PROJECTING LATTICE POLYTOPES ACCORDING TO THE MINIMAL MODEL PROGRAM

VICTOR V. BATYREV

ABSTRACT. The Fine interior $F(P)$ of a d-dimensional lattice polytope $P \subset \mathbb{R}^d$ is the set of all points $y \in P$ having integral distance at least 1 to any integral supporting hyperplane of P. We call a lattice polytope F-hollow if its Fine interior is empty. The main theorem claims that up to unimodular equivalence in each dimension d there exist only finitely many d-dimensional F-hollow lattice polytopes P, so called *sporadic*, which do not admit a lattice projection onto a k-dimensional F-hollow lattice polytope P' for some $1 \leq k \leq d-1$. The proof is purely combinatorial, but it is inspired by $\mathbb{Q}$-Fano fibrations in the Minimal Model Program, since we show that non-degenerate toric hypersurfaces $Z \subset (\mathbb{C}^*)^d$ defined by zeros of Laurent polynomials with a given Newton polytope P have negative Kodaira dimension if and only if P is F-hollow. The finiteness theorem for d-dimensional sporadic F-hollow Newton polytopes P gives rise to finitely many families $\mathcal{F}(P)$ of $(d-1)$-dimensional $\mathbb{Q}$-Fano hypersurfaces with at worst canonical singularities.

1. INTRODUCTION

Let $P \subset \mathbb{R}^d$ be an arbitrary d-dimensional convex polytope.

Definition 1.1. For a nonzero lattice vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d \setminus \{0\}$ consider the **integral supporting hyperplane** of P:

$$H_{\mathbf{a},P} := \left\{ \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{i=1}^{d} a_i x_i = \mathrm{Min}_P(\mathbf{a}) := \min_{\mathbf{y} \in P} \left(\sum_{i=1}^{n} a_i y_i \right) \right\}.$$

If $\gcd(a_1, \ldots, a_d) = 1$ we call the number

$$\mathrm{dist}_{\mathbb{Z}}(\mathbf{y}, H_{\mathbf{a},P}) := \sum_{i=1}^{n} a_i y_i \; - \; \mathrm{Min}_P(\mathbf{a}) \geq 0$$

the **integral distance** between a point $\mathbf{y} = (y_1, \ldots, y_d) \in P$ and integral supporting hyperplane $H_{\mathbf{a},P}$.

Definition 1.2. [10, 21] Let $P \subset \mathbb{R}^d$ be an arbitrary d-dimensional convex polytope. The set $F(P)$ of all points in P having integral distance at least 1 to any integral supporting hyperplane $H_{\mathbf{a},P}$ is called the **Fine interior of** P, i.e.,

$$F(P) := \{\mathbf{y} \in \mathbb{R}^d \mid \mathrm{dist}_{\mathbb{Z}}(\mathbf{y}, H_{\mathbf{a},P}) \geq 1, \;\; \forall \mathbf{a} \in \mathbb{Z}^d \setminus \{0\}\}.$$

Remark 1.3. If the affine span of every facet of P is a integral supporting hyperplane, then Gordan's lemma shows that among countably many inequalities $\mathrm{dist}_{\mathbb{Z}}(\mathbf{y}, H_{\mathbf{a},P}) \geq 1$ defining $F(P)$ only finitely many $\mathbf{a} \in \mathbb{Z}^d$ are necessary. In particular, $F(P) \subset P$ is a rational polytope (or empty set) if all vertices of P belong to $\mathbb{Q}^d$. We explain more details concerning this fact in Proposition 3.1. **Lattice**

polytopes P, i.e., polytopes having vertices in $\mathbb{Z}^d$, are main objects of our study. However, for some technical reasons it will be convenient to consider the Fine interior $F(P)$ of rational polytopes P and the Fine interior $F(\lambda P)$ of their arbitrary real positive multiples λP ($\lambda \in \mathbb{R}_{>0}$).

Remark 1.4. If $P \subset \mathbb{R}^d$ is a d-dimensional lattice polytope, then every interior lattice point $\mathbf{m} \in Int(P) \cap \mathbb{Z}^d$ necessarily belongs to $F(P)$, and we obtain the inclusion

$$\text{Conv}\,(Int(P) \cap \mathbb{Z}^d) \subset F(P)$$

which is in fact equality for lattice polytopes P of dimension $d \in \{1, 2\}$ [8].

Recall some standard definitions.

Definition 1.5. Two lattice polytopes $P_1, P_2 \subset \mathbb{R}^d$ are called **unimodular equivalent** if there exists a lattice-preserving affine isomorphism $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\varphi(P_1) = P_2$.

Definition 1.6. A k-dimensional lattice polytope $P' \subset \mathbb{R}^k$ $(1 \leq k < d)$ is called **lattice projection**, or **$\mathbb{Z}$-projection**, of a d-dimensional lattice polytope $P \subset \mathbb{R}^d$ if there exists an affine map $\pi : \mathbb{R}^d \to \mathbb{R}^k$ inducing a surjective map of lattices $\pi : \mathbb{Z}^d \to \mathbb{Z}^k$ and $\pi(P) = P'$.

Remark 1.7. Since every epimorphism $\pi : \mathbb{Z}^d \to \mathbb{Z}^k$ splits, we can choose a splitting $\mathbb{Z}^d \cong \mathbb{Z}^k \oplus \mathbb{Z}^{n-k}$ such that the $\mathbb{Z}$-projection π has the standard form:

$$\pi(x_1, \ldots, x_d) = (x_1, \ldots, x_k) \in \mathbb{R}^k \;\; \forall (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

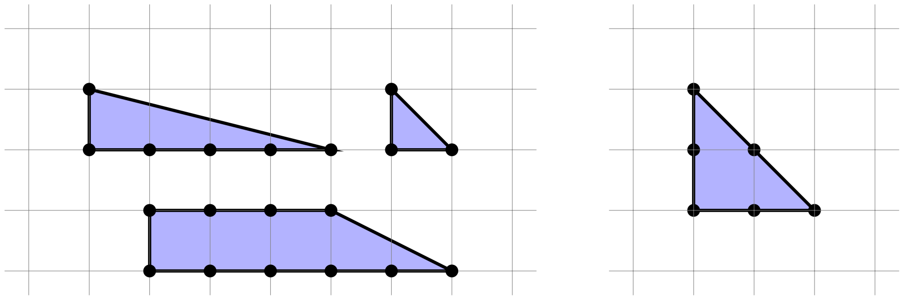
Definition 1.8. We call a d-dimensional polytope $P \subset \mathbb{R}^d$ **F-hollow** if $F(P) = \emptyset$.

Remark 1.9. Let $P' \subset \mathbb{R}^k$ be a lattice projection of a d-dimensional polytope $P \subset \mathbb{R}^d$, then the lattice epimorphism $\pi : \mathbb{Z}^d \to \mathbb{Z}^k$ allows to lift every integral supporting hyperplane of P' to an integral supporting hyperplane of P. In particular, the condition $F(P') = \emptyset$ implies $F(P) = \emptyset$. This fact allows to construct infinitely many pairwise unimodular distinct d-dimensional F-hollow lattice polytopes P whose lattice projections are equal to a given lower-dimensional F-hollow lattice polytope P'.

Definition 1.10. We call a d-dimensional F-hollow lattice polytope P **sporadic**, if P does not admit any lattice projection $\pi : P \to P'$ onto a k-dimensional F-hollow lattice polytope P' $(1 \leq k \leq d-1)$.

The present paper shows that in any fixed dimension d, apart from finitely many unimodular equivalence classes of d-dimensional sporadic F-hollow lattice polytopes, every d-dimensional F-hollow lattice polytope P admits a $\mathbb{Z}$-projection to some lower-dimensional F-hollow lattice polytope P':

Theorem 1.11. *In each dimension d there exist up to unimodular transformations only finitely many d-dimensional sporadic F-hollow lattice polytopes P.*



countably many F-hollow polygons a sporadic polygon

Remark 1.12. Theorem 1.11 is a purely combinatorial contribution to the theory of lattice polytopes but its motivation comes from birational algebraic geometry [8, 22]. Recall that a lattice P is called **hollow** if P contains no lattice points in its interior. By 1.4, any F-hollow lattice polytope P is hollow. In dimension $d \in \{1, 2\}$ the inverse statement is also true. Therefore, for $d \leq 3$, Theorem 1.11 follows from a result of Treutlein [22] which was later generalized by Nill and Ziegler for hollow lattice polytopes in arbitrary dimension d [20]. In case $d \geq 4$, Theorem 1.11 does not follow from Theorem of Nill and Ziegler because hollow lattice polytopes P of dimension ≥ 3 need not be F-hollow. For instance, there exist up to unimodular equivalence exactly 9 examples of 3-dimensional hollow lattice polytopes which are not F-hollow [11, Appendix B].

2. The proof of Theorem 1.11

The proof of Theorem 1.11 uses standard combinatorial notions from the theory of toric varieties [13]. We consider two dual to each other lattices $M \cong \mathbb{Z}^d$ and $N := Hom(M, \mathbb{Z})$ together with their scalar extensions $M_{\mathbb{R}} := M \otimes \mathbb{R}$, $N_{\mathbb{R}} := N \otimes \mathbb{R}$ and the natural pairing

$$\langle *, * \rangle \ : \ M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}.$$

Every d-dimensional rational polytope P defines a convex piecewise linear function

$$\mathrm{Min}_P \ : \ N_{\mathbb{R}} \to \mathbb{R}, \ \ y \mapsto \min_{u \in P} \langle u, y \rangle$$

whose domains of linearity form a complete rational polyhedral fan Σ_P, a collection of rational polyhedral cones σ in $N_{\mathbb{R}}$, which is called the **normal fan of** P.

Using the above notations, the Fine interior $F(P)$ of a polytope $P \subset M_{\mathbb{R}}$ can be equivalently reformulated as follows:

Definition 2.1.

$$F(P) := \{x \in M_{\mathbb{R}} \ | \ \langle x, \nu \rangle \geq \mathrm{Min}_P(\nu) + 1, \forall \nu \in N \setminus \{0\}\}.$$

Our first idea in the proof of 1.11 is to consider a positive number $\mu = \mu(P)$ attached to P:

Definition 2.2. Let $P \subset M_{\mathbb{R}}$ be a d-dimensional rational polytope. We call the number

$$\mu(P) := \inf \{\lambda \in \mathbb{R}_{>0} \ | \ F(\lambda P) \neq \emptyset\}$$

the **minimal multiplier of** P.

Remark 2.3. If $P \subset M_{\mathbb{R}}$ is a d-dimensional lattice polytope, then it follows from standard properties of Ehrhart polynomials that the lattice polytope $(d+1)P$ always contains at least one interior lattice point and hence $F((d+1)P) \neq \emptyset$. This implies the inequality

$$\mu(P) \leq d+1$$

for all d-dimensional lattice polytopes P. Note that the inequality is sharp, because $\mu(P) = \dim P + 1$ if P is the d-dimensional lattice simplex spanned by $0 \in \mathbb{Z}^d$ and by the standard lattice basis $e_1, \ldots, e_d$ of $\mathbb{Z}^d$.

Remark 2.4. It is clear that $\mu(P) \leq 1$ if and only if P is not F-hollow.

We will use the following property of the minimal multiplier $\mu(P)$ which will be proved later in Propositions 3.2 and 3.4:

Proposition 2.5. *Let $P \subset M_{\mathbb{R}}$ be a d-dimensional rational polytope. Then the following statements hold:*
(i) *The number $\mu(P)$ is rational.*
(ii) *For a positive rational number λ, one has $\lambda = \mu(P)$ if and only if*

$$0 \leq \dim F(\lambda P) \leq d-1.$$

(iii) *If $\dim F(\mu P) = 0$, then the convex hull $Q := \mathrm{Conv}(S_F(\mu P))$ of the set*

$$S_F(\mu P) := \{\nu \in N \ \mid \mathrm{Min}_{F(\mu P)}(\nu) = \mathrm{Min}_{\mu P}(\nu) + 1\} \subset N$$

is a d-dimensional lattice polytope in $N_{\mathbb{R}}$ having a unique interior lattice point $0 \in N$.

Finally, we need Theorem of Hensley [15] and some its generalizations due to Lagarias and Ziegler [19].

Theorem 2.6. [15] *For any given positive integers k, d, there exists a constant $C(k,d)$ depending only on k and d such that the volume $vol(P)$ of any d-dimensional lattice polytope having exactly k interior lattice points is bounded from above by $C(k,d)$.*

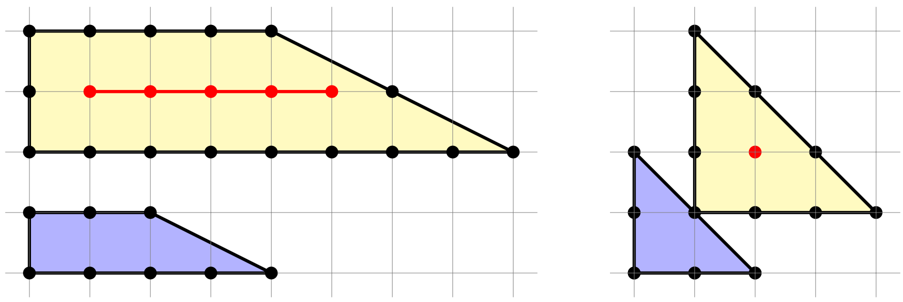
Theorem 2.7. [19, Thm. 2] *If a d-dimensional lattice polytope P has volume $vol(P) \leq V$, then P is unimodular equivalent to a lattice polytope in the d-dimensional lattice cube*

$$\{\mathbf{x} \in \mathbb{R}^d \ \mid \ 0 \leq x_i \leq n \cdot n! V, \ i = 1, \ldots, d\}.$$

In particular, a family of d-dimensional lattice polytopes P_i $(i \in I)$ contains only finitely many unimodular equivalence classes if and only if there exists a constant $C > 0$ such that

$$vol(P_i) < C \ \ \forall i \in I.$$

Proof of Theorem 1.11. Let $P \subset M_{\mathbb{R}}$ be a d-dimensional F-hollow lattice polytope. We put $\mu := \mu(P)$. By 2.5(ii), we have $0 \leq \dim F(\mu P) \leq d-1$. Consider two cases (two pictures below illustrate the case $d = 2$).



$\mu = 2$ $\mu = \frac{3}{2}$

Case 1. $1 \leq \dim F(\mu P) \leq d - 1$**.** We define the sublattice $N' \subset N$ consisting of all $\nu \in N$ such that $\langle x, \nu \rangle = \langle x', \nu \rangle$ for any two points $x, x' \in F(\mu P)$. Then N/N' has no torsion elements and N', because if $l\nu \in N'$ for some positive integer l, then $\langle x, l\nu \rangle = \langle x', l\nu \rangle$ for any two points $x, x' \in F(\mu P)$, hence $\langle x, \nu \rangle = \langle x', \nu \rangle$, i.e., $\nu \in N'$. Therefore N' is is a direct summand of N, and the embedding $N' \hookrightarrow N$ defines a lattice projection $\pi : M \to M' := Hom(N', \mathbb{Z})$, where

$$1 \leq \mathrm{rk}(M') = d - \dim F(\mu P) \leq d - 1.$$

Consider the lattice polytope $P' := \pi(P)$. It remains to show that $F(P') = \emptyset$. By definition of N', $\pi(F(\mu P))$ is some rational point $q \in M'_{\mathbb{Q}}$. Moreover, one has $q = F(\mu\pi(P)) = F(\mu P')$. Since $\mu > 1$, by monotonicity of Fine interior [8, Remark 3.7], the polytope $F(P')$ must be strictly smaller than the point $q = F(\mu P')$. Hence $F(P') = \emptyset$, i.e., P' is F-hollow. In fact we have shown that $\mu(P') = \mu(P) = \mu$ and $\dim F(\mu P') = 0$.

Case 2. $\dim F(\mu P) = 0$**.** By 2.5(i), $F(\mu P)$ is a rational point $p \in M_{\mathbb{Q}}$ and for any $\nu \in S_F(\mu P)$ we have

$$\mathrm{Min}_{F(\mu P)}(\nu) = \langle p, \nu \rangle = 1 + \mathrm{Min}_{\mu P}(\nu) = 1 + \min_{x \in \mu P} \langle x, \nu \rangle.$$

Equivalently, we have $\langle x, \nu \rangle \geq -1 + \langle p, \nu \rangle$ for all $x \in \mu P$ and for all $\nu \in S_F(\mu P)$, or

$$\langle x, \nu \rangle \geq -1 \;\; \forall x \in \mu P - p, \, \forall \nu \in S_F(\mu P).$$

The last conditions imply the inequalities

$$\langle x, y \rangle \geq -1 \;\; \forall x \in \mu P - p, \, \forall y \in Q := \mathrm{Conv}(S_F(\mu P)).$$

and we obtain the inclusion $\mu P - p \subseteq Q^*$, where

$$Q^* := \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geq -1, \, \forall y \in Q\}$$

is the rational polar dual polytope of Q. By 2.5(iii), $Q \subset N_{\mathbb{R}}$ is a d-dimensional lattice polytope having only one interior lattice point 0. By theorems of Hensley 2.6 and Lagarias-Ziegler 2.7, up to unimodular transformations there exist only finitely many possibilities for the d-dimensional lattice polytope Q. Therefore, there exists only finitely many possible values for volumes $vol(Q^*)$ of the dual polytope of Q, i.e., $vol(Q^*)$ is bounded by some constant $C(d)$ depending only on d. Since $\mu > 1$, it follows from the inclusion $\mu P - p \subseteq Q^*$ that

$$C(d) \geq vol(Q^*) \geq vol(\mu P - p) = vol(\mu P) = \mu^d vol(P) > vol(P).$$

Hence, by Theorem 2.7 of Lagarias and Ziegler, up to unimodular equivalence we obtain only finitely many of d-dimensional lattice polytopes P such that $\dim F(\mu P) = 0$. □

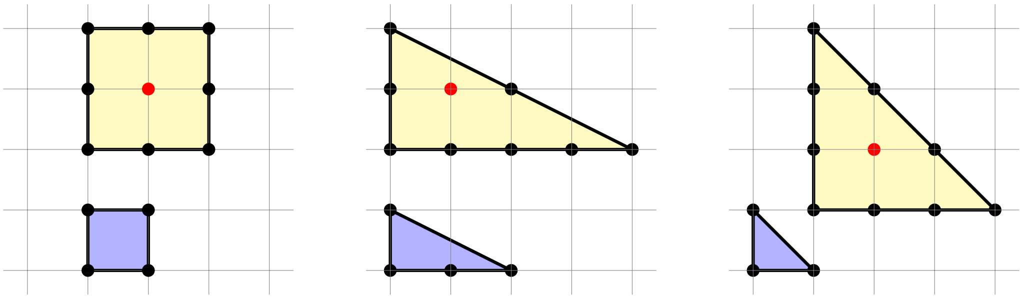
Definition 2.8. We call a d-dimensional F-hollow lattice polytope $P \subset \mathbb{R}^d$ **weakly sporadic**, if $\dim F(\mu P) = 0$, where μ is the minimal multiplier $\mu(P)$ of P.

Corollary 2.9. *In each dimension d there exist up to unimodular equivalence only finitely many weakly sporadic d-dimensional F-hollow lattice polytopes. Moreover, if a F-hollow lattice polytope P with minimal multiplier μ is not sporadic, then P admits a canonical lattice projection $\pi : P \to P'$, where P' is a k-dimensional weakly sporadic F-hollow lattice polytope* $(1 \leq k \leq d-1)$ *with the same minimal multiplier $\mu = \mu(P) = \mu(P')$.*

Proof. The statemen immediately follows from Cases 1 and 2 in the proof of Theorem 1.11. □

Remark 2.10. If $d \geq 2$, one can easily find examples of d-dimensional weakly sporadic F-hollow lattice polytopes P which are not sporadic F-hollow polytopes, i.e. P admitting lattice projections onto a k-dimensional F-hollow polytope P' $(1 \leq k \leq d-1)$.

Example 2.11. Up to unimodular equivalence there exist exactly three weakly sporadic F-hollow lattice polygons P which are not sporadic F-hollow lattice polygons:



$\mu = 2$ $\mu = 2$ $\mu = 3$

Example 2.12. It follows from the combinatorial classification of all maximal hollow 3-dimensional lattice polytopes obtained by Averkov et. al. [1, 2] that the following three 3-dimensional weakly sporadic F-hollow lattice with the minimal multipliers $\mu \in \{\frac{7}{6}, \frac{5}{4}, \frac{4}{3}\}$:

$$\Delta_1 := \text{Conv}\{(0,0,0),(2,0,0),(0,3,0),(0,0,6)\},\ \mu = \frac{7}{6};$$

$$\Delta_2 := \text{Conv}\{(0,0,0),(2,0,0),(0,4,0),(0,0,4)\},\ \mu = \frac{5}{4};$$

$$\Delta_3 := \text{Conv}\{(0,0,0),(3,0,0),(0,3,0),(0,0,3)\},\ \mu = \frac{4}{3}$$

are in fact sporadic F-hollow polytopes.

Remark 2.13. We will see in the last section that every d-dimensional weakly sporadic F-hollow polytope P defines a family $\mathcal{F}(P)$ of non-degenerate $(d-1)$-dimensional $\mathbb{Q}$-Fano toric hypersurfaces with at worst canonical singularities. For example, three lattice tetrahedra Δ_i, $i \in \{1,2,3\}$ from Example 2.12 are Newton polytopes of smooth Del Pezzo surfaces of the anticanonical degree $i \in \{1,2,3\}$ naturally imbedded into 3-dimensional toric weighted projective spaces $\mathbb{P}(1,1,2,3)$, $\mathbb{P}(1,1,1,2)$, and $\mathbb{P}(1,1,1,1)$ respectively. We expect that the complete list of all unimodular classes of 3-dimensional weakly sporadic F-hollow lattice polytopes P must have reasonable length. We draw attention to the fact that this list includes not only $\Delta_1, \Delta_2, \Delta_3$ but also 31 more 3-dimensional weakly sporadic F-hollow polytopes P with $\mu(P) = 2$ arising from 3-dimensional Gorenstein polytopes of index 2 which were classified in [7].

3. The minimal multiplier $\mu(P)$

Let $P \subset M_{\mathbb{R}}$ be a d-dimensional rational polytope. Theory of toric varieties associates with the normal fan Σ_P a d-dimensional projective toric variety X_P together with the ample $\mathbb{Q}$-Cartier divisor

$$L_P := \sum_{\nu \in \Sigma_P[1]} -\mathrm{Min}_P(\nu) D_\nu,$$

where D_ν ($\nu \in \Sigma_P[1]$) are torus invariant divisors on X_P corresponding to primitive lattice generators ν of 1-dimensional cones in Σ_P, and we have

$$P = \{x \in M_{\mathbb{Q}} \mid \langle x, \nu \rangle \geq \mathrm{Min}_P(\nu),\ \forall \nu \in \Sigma_P[1]\}.$$

Theory of toric varieties allows to describe the Fine interior $F(P)$ as rational polytope associated with the adjoint divisor on some smooth projective toric variety Y_Σ obtained by a regular refinement Σ of the normal fan Σ_P. More precisely, one has

Proposition 3.1. *Let $\rho : Y = Y_\Sigma \to X_P$ be a projective desingularization of the toric variety X_P corresponding to a regular simplicial refinement Σ of the normal fan Σ_P. Denote by $D_{\nu_1}, \ldots, D_{\nu_s}$ the torus invariant divisors corresponding to primitive lattice vectors in $\Sigma[1] = \{\nu_1, \ldots, \nu_s\}$. Then the Fine interior $F(P)$ is the rational polytope defined by s inequalities*

$$\langle x, \nu_i \rangle \geq \mathrm{Min}_P(\nu_i) + 1, \quad i \in \{1, \ldots, s\}$$

corresponding to the adjoint divisor on Y:

$$K_Y + \rho^*(L_P) = \sum_{\nu \in \Sigma[1]} \left(-1 - \mathrm{Min}_P(\nu)\right) D_\nu.$$

Proof. Let $\nu \in N$ an arbitrary nonzero lattice vector. Then there exists a minimal regular simplicial cone $\sigma \in \Sigma$ containing ν. Without loss of generality, we can assume that $\nu_1, \ldots, \nu_r \in \Sigma[1]$ ($r \leq d < s$) are generators of σ. Then $\nu = \sum_{i=1}^r l_i \nu_i$ for some positive integer coefficients $l_1, \ldots, l_r$. Since Σ is a refinement of Σ_P, there exists a minimal r'-dimensional cone $\sigma' \in \Sigma_P$ containing σ ($r \leq r' \leq d$). The cone σ' is dual to some $(d - r')$-dimensional face $\Theta' \prec P$ and we have

$$\mathrm{Min}_P(\nu_i) = \langle x, \nu_i \rangle, \quad \forall x \in \Theta',\ \forall i = 1, \ldots, r.$$

Since $\mathrm{Min}_P(\cdot)$ is linear on $\sigma \subset \sigma'$, we obtain

$$\mathrm{Min}_P(\nu) = \mathrm{Min}_P\left(\sum_{i=1}^{r} l_i\nu_i\right) = \sum_{i=1}^{r} l_i \mathrm{Min}_P(\nu_i),$$

and r inequalities appearing in definition of $F(P)$

$$\langle x, \nu_i\rangle \geq \mathrm{Min}_P(\nu_i) + 1, \ \ i \in \{1, \ldots, r\}$$

imply that

$$\langle x, \nu\rangle = \sum_{i=1}^{r} l_i\langle x, \nu_i\rangle \geq \sum_{i=1}^{r} l_i\mathrm{Min}_P(\nu_i) + \sum_{i=1}^{r} l_i = \mathrm{Min}_P(\nu) + \sum_{i=1}^{r} l_i \geq \mathrm{Min}_P(\nu) + 1,$$

and the equality $\langle x, \nu\rangle = \mathrm{Min}_P(\nu) + 1$ for some $x \in F(P)$ can happen only if $r = 1$ and $\nu = \nu_1 \in \Sigma[1]$. Therefore, s inequalities

$$\langle x, \nu_i\rangle \geq \mathrm{Min}_P(\nu_i) + 1, \ \ i \in \{1, \ldots, s\}$$

are already sufficient to obtain the Fine interior $F(P)$. Finally, we note that the canonical divisor of toric variety $Y = Y_\Sigma$ equals $-\sum_{i=1}^{s} D_{\nu_i}$ and $\mathbb{Q}$-Cartier divisor $\rho^* L_P$ equals $-\sum_{i=1}^{s} \mathrm{Min}_P(\nu_i) D_{\nu_i}$. Therefore, the rational polytope $F(P)$ corresponds to the adjoint divisor $K_Y + \rho^* L_P$ of Y. □

Proposition 3.2. *Let X_P be projective toric variety corresponding to a d-dimensional rational polytope P. Consider any projective toric desingularization $\rho : Y = Y_\Sigma \to X_P$ as in 3.1. Denote by $\Lambda_{\mathrm{eff}}(Y) \subset \mathrm{Pic}(Y)_{\mathbb{R}}$ the closed cone of effective divisors of the smooth projective toric variety Y. Let $L := \rho^* L_P \in \mathrm{Pic}(Y)_{\mathbb{Q}}$ be the pullback of the ample $\mathbb{Q}$-Cartier divisor L_P. Then*

$$\mu(P) = \inf\{\lambda \in \mathbb{R}_{>0} \mid [K_Y] + \lambda[L] \in \Lambda_{\mathrm{eff}}(Y)\},$$

is a rational number and for $\mu := \mu(P)$ one has

$$0 \leq \dim F(\mu P) < d.$$

Proof. Using Cox coordinates on Y one easily obtains that the cone of effective divisors $\Lambda_{\mathrm{eff}}(Y) \subset \mathrm{Pic}(Y)_{\mathbb{R}}$ is a rational polyhedral cone generated by the classes $[D_\nu]$ ($\nu \in \Sigma[1]$) of torus invariant divisors. The class $[L]$ represenst the class of a semiample big $\mathbb{Q}$-Cartier divisor $\rho^*(L_P)$ on Y which defines a rational point $[\rho^*(L_P)] \in \mathrm{Pic}(Y)_{\mathbb{R}}$ in the interior of the cone $\Lambda_{\mathrm{eff}}(Y)$. On the other hand, the canonical class $[K_Y] \in \mathrm{Pic}(Y)_{\mathbb{R}}$ of toric variety Y does not belong to the cone $C_{\mathrm{eff}}(Y)$, because Y is a rational toric variety. Therefore, the ray

$$\{[K_Y] + \lambda[\rho^* L_P] \mid \lambda \in \mathbb{R}_{\geq 0}\} \subset \mathrm{Pic}(Y)_{\mathbb{R}}$$

with the rational origin $[K_Y]$ having the rational direction $[\rho^* L_P]$ must hit the rational polyhedral cone $\Lambda_{\mathrm{eff}}(Y)$ in some rational point $[K_Y] + \mu[\rho^* L_P] \in \Lambda_{\mathrm{eff}}(Y)$ located at the polyhedral boundary $\partial\Lambda_{\mathrm{eff}}(Y)$ contained in some proper rational polyhedral facet $\Gamma \prec \Lambda_{\mathrm{eff}}(X)$. Since the intersection point $[K_Y] + \mu[\rho^* L_P]$ of the rational ray $[K_Y] + \lambda[\rho^* L_P]$ with the facet Γ has rational coordinates, the number μ must be rational. Now we use the fact that the class $[D]$ of a $\mathbb{Q}$-divisor

$$D = \sum_{i=1} b_i[D_{\nu_i}] \in \mathrm{Pic}(Y)_{\mathbb{Q}}, \ \ b_i \in \mathbb{Q},$$

represents a point in $\Lambda_{\rm eff}(Y)$ if and only if the rational polytope

$$P_D := \{x \in M_{\mathbb{R}} \mid \langle x, \nu_i \rangle \geq -b_i, \ \ \forall\, i \in \{1, \ldots, s\}\}$$

is not empty. Moreover, $[D]$ represents an interior point in $\Lambda_{\rm eff}(Y)$ if and only if the rational polytope P_D has maximal dimension d. Applying 3.1 to the adjoint $\mathbb{Q}$-divisor

$$D := K_Y + \mu\rho^* L_P = K_Y + \rho^* \mu L_P = K_Y + \rho^* L_{\mu P},$$

we obtain that $F(\mu P)$ is not empty and $\dim F(\mu P) < d$. □

Definition 3.3. Let $P \subset M_{\mathbb{R}}$ be an arbitrary d-dimensional rational polytope. Assume $F(P) \neq \emptyset$. Then we call the set

$$S_F(P) := \{\nu \in N \ \mid \ {\rm Min}_{F(P)}(\nu) = {\rm Min}_P(\nu) + 1\}$$

the **support of the Fine interior** of P. It follows from 3.1 that $S_F(P)$ is always a finite set whose positive convex span $\mathbb{R}_{\geq 0} S_F(P)$ equals $N_{\mathbb{R}}$ since

$$F(P) = \{x \in M_{\mathbb{R}} \ \mid \ \langle x, \nu \rangle \geq {\rm Min}_P(\nu) + 1 \ \ \forall \nu \in S_F(P)\}$$

is compact.

Proposition 3.4. *Let P be a d-dimensional rational polytope with $\dim F(P) = 0$. Then the convex hull $Q := {\rm conv}(S_F(P)) \subset N_{\mathbb{R}}$ is a d-dimensional lattice polytope containing in its interior only one lattice point $0 \in N$.*

Proof. Let $p := F(P) \in M_{\mathbb{Q}}$. Note that the shifted polytope $P_0 := P - p$ has the Fine interior $0 \in M$, and the sets $S_F(P)$ and $S_F(P_0)$ are the same, since $F(P_0) = F(P) - p$. Hence we can assume $F(P) = p = \{0\} \subset Int(P)$. The lattice polytope $Q = {\rm conv}(S_F(P))$ is d-dimensional, since

$$0 = F(P) = \{x \in M_{\mathbb{R}} \ \mid \ \langle x, \nu \rangle \geq 0 \ \ \forall \nu \in S_F(P)\}.$$

Using the upper convex piecewise linear function ${\rm Min}_P : N_{\mathbb{R}} \to \mathbb{R}$, we obtain the dual to P rational polytope

$$P^* := \{y \in N_{\mathbb{R}} \ \mid \ {\rm Min}_P(y) \geq -1\}.$$

Since ${\rm Min}_{F(P)}(\nu) = \langle 0, \nu \rangle = 0$ for all $\nu \in N$, we obtain $S_F(P) = \partial P^* \cap N$, where $\partial P^* := \{y \in N_{\mathbb{R}} \ \mid \ {\rm Min}_P(y) = -1\}$ is the boundary of P^*. The d-dimensional rational polytope P^* has only $0 \in N$ as interior lattice point, because ${\rm Min}_P(\nu) \leq -1$ for all $\nu \in N \setminus \{0\}$. Hence the d-dimensional lattice subpolytope $Q \subseteq P^*$ has also only $0 \in N$ as its interior lattice point. □

Remark 3.5. We note that the minimal multiplier $\mu(P)$ has naturally appeared in the arithmetical problem of counting rational points of bounded height on algebraic varieties [3]. The close relation between the boundary point $[K_Y + \mu L] \in \Lambda_{\rm eff}(Y)$ and the Minimal Model Program was observed in [4]. Fano fibrations of smooth toric varieties Y associated with adjoint divisors $K_Y + \mu L$ were considered in [5, 6].

4. Toric hypersurfaces with Newton polytope P

Let $A \subset M \cong \mathbb{Z}^d$ be a finite subset such that the convex hull $P := \mathrm{Conv}(A) \subset M_{\mathbb{R}} \cong \mathbb{R}^d$ is a d-dimensional lattice polytope. Take an arbitrary field K and consider P as **Newton polytope** of a Laurent polynomial

$$f(\mathbf{t}) = \sum_{\mathbf{m} \in A} c_{\mathbf{m}} \mathbf{t}^{\mathbf{m}} \in K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}],$$

that is, $c_{\mathbf{m}} \neq 0$ for all vertices $\mathbf{m} \in P$. The zero locus

$$Z := \{f(\mathbf{t}) = 0\} \subset \mathbb{T}^d_K := \mathrm{Spec}\, K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$$

we call **affine toric hypersurface**.

Remark 4.1. Note that the affine toric hypersurface $Z := \{f(\mathbf{t}) = 0\} \subset \mathbb{T}^d$ determines its defining non-constant Laurent polynomial f uniquely up to multiplication by a nonzero monomial $a\mathbf{t}^{\mathbf{m}} (a \neq 0)$ which shifts the Newton polytope of f by lattice vector $\mathbf{m} \in M$. So it will be convenient to refer to P as **Newton polytope** of the toric hypersurface $Z \subset \mathbb{T}^d$. Moreover, it is natural to consider Newton polytopes P of toric hypersurfaces $Z \subset \mathbb{T}^d$ up unimodular equivalence, since the group of affine linear transformations $\mathrm{Aff}(\mathbb{Z}^d) = GL(d, \mathbb{Z}) \rtimes \mathbb{Z}^d$ acts on Laurent polynomials f via automorphisms $\mathrm{Aut}(\mathbb{T}^d) \cong GL(n, \mathbb{Z})$ of the algebraic torus $\mathbb{T}^d$, and via multiplication by monomials $\mathbf{t}^{\mathbf{m}} = t_1^{m_1} \ldots t_d^{m_d}$. An unimodular isomorphism $\varphi \in \mathrm{Aff}(\mathbb{Z}^d) = GL(d, \mathbb{Z}) \rtimes \mathbb{Z}^d$ transforms an affine toric hypersurface $Z_1 \subset \mathbb{T}^d$ with the Newton polytope P_1 into the isomorphic affine hypersurface $Z_2 \subset \mathbb{T}^d$ with the Newton polytope $P_2 = \varphi(P_1)$.

Now let us consider the geometric meaning of lattice projections $\pi : P \to P'$ from view point of toric hypersurfaces $Z \subset \mathbb{T}^d$ with the Newton polytope P.

Remark 4.2. Let P be the Newton polytope of a Laurent polynomial f. Assume that P admits a standard lattice projection onto a k-dimensional lattice polytope $P' \subset \mathbb{R}^k$ $(0 < k < d)$. Then we can view P' as Newton polytope of the Laurent polynomial f considered as element of the Laurent polynomial ring $R[t_1^{\pm}, \ldots, t_k^{\pm}]$ whose coefficients ring is another Laurent polynomial ring $R := \mathbb{C}[t_{k+1}^{\pm 1}, \ldots, t_d^{\pm 1}]$. Using the splitting $\mathbb{T}^d \cong \mathbb{T}^k \times \mathbb{T}^{d-k}$ and the ring embedding $R \hookrightarrow R[t_1^{\pm}, \ldots, t_k^{\pm}]$, we obtain the surjective morphism $\mathbb{T}^d \to \mathbb{T}^{d-k} = \mathrm{Spec}(R)$ whose restriction to Z is a dominant morphism $Z \to \mathbb{T}^{d-k}$ such that general fibers are affine toric hypersurfaces in k-dimensional torus $\mathbb{T}^k$ having P' as Newton polytope.

Consider some examples of lattice projections.

Example 4.3. Assume that a d-dimensional lattice polytope P has **width 1**, that is, P has a lattice projection on the unique hollow (also F-hollow) lattice segment $[0, 1] \subset \mathbb{R}$. The lattice projection $\pi : P \to P' = [0, 1]$ means that the d-dimensional lattice polytope $P \subset \mathbb{R}^d$ is unimodular equivalent to a lattice polytope in $\mathbb{R}^d$ contained between two parallel integral affine hyperplanes $\{x_d = 0\}$ and $\{x_d = 1\}$. Up to this unimodul isomorphism, we obtain the corresponding Laurent polynomial $f(t_1, \ldots, t_d) \in K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ in the form:

$$f(t_1, \ldots, t_d) = g_0(t_1, \ldots, t_{d-1}) + t_d g_1(t_1, \ldots, t_{d-1}),$$

for some Laurent polynomials $g_0, g_1 \in K[t_1^{\pm 1}, \ldots, t_{d-1}^{\pm 1}]$. Since the polynomial $f(\mathbf{t})$ defining $Z \subset \mathbb{T}^d$ is linear with respect to the last variable t_d, we can rationally eliminate t_d from this equation $f = 0$ by the formula

$$t_d = -\frac{g_0(t_1, \ldots, t_{d-1})}{g_1(t_1, \ldots, t_{d-1})}$$

and obtain a birational isomorphism $Z \overset{bir}{\sim} \mathbb{A}_K^{d-1}$ over K, i.e., Z is an irreducible K-rational algebraic variety.

The following conjecture proposes a natural "inverse statement" to last example.

Conjecture 4.4. *Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytope. Assume that for any field K any toric hypersurface $Z \subset (K^*)^d$ with the Newton polytope P is irreducible and birational to $\mathbb{A}_K^{d-1}$ over K. Then the Newton polytope P admits a lattice projection onto* $[0, 1]$.

Example 4.5. In case $d = 2$, the conjecture can be easily verified. Indeed, the rationality of general curve $Z \subset \mathbb{T}^2$ with Newton polygone P implies that P has no interior lattice points, i.e., P is hollow. Up to unimodular isomorphisms, the unique sporadic hollow lattice polygon is the triangle

$$Q := \mathrm{Conv}((0,0), (2,0), (0,2)) \subset \mathbb{R}^2.$$

which is the Newton polytope of a general conic $C \subset \mathbb{A}_K^2$ defined by a quadratic equation

$$a_{0,0} + a_{1,0}t_1 + a_{0,1}t_2 + a_{2,0}t_1^2 + a_{1,1}t_1t_2 + a_{0,2}t_2^2 = 0, \quad (a_{i,j} \in K).$$

If $K = \mathbb{C}$, then C is birational to $\mathbb{A}_{\mathbb{C}}^1$. However, we can take $K = \mathbb{R}$ consider the conic

$$1 + t_1^2 + t_2^2 = 0$$

is not birational to $\mathbb{A}_{\mathbb{R}}^1$. Moreover, we can consider the conic $(1 + t_1)^2 - t_2^2 = 0$ over any field K with the Newton polygone T consisting of two irreducible components.

More generally, if a d-dimensional Newton polytope P hypersurface $Z \subset \mathbb{T}^d$ has a lattice projection onto Q, then the toric hypersurface Z becomes birational to a conic bundle over $(d-2)$-dimensional algebraic torus. Note that the corresponding toric hypersurface $Z \subset \mathbb{T}^d$ might be non-rational variety even over the algebraically closed field $\mathbb{C}$ (see Example 4.6 below).

Example 4.6. Let $\mathcal{K}_d \subset \mathbb{P}^{d+1}$ be smooth projective d-dimensional Klein cubic given by the homogeneous equation

$$z_0z_1^2 + z_1z_2^2 + \cdots + z_dz_{d+1}^2 + z_{d+1}z_0^2 = 0, \quad d \geq 2,$$

which is invariant under the cyclic permutation of the homogeneous coordinates $z_0, z_1, \ldots, z_{d+2}$. The Newton polytope P_{d+1} of $\mathcal{K}_d$ is a weakly sporadic $(d+1)$-dimensional F-hollow lattice simplex with the minimal multiplier $\mu = (d+2)/3$ having no other lattice points besides its vertices. The affine equation of $\mathcal{K}_d \cap \mathbb{A}^{d+1} \subset \mathbb{A}^{d+1} = \{z_0 = 1\}$:

$$t_1^2 + t_1t_2^2 + t_2t_3^2 + t_3t_4^2 + \cdots + t_{d+1} = 0$$

considered with respect to the pair of variables x_1 and x_3 defines a conic C from Example 4.5. Note that the lattice $(d+1)$-simplex P_{d+1} admits several different lattice projections onto 2-dimensional simplex $Q := \mathrm{Conv}((0,0),(2,0),(0,2))$ defining corresponding different conic bundle structures on the Klein cubic $\mathcal{K}_d$. If the dimension $d = 2k$ is even, then P_{2k+1} admits a lattice projection onto $[0,1]$ which sends $k+1$ lattice vertices of P_{2k+1} corresponding to cubic monomials $z_{2i}z_{2i+1}^2$ $(0 \leq i \leq k)$ to the lattice point 0 and the remaining $k+1$ lattice vertices of P_{2k+1} to the lattice point 1. By 4.3, we see that any even-dimensional Klein cubic $\mathcal{K}_{2k}$ is rational over any field K. Note that it is rather nontrivial to show that the 3-dimensional Klein cubic $\mathcal{K}_3$ is not rational over $\mathbb{C}$ (see [12]). Moreover, a still open conjecture claims that the Klein cubic $\mathcal{K}_{2k+1}$ is not rational over $\mathbb{C}$ in all odd dimensions $d = 2k+1 \geq 3$ (see [18]). This conjecture is supported by the fact that the $2k$-dimensional empty lattice simplex P_{2k} has no a lattice projection onto $[0,1]$ for all integers $k \geq 2$. We note that any d-dimensional toric hypersurface $Z \subset \mathbb{T}^{d+1}$ with the Newton polytope P_{d+1} is irreducible and smooth for $d \geq 3$.

5. Non-degenerate toric hypersurfaces and $\mathbb{Q}$-Fano fibrations

Definition 5.1. Let $Z \subset \mathbb{T}_K^d$ be an affine toric hypersurface given as zero locus of a Laurent polynomial

$$f(\mathbf{t}) = \sum_{\mathbf{m} \in A} c_{\mathbf{m}} \mathbf{t}^{\mathbf{m}} \in K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$$

with the Newton polytope $P = \mathrm{Conv}(A)$. The toric hypersurface Z is called **non-degenerate** if Z is smooth over the algebraic closure $\overline{K}$, and for any face $\Theta \prec P$ $(0 < \dim \Theta < d)$ the affine toric hypersurface $Z_\Theta \subset \mathbb{T}^d$ defined as zero locus of the Laurent polynomial

$$f_\Theta(\mathbf{t}) := \sum_{\mathbf{m} \in A \cap \Theta} c_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}$$

is reduced and smooth over $\overline{K}$ as well.

Example 5.2. Let P be a d-dimensional lattice simplex. Denote by A the set of its $d+1$ vertices. Assume that $\mathrm{char}\, K = 0$. Then the non-degeneracy of $Z \subset \mathbb{T}^d$ given by $\sum_{\mathbf{m} \in A} c_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}$ is equivalent to the nonvanishing condition

$$\prod_{\mathbf{m} \in A} c_{\mathbf{m}} \neq 0.$$

Now we want to explain why Theorem 1.11 is inspired by the Minimal Model Program for non-degenerate toric hypersurfaces $Z \subset \mathbb{T}^d$ with Newton polytope P.

It was proved in [8] that if the Fine interior $F(P)$ of a d-dimensional lattice polytope P is not empty, then every non-degenerate toric hypersurface $Z \subset \mathbb{T}^d$ with the Newton polytope P has a minimal model, i.e., a projective model $\widehat{Z}$ with at worst $\mathbb{Q}$-factorial terminal singularities and semi-ample canonical $\mathbb{Q}$-divisor. Moreover, the Fine interior $F(P)$ allows to compute the Kodaira dimension $\kappa(\widehat{Z})$ by the formula:

$$\kappa(\widehat{Z}) = \min\{d-1, \dim F(P)\}.$$

Now we are interested in birational geometry of non-degenerate toric hypersurfaces $Z \subset \mathbb{T}^d$ with F-hollow Newton polytope P.

Our main result is the following.

Theorem 5.3. *Let P be a d-dimensional weakly sporadic F-hollow lattice polytope P with the minimal multiplier $\mu = \mu(P) > 1$. Denote by $\mathbb{P}_{Q^*}$ the canonical toric $\mathbb{Q}$-Fano variety whose defining fan is spanned by faces of the canonical Fano polytope*

$$Q := \mathrm{Conv}(S_F(\mu P)).$$

We denote by $\widetilde{Z}$ the Zariski closures in $\mathbb{P}_{Q^*}$ of a non-degenerate toric hypersurface $Z \subset \mathbb{T}^d$ with the Newton polytope P. Then $\widetilde{Z}$ is a normal variety and the following adjunction formula holds:

$$K_{\widetilde{Z}} = (K_{\mathbb{P}_{Q^*}} + \widetilde{Z})|_{\widetilde{Z}} = \left(\frac{\mu - 1}{\mu}\right) K_{\mathbb{P}_{Q^*}}|_{\widetilde{Z}}.$$

Furthermore, $\widetilde{Z}$ has at worst canonical singularities and ample anticanonical $\mathbb{Q}$-divisor, i.e., $\widetilde{Z}$ is a $(d-1)$-dimensional canonical $\mathbb{Q}$-Fano hypersurface in $\mathbb{P}_{Q^*}$.

In the proof we need the following technical statement.

Lemma 5.4. *Let P be as in 5.3. Denote by $L(P) \subset \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ the linear system on the canonical toric $\mathbb{Q}$-Fano variety $\mathbb{P}_{Q^*}$ spanned by all monomials $\mathbf{t}^{\mathbf{m}}$ corresponding to lattice points $\mathbf{m} \in P \subseteq Q^* \cap M$. Then no codimension-2 torus orbit in $\mathbb{P}_{Q^*}$ is contained in the base locus of $L(P)$.*

Proof of Lemma 5.4. Take an arbitrary 1-dimensional edge $E \prec Q$ of the canonical Fano polytope $Q \subset N_R$. Let $\sigma_E \subset N_{\mathbb{R}}$ be the 2-dimensional cone over E in the fan Σ_{Q^*} defining the canonical toric $\mathbb{Q}$-Fano variety $\mathbb{P}_{Q^*}$. Since the lattice polytope Q has only one interior lattice point $0 \in N$, the lattice polygon $\mathrm{Conv}(0, E) \subset Q$ has no lattice points other than 0 and $E \cap N$. This means that σ_E is unimodular equivalent to the 2-dimensional cone $\sigma_k \subset \mathbb{R}^2$ spanned by lattice vectors $(1, 0)$ and $(1, k)$ for some $k \geq 1$. Let $\{e_0, e_1, \dots, e_k\} := E \cap S_F(\mu P)$ be the lattice vectors corresponding under the above equivalence to lattice vectors

$$(1, 0), (1, 1), \dots, (1, k) \in \sigma_k.$$

Denote by $N' \subset N$ the sublattice in N spanned by $e_0, e_1, \dots, e_k$. Then N' is a direct summand of N since each pair e_i, e_{i+1} forms a part of $\mathbb{Z}$-basis of N for every $i \in \{0, 1, \dots, k-1\}$. Take a $\mathbb{Z}$-basis $v_1, v_2, \dots, v_d$ of N such that $v_1 = e_0$, $v_2 = e_1$ and consider the standard lattice projection

$$\pi_E \,:\, M \to M' := Hom(N', \mathbb{Z}) \cong \mathbb{Z}^2$$

together the identification of lattice vectors $e_i \in N'$ with $(1, i) \in \mathbb{Z}^2$ for all $i \in \{0, 1, \dots, k\}$. Denote by $P'_E \subset \mathbb{R}^2$ the lattice polygon $\pi_E(P)$. Using a shift by an appropriate lattice vector $\mathbf{m} \in M$, we can assume without loss of generality that $\mathrm{Min}_P(e_0) = \mathrm{Min}_P(e_1) = 0$, that is, both hyperplanes $\langle x, e_0 \rangle = 0$ and $\langle x, e_1 \rangle = 0$ are integral supporting hyperplanes for the lattice polytope P and for the rational polytope μP. Since $F(\mu P)$ is just a rational point $p \in M_{\mathbb{Q}}$, we obtain

$$\langle p, e_0 \rangle = \mathrm{Min}_{F(\mu P)}(e_0) = \mathrm{Min}_{F(\mu P)}(e_1) = \langle p, e_1 \rangle = 1,$$

and by linearity of $\langle p, * \rangle$ we obtain

$$\langle p, e_i \rangle = \mathrm{Min}_{F(\mu P)}(e_i) = 1, \quad \forall i \in \{0, 1, \dots, k\}.$$

Since all lattice points $e_0, e_1, \ldots, e_k \in E \prec Q$ are contained in $S_F(\mu P)$, we obtain

$$\mathrm{Min}_{\mu P}(e_i) = 0 \ \ \forall i \in \{0, 1, \ldots, k\},$$

i.e., the linear equations $\langle x, e_i \rangle = 0$ define integral supporting hyperplanes for μP and for P.

In order to show that the codimension-2 torus orbit corresponding to σ_E is not contained in the base locus of $L(P)$, we have to show that the $(d-2)$-dimensional linear subspace $l_E \subset M_{\mathbb{R}}$,

$$l_E := \langle x, e_i \rangle = 0, \ \ \forall i \in \{0, 1, \ldots, k\},$$

contains at least one lattice vertex of P, or equivalently, the lattice π_E-projection $P'_E = \pi_E(P) \subset \mathbb{R}^2$ contains the origin $(0,0) = \pi_E(l_E) \in \mathbb{R}^2$. We consider two cases.
Case 1: $k \geq 2$. Then π_E-projection of P is a lattice polygon $P'_E \subset \mathbb{R}^2$ contained in the cone

$$C_k := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_1 + kx_2 \geq 0\}.$$

The supporting integral hyperplane $\langle x, e_1 \rangle = 0$ for P must contain at least one vertex of P. On the other hand, π_E-projection of this hyperplane is the line in $\mathbb{R}^2$ with the equation $x_1 + x_2 = 0$, and it has only the origin $(0,0)$ as common point with the cone C_k.
Case 2: $k = 1$. Then 2-dimensional cone $\sigma_1 \subset \mathbb{R}^2$ is regular, i.e., its generators $e_0, e_1 \in S_F(\mu P)$ form a $\mathbb{Z}$-basis of N'. Assume that $(0,0) \notin P'_E = \pi_E(P)$. Then one of two generators $(0,1)$ and $(1,-1)$ of σ_1 must belong to P'_E, since otherwise the lattice polytope P'_E would be contained in the convex set

$$B := \mathrm{Conv}(\{\sigma_1 \cap \mathbb{Z}^2\} \setminus \{(0,0), (0,1), (1,-1)\})$$

and the lattice point $(1,0) = \pi_E(F(\mu P))$ would not be in the interior of $\mu P'_E = \pi_E(\mu P) \subset \mu B$ for $\mu > 1$. So we obtain $\mathrm{Min}_P(e_0+e_1) = 1$ and $\mathrm{Min}_{\mu P}(e_0+e_1) = \mu > 1$. However, we have $\mathrm{Min}_{F(\mu P)}(e_0 + e_1) = \mathrm{Min}_{F(\mu P)}(e_0) + \mathrm{Min}_{F(\mu P)}(e_1) = 1 + 1 = 2$. This implies

$$\mathrm{Min}_{F(\mu P)}(e_1 + e_2) - \mathrm{Min}_{\mu P}(e_1 + e_2) = 2 - \mu < 1.$$

The latter contradicts the definition of Fine interior $F(\mu P)$.

Proof of Theorem 5.3. Let $\mathbb{P}^{(2)}_{Q^*} \subset \mathbb{P}_{Q^*}$ be the Zariski open toric subvariety obtained from $\mathbb{P}_{Q^*}$ by deleting all torus orbits of codimension at least 3 in $\mathbb{P}_{Q^*}$. The Gorenstein toric variety $\mathbb{P}^{(2)}_{Q^*}$ has a minimal crepant toric desingularization $\rho : \widehat{\mathbb{P}}^{(2)}_{Q^*} \to \mathbb{P}^{(2)}_{Q^*}$ by resolving of possible A_{k-1}-singularities along codimension-2 strata in $\mathbb{P}^{(2)}_{Q^*}$. Applying Lemma 5.4, we obtain that the Zariski closure $\widehat{Z}^{(2)}$ of the non-degenerate toric hypersurface Z in $\widehat{\mathbb{P}}^{(2)}_{Q^*}$ is smooth, and the Zariski closure $\widetilde{Z}^{(2)}$ of Z in $\mathbb{P}^{(2)}_{Q^*}$ is Gorenstein, and it has at worst A_{k-1}-singularities along the transversal intersections of the non-degenerate quasi-projective toric hypersurface $\widetilde{Z}^{(2)} \subset \mathbb{P}^{(2)}_{Q^*}$ with codimension-2 torus orbits in $\mathbb{P}^{(2)}_{Q^*}$. Thus we obtain that the singular locus of $(d-1)$-dimensional projective hypersurfaces $\widetilde{Z} \subset \mathbb{P}_{Q^*}$ has codimension at least 2 and, by Serre's criterion for normality, $\widetilde{Z}$ is normal. Now we can relate the canonical classes of $\widetilde{Z}$ and $\mathbb{P}_{Q^*}$ by the adjunction formula on the Gorenstein quasi-projective toric variety $\mathbb{P}^{(2)}_{Q^*}$.

Let $\dim F(\mu P) = 0$ and $p = F(\mu P)$. We consider the shifted rational polytope $P_0 := \mu P - p$ such that $F(P_0) = \{0\}$. Let $\Sigma_{Q^*}[1]$ be the set of generators of 1-dimensional cones in the fan defining the canonical toric $\mathbb{Q}$-Fano variety $\mathbb{P}_{Q^*}$. Then $\Sigma_{Q^*}[1] \subseteq S_F(\mu P)$ and we have $\mathrm{Min}_{P_0}(\nu) = -1$ for all $\nu \in \Sigma_{Q^*}[1]$.

The lattice polytope P defines on $\mathbb{P}_{Q^*}$ a toric Weil divisor

$$L_P := \sum_{\nu \in \Sigma_{Q^*}[1]} (-\mathrm{Min}_P(\nu)) D_\nu$$

such that $\mathbb{Q}$-Cartier divisor μL_P is rationally equivalent to the anticanonical class $-K_{\mathbb{P}_{Q^*}}$ of $\mathbb{P}_{Q^*}$, since

$$K_{\mathbb{P}_{Q^*}} = \sum_{\nu \in \Sigma_{Q^*}[1]} -D_\nu = \sum_{\nu \in \Sigma_{Q^*}[1]} \mathrm{Min}_{P_0}(\nu) D_\nu = \sum_{\nu \in \Sigma_{Q^*}[1]} \left(\mu \mathrm{Min}_P(\nu) - \langle p, \nu \rangle\right) D_\nu,$$

and for any $p \in M_{\mathbb{Q}}$ the $\mathbb{Q}$-divisor

$$\sum_{\nu \in \Sigma_{Q^*}[1]} \langle p, \nu \rangle D_\nu$$

is a principal $\mathbb{Q}$-divisor of $\mathbb{P}_{Q^*}$. We write the adjunction formula on the Gorenstein toric variety $\mathbb{P}^{(2)}_{Q^*}$ in the form

$$K_{\widetilde{Z}} = (K_{\mathbb{P}_{Q^*}} + L_P)|_{\widetilde{Z}} = (-\mu L_P + L_P)\,|_{\widetilde{Z}} = \left(\frac{\mu - 1}{\mu}\right) K_{\mathbb{P}_{Q^*}}|_{\widetilde{Z}}.$$

In particular, the canonical divisor of $\widetilde{Z}$ is a $\mathbb{Q}$-Cartier divisor. It order to show that singularities of $\widetilde{Z}$ are at worst canonical, we use a result of Khovanskiî that a non-degenerate toric hypersurface with Newton polytope P always admits smooth projective birational model W in a smooth toric variety defined by a regular refinement Σ of the normal fan Σ_P [17]. We can always choose a regular refinement Σ of Σ_P which is simultaneously a regular refinement of Σ_{Q^*}, so that we obtain for smooth toric hypersurface $W \subset \mathbb{P}_\Sigma$ with a birational morphism $\rho : W \to \widetilde{Z}$ together with a formula

$$K_W = \rho^* K_{\widetilde{Z}} + \sum_{\nu \in \Sigma[1]} a_\nu (D_\nu \cap W),$$

where $a_\nu = 0$ if $\nu \in S_F(\mu P)$, or if $D_\nu \cap W = \emptyset$, and $a_\nu = (-1 - \mathrm{Min}_{P_0}(\nu)) > 0$ for all $\nu \notin S_F(\mu P)$. Thus $\widetilde{Z}$ has at worst canonical singularities. □

Corollary 5.5. *Let P be a d-dimensional F-hollow lattice polytope and* $\dim F(\mu P) = k \geq 1$. *Denote by $\pi : P \to P'$ the lattice projection onto $(d-k)$-dimensional weakly sporadic F-hollow lattice polytope P' with $\mu(P') = \mu(P) = \mu$ constructed in 1.11. Then π defines a dominant morphism $\varphi : Z \to \mathbb{T}^k$ whose general fibers are non-degenerate hypersurfaces with the $(n-k)$-dimensional weakly sporadic F-hollow lattice polytope P' admitting $\mathbb{Q}$-Fano projective compactifications.*

Proof. We use Theorem 5.3 in relative situation by considering the $(n-k)$-dimensional fan $\Sigma' \subset N'_{\mathbb{R}}$, where $N' \subset N$ is the $(n-k)$-dimensional sublattice from Case 1 in the proof of Theorem 1.11. The fan Σ' is spanned by faces of $(n-k)$-dimensional canonical Fano polytope Q', where $Q' = \mathrm{Conv}(S_F(\mu P)) \cap N')$. We can identify the finite set $S_F(\mu P)) \cap N'$ with the support $S_F(\mu P')$ of the $(n-k)$-dimensional weakly sporadic F-hollow lattice polytope $P' = \pi(P) \subset M_{\mathbb{R}}$ with the minimal multiplier $\mu = \mu(P') = \mu(P)$. By Theorem 5.3, each general fiber of the dominant morphism $\varphi Z \to \mathbb{T}^k$ admits a natural $\mathbb{Q}$-Fano projective compactification which can be obtained in the following way.

Using [8, Theorem 6.3], we embed the non-degenerate toric hypersurface $Z \subset \mathbb{T}^d$ into d-dimensional $\mathbb{Q}$-Gorenstein toric variety $\widetilde{\mathbb{P}}$ associated with the Minkowski sum $F(\mu P) + C(\mu P)$, where $C(\mu P)$ is the d-dimensional rational polytope containing μP defined by the inequalities:

$$C(\mu P) := \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geq \mathrm{Min}_{\mu P}(\nu), \ \ \forall \nu \in S_F(\mu P)\}.$$

Since $F(\mu P)$ is a Minkowski summand of $\widetilde{P}$, the normal fan $\Sigma_{\widetilde{P}}$ contains the $(n-k)$-dimensional fan Σ' as a subfan describing toric $\mathbb{Q}$-Fano fibers of the toric morphism $\alpha : \widetilde{\mathbb{P}} \to \mathbb{P}_{F(\mu P)}$, where $\mathbb{P}_{F(\mu P)}$ is k-dimensional toric variety corresponding to k-dimensional rational polytope $F(\mu P)$. By restricting α to the Zariski closure $\widetilde{Z}$ of Z in $\widetilde{\mathbb{P}}$, we obtain the $\mathbb{Q}$-Fano fibration $\widetilde{\varphi} : \widetilde{Z} \to \mathbb{P}_{F(\mu P)}$ which extends the dominant morphism $\varphi : Z \to \mathbb{T}^k \subset \mathbb{P}_{F(\mu P)}$ to the hypersurface $\widetilde{Z} \subset \widetilde{\mathbb{P}}$. □

Corollary 5.6. *A non-degenerate hypersurface $Z \subset \mathbb{T}^d$ with Newton polytope P has negative Kodaira dimension if and only if P is F-hollow.*

Proof. If $F(P) \neq \emptyset$, then $Z \subset \mathbb{T}^d$ has a minimal model which can be constructed as Zariski closure $\widehat{Z}$ in some simplicial torus embedding $\mathbb{T}^d \subset \widehat{V}$. In particular, the Kodaira dimension of Z is non-negative [8].

If $F(P) = \emptyset$, then, by Theorems 1.11 and 3.4, if P is either weakly sporadic and $Z \subset \mathbb{T}^d$ is birational to a $\mathbb{Q}$-Fano hypersurface in the canonical $\mathbb{Q}$-Fano variety $\mathbb{P}_{Q^*}$, or, by 5.5, Z is birational to a $\mathbb{Q}$-Fano fibration $\widetilde{Z} \to \mathbb{P}_{F(\mu P)}$ over a k-dimensional toric variety. Therefore the projective toric hypersurface $\widetilde{Z}$ has negative Kodaira dimension. □

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Fachbereich Mathematik, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany

Email address: batyrev@math.uni-tuebingen.de