *Suppose that $\mathcal{T} \to Y^0$ is a universal torsor. Then so is $\mathcal{T} \times_{Y^0} X^0 \to X^0$. Moreover, every universal torsor on X^0 arises in this way.*

PROOF: Let S_0 denote the torus whose character group $\hat{S}_0$ is $\mathrm{Pic}(Y^0)$. There is a commutative diagram (cf. [CT–San2])

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^1(K, S_0) & \longrightarrow & H^1(Y^0, S_0) & \xrightarrow{\chi} & \mathrm{Hom}_{\mathrm{Gal}_K}(\hat{S}_0, \mathrm{Pic}\, Y^0) \\
& & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} \\
0 & \longrightarrow & H^1(K, S_0) & \longrightarrow & H^1(X^0, S_0) & \xrightarrow{\chi} & \mathrm{Hom}_{\mathrm{Gal}_K}(\hat{S}_0, \mathrm{Pic}\, X^0).
\end{array}
$$

By definition, a torsor under S_0 is universal if the image of its class under χ is the identity, and now the result is immediate. □

Now assume that X has K_v–points for all places v of K. That is, $X(\mathbb{A}_K)$ is not empty, where $\mathbb{A}_K$ is the ring of adèles of K.

Proposition 4.2 *There is a universal torsor over X^0.*

PROOF: We know that X is a section of Y; since Y satisfies the Hasse principle it has a K–point, and so there is a universal torsor over Y^0. Now use Lemma 4.1. □

Proposition 4.3 *Every universal torsor over X^0 is K–birational to the cone over a cubic 7–fold that is singular along a double–three.*

PROOF: Immediate from Proposition 2.4. □

Proposition 4.4 *The universal torsors $\mathcal{T}$ over X^0 satisfy the Hasse principle.*

PROOF: Say that $\mathcal{T}$ is K–birational to the cone over the cubic 7–fold Z. Since quadratic extensions are harmless, we can suppose that the given double–three along which Z is singular splits into two threes. Then Z has three conjugate singular points, and so [CT–Sal] satisfies the Hasse principle. □

For any K-variety V, there is a pairing

$$
\mathrm{Br}(V) \times \prod V(\mathbb{A}_K) \to \mathbb{Q}/\mathbb{Z}
$$

given by

$$(\mathcal{A}, (P_v)) \mapsto \sum_v \mathrm{inv}_v(\mathcal{A}(P_v)).$$

We denote by $V(\mathbb{A}_K)^{\mathrm{Br}}$ the subset of $V(\mathbb{A}_K)$ that is the kernel of this pairing. The set $V(K)$ lies naturally in $V(\mathbb{A}_K)^{\mathrm{Br}}$. If the non-emptiness of $V(\mathbb{A}_K)^{\mathrm{Br}}$ implies that of $V(K)$ then we say that "the only obstruction to the Hasse principle on V is the Brauer–Manin obstruction".

Theorem 4.5 *The Brauer–Manin obstruction to the Hasse principle on X is the only one.*

PROOF: By definition [CT–San2], the Brauer–Manin obstruction to the existence of a K–point on X (equivalently, since X is a cubic, on X^0) is the existence, for all $(x_v) \in X^0(\mathbb{A}_K)$, of an element A of $\mathrm{Br}(X^0)$ such that $\sum_v \mathrm{inv}_v(A) \ne 0$. Assume that this obstruction vanishes; then the proof of Théorème 3.8.1 of *loc. cit.* shows that there is a universal torsor over X^0 with a point everywhere locally. By Proposition 4.4, we are done.   □

Colliot–Thélène points out that the following variant of Th. 3.8.1 of *loc. cit.* is valid, where $\widetilde{X}$ denotes a smooth compactification of X^0 (for example, the blow–up of the nodes of X).

Lemma 4.6 *Suppose that Z is a projective variety over K with only nodes and that $\dim Z \ge 2$. Denote by Z^0 its smooth locus and $\widetilde{Z} \to Z$ the blow–up of the nodes. Then the natural map $\mathrm{Br}(\widetilde{Z}) \to \mathrm{Br}(Z^0)$ is an isomorphism.*

PROOF: We first prove this when $K = \overline{K}$.

Brauer groups are torsion, so it is enough to prove this for the n–torsion subgroups. The Kummer sequence shows that then it is enough to prove the surjectivity of $H^2(\widetilde{Z}, \mu_n) \to H^2(Z^0, \mu_n)$. Let $E = \sum E_i$ be the exceptional locus in $\widetilde{Z}$ and $j : Z^0 \to \widetilde{Z}$ the inclusion. Then the purity theorem shows that $j_* \mu_n = (\mu_n)_{\widetilde{Z}}$, $R^1 j_* \mu_n$ is locally isomorphic to $(\mu_n)_E$ and that $R^q j_* \mu_n = 0$ for $q \ge 2$. Since each E_i is simply connected, it follows that $H^1(\widetilde{Z}, R^1 j_* \mu_n) = 0$. Now the Leray spectral sequence $E_2^{pq} = H^p(\widetilde{Z}, R^q j_* \mu_n) \Rightarrow H^{p+q}(Z^0, \mu_n)$ gives the result.

The general case then follows from the facts that $\mathrm{Br}(V)/\mathrm{Br}(K)$ is naturally isomorphic to $H^1(\mathrm{Gal}_K, \mathrm{Pic}(V_{\overline{K}}))$ and that the homomorphism $\mathrm{Br}(\widetilde{Z}_{\overline{K}}) \to \mathrm{Br}(Z^0_{\overline{K}})$ is Gal_K–equivariant.   □

Proposition 4.7 (Colliot-Thélène.) *If there is no Brauer–Manin obstruction using $\mathrm{Br}(\widetilde{X})$ to the existence of a K–point on X, then there is none using $\mathrm{Br}(X^0)$.*

PROOF: Choose a finite set $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of representatives of the elements of the finite group $\mathrm{Br}(X^0)/\mathrm{Br}(K)$. Over an open subscheme $\mathrm{Spec}(\mathcal{O})$ of the spectrum of the ring of integers of K, the variety X^0/K has a model $\mathcal{X}^0/\mathcal{O}$ such that $\mathcal{X}^0(\mathcal{O}_v) \ne \emptyset$ for each $v \in \mathrm{Spec}(\mathcal{O})$, and such that each $\mathcal{A}_i$ extends to an element of $\mathrm{Br}(\mathcal{X}^0)$. Using Harari's "formal lemma" (Lemme 2.6.1 of [H], but see also p.

225 of [CT–San2]) and the hypothesis that there is no Brauer-Manin obstruction on $\widetilde{X}$, we find a finite set S of places, which we may assume contains all places not in $\mathrm{Spec}(\mathcal{O})$, and local points $M_v \in X^0(K_v)$ for $v \in S$, such that

$$\sum_{v \in S} \mathrm{inv}_v(\mathcal{A}_i(M_v)) = 0$$

for each $i = 1, \ldots, n$.

Now pick any set of integral points $M_v \in \mathcal{X}^0(\mathcal{O}_v)$ for $v \notin S$. Then for each $i \in \{1, \cdots, n\}$, the sum $\sum_v \mathrm{inv}_v(\mathcal{A}_i(M_v))$ vanishes (the sum is over all places of K), thus completing the proof. □

5 Weak approximation

Recall that a variety V over K satisfies *weak approximation* (WA) if for every finite set S of places of K, $V(K)$ is dense in $\prod_{v \in S} V(K_v)$. If V is complete, this is equivalent to $V(K)$ being dense in $V(\mathbb{A}_K)$. Moreover, we say that the Brauer–Manin obstruction to WA on V is the only one if $V(K)$ is dense in $V(\mathbb{A}_K)^{\mathrm{Br}}$. This is equivalent to the density of $V(K)$ in the image of $V(\mathbb{A}_K)^{\mathrm{Br}}$ under every projection $V(\mathbb{A}_K) \to \prod_{v \in S} V(K_v)$, for every finite set S.

Now assume that X and K are as in Section 4.

Lemma 5.1 $X^0(\mathbb{A}_K)^{\mathrm{Br}}$ *is dense in* $X(\mathbb{A}_K)^{\mathrm{Br}}$.

PROOF: This is Corollary 1.2 of [CT–Sk]. □

Proposition 5.2 *Assume that W lies in the index 2 subgroup $S_3 \times S_3$ of Γ. Then the only obstruction to weak approximation on X is the Brauer–Manin one.*

PROOF: Take a point $(P_v) \in X(\mathbb{A}_K)^{\mathrm{Br}}$. By the lemma, we can approximate (P_v) by $(M_v) \in X^0(\mathbb{A}_K)^{\mathrm{Br}}$. By the version of descent given in Proposition 1.3 of [CT–Sk], there is a universal torsor $\pi : T \to X^0$ such that (M_v) lifts to a point $(\widetilde{M_v}) \in T(\mathbb{A}_K)$. We know that T is K–birational to the cone over a cubic 7–fold Z, and that Z is singular along a double–three. The hypothesis on the Galois group means that the double–three splits into two threes, so that Z has three conjugate singularities. Since this gives WA for Z, by [CT–Sal], T then satisfies WA, so that there exists $t \in T(K)$ close to $(\widetilde{M_v})$. Then $x := \pi(t) \in X^0(K)$ is close to (M_v), and then x is close to (P_v). □

6 Computing the Brauer group and obstructions

Suppose that X is a 9–nodal cubic 3–fold over K and $\widetilde{X}$ its blow-up at the nodes. Put $\mathrm{Pic}(X^0 \otimes \overline{K}) = P$. Then, by Lemma 4.6 and the Hochschild–Serre spectral sequence, $\mathrm{Br}(X^0)/\mathrm{Br}(K)$ is isomorphic to $H^1(\mathrm{Gal}_K, P) = H^1(W, P)$.

Proposition 6.1 $\mathrm{Br}(X^0)/\mathrm{Br}(K)$ *is either trivial or of order* 3.

PROOF: Pick a general projection $\pi : X \to \mathbb{P}^1_K$, so that the generic fibre X_η is a smooth cubic surface and all geometric fibres are irreducible. Put $K(\mathbb{P}^1) = L$. The main result of [SD1] is that $\mathrm{Br}(X_\eta)/\mathrm{Br}(L)$ is a subgroup of either $(\mathbb{Z}/2\mathbb{Z})^2$ or $(\mathbb{Z}/3\mathbb{Z})^2$.

 We show first that the natural homomorphism

$$\phi : \mathrm{Br}(X^0)/\mathrm{Br}(K) \to \mathrm{Br}(X_\eta)/\mathrm{Br}(L)$$

is injective. For this, suppose that $\alpha \in \mathrm{Br}(X^0)$ restricts to $\pi^*\beta$ in $\mathrm{Br}(X_\eta)$. For any closed point M of $\mathbb{P}^1_K$, the image of α under the residue map $\partial_{\pi^{-1}(M)} :$ $\mathrm{Br}(K(X)) \to H^1(K(\pi^{-1}(M)), \mathbb{Q}/\mathbb{Z})$ is zero. Since $K(M)$ is algebraically closed in $K(\pi^{-1}(M))$, it follows that $\partial_M(\beta) = 0$, so that $\beta \in \mathrm{Br}(\mathbb{P}^1_K) = \mathrm{Br}(K)$. So ϕ is injective.

 Suppose next that there is 2–torsion in $\mathrm{Br}(X^0)/\mathrm{Br}(K)$. Then the same is true after any base extension of odd degree, so that we can assume that X has a K–rational node. Then X is K-rational, so that, by Lemma 4.6, $\mathrm{Br}(X^0)/\mathrm{Br}(K) = 0$, contradiction. So $\mathrm{Br}(X^0)/\mathrm{Br}(K)$ has odd order.

 Since $\Gamma \cong S_3 \wr S_2$, we can assume that W is non-trivial and lies in the unique Sylow 3-subgroup of Γ, which is $A_3 \times A_3$. So, up to Γ–conjugacy, there are three possibilities for W: $A_3 \times 1$; the diagonal copy Δ of A_3 in $A_3 \times A_3$; and $A_3 \times A_3$.

 Recall that P is generated by the L_{ij} subject to the relations $R_i - C_j = 0$. We consider the three possibilities separately.

(1) $W = A_3 \times 1$. Put $P_1 = \oplus_{j \neq 3}\mathbb{Z}.L_{ij}$. This is a permutation W–module and there is a short exact sequence

$$0 \to \mathbb{Z}.(C_1 - C_2) \to P_1 \to P \to 0.$$

   Since $H^2(W, \mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ and $H^i(W, P_1) = 0$ for $i = 1, 2$, it follows that $H^1(W, P)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

(2) $W = \Delta$. Put $T = s - 1$, where $s = ((123), (123))$ is a generator of Δ, and $N = \sum s^i$. So, for any W–module B, $H^1(W, B) \cong \ker N_B/\operatorname{im} T_B \cong$ $\mathrm{Tors}(\operatorname{coker} T_B)$. Let F denote the free $\mathbb{Z}$-module on the L_{ij}, so that there is a short exact sequence

$$0 \to Q \to F \to P \to 0$$

   of W-modules, which defines the submodule Q of F. Inspection shows that for $i \neq j$ $L_{ii} - L_{jj} = \pm T(L_{ii})$ or $\pm T(L_{jj})$ and $L_{ik} - L_{kj} = \pm T(L_{ik})$ or $\pm T(L_{kj})$. Hence $Q \subseteq \operatorname{im}(T_F)$, so that $\operatorname{coker} T_P \cong F/(Q+\operatorname{im} T_F) \cong \operatorname{coker} T_F$. Since $H^1(W, F)$ vanishes, so does $H^1(W, P)$.

(3) $W = A_3 \times A_3$. By Lemma 5 of [SD1] $\mathrm{Br}(X_\eta)/\mathrm{Br}(L)$ is then of order at most 3, so the same is true of $\mathrm{Br}(X^0)/\mathrm{Br}(K)$.

$\square$

Proposition 6.2 *Suppose that $W \subseteq A_3 \times A_3$. Then $\mathrm{Br}(X^0)/\mathrm{Br}(K) \cong \mathbb{Z}/3\mathbb{Z}$ if W is Γ–conjugate to either $A_3 \times A_3$ or $A_3 \times 1$, and is trivial otherwise. Moreover, when $\mathrm{Br}(X^0)/\mathrm{Br}(K)$ is non–trivial and K contains a cube root of unity, there is an explicit description (given in the course of the proof) of a non–trivial element.*

PROOF: We use the results and notation of [SD2], especially Lemma 2 of *loc. cit.*

Assume that $\mathrm{Br}(X^0)/\mathrm{Br}(K)$ is non–trivial. There is a cubic extension K_1/K over which the divisors L_{11}, L_{22}, L_{33} are defined. The image W' of Gal_{K_1} in Γ is of order 3 and is generated by $\sigma = ((123), 1)$. Choose a hyperplane section H defined over K and put $D = \sum L_{ii} - H$. Then $\sum_j \sigma^j(D)$ is principal; say $\sum \sigma^j(D) = (f)$, with $f \in K(X)$. Then, according to *loc. cit.*, there is a non–trivial element $\mathcal{A}$ of $\mathrm{Br}(X^0)$ such that, for every adelic point (P_v) on X^0, with no P_v in the support of (f), $\sum_v \mathrm{inv}_v(\mathcal{A}(P_v)) = \sum_v (f(P_v), K_1/K)_v$, where the summands on the right are the norm residue symbols. $\square$

7   An aside: weak approximation for 10 nodes

Coray has shown [C] that a 10-nodal cubic threefold X has K-points. We show here that also weak approximation holds.

We start by recalling a construction from [D–O]. There the ground field is $\mathbb{C}$, but this part of *loc. cit.* is valid over any field K, or even over $\mathbb{Z}$.

Regard $T = \mathbb{G}_{m,K}^6$ as the group of 6×6 diagonal matrices acting in the obvious way on the 6–dimensional K–vector space V. Let U be the standard 2–dimensional representation of SL_2 over K. Then $G = (SL_2 \times T)/\mu_2$ acts on the space $M = U \otimes V \cong \mathbb{A}^{12}$ of 2×6 matrices, where μ_2 is embedded diagonally. Regard SL_2 as acting on the left and T on the right. Let M^0 be the locus of matrices of rank 2; then SL_2 acts freely on M^0 and there is a geometric quotient $SL_2 \backslash M^0$ isomorphic to the punctured cone $\widetilde{Gr}$ over the Grassmannian $Gr(2, V)$ in its Plücker embedding. If M^{00} is the locus of matrices where no column is zero and no 3 columns are proportional, then G acts freely on M^{00}. There is a geometric quotient M^{00}/G, which is isomorphic to the smooth locus Σ^0 of the Segre cubic 3-fold Σ given by $\sum_1^6 x_i^3 = \sum_1^6 x_i = 0$. We can identify $\Sigma^0 = M^{00}/G$ with $(SL_2 \backslash M^{00})/S$, where $S = T/\mu_2$ with μ_2 embedded in the diagonal copy of $\mathbb{G}_m$ in T. It follows that the S-torsor $\widetilde{Gr} \to \Sigma^0$ is a universal torsor over Σ^0. The basic geometry of Σ is described in [S–R], p.169. It is the image of $\mathbb{P}^3$ under the rational map defined by the linear system of quadrics that pass through five given points in general position. It has 10 nodes and 15 planes, all defined over

$\mathbb{Q}$, and over $\overline{\mathbb{Q}}$ it is the unique 10-nodal cubic threefold. That is, every 10-nodal cubic threefold X over K is a twist of Σ.

Lemma 7.1

(1) $\mathrm{Aut}(\Sigma) = \mathrm{Aut}(\Sigma \otimes \bar{\mathbb{Q}}) \cong S_6$.

(2) *Every 1-cocycle* $\mathrm{Gal}_Q \to \mathrm{Aut}(\Sigma)$ *is a homomorphism.*

(3) *Every 10-nodal cubic threefold* X *over* K *is the twist of* Σ *by a homomorphism* $\mathrm{Gal}_K \to S_6$.

(4) *Given such an* X *over* K, *there is a separable sextic* K-*algebra* L *such that* X *is defined in* $\mathbb{P}(L)$ *by the equation* $\mathrm{Tr}(z) = \mathrm{Tr}(z^3) = 0$.

PROOF: (1) The projective dual of $\Sigma \otimes \overline{\mathbb{Q}}$ is a quartic threefold T that is the Satake compactification of the moduli space of principally polarized Abelian surfaces with level 2 structure. The Satake boundary is $\mathrm{Sing}\, T$, which is a $(15_3, 15_3)$ configuration of points and lines. The automorphism group of this is clearly $Sp_4(\mathbb{F}_2) \cong S_6$, so that there is a homomorphism $f : \mathrm{Aut}(\Sigma \otimes \overline{\mathbb{Q}}) \to S_6$. Restrict f to the copy of S_6 in $\mathrm{Aut}(\Sigma) \subseteq \mathrm{Aut}(\Sigma \otimes \overline{\mathbb{Q}})$; this is clearly an isomorphism, so that f is split. Since the lines in $\mathrm{Sing}\, T$ are the images of the planes in Σ, any $s \in \ker f$ preserves every plane in Σ. Since each node of Σ is (in many ways) the intersection of two of these planes, s fixes every node. It is then clear that $s = 1$.

(2) This follows at once from Lemma 2.1 and the definition of a 1-cocycle.

(3) and (4) are now immediate.   □

Now suppose that X/K is a 10–nodal cubic 3–fold with smooth locus X^0. We have just seen that X is K–isomorphic to a twist Σ_ρ of Σ by a homomorphism $\rho : \mathrm{Gal}_K \to S_6$. Then the construction above can be twisted to show that X^0 is the geometric quotient M_ρ^{00}/G_ρ.

Proposition 7.2 *There is a universal torsor* T_0 *over* X^0 *whose total space is isomorphic to* $\widetilde{Gr}$.

PROOF: The morphism $\widetilde{Gr} \to \Sigma^0$ can be twisted by ρ. The twist of Gr by ρ is $Gr(2, V_\rho)$, which is isomorphic to Gr, by Hilbert's Theorem 90. It is also possible to twist the action of the torus S on $\widetilde{Gr}$ by ρ, and we are done, since $\widetilde{Gr}$ has trivial Picard group.   □

Corollary 7.3 *Every universal torsor* T *over* X^0 *is* K-*birational to a line bundle over a twist of* Gr.

PROOF: Given T_0, the other universal torsors over X^0 are classified by $H^1(K, S)$. Since S acts by right multiplication on Gr and the Plücker line bundle $\mathcal{O}(1)$ over Gr is S-linearized, the result follows.   □

Corollary 7.4 *Both the Hasse principle and weak approximation hold for every universal torsor* T *over* X^0.

PROOF: Every twist V of Gr is homogeneous under its automorphism group, and the point stabilizers are connected. Hence the Hasse principle and weak approximation hold for V, and then for line bundles over it.   □

Theorem 7.5 *The only obstruction to weak approximation on X is the Brauer–Manin one.*

PROOF: Exactly as for the class of 9–nodal cubics that we handled earlier, in 5.2. □

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