Dedicated to the memory of Yuri Ivanovich Manin

ZETA-POLYNOMIALS, SUPERPOLYNOMIALS, DAHA AND PLANE CURVE SINGULARITIES

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Abstract. We begin with modular form periods, a focal point of several Yuri Manin's works. The similarity between the corresponding zeta-polynomials and superpolynomials in the theory of refined knot invariants is discussed. We present 3 constructions (conjecturally coinciding) of superpolynomials: via DAHA, compactified Jacobians and L-functions of plane curve singularities, and provide some super-analogs of $\rho_{ab}-$ invariants (which is new). They conjecturally satisfy the Riemann Hypothesis in some sectors of the parameters. Presumably, they can be interpreted as partition functions of certain Landau-Ginzburg models, and there is a remarkable similarity with the Lee-Yang theorem. General perspectives of the passage to isolated curve and surface singularities are discussed, including possible implications in number theory.

Key words: double affine Hecke algebras; HOMFLY-PT polynomials; Alexander polynomials; rho-invariants; plane curve singularities; compactified Jacobians; affine Springer fibers; iterated torus links; knot invariants; Riemann hypothesis; Khovanov-Rozansky homology; Macdonald polynomials; Hasse-Weil zeta functions; Dirichlet L-functions; modular forms; Riemann hypothesis; Iwasawa theory; Landau-Ginzburg models; Witten index

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Contents

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1. Modular form periods

1.1. **Manin, my teacher.** In 1965-67, Yuri Ivanovich Manin and Ernest Borisovich Vinberg delivered special courses at Moscow School no 2 for senior students. This is when I met Yu.I. With some stretch, I can say that Manin and Vinberg were my high school teachers and Vasilii Iskovskikh and Victor Kac were our tutors (teaching assistants of Yu.I. and E.B).

Our regular relations with Yu.I. began about 1968, when he took me as his student at Moscow State University. My first assignment was reading Serre's "Corps locaux"; I learned Herbrand theory (the higher ramification), but cannot say the same about French.

Let me omit 50 years and go to 2017, the Arbeitstagung devoted to his 80th birthday. It was a great meeting! Yu.I. and Ksenia Glebovna were terrific hosts, many people were around, a perfect view of Rein from their apartment etc.

Mostly we discussed anything but mathematics, though something came up: *zeta-polynomials*, certain combinations of modular form periods satisfying Riemann Hypothesis, predicted by Manin.

My talk was mostly about DAHA *superpolynomials* $\mathcal{H}(q,t,a)$ for *double affine Hecke algebras*. Superpolynomials have several interpretations; the major one is via *Khovanov-Rozansky triply graded homology*.

This direction is in progress. Conjecturally, topological superpolynomials, those from the BPS states (SCFT), DAHA superpolynomials, motivic ones and L-functions of plane curve singularities coincide (when these theories overlap). I will focus on the latter three below; this note is introductory, with very few names and references.

1.2. **Modular periods.** Let us begin with Manin's well-known paper *"Periods of parabolic forms and p-adic Hecke series"* (1973). Basically, you consider a parabolic (cusp) form $\Phi(z)$ of even weight w and calculate its *periods* $r_k(\Phi) = \int_0^{\imath\infty} \Phi(z) z^k dz$ for $0 \le k \le w-2$. Then the ratios of r_k for even k or those for odd k are rational numbers, which can be calculated (Manin's theorem).

For instance, such Φ are proportional to $\Delta = e^{2\pi\imath z} \prod_{n=1}^{\infty}(1-e^{2\pi\imath n z})^{24}$ for $w=12$. Then $r_2/r_0 = -\frac{2^2 3^4 5}{691}$, $r_3/r_1 = -\frac{2^4 3}{5^2}$, etc. We note a relation to the Ramanujan's $\tau(n) = \sigma_{11} \mod 691$ (1916); Manin reproved it.

The periods are essentially the values $L_\Phi(s)$ of the corresponding L-function for integer s inside the critical strip. This can be extended to

$L_\Phi(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\lambda_n n^{-s}$ for suitable Dirichlet characters χ if Φ is an eigenfunction of the Hecke operators T_n with eigenvalues λ_n.

The second part of his paper was on the p-adic extrapolations of the ratios of the periods, which is closely related to the *Kubota-Leopoldt p-adic zeta function* and *eigenvarieties*. Concerning the origins of this direction, let me mention at least Barry Mazur and Nicholas Katz.

The periods are generally for any paths $\gamma[0,\imath\infty]$ for $\gamma\in SL(2,\mathbb{Z})$, but $[0,\imath\infty]$ is sufficient due to the modularity of Φ. However, more general paths do occur in the Manin's paper in process of calculations.

The p-adic extrapolations and the Kubota-Leopoldt zeta (1964) are closely related to the *Iwasawa invariants* of Γ-extensions due to Mazur and Wiles in full generality. The examples of Γ-extensions are some towers of cyclotomic fields, where we monitor the class numbers. Following Mazur, the Iwasawa invariants are parallel to the *Alexander polynomials*. The covers of the $S^3\setminus K$ for knots K in their theory are similar to Γ-extensions; those of $\mathbb{P}^2$ minus the corresponding plane curve singularity are sufficient for algebraic K due to Libgober and others.

The Alexander polynomials are $\mathcal{H}(q=t,t,a=-1)$ for the DAHA superpolynomials $\mathcal{H}$, and there is a relation to $\rho(q,t)$, refined *quasi-ρ-invariants* introduced and discussed below (for algebraic knots).

1.3. **Using DAHA.** The modular periods and DAHA are not really connected at the moment, but there is a clear common denominator: the action of $SL(2,\mathbb{Z})$. The main feature of DAHA is that it provides a universal formalization of *Fourier transforms* and the action of (projective) $SL(2,\mathbb{Z})$ in algebra, harmonic analysis and physics.

To be more exact, DAHA serve the theories with the Fourier transform and the Gaussian, where the latter is an eigenfunction of the former. The classical Fourier transform, its q-counterparts, the Hankel transform and the Verlinde S,T-operators are basic examples. DAHA is a universal (flat) deformation of the Heisenberg and Weyl algebras, so its role in Fourier analysis is not surprising.

Moreover, it appeared that DAHA provides invariants of iterated torus links. This is not very surprising because the *Verlinde algebras* are closely connected with the invariants of links and 3-folds. In DAHA theory, these algebras become *perfect quotients* of the polynomial representations. Hopefully, DAHA and the theory of modular forms and L-functions can eventually merge into one, but this will require efforts.

Number theory already provided some framework for quite a bunch of similar directions, which is hardly accidental. Let me quote from the Manin's paper: "... any points of contact with concrete number-theoretical facts, whether old or new, take on especial significance. They discipline the imagination, and they provide a breathing space and the opportunity to evaluate the stunning beauty of past discoveries." This is very much applicable now to the relations between number theory and physics (in both directions).

1.4. **Zeta-polynomials.** Next, in his *"Local zeta factors and geometries under Spec $\mathbb{Z}$"* (2014), Yu.I. conjectured that a certain combination of $L_\Phi(1), \ldots, L_\Phi(w-1)$ is a *zeta-polynomial*: satisfies the functional equation $s \mapsto 1-s$ and the Riemann Hypothesis. This was fully confirmed for $w \geq 4$ by Ken Ono, Larry Rolen and Florian Sprung in their paper *"Zeta-polynomials for modular form periods"* (2016).

Let $M_\Phi(m) \stackrel{\mathbf{def}}{=\!=} \sum_{j=0}^{w-2} \left(\frac{\sqrt{N}}{2\pi}\right)^{j+1} \frac{L_\Phi(j+1)}{(w-2-j)!} j^m$ for a $\Gamma_0(N)$-modular Φ. Then their zeta-polynomial $Z_\Phi(s)$ is a linear combination of $M_\Phi(m)$ for $m = 0, \ldots, w-2$ with the coefficients given in terms of Stirling polynomials of the 1st kind and the Fernando Rodriguez-Villegas transform. Manin used the latter too. There is a relation of zeta-polynomials to the Bloch-Kato Conjecture, which is a Galois cohomological interpretation of the periods, and related advanced number-theoretical problems.

What is important for us is that there is some "canonical" way to combine the modular periods in a zeta-polynomial, which resembles very much the theory of Witten-Reshetikhin-Turaev invariants of 3-folds invariants and their relations to knot invariants. DAHA invariants are "canonical" combinations of basic coinvariants in a similar way.

There are various connections of the *WRT-invariants* with modular forms. Let us at least mention *"Quantum invariants, modular forms, and lattice points II"* (K.Hikami, 2006). A challenge is to connect the inequalities $0 \leq k \leq w-2$ with those in *Verlinde algebras*, more exactly with the range of Macdonald polynomials in *prefect DAHA modules* at roots of unity, but this is fully open at the moment.

2. Basic DAHA theory

2.1. **Main definitions.** DAHA, denoted by $\mathcal{H}$, were initially introduced to complete the theory of Knizhnik-Zamolodchikov equations and Quantum Many Body Problem. It is a universal flat deformation of $\mathcal{W}$, the Weyl algebra extended by the Weyl group W; this is for any reduced root systems. The projective $SL(2,\mathbb{Z})$ due to Steinberg acts in $\mathcal{H}$. This is actually the braid group B_3; the notation will be $\widetilde{SL}(2,\mathbb{Z})$.

For A_1, $\mathcal{H}$ is generated by $X^{\pm1}, Y^{\pm1}, T$ subject to group relations $TXTX = 1 = TY^{-1}TY^{-1}, Y^{-1}X^{-1}YXT^2 = q^{-1/2}$ and the quadratic one $(T - t^{1/2})(T + t^{-1/2}) = 1$. The action of the $\widetilde{SL}(2,\mathbb{Z})$ is:

$$\tau_+ : Y \mapsto q^{-\frac{1}{4}}XY,\ X \mapsto X,\ T \mapsto T,\ \tau_- : X \mapsto q^{\frac{1}{4}}YX,\ Y \mapsto Y,\ T \mapsto T,$$

exactly as for $\mathcal{W}$ (there is no q here). The automorphisms $\tau_\pm$, the generators of $\widetilde{SL}(2,\mathbb{Z})$, are the preimages of $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, the standard generators of $SL(2,\mathbb{Z})$. The defining relation of $\widetilde{SL}(2,\mathbb{Z})$ is simple: $\tau_+\tau_-^{-1}\tau_+ = \sigma = \tau_-^{-1}\tau_+\tau_-^{-1}$, which formally gives that σ^2 is central. The element σ^{-1} is the (operator) *DAHA-Fourier transform*.

When $t^{1/2} = 1$, T becomes the inversion s of X and Y, and we arrive at $\mathcal{W}$. Upon $t = q$, DAHA is closely related to quantum groups. The case $t = q^k$ as $q \to 1$ serves the Harish-Chandra theory and its k-generalization, called Heckman-Opdam theory (in mathematics), including spherical functions and Jack polynomials. Also, $q \to 0$ is the p-adic limit and $t \to 0$ is the Kac-Moody limit. The action of $\widetilde{SL}(2,\mathbb{Z})$ generally collapses in the limits. However it survives in the important limit to rational DAHA and Hankel transforms, which is when $X = q^x, Y = q^{-y}$, $t = q^k$ and $q \to 1$ (for A_1).

Also, $\widetilde{SL}(2,\mathbb{Z})$ acts in the nonsymmetric *Verlinde algebras*, which are *perfect representations* of $\mathcal{H}$ when $t = q$ and q is a root of unity (below).

For an arbitrary reduced root system of rank n, $\mathcal{H}$ is generated by pairwise commutative X_λ for $\lambda \in P$, pairwise commutative Y_λ and T_i for $1 \le i \le n$ such that $(T_i - t^{1/2})(T_i + t^{-1/2}) = 0$, where different t can be generally used for long and short simple roots.

2.2. **Polynomial representation.** The key property of DAHA is the PBW theorem: any element $H \in \mathcal{H}$ can be uniquely represented as $H = \sum c_{\lambda,w,\mu} X_\lambda T_w Y_\mu$ for $\lambda, \mu \in P$ and $w \in W$ (the non-affine Weyl group). Equivalently, there is a faithful action of $\mathcal{H}$ in the *polynomial representation* $\mathscr{X} = \mathbb{C}[X_\lambda]$: the one induced from the character

$Y_\lambda \mapsto t^{(\rho,\lambda)}, T_i \mapsto t_i^{1/2}$. It is a deformation of the classical Fock representation of the corresponding Heisenberg-Weyl algebra. We will need below the *DAHA coinvariant*: $\{\,H\,\} = \sum c_{\lambda,w,\mu} t^{(\mu-\lambda,\rho)+\text{length}(w)/2}$, which is $H(1)(X_\lambda \mapsto t^{-(\rho,\lambda)})$, where $H(1)$ is the action of $H \in \mathcal{H\!\!H}$ at $1 \in \mathscr{X}$.

Comments. Technically, the simplest definition of $\mathcal{H\!\!H}$ is with T_0 instead of Y_λ. It is from the $T\times\mathbb{C}^*-$ equivariant K-theory of affine flag varieties, one of the interpretations of DAHA. Then Y_λ are defined as some products in terms of T_i for $i \geq 0$ and their commutativity is some proposition. The construction of Y_λ via $\{T_i\}$ is essentially due to Bernstein-Zelevinsky and Lusztig.

Finding explicit defining relations between X_λ and Y_μ in the approach via Y is somewhat involved unless for A_n (and in small ranks). Also, the action of $\widetilde{SL(2,\mathbb{Z})}$ is far from obvious from the K-theoretical viewpoint; the Fourier transform is a challenge in algebraic geometry. The key property of any Fourier transforms is that they send polynomials to delta-functions, which is not simple to incorporate.

DAHA can be defined via the orbifold fundamental group of the *elliptic configuration space*: $\pi_1^{orb}\Big(\big(x \in E^n \mid \prod_\alpha (x,\alpha) \neq 0\}\big)/W\Big)$, where we use that the roots α are with integer coefficients in terms of the fundamental weights. Here E is an elliptic curve and W is a nonaffine Weyl group. Then we take the group algebra and impose the quadratic relations for T_i as above, which are topologically the half-turns corresponding to simple reflections. This configuration space is the "big cell" in $Bun_G(E)$ for the corresponding simple Lie group G. The element σ becomes basically the transposition of the periods of E.

The existence of the action of $\widetilde{SL(2,\mathbb{Z})}$ is straightforward from this definition. However, the polynomial representation is far from immediate, which is almost by construction via the K-theory of affine flag varieties. I used both approaches. The exact connection with affine flag varieties was clarified somewhat later: Garland-Grojnowski and Ginzburg-Kapranov-Vasserot (1995).

The action of $\widetilde{SL(2,\mathbb{Z})}$ can be introduced in other approaches. The main other ones are (3): via the harmonic analysis (σ becomes the Fourier transform), (4): *elliptic Hall algebras* (Schiffmann-Vasserot, ...), and (5): in terms of the *shuffle algebras*. However, some algebraic verifications (including (4)) are needed and (4,5) are for GL_N.

Given a reduced root system, the *nonsymmetric Macdonald polynomials* E_λ for $\lambda \in P$ generalize the monomials X_λ for the corresponding Weyl algebra and form a basis of $\mathscr{X}$. They are eigenfunctions of Y_μ normalized by the conditions $E_\lambda = X_\lambda +$ (lower terms). The action of $\mathcal{HH}$ in $\mathscr{X}$ is reasonably explicit. For A_1: $T \mapsto t^{1/2}s + \frac{t^{1/2}-t^{-1/2}}{X^2-1}(s-1)$, $X \mapsto X,\ Y \mapsto spT$, where $s(X) = X^{-1},\ p(X) = q^{1/2}X$. The divided differences are very standard in the theory of affine Hecke algebras and are quite common in related geometry and algebraic combinatorics.

For GL_n, the corresponding $\mathcal{HH}$ is generated by $X_i^{\pm 1}, Y_j^{\pm 1}, T_k$, where $1 \le i,j \le n$ and $1 \le k \le n-1$ for pairwise commutative $\{X_i\}$ and $\{Y_j\}$. One has: $\tau_+(Y_1) = q^{-1/2}X_1Y_1,\ \tau_-(X_1) = q^{+1/2}Y_1X_1$ and so on. The action of Y_1 in $\mathscr{X}$ is via the formula $Y_1 = \pi T_{n-1}\dots T_1$, where $\pi : X_1 \mapsto X_2, X_2 \mapsto X_3, \dots, X_n \mapsto q^{-1}X_1$. These formulas are quite similar for any Y_i. We will use them below when calculating the DAHA-superpolynomial of trefoil (as an example).

Going back to the modular periods, the evaluation map $X \mapsto t^{-\rho}$, *the DAHA coinvariant*, plays a role of integration $\int_0^{\imath\infty}\{\cdot\}\Phi dz$, E_λ replace z^k, and the action of $\widetilde{\gamma}$ corresponds to the change of the integration path to $\gamma[0,\imath\infty]$. The main deviation is that the action of γ plays a much more significant role for DAHA superpolynomials versus that for the periods. In contrast to $\int_0^{\imath\infty}\{\cdot\}\Phi dz$, the DAHA coinvariant is not stable in any way with respect to the action of $\widetilde{SL(2,Z)}$.

When switching to the zeta-polynomials $Z_\Phi(s)$, special linear combinations of z^k-momenta must be considered, which resembles our usage of E_λ. Both constructions are "canonical" in a sense; the restriction $0 \le k \le w-2$ seems somewhat similar to those in Verlinde algebras.

The next topic, *DAHA-Verlinde algebras*, gives a direct link of DAHA at roots of unity to number theory. They are some counterparts of *Tate modules*, where the covers of elliptic curves are ramified at one point; the *absolute Galois group* acts there. The Verlinde algebras are one of the key ingredients of the *Witten-Reshetikhin-Turaev invariants*, generalize K_0 of the *reduced category* in representation theory of *quantum groups* at roots of unity, and that of integrable modules of *Kac-Moody algebras* (Kazhdan-Lusztig, Finkelberg).

2.3. **Refined Verlinde algebras.** These algebras are *perfect* finite-dimensional quotients of the polynomial representations $\mathscr{X}$: those with the action of $\widetilde{SL(2,\mathbb{Z})}$ and invariant non-degenerate quadratic forms. By construction, they are commutative algebras, but can be non-semisimple, related to *logarithmic* CFT in examples. Technically, we divide $\mathscr{X}$ by the radical of the evaluation pairing. Such modules exist either when q is a root of unity or for *singular* k, where $t=q^k$.

In the case of A_1, let $q=\exp(\frac{2\pi i}{N}), k\in\frac{\mathbb{Z}_+}{2}$ and $k<N/2$. The map $X(z)=q^z$ can be naturally extended to an $\mathcal{H}$-homomorphism $\mathbb{C}[X^{\pm1}]\to V_{2N-4k}$, the *nonsymmetric Verlinde algebras*, which is the space of functions $f:\{-\frac{N+k+1}{2},...,-\frac{k+1}{2},-\frac{k}{2},\frac{k+1}{2},...,\frac{N-k}{2}\}\to\mathbb{C}$ with pointwise multiplication. The formulas for the action of X,T,Y in $\mathscr{X}$ are compatible with this map. These modules are *rigid*, which readily gives an action of $\widetilde{PSL_2(\mathbb{Z})}$ there, and in $V^{sym}_{N-2k+1}\stackrel{\textbf{def}}{=\!=}\{v\in V\,|\,Tv=t^{\frac{1}{2}}v\}$. The indices are the dimensions: $\dim V_{2N-4k}=2N-4k$, $N-2k+1$ is $\dim(V_{2N-4k})^{sym}$. The operators X,Y,T become unitary in V_{2N-4k} if the "minimal" primitive Nth root $q=e^{\frac{2\pi\imath}{N}}$ is taken.

We classified *rigid* modules for A_1 in *"On Galois action in rigid DAHA modules"* (2017). They are: $(\boldsymbol{\alpha})$ V_{2N-4k} as above, $(\boldsymbol{\beta})$ non-semisimple $V_{2N+4|k|}$ for $k\in-\mathbb{Z}_+$ such that $-N/2<k<0$, and $(\boldsymbol{\gamma})$ $V_{2|k|}$ for $k=-\frac{1}{2}-m>-N/2$, where $m\in\mathbb{Z}_+$. There is a similar list for the *little* DAHA $\mathcal{H}'=\langle X^{\pm2},Y^{\pm2},T\rangle\subset\mathcal{H}$. Importantly, families $(\boldsymbol{\alpha},\boldsymbol{\gamma})$ have *flat* q-deformations, where q is arbitrary. The unimodular q such that $\arg q\le\frac{2\pi}{N}$ result in the positivity of the invariant form in type $(\boldsymbol{\alpha})$. Such a deformation leads to some relations between V defined for different N (similar to those for the Tate modules).

The usual *Verlinde algebra* is V^{sym}_{N-1} of type $(\boldsymbol{\alpha})$, which is for $k=1$, i.e. for $t=q$. Then τ_+ becomes the T-operator, and $\sigma=\tau_+\tau_-^{-1}\tau_+$ becomes the Verlinde S-operator. The *reduced characters* in Verlinde algebras are replaced by eigenfunctions of Y in V, the images of the corresponding Macdonald E-polynomials (symmetric ones for V^{sym}).

Perfect representations are quotients of the ones obtained from $\mathscr{X}$ by fixing the corresponding central characters, which are of dimension $4N$. For $k=1$, the symmetrizations V^{sym} of the latter are connected with the category of representations of *small quantum group*. For instance, V^{sym}_{N-1} is the Grothendieck ring K_0 of the so-called *reduced category* for A_1. The perfect representations for $\mathbb{Z}/2\ni k\neq 1$ are generally beyond quantum groups, though the ones of type $(\boldsymbol{\beta})$ are connected with *logarithmic* conformal field theories and there are other links.

2.4. **The Galois action.** The rigidity provides that the *absolute Galois group* acts in the modules above (including the usual Verlinde algebras). We use that *elliptic braid group* $\mathcal{B}_q$ generated by X, T, Y subject to the group relations in the definition of $\mathcal{HH}$ of type A_1 is a renormalization of the orbifold fundamental group $\pi_1^{orb}(E/\{1, s\})$, where E is an elliptic curve, $s : x \mapsto -x$. If E and its origin o are defined over some field $\mathbb{Q}[q^{1/4}] \subset K \subset \overline{\mathbb{Q}}$, then $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts projectively in these modules.

More exactly, setting $A = XT$, $B = XTY$, $C = T^{-1}Y$, the relations of $\mathcal{B}_q$ and the action of $\tau_\pm$ there become as follows:

$$A^2 = 1 = C^2 = q^{1/2}B^2, \text{ where } ABC = A^2YT^{-1}Y = YY^{-1}T = T,$$

$$\tau_+ : A \mapsto A, \; B \mapsto q^{-1/4}C, \; C \mapsto q^{1/4}C^{-1}BC,$$

$$\tau_- : A \mapsto q^{1/4}ABA^{-1}, \; B \mapsto q^{-1/4}A, \; C \mapsto C.$$

Thus, the classification of $\mathcal{HH}$-modules at roots of unity q, t becomes equivalent to the corresponding multiplicative *Deligne-Simpson problem* with specific quadratic relations for $A, B, C, D = T^{-1}$. They can be arbitrary quadratic for DAHA of type $C^\vee C_1$ (Sahi, Noumi-Stokman); let me also mention Oblomkov-Stoica (2009). It is generated by A, B, C, D satisfying any quadratic relations such that $ABCD = 1$, those for the monodromy of the *Heun equation*. There are links to SCFT.

The images of $\mathcal{B}_q$ in type ($\boldsymbol{\alpha}$) rigid modules with positive-definite invariant forms are finite and we obtain finite covers of $\mathbb{P}^1$ ramified at $0, 1, \infty$ and $o \in E(K)$, where A, B, C, D are the corresponding monodromies. When $t = 1$, we arrive at unramified covers of E.

The case of the Hermitian invariant forms with one minus is interesting. Then the images of $\mathcal{B}_q$ are discrete groups. The smallest non-trivial such V is for *little* $\mathcal{HH}$; its dimension is 3. We obtain then *all Livné lattices* in $PU(2, 1)$, which are examples of the *Mostow groups*. They occurred in his thesis via a certain branched cover of degree 2 of the *universal elliptic curve*.

Also, there is a direct connection with the theory of equilateral *triangle groups* in $PU(2, 1)$; for instance, see *"Complex hyperbolic triangle groups"* (R.E. Schwartz, 2002) and *"Cone metrics on the sphere and Livné's lattices"* (Parker, 2006).

We mention here that the (regular) *Inverse Galois Problem* is based on *rigid triples*, which are $\{a, b, c\}$ generating a group G and satisfying $abc = 1$. They are assumed from given conjugacy classes in G and the rigidity means essentially the uniqueness of such $\{a, b, c\}$ up to (simultaneous) conjugations in G. We need $\{a, b, c, d\}$ here and 4 points in $\mathbb{P}^1$. Also, the so-called *linear rigidity* (in matrices) based on Katz' theory of rigid systems is required (M. Dettweiler and others).

Such covers extend the Belyi's theorem and Grothendieck's program of *dessins d'enfants* to E; let us mention Beilinson-Levin (1991). However, we deal only with very "small" covers: those from DAHA modules. This is similar to Tate modules, though the ramified counterpart of $T_p(E) = \varprojlim(E/E_{p^n})$ as $n \to \infty$ is not a module over p-adic numbers.

3. Knot invariants via DAHA

3.1. **DAHA-Jones polynomials.** The definitions above are sufficient to introduce the *refined invariants* of torus knots $T(r, s)$. The algebraic knots are with $r, s > 0$ and such that $\gcd(r, s) = 1$. They can be represented as $T(r, s) = \{x^r = y^s\} \cap S^3$ for a small sphere S^3 centered at 0. The formula for the corresponding *DAHA-Jones invariant* is:

$$J^{\lambda}_{r,s}(q, t) = \left(\widetilde{\gamma}\big(\frac{E_\lambda}{E_\lambda(t^{-\rho})}\big)(1)\right)(X \mapsto t^{-\rho}),$$

where $(r, s)^{tr}$ is the 1st column of $\gamma \in SL(2, \mathbb{Z})$, $\widetilde{\gamma}$ is its action in $\mathcal{H}$, and the Laurent polynomial $\widetilde{\gamma}(E_\lambda)(1)$ is $\widetilde{\gamma}(E_\lambda) \in \mathcal{H}$ applied to $1 \in \mathscr{X}$. It is not necessary to assume that $r, s > 0$ in this definition; the corresponding torus knots will be non-algebraic for $rs < 0$.

An important theorem is that it is always a q, t-polynomial (up to some fractional power $q^\bullet t^\bullet$) in spite of the q, t-singularities of E and $E_\lambda(t^{-\rho})$ in the denominator. The latter is given by the nonsymmetric evaluation formula in the DAHA theory. At roots of unity q, $\frac{E_\lambda}{E_\lambda(t^{-\rho})}$ are generally singular for λ beyond the *perfect representations*; however $J^{\lambda}_{r,s}(q, t)$ are well-defined for any λ.

The usage of the *symmetric Macdonald polynomial* P_λ for dominant λ is sufficient here. Namely, E_λ can be replaced by its t-symmetrization (applying the t-symmetrizer), which results in P_λ. The latter become the Weyl characters when $q = t$; they do not depend on t in this case. In the uncolored A_n-case, the usage of $E_{\omega_1} = X_{\omega_1}$ is the key to connect $J^{\omega_1}_{r,s}(q, t)$ with the Shuffle Conjecture (a theorem now due to Carlsson-Mellit). *We will omit $\lambda = \omega_1$ below for uncolored invariants.*

The colored *Jones polynomials* for $T(r, s)$ are obtained by this formula when $q = t$ for A_1; the weights are $m\omega_1$ in this case. To be more exact, this identification is up to some factor $q^\bullet t^\bullet$; both theories come with their own normalization (framing).

The last step is the A_n-stabilization of $J_K^\lambda(q,t)$ for $K=T(r,s)$. For any $P\in q^u t^v\mathbb{C}[q,t]$ for rational u,v, let $P^\circ \stackrel{\mathbf{def}}{=\!=} q^\bullet t^\bullet P\in 1+q\mathbb{C}[[q,t]]+t\mathbb{C}[[q,t]]$. If P also depends on $a^{\pm 1}$ (in the next paragraphs), then $P^\circ \stackrel{\mathbf{def}}{=\!=} q^\bullet t^\bullet a^\bullet P\in 1+q\mathbb{C}[[q,t]]+t\mathbb{C}[[q,t]]+a\mathbb{C}[[q^{\pm1},t^{\pm1}]]$.

For the a-stabilization, $\lambda=\sum_{i=1}^n m_i\omega_i$ are considered as Young diagrams (partitions of $|\lambda|$); m_i is the number of columns with i boxes.

The claim is that there exists a unique polynomial $\mathcal{H}_K(q,t,a)$ in terms of q,t,a such that $J_K^\lambda(q,t)^\circ=\mathcal{H}_K^\lambda(q,t,a=-t^{n+1})$ for J_K^λ constructed for the root system A_n, where $K=T(r,s)$ Automatically, $\mathcal{H}_K(q,t,a)^\circ=\mathcal{H}_K(q,t,a)\subset\mathbb{C}[q,t^{\pm1},a]$. This construction was extended to any colored iterated torus links (I.Ch. and Danilenko).

The starting point of this theory was due to Aganagic-Shakirov (2011) and the author (2011). Concerning the related physics, let me mention at least the paper by Gukov, Iqbal, Kozcaz and Vafa (2010). The stabilization of J_K^λ for A_n is based on a DAHA theorem due to Schiffmann-Vasserot (2012). Let me also mention Gorsky-Negut, concerning the proof of the DAHA-superduality for torus knots.

3.2. **The case of trefoil.** Let us calculate $\mathcal{H}_{3,2}$ for uncolored trefoil; we begin with A_1. As above, $\{H\}\stackrel{\mathbf{def}}{=\!=}H(1)(X\mapsto t^{-\rho})$, where $t^{-\rho}=t^{-\frac12}$ for A_1. By $\sim$, we mean "up to $q^\bullet t^\bullet$". One has:

$$\begin{aligned}J_{3,2}&=\{\tau_+\tau_-^2(X)\}\sim\{(XY)(XY)X(1)\}\sim\{Y(X^2)\}\\&=t^{-\frac12}q^{-1}X^2-t^{\frac12}+t^{-\frac12}|_{X^2\mapsto t^{-1}}\sim 1+qt-qt^2.\end{aligned}$$

When $q=t$, we obtain the *Jones polynomial*: $J_{3,2}(q\mapsto t)^\circ=1+t^2-t^3$.

We use that $E_1=X$: $Y(X)=(qt)^{-\frac12}X$. Using the formula for Y_1 for A_n above and the action of $\tau_\pm$ on X_1,Y_1, we obtain that $J_{3,2}^\circ=1+qt-qt^{n+1}$, which gives that $\mathcal{H}_{3,2}=1+qt+aq$ for $a=-t^{n+1}$. The relations $\mathcal{H}_{3,2}(a\mapsto -t)=1$ and $\mathcal{H}_{3,2}(a\mapsto -t^2)=1+qt-qt^2$ are sufficient to fix it uniquely if it is known that $deg_a\mathcal{H}=1$. Generally, $deg_a\mathcal{H}_{r,s}^\lambda=|\lambda|\big(\mathrm{Min}(r,s)-1\big)$. A remarkable simplicity of $\mathcal{H}_{3,2}$ is fully clarified in the approach via motivic superpolynomials (below).

A similar manipulation works for $T(2p+1,2)$ for $p=0,1,2,\ldots$. One obtains: $\mathcal{H}_{2p+1,2}=1+qt+q^2t^2+\cdots+q^pt^p+aq(1+qt+\cdots+$

$(qt)^{p-1})$. These knots are the simplest, including direct calculations with Khovanov-Rozansky polynomials. The formula becomes significantly more involved with colors. Let $\lambda = m\omega_1 = \square\square\cdots\square$ (m boxes) for $m = 1, 2, \ldots$. Then:

$$\mathcal{H}^{\lambda}_{2p+1,2} = \frac{(q;q)_m}{(-a;q)_m(1-t)}\sum_{k=0}^{m}(-1)^{m-k}$$
$$(qt)^{\frac{m-k}{2}}\big((q^{\frac{m(m+1)}{2}} - q^{\frac{k(k+1)}{2}})(t/q)^{\frac{m-k}{2}}\big)^{2p+1}$$
$$\frac{(t;q)_k(-a;q)_{m+k}(-a/t;q)_{m-k}(1-q^{2k}t)}{(q;q)_k(qt;q)_{m+k}(q;q)_{m-k}},$$

where $(a;q)_n = (1-a)\cdots(1-aq^{n-1})$. This formula was proposed by Dunin-Barkowski-Mironov-Morozov-Sleptsov-Smirnov (2011-12), and, independently, by Fuji-Gukov-Sulkowsky (2012). A somewhat different formula is $\mathcal{H}^{\lambda}_{3,2} = \sum_{k=0}^{m} q^{mk}t^k \frac{(q;q)_m(-a/t;q)_k}{(q;q)_k(q;q)_{m-k}}$ (only for trefoil). The justifications were obtained via DAHA, i.e. for the DAHA superpolynomials. The Habiro's formula (2000) is for $p = 1, a = -t^2, t = q$.

Let us mention here *"Torus knots and quantum modular forms"* devoted to color Jones polynomials for $T(2p+1, 2)$ (K.Hikami-Lovejoy, 2014), and the *Kontsevich-Zagier series* from *"Vassiliev invariants and a strange identity related to the Dedekind eta-function"* (Zagier, 2001). Presumably, our refined formulas above can be used in a similar way. See also Example 5 from *"Quantum modular forms"* (Zagier, 2010).

3.3. **Iterated knots.** For any sequence $\gamma_1, \gamma_2, \ldots, \gamma_\ell \in SL(2,\mathbb{Z})$, we set $J^{\lambda}_K = \Big(\cdots \widetilde{\gamma}_{\ell-1}\Big(\widetilde{\gamma}_\ell\big(\frac{E_\lambda}{E_\lambda(t^{-\rho})}\big)(1)\Big)(1)\cdots\Big)(t^{-\rho})$. The knot K here is the corresponding *torus iterated knot*; see an example below.

This is somewhat similar to Manin's work *"Iterated integrals of modular forms and noncommutative modular symbols"* (2005). Basically, $\int_0^\infty$ is replaced by $\int_p^q$ for rational p, q in this paper and multiple zeta values occur. When the coinvariant is replaced by the corresponding integral formula (a DAHA theorem), the similarity becomes less speculative.

For instance, one obtains for $K = Cab(53,2)Cab(13,2)Cab(2,3)$:

$$J^{\lambda}_{\mathcal{L}} = \{\mathcal{P}_\lambda\},\ \mathcal{P}_\lambda = \Downarrow\begin{pmatrix}3 & * \\ 2 & *\end{pmatrix} \Downarrow\begin{pmatrix}2 & * \\ 1 & *\end{pmatrix} \Downarrow\begin{pmatrix}2 & * \\ 1 & *\end{pmatrix}\Big(\frac{E_\lambda(X)}{E_\lambda(t^{-\rho})}\Big),$$

where the γ-matrices act via their lifts to $\mathrm{Aut}(\mathcal{HH})$, $\Downarrow H \stackrel{\mathbf{def}}{=} H(1)$, $\{H\} \stackrel{\mathbf{def}}{=} H(1)(t^{-\rho})$ is the coinvariant, and E_λ is the E-polynomial for dominant λ. Basically, $Cab(a,b)K$ is $T(a,b)$ at the boundary of the solid torus around a given knot K for the standard framing.

Generally, given a sequence $\begin{pmatrix} r_1 & r_2 & r_3 & \cdots \\ s_1 & s_2 & s_3 & \cdots \end{pmatrix}$ of the 1st columns of $\gamma_1, \gamma_2, \gamma_3, \ldots$, which is $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$ in the example above, the corresponding cable is $\cdots Cab(a_3, r_3)Cab(a_2, r_2)Cab(a_1, r_1)$ for $a_1 = s_1$, $a_2 = r_1 s_1 r_2 + s_2$, $a_3 = a_2 r_2 r_3 + s_3$ and so on. Technically, $Cab(a_1, r_1) = T(r_1, s_1)$, which can be changed to $Cab(r_1, a_1)$ due to the isotopy (only for (a_1, r_1)).

Generally, $\deg_a(\mathcal{H}^\lambda) = |\lambda|(mult - 1)$ for the multiplicity of the corresponding singularity; here $mult = \mathrm{Min}(r_1, s_1) \cdot r_2 \cdot r_3 \cdots$.

3.4. **Superduality and RH.** The *superduality* is associated with the transformation $q \leftrightarrow t$ for type-A stable Macdonald polynomials; in physics, it is related to the S-*duality* in SCFT via the BPS states and the *CPT symmetry*. We note that it is difficult to interpret for the Khovanov-Rozansky triply-graded homology. Generally, arbitrary colors and links are significantly simpler to manage via DAHA (for iterated torus links) than in the *categorification theory*. Also, DAHA-Jones polynomials are defined for any reduced root systems and $C^\vee C_1$. Some a-stabilization is expected for B, C, D.

The superduality was suggested by Gukov-Stosic and the author. It was justified using some DAHA facts by Gorsky-Negut (torus knots) and Cherednik-Danilenko. It is as follows in terms of the standard DAHA parameters: $\mathcal{H}_K^\lambda(q,t,a) = q^\bullet t^\bullet \mathcal{H}_K^{\lambda'}(\frac{1}{t}, \frac{1}{q}, a)$, were by $q^\bullet t^\bullet$ we mean "up to some power of q,t"; λ' is the transposition of λ.

Many formulas and properties of superpolynomials were obtained and conjectured by physicists. Their works are mostly experimental, though the BPS states can be defined rigorously.

We note here that there are some restrictions when comparing DAHA $\mathcal{H}$ with motivic ones below. *Motivic superpolynomials* are by now for algebraic knots and the colors are "columns" by now. This is the same for the *flagged L-functions* (below), which are conjectured to coincide with superpolynomials. The latter provide a simple proof of superduality.

The a-stabilization of $J^\lambda(q,t)$ for $q = t$ corresponds to that in the theory of HOMFLY-PT polynomials, $HOM(t,a)$. The quantum group invariants for A_n are essentially $HOM(t, a = t^{n+1})$. This stabilization is connected with the *Deligne category* $Rep(GL(t))$.

HOMFLY-PT polynomials. The definition of $HOM(t, a; \lambda)$ is especially simple in the uncolored case, which is for $\lambda = \square$ (i.e. for $\lambda = \omega_1$ for A_n). The following *skein relation* is sufficient to define them:

$$a^{1/2} HOM(\nwarrow\!\!\!\!\nearrow) - a^{-1/2} HOM(\nwarrow\!\!\!\!\nearrow) = (t^{1/2} - t^{-1/2}) HOM(\uparrow\uparrow),\ HOM(\bigcirc) = 1.$$

Given λ (type A), $HOM^{\circ}_K(t, a; \lambda) = \mathcal{H}^{\lambda}_K(q = t, t, a \mapsto -a)$ for *iterated torus links* K under the normalization $P^{\circ} \stackrel{\mathbf{def}}{=\!=} q^{\bullet} t^{\bullet} a^{\bullet} P \in 1 + q\mathbb{C}[[q, t]] + t\mathbb{C}[[q, t]] + a\mathbb{C}[[q^{\pm 1}, t^{\pm 1}, a]]$. This is due to Cherednik (torus knots), Morton-Samuelson (iterated torus knots), and Cherednik-Danilenko (iterated torus links). Algebraic links are torus iterated links, but the latter constitute a significantly larger class.

In the case of the uncolored trefoil $K = T(3, 2)$ (when $\lambda = \omega_1$): $HOM = a(t + t^{-1} - a)$, $HOM^{\circ} = 1 + t^2 - ta$; recall that $\mathcal{H} = 1 + qt + qa$. The Alexander polynomials $Al(q)$ are generally $HOM(t, a = 1)$ for knots; in particular, $Al = t^{-1} - 1 + t$, $Al^{\circ} = 1 - t + t^2$ for trefoil.

The superduality becomes $t^{\frac{1}{2}} \to -t^{-\frac{1}{2}}, a^{\frac{1}{2}} \to a^{-\frac{1}{2}}$; it obviously preserves HOM_K due to the skein relations above (in the uncolored case). Generally, the Young diagram λ goes to its transpose. This holds for Al too, but the (quantum group) A_n-invariants, which are basically $HOM_K(t, a = t^{n+1}; \lambda)$, do not have such a symmetry under $t \to t^{-1}$. Here and in DAHA-Jones polynomials, we need to "separate" a.

RH for superpolynomials. After our talks with Yu.I. in 2017, I focused on RH for DAHA superpolynomials. Let $\mathbf{H}(q, t, a) \stackrel{\mathbf{def}}{=\!=} \mathcal{H}(qt, t, a)$, i.e. we switch to $q_{new} = q/t$, which is fixed under the superduality. Then $\mathbf{H}(q, 1/(qt), a) = q^{\bullet} t^{\bullet} \mathbf{H}(q, t, a)$ and the *qualitative* RH is that $|\xi| = 1/\sqrt{q}$ for the t-zeros ξ of $\mathbf{H}(q, t, a)$ for *for sufficiently small* q.

Such "weak" RH is not too difficult to verify for (uncolored) *motivic* superpolynomials, conjecturally coinciding with DAHA ones. A strong version for $a = 0$ is that RH holds for $0 < q \leq 1/2$ for any uncolored algebraic knots. Arbitrary $q > 0$ can be taken for $T(2p + 1, 2)$: $\mathbf{H}(q, t, a = 0) = \frac{1 - (qt^2)^{p-1}}{1 - qt^2}$. However, this holds only for them. For torus knots, the value $q = 1$ is exceptional: $\mathbf{H}(q = 1, t, a = 0)$ becomes a product of cyclotomic polynomials due to the Shuffle Conjecture (now a theorem). However, we need the lower bound for *all* q where RH does *not* hold, which is smaller than 1 for sufficiently large torus knots.

Numerically, this bound tends to $\frac{1}{2}$ for $Cab(13+2m,2)Cab(2,3)$ as $m \to \infty$. They are algebraic knots corresponding to the singularity rings $\mathcal{R} = \mathbb{C}[[z^4, z^6+z^{7+2m}]]$ (see below). Interestingly, this bound can become greater (better) for multiple cables or if the cables begin with torus knots different from $T(3,2)$. For instance, it is somewhat better for $Cab(53,2)Cab(13,2)Cab(2,3)$ versus $Cab(13,2)Cab(2,3)$; numerically, 0.6816 versus 0.6686 for $a=0$.

This is from my paper *"Riemann hypothesis for DAHA superpolynomials and plane curve singularities"* (2018). RH can be stated for algebraic links too; there can be exceptional non-RH zeros, but their number depends on the number of connected components. Colors can be added too (see the paper). Generally, RH does not hold for *non-algebraic* knots/links; it seems a really algebraic phenomenon.

The substitution $q \mapsto qt$ in the passage from $\mathcal{H}$ to $\mathbf{H}$ in RH is a convenient technicality: we replace the DAHA duality $q \leftrightarrow t^{-1}$ by $q \mapsto q, t \mapsto 1/(qt)$. However this is not accidental. The supersymmetry of $\mathbf{H}$ was connected in the "RH paper" with the Hasse-Weil symmetry $t \to 1/(qt)$ for plane curve singularities (below). Some program toward Riemann's zeta and the Dirichlet L-functions was outlined there. It is on its way, at least for the lens spaces.

4. Plane curve singularities

The (uncolored) Alexander polynomials of algebraic knots can be described in a very explicit way via the corresponding plane curve singularities. Let us provide the necessary definitions.

Algebraic links are defined as intersections of singularities at $(0,0) \in \mathbb{C}^2$ with small $S^3 \subset \mathbb{C}^2$ centered at $(0,0)$; they are algebraic *knots* for irreducible (unibranch) singularities. For such *knots*, the corresponding (local) singularity rings can be considered inside $\mathbb{C}[[z]]$, where z is the uniformizing parameter. By definition, irreducible *plane curve singularities* are those for any local rings $\mathcal{R} \subset \mathbb{C}[[z]]$ with 2 generators and the localization $\mathbb{C}((z))$. They are all *Gorenstein* (see below).

The simplest topological invariants of a singularity are its multiplicity $\dim \mathbb{C}[[z]]/\mathbb{C}[[z]]\mathfrak{m}$ for the maximal ideal $\mathfrak{m} \subset \mathcal{R}$, and the arithmetic genus $\delta = \dim \mathbb{C}[[z]]/\mathcal{R}$, the Serre number.

The rings $\mathcal{R} = \mathbb{C}[[x=z^r, y=z^s]]$ for $r, s \in \mathbb{N}$ such that $\gcd(r,s)=1$ correspond to *quasi-homogeneous singularities* $x^s = y^r$ and torus knots

$T(r,s)$. The multiplicity is $\mathrm{Min}(r,s)$ and $\delta = \frac{(r-1)(s-1)}{2}$, which is actually due to Sylvester (the Frobenius coin problem). The simplest "non-torus" plane curve singularity is for $\mathcal{R} = \mathbb{C}[[z^4, z^6+z^7]]$.

The families $\mathcal{R} = \mathbb{C}[[z^r, z^{s+mr}]]$ for $m \in \mathbb{Z}_+$ and similar ones for other $\mathcal{R}$ are natural to consider. This is related to the theory of *Drinfeld-Vléduts bound* (1983) and the paper by Manin-Vléduts *"Linear codes and modular curves"* (1985). There is parallel to Γ-extensions in the Iwasawa theory if Puiseux theory is used to interpret these families as towers of extensions of $\mathbb{C}[[x,y]]$. Generalizing δ, a counterpart of the class number, we try to obtain superpolynomials for the whole family, which is connected with Rosso-Jones iteration formulas for knots.

Due to Mazur, the Γ-extensions are counterparts of abelian coverings of S^3 ramified at links K in the theory of Alexander polynomials. For algebraic K, coverings of $\mathbb{C}^2$ ramified at the corresponding singularity are sufficient to consider (Libgober and others). The *valuation semigroups* are very suitable here.

4.1. **Valuation semigroups.** The following definition is one of the key in the theory: $\Gamma \stackrel{\mathbf{def}}{=\!=} \{\nu_z(f), f \in \mathcal{R} \subset \mathcal{O} \stackrel{\mathbf{def}}{=\!=} \mathbb{C}[[z]]\}$, where ν_z is the valuation, the order of z. This is a semigroup. It gives the topological type of the corresponding algebraic knot (considered up to isotopy), which is due to Zariski and others. One has: $\delta = |\mathbb{Z}_+ \setminus \Gamma|$.

The Alexander polynomial, its normalization Al° to be precise, is $(1-t)\sum_{\nu\in\Gamma} t^\nu$ for any $\mathcal{R} \subset \mathcal{O}$. For instance, it is $(1-t)(\frac{1}{1-t} - t) = 1 - t + t^2$ for trefoil $T(3,2)$. We note that the theory of topological equivalence of algebraic *links* is known too, but it is significantly more ramified; *splice diagrams* are generally needed.

For $Cab(53,2)Cab(13,2)Cab(3,2)$ above (note the change $(2,3) \mapsto (3,2)$), the ring is $\mathcal{R}=\mathbb{C}[[x=z^8, y=z^{12}+z^{14}+z^{15}]]$. The Newton's pairs are generally $\{r_1,s_1\},\{r_2,s_2\},\cdots$, and the Puiseux-type equation is $y = x^{\frac{s_1}{r_1}}\Big(1+c_1x^{\frac{s_2}{r_1r_2}}\big(1+c_2x^{\frac{s_3}{r_1r_2r_3}}(\cdots)\big)\Big)$ for generic c_i. We assume that $r_1 < s_1$; this can be always imposed in the corresponding cable.

The arithmetic genus is $\delta = 42$, and the valuation semigroup $\Gamma = \langle 8, 12, 26, 53\rangle$. Generally, $\Gamma = \langle r_1r_2r_3,\, a_1r_2r_3,\, a_2r_3,\, a_3\rangle$ and so on for the cable parameters (a_i, r_i) above (here $r_1 < s_1$ is used).

The passage from $\mathbb{C}$ to finite fields $\mathbb{F}_q$ for $q = p^k$ is sufficiently straightforward; we will need it below. A prime number p is called a *prime of good reduction* if Γ remains unchanged over $\mathbb{F}_p$. Namely, we begin with $\mathcal{R}$ over $\mathbb{C}$, define it over $\mathbb{Z}$, which is doable within a given topological type, and then consider $\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{F}_p$.

All primes p are good for quasi-homogeneous rings $\mathbb{C}[[x=z^r, y=t^s]]$ (for torus knots). Presumably, there are no prime p of bad reduction within a given topological type for any plane curve singularity.

For instance, consider $\mathcal{R}=\mathbb{Z}[[x=t^4, y=t^6+t^7]]$. One has: $\Gamma=\{0,4,6,8,10,13,14,16,17,18,\ldots\}$ and $\delta=8$. This $\mathcal{R}$ has one prime of bad reduction, which is $p=2$. Indeed, $\nu_z(y^2-x^3)=14$ in $\mathbb{F}_2$, which is 13 for $p\neq 2$. However, this singularity is in the same isotopy class as the one for $\mathbb{Z}[[t^4+t^5,t^6]]$, where bad p becomes 3. We conclude that the corresponding algebraic knot, which is $Cab(13,2)Cab(2,3)$, has no primes of bad reduction.

4.2. **Compactified Jacobians.** Let $\mathcal{R}\subset\mathcal{O}\stackrel{\mathbf{def}}{=\!=}\mathbb{F}[[z]]$ be the ring of an irreducible plane curve singularity over any field $\mathbb{F}$. The corresponding *flagged compactified Jacobian* $\mathcal{J}_\ell$, considered as a set of $\mathbb{F}$-points by now, is formed by *standard flags* $\overrightarrow{M}=M_0\subset M_1\subset\cdots\subset M_\ell\subset\mathcal{O}=\mathbb{F}[[z]]$ of $\mathcal{R}$-submodules M_i of $\mathcal{O}$ such that (a) $M_0\ni\phi=1+z(\cdot)$ (where $(\cdot)\in\mathcal{O}$), (b) $\dim M_i/M_{i-1}=1$ and $M_i=M_{i-1}\oplus\mathbb{C}\,z^{g_i}(1+z(\cdot))$, and (c) (important) $g_i<g_{i+1}$, where $i\geq 1$. We will call them ℓ-flags.

When $\ell=0$ (0-flags), there is only one condition: $\mathcal{O}\supset M\ni\phi=1+z(\cdot)$. Equivalently, $\Delta(M)\ni 0$, where $\Delta(M)\stackrel{\mathbf{def}}{=\!=}\{\nu_z(v)\mid v\in M\}$.

Generally, $\Delta(M)$ are Γ-modules for any $\mathcal{R}$-modules M, i.e. $\Gamma+\Delta\subset\Delta$. *Standard* Δ are those in $\mathbb{Z}_+$ containing 0 and, therefore, containing the whole Γ. Thus, standard M are those with standard $\Delta(M)$.

For quasi-homogeneous singularities $\mathcal{R}=\mathbb{F}[[x=z^r, y=z^s]]$, where $\gcd(r,s)=1$, $r,s>1$, all standard Γ-modules Δ come from some standard M. However, this holds only for such $\mathcal{R}$. For instance, for $\mathbb{F}[[z^4,z^6+z^7]]$, two from 25 such Δ are not in the form $\Delta(M)$ for any standard M, which phenomenon is due to Piontkowski.

Let us supply $\mathcal{J}_0$ with a structure of a *projective* variety. Generally, it is a scheme (we will reduce it). The main steps are as follows.

First, any standard M contains the ideal $(z^{2\delta})=z^{2\delta}\mathcal{O}$. Indeed, the latter is the *conductor* of $\mathcal{R}$ for any Gorenstein $\mathcal{R}$, the greatest ideal in $\mathcal{O}$ that belongs to $\mathcal{R}$. Using this, $\phi=1+z(\cdot)\in M$ (it is standard) implies that $\phi\cdot(z^{2\delta})=(z^{2\delta})\subset M$.

Second, let $dev(M) \stackrel{\textbf{def}}{=\!=} \delta - \dim(\mathcal{O}/\mathcal{R})$ (its deviation from $\mathcal{R}$); this is for any $\mathcal{R}$-modules in $\mathcal{O}$. Then $dev(M) \geq 0$ for standard M and it is 0 if and only if $M = \phi\mathcal{R}$ for some ϕ as above. The latter modules are called *invertible*. They form the *generalized Jacobian variety* of this singularity. Third (the key), $z^{dev(M)}M \supset (z^{2\delta})$ for standard M due to Pfister-Steenbrink. Equivalently, $dev(M) + \Delta(M) \supset 2\delta + \mathbb{Z}_+$.

Finally, consider the map $M \mapsto M' = z^{dev(M)}M$. Then $dev(M') = dev(M) - dev(M) = 0$. It establishes an identification of standard M with $\mathcal{R}$-modules $(z^{2\delta}) \subset M' \subset \mathcal{O}$ such that $dev(M') = 0$. The inverse map is $M' \mapsto z^{-d}M'$ for $d = \mathrm{Min}\{\nu_z(m) \mid m \in M'\}$. Then $\{M'\}$ is an algebraic subvariety of the Grassmannian of the subspaces of the middle dimension in $\mathcal{O}/(z^{2\delta})$, i.e. it is a projective variety. This is the *compactified Jacobian*. The spaces $\mathcal{J}_\ell$ are natural fiber spaces over $\mathcal{J}_0$.

Isotopy. For torus knots, the relations are: $T(r,s)$, $T(s,r)$ and $T(-s,-r)$ are isotopic, $T(1,s)$ is unknot, and $T(-s,r)$ is the mirror image of $T(s,r)$. The corresponding identities for $\mathcal{H}^\lambda_{r,s}$ are some DAHA facts. The symmetry $\mathcal{H}^\lambda_{r,s} = \mathcal{H}^\lambda_{s,r}$ can be a challenge for other algebraic interpretations and modifications of HOMFLY-PT polynomials.

The topological invariance of the DAHA superpolynomials $\mathcal{H}^\lambda_K$ was proven for any iterated torus links K, which follows from some properties of the DAHA *coinvariant*, not too difficult. Iterated torus links form a small class. Their isotopy invariance is well-known, though the case of iterated torus *links* is somewhat ramified.

4.3. **Motivic superpolynomials.** Let the field $\mathbb{F}$ above be $\mathbb{F}_q$. Given $\mathcal{R} \subset \mathcal{O}$ over $\mathbb{F}_q$, its *motivic superpolynomial* is defined as follows:

$\mathcal{H}^{mot} \stackrel{\textbf{def}}{=\!=} \sum_{\{M_0 \subset \cdots \subset M_\ell\} \in \mathcal{J}_\ell(\mathbb{F})} t^{\dim(\mathcal{O}/M_\ell)} a^\ell$ for ℓ-flags $\overrightarrow{M} \subset \mathcal{O}$, where $\ell \geq 0$.

It can be presented in terms of $rk_q(M) \stackrel{\textbf{def}}{=\!=} \dim_{\mathbb{F}_q} M/\mathfrak{m}M$ for the maximal ideal $\mathfrak{m}$ of $\mathcal{R}$: $\mathcal{H}^{mot} = \sum_M t^{\dim(\mathcal{O}/M_\ell)}(1+aq)\cdots(1+aq^{rk_q(M)-1})$, where the summation is over all *standard* $M \subset \mathcal{O}$.

This is from *"DAHA and plane curve singularities"* (Cherednik-Philipp, 2017); the latter presentation is a combination of Proposition 2.3 in this paper and the q-binomial theorem.

We conjectured there that $\mathcal{H}^{mot} = \mathcal{H}$, i.e. the motivic one for $\mathcal{R}$ coincides with the uncolored DAHA superpolynomial $\mathcal{H}_K(a,q,t)$ for the knots K associated with the singularity for $\mathcal{R}$. The definition of

$\mathcal{H}^{mot}$ and this conjecture were extended later (with Philipp) to torsion free sheaves of any ranks over irreducible plane curve singularities. This includes the (conjectural) claims that $\mathcal{H}^{mot}$ polynomially depends on q and that it is a *topological* invariant.

We note that the motivic construction is faster than the DAHA one for $T(r,s)$. However, the later is faster for cables and works with any colors and for other root systems. Also, motivic superpolynomials are significantly faster to calculate than flagged L-functions (below).

Affine Springer fibers. Our $\mathcal{J}_0$ can be interpreted as a (parahoric) affine Springer fiber $\mathcal{X}_\gamma$ for GL_n or GL_m, where n and m are the top x-degree and y-degree in the equation of the corresponding plane curve singularity. Generally, AFS are due to Kazhdan-Lusztig (1988). Their description entirely in terms of $\mathcal{R}$ is a remarkable feature of type A. Using GL_n or GL_m, the corresponding AFS coincide, which is not immediate from their definition (below). The standard modules M and $\mathcal{J}_0$ depend only on $\mathcal{R}$ by definition.

For semisimple Lie algebra $\mathfrak{g}$ and any field $\mathbb{F}$, let $\mathfrak{g}[[x]] = \mathfrak{g}\otimes_{\mathbb{F}}\mathbb{F}[[x]]$ and $\mathfrak{g}((x)) = \mathfrak{g}\otimes_{\mathbb{F}}\mathbb{F}((x))$. Accordingly, we define $G[[x]]$ and $G((x))$ for simply-connected G with $\mathrm{Lie}(G)=\mathfrak{g}$.

Given $\gamma\in\mathfrak{g}[[x]]$, $\mathcal{X}_\gamma \stackrel{\mathbf{def}}{=\!=} \{g\in G((x))/G[[x]] \mid g^{-1}\gamma g\in\mathfrak{g}[[x]]\}$, where we assume that the centralizer of γ in $G((x))$ is *anisotropic* (the nil-elliptic case). Then $\mathcal{X}_\gamma\cong\mathcal{J}_0$ in type A, where the singularity is $P(x,y)=0$ for the characteristic polynomial $P(x,y)=\det(\mathbf{1}y-\gamma)$. For instance, $e(\mathcal{X}_\gamma)=e(\mathcal{J}_0)=\mathcal{H}^{mot}(q\!=\!1,t\!=\!1,a\!=\!0)$ (conjecturally) for the Euler characteristic $e(\cdot)$. The corresponding p-adic orbital integral is $\mathcal{H}^{mot}(q,t\!=\!1,a\!=\!0)$, where $\mathbb{F}=\mathbb{F}_q$.

Also, our compactified Jacobians occur as *Jacobian factors* if projective rational singular curves are considered, which is related to *Hitchin fibers*. Generally, they are formed by families of subtori $\mathcal{T}\subset\mathcal{G}$ with fixed characteristic polynomials in the group schemes corresponding to *factorizable Lie algebras* $\mathfrak{G}$ (Cherednik, 1983).

Such $\mathfrak{G}$ are schemes of Lie algebras over smooth projective E with $\mathfrak{g}$ as the generic fiber. They are vector bundles but can have degenerate fibers as Lie algebras. The factorization conditions are $H^0(E,\mathfrak{G})=\{0\}=H^1(E,\mathfrak{G})$ for Čech cohomology. Such $\mathfrak{G}$ are in 1-1 correspondence with *not necessarily unitary classical r-matrices* $r(u,v)\in\mathfrak{g}^{\otimes 2}$: those satisfying the identity $[r^{12},r^{13}+r^{23}]=[r^{13},r^{32}]$, where r^{ij} is

$r(u_i, u_j)$ taking values in the components i, j of $\mathfrak{g}^{\otimes 3}$, u_i are near 0. We assume that $r - \Omega/(u-v)$ is regular for the "permutation" $\Omega \in \mathfrak{g}^{\otimes 2}$.

The link to AFS is basically as follows. We start with a subscheme $\mathcal{T} \subset G$, which is a maximal subtorus at the generic point of E. Since $H^1(E, \mathcal{G}) = \{0\}$, any cocycle ϕ in the *generalized Jacobian* $H^1(E, \mathcal{T})$ becomes the boundary $\{\phi_i \phi_j^{-1}\}$ for an open cover $E = \cup_i U_i$ and $\phi_i \in H^0(U_i, \mathcal{G})$. We obtain $\mathcal{T}_\phi = \phi_i^{-1} \mathcal{T} \phi_i \subset \mathcal{G}$ with the same characteristic polynomial as $\mathcal{T}$. Then consider $\mathcal{T} = \mathbb{G}_m(C)$ for rational projective C over $E = \mathbb{P}^1$ with one singularity and perform the compactification.

Let us mention that the connection between the dimensions of cells of $\mathcal{J}_0$ and the deviations (much simpler to find) was observed by Lusztig-Smelt (1995) for $\mathcal{R} = \mathbb{F}_q[[z^r, z^s]]$, a preimage of our superduality.

The coincidence $\mathcal{H}^{mot} = \mathcal{H}$ is checked reasonably well (mostly numerically), including the cases when the *Piontkowski cells* (2007) are not all affine spaces $\mathbb{A}^N$. These cells are defined as follows.

Let $\Delta(\overrightarrow{M}) = \{\Delta(M_i)\}$. We call $\overrightarrow{\Delta} = \{\Delta_0 \subset \cdots \subset \Delta_\ell \subset \mathbb{Z}_+\}$ *standard* if Δ_0 contains Γ, $\Delta_i = \Delta_{i-1} \cup \{g_i\}$ and $g_i < g_{i+1}$ for $1 \le i \le \ell$. Given a standard $\overrightarrow{\Delta}$, $\mathcal{J}_\ell(\overrightarrow{\Delta}) = \big\{\overrightarrow{M} \in \mathcal{J}_\ell \mid \Delta(\overrightarrow{M}) = \overrightarrow{\Delta}\big\}$. Then $\mathcal{J}_\ell = \cup\, \mathcal{J}_\ell(\overrightarrow{\Delta})$, where the union is disjoint. Some $\mathcal{J}_\ell(\overrightarrow{\Delta})$ can be empty; actually, this is always the case unless for quasi-homogeneous singularities $x^s = y^r$.

Generally, they are conjectured to be certain configurations of affine spaces: some their unions in proper $\mathbb{A}^N$ with nonzero intersections, which can be disconnected and non-equidimensional in examples.

If all (nonempty) cells of $\mathcal{J}_0$ are affine spaces, which holds for torus knots and some other exceptional families, then we readily obtain that the coefficient of q^i in $\mathcal{H}^{mot}(q, t=1, a=0)$ is the Betti number $b_{2i} = \mathrm{rk}\, H_{2i}(\mathcal{J}_0)$. For instance, $\mathcal{H}^{mot}(q=1, t=1, a=0)$ is the Euler number $e(\mathcal{J}_0)$. The latter is the rational Catalan number $\frac{1}{r+s}\binom{r+s}{r}$ for $\mathcal{R} = \mathbb{F}[[z^r, z^s]]$ (Beauville). This is the number of standard Δ for such $\mathcal{R}$, which is provided by classical combinatorics of Dyck paths.

One can replace here $\mathcal{H}^{mot}$ by $\mathcal{H}$. We conjectured with Ivan Danilenko that the relation to Betti numbers of $\mathcal{J}_0$ always holds for $\mathcal{H}$. More generally, the conjecture was that the *geometric superpolynomials* defined in terms of *Borel-Moore homology* of $\mathcal{J}_0$ coincide with the DAHA ones.

4.4. **Some examples.** Let us consider $\mathcal{J}_\mathbf{0}$ for $\mathcal{R} = \mathbb{F}[[z^4, z^6 + z^7]]$ with $K = Cab(13,2)Cab(2,3)$ discussed above. All cells are affine spaces and we show only dim =dim $J_0(\Delta)$ in the table below for the corresponding sets of gaps $D \stackrel{\mathbf{def}}{=\!=} \Delta\setminus\Gamma$ for standard Δ. One has $dev(D)=|D|$ and $\dim \mathcal{O}/M=\delta-|D|$, which gives the power of t. Two standard Δ from 25 have no standard M, namely for $D=[2,15]$ and $D=[2,11,15]$. The table of D and the corresponding dimensions of the cells $\mathcal{J}_0(\Delta)$ is:

D-sets	*dim*	D-sets	*dim*
∅	8	1,3,5,7,9,11,15	2
15	7	2,7,11,15	6
11,15	6	2,9,15	7
7,11,15	6	2,9,11,15	6
9,15	7	2,7,9,11,15	5
9,11,15	5	2,3,7,9,11,15	4
7,9,11,15	4	2,5,9,11,15	5
3,7,9,11,15	4	2,5,7,9,11,15	3
5,9,11,15	5	2,3,5,7,9,11,15	1
5,7,9,11,15	3	1,2,5,7,9,11,15	3
3,5,7,9,11,15	2	1,2,3,5,7,9,11,15	0
1,5,7,9,11,15	4	2,15 and 2,9,15	∅

The whole (uncolored) superpolynomial is: $\mathcal{H}_K(q,t,a) = \ 1+qt+q^8t^8+q^2\big(t+t^2\big)+q^3\big(t+t^2+t^3\big)+q^4\big(2t^2+t^3+t^4\big)+q^5\big(2t^3+t^4+t^5\big)+q^6\big(2t^4+t^5+t^6\big)+q^7\big(t^5+t^6+t^7\big)+a\big(q+q^2\big(1+t\big)+q^3\big(1+2t+t^2\big)+q^4\big(3t+2t^2+t^3\big)+q^5\big(t+4t^2+2t^3+t^4\big)+q^6\big(t^2+4t^3+2t^4+t^5\big)+q^7\big(t^3+3t^4+2t^5+t^6\big)+q^8\big(t^5+t^6+t^7\big)\big)+a^2\big(q^3+q^4\big(1+t\big)+q^5\big(1+2t+t^2\big)+q^6\big(2t+2t^2+t^3\big)+q^7\big(2t^2+2t^3+t^4\big)+q^8\big(t^3+t^4+t^5\big)\big)+a^3\big(q^6+q^7t+q^8t^2\big)$.

For instance, there are 3 cells of dimensions 7 in $\mathcal{J}_0$ (for $a=0$). Namely, those with $D=[15],[9,15],[2,9,15]$ and t^7,t^6,t^5. Generally, the number of cells of dim$=\delta-1$ is the multiplicity of singularity; it equals the coefficient of t for $q=1, a=0$ due to the superduality, which is for $\mathbb{Z}_+\setminus\Delta=\{1\},\{2\},\{3\}$ in this example. Only $\{1\}$ from them has dim$=1$, which is for qt; its reflection is $D=[2\delta-1]$ for $(qt)^{\delta-1}$.

In the case of $K=Cab(53,2)Cab(13,2)Cab(2,3)$ and the ring $\mathcal{R}=\mathbb{F}[[z^8,z^{12}+z^{14}+z^{15}]]$ discussed above, one has: $\mathcal{H}_K(q,t=1,a=0)= q^{42}+7q^{41}+24q^{40}+56q^{39}+104q^{38}+166q^{37}+236q^{36}+306q^{35}+370q^{34}+424q^{33}+465q^{32}+492q^{31}+507q^{30}+510q^{29}+504q^{28}+488q^{27}+466q^{26}+437q^{25}+406q^{24}+370q^{23}+335q^{22}+298q^{21}+264q^{20}+230q^{19}+199q^{18}+168q^{17}+143q^{16}+118q^{15}+97q^{14}+78q^{13}+63q^{12}+48q^{11}+38q^{10}+28q^{9}+21q^8+15q^7+11q^6+7q^5+5q^4+3q^3+2q^2+q+1.$

Here $\delta = 42$, which corresponds to q^{42} (invertible modules). The coefficients of q^i are the Betty number b_{2i} (the odd ones vanish), and the Euler number $e(\mathcal{J}_0)$ is 8512 (which is for $a=0, t=1, q=1$).

The simplest $\mathcal{H}^{mot}$ is for trefoil $T(3,2)$. Its singularity ring is $\mathcal{R} = \mathbb{F}_q[[z^2, z^3]]$ with $\Gamma = \mathbb{Z}_+ \setminus \{1\}$. There are no primes of bad reduction for this and any torus knots. The standard modules are $M_\lambda = (1+\lambda z)$ (invertible ones) of $\dim \mathcal{O}/M = 1$ and $M = \mathcal{O}$, where there are 2 generators (dim$=0$). The standard flags for $\ell = 1$ are $\{M_0 = M_\lambda \subset M_1 = \mathcal{O}\}$; the dimension is $\dim \mathcal{O}/M_1 = 0$ for them. Thus $\mathcal{H}^{mot} = 1\,(\text{for}\,\mathcal{O}) + qt\,(\text{counting invertible modules}) + aq\,(\text{counting flags})$. This calculation is almost equally simple for $T(2p{+}1, 2)$.

5. Zetas for singularities

5.1. **Galkin's zeta.** V.M. Galkin studied in 1973 zeta- and L-functions for Gorenstein rings in dimension one. Plane curve singularities are an important particular case. We will consider the unibranch case: for $\mathcal{R} \subset \mathcal{O} = \mathbb{F}_q[[z]]$ with 2 generators and the localization $\mathbb{F}_q((z))$. Recall that $\delta = \dim_{\mathbb{F}_q} \mathcal{O}/\mathcal{R} = |\,\mathbb{Z}_+ \setminus \Gamma\,|$.

The *admissible flags* of ideals in $\mathcal{R}$ are $\overrightarrow{M} = \{M_0 \subset M_1 \subset \cdots \subset M_\ell \subset \mathcal{R}\}$ such that $\{z^{-m_0} M_i \subset \mathcal{O}\}$ for $m_0 = \mathrm{Min}(\nu_z(M_0))$ are *standard flags* in $\mathcal{O}$ as in Section 4.2. The *flagged zeta function* is:

$$Z(q,t,a) \overset{\mathbf{def}}{=\!=} \sum_{\overrightarrow{M}} a^\ell t^{\,\dim(\mathcal{R}/M_\ell)} = \sum_{M} t^{\,\dim(\mathcal{R}/M)} (1{+}aq) \cdots (1{+}aq^{rk_q(M)-1}),$$

where the summation is over all admissible flags $\overrightarrow{M} \subset \mathcal{R}$ and over all ideals $M \subset \mathcal{R}$ in the 2nd formula; dim$=$dim$_{\mathbb{F}_q}$, $rk_q(M) = \dim M/\mathfrak{m}M$.

The *flagged L-function* is then $L(q,t,a) \overset{\mathbf{def}}{=\!=} (1-t)Z(q,t,a)$; it is a polynomial in terms of t, a, and $t^{-\delta}L(q,t,a)$ is invariant under $t \mapsto 1/(qt)$, which is the *functional equation*. In contrast to the smooth case, the Riemann Hypothesis generally fails.

The definition of the Galkin zeta, which is $Z(q,t,a)$ for $a = 0$ (no flags), is sufficiently standard: a Dirichlet series. The functional equation for L is actually surprising because generally there is no Poincaré duality for singular varieties (unless intersection cohomology is used or so). Stöhr found a short entirely combinatorial proof of this fact, a significant simplification of that due the John Tate's thesis. Tate's p-adic proof works well for curve singularities.

The key here is the following property of Γ, which is actually the defining property of *Gorenstein rings*: the map $g \mapsto g' = 2\delta - 1 - g$ identifies $\{g \in \mathbb{Z}_+ \setminus \Gamma\}$ (the set of "gaps") with $\{g' \in \Gamma \setminus \{2\delta + \mathbb{Z}_+\}\}$. For instance, the last gap, which is $2\delta - 1$, maps to $g' = 0$.

Let $\mathbf{H}^{mot}(q,t,a) \stackrel{\mathbf{def}}{=\!=} \mathcal{H}^{mot}(qt,t,a)$ for motivic $\mathcal{H}^{mot}$ above: we switch to $q_{new} = q/t$ as we did for $\mathbf{H}(q,t,a)$. Then, conjecturally:

$$\mathbf{H}^{mot}(q,t,a) = L(q,t,a) \text{ for any plane curve rings } \mathcal{R} \subset \mathcal{O},$$

$$\mathbf{H}^{mot}(q,t,a{=}-1/q) = L_{\text{prncpl}}(q,t) \text{ for Gorenstein } \mathcal{R} \subset \mathcal{O}.$$

. The latter is the *Zúñiga zeta function*: for $a = 0$ and when the summation is only over *principle* $M \subset \mathcal{R}$. It is indeed $L(q,t,a{=}-1/q)$; see the alternative formula for flagged L provided above.

When $q \to 1$ (for the "field" with 1 element): $Z_{\text{prncpl}} = \sum_{\nu\in\Gamma} t^\nu$ and $L_{\text{prncpl}} = (1-t)Z_{\text{prncpl}}$ is the Alexander polynomial $Al^\circ(t)$. Namely,

$$\lim_{q\to 1} L_{\text{prncpl}} = (1-t)\big(\sum_{i=1}^{\delta} t^{g_i}\big) + t^{2\delta} \text{ for } \{g_i\} = \Gamma \setminus (2\delta + \mathbb{Z}_+).$$

We obtain that $\big(L_{\text{prncpl}} - t^{2\delta}\big)/(1-t)$ becomes δ when $q \to 1$ and $t = 1$.

The conjectural coincidence of $\mathbf{H}(q,t,a)$ (DAHA), $\mathbf{H}^{mot}(q,t,a)$ and $L(q,t,a)$ identifies the superduality for the former with the functional equation for the later. The conjectural RH-bound for $\mathbf{H}(q,t,a{=}0)$ is $q \le 1/2$, i.e. far from "arithmetic" $q = p^m$ for $L(q,t,0)$.

The coincidence $\mathbf{H}^{mot}(q,t,a)$ and $L(q,t,a)$ is simple for $t = 1$ (for any rings $\mathcal{R} \subset \mathcal{O}$, not only Gorenstein ones). Indeed, any admissible flag of ideals $\overrightarrow{M}' \subset \mathcal{R}$ is $z^{m'}\overrightarrow{M}$ for standard $\overrightarrow{M} \subset \mathcal{O}$, where $m', \overrightarrow{M}$ are uniquely determined by $\overrightarrow{M}'$. Vice versa, given a standard flag $\overrightarrow{M}$, let:

$$\{m \mid z^m\overrightarrow{M} \subset \mathcal{R}\} = \{0 \le m_1 < m_2 < \cdots < m_k < 2\delta\} \cup \{2\delta + \mathbb{Z}_+\}$$

for some $k = k_M$ and $\{m_i\}$. The contribution of $z^m\overrightarrow{M} \subset \mathcal{R}$ for such m to $L(q,t,a) = (1-t)Z(q,t,a)$ is $(1-t)t^{m-dev(M_\ell)}$. Recall: $dev(M) = \delta - \dim \mathcal{O}/M$. Thus, all such $z^m\overrightarrow{M}$ contribute $(1-t)t^{-dev}\big(\sum_{i=1}^{k_M} t^{m_i} + t^{2\delta}/(1-t)\big)$. This will be 1 for $t \to 1$ (and any q). We obtain that $L(q,t=1,a) = \sum_{\ell=0}^{2\delta-1} |\mathcal{J}_\ell(\mathbb{F}_q)| a^\ell$, which is $\mathbf{H}^{mot}(q,t=1,a)$.

5.2. **Hilbert schemes.** For a *rational* projective curve $C \subset \mathbb{P}^2$, the following identity is a natural object for physicists and mathematicians

(Gopakumar-Vafa and Pandharipande-Thomas): $\sum_{n\geq 0} q^{n+1-\delta} e(C^{[n]}) = \sum_{0\leq i\leq \delta} n_C(i)\bigl(\frac{q}{(1-q)^2}\bigr)^{i+1-\delta}$, for the Euler numbers of *Hilbert schemes* $C^{[n]}$ of ideals at some points (zero-cycles) of C of the (total) colength n. Here δ is the arithmetic genus of C, $n_C(i)$ are some numbers. The passage from a series to a polynomial is far from obvious (even in this relatively simple case). It is much more subtle to prove that $n_C(i)\in\mathbb{Z}_+$ (Göttsche and then Shende for all i); the usage of versal deformations of singularities appeared necessary in the Shende's proof.

Switching to local rings $\mathcal{R}$ of singularities, the following conjecture is for *nested* Hilbert schemes $Hilb^{[l\leq l+m]}$ formed by pairs of ideals $\mathfrak{m}I'\subset I\subset I'\subset\mathcal{R}$ of colengths $l, l+m$ for the maximal ideal $\mathfrak{m}\subset\mathcal{R}$. One needs the *weight t-polynomial* $\mathfrak{w}(Hilb^{[l\leq l+m]})$ defined for the weight filtration of $Hilb^{[l\leq l+m]}$ due to Serre and Deligne. The *Oblomkov-Rasmussen-Shende conjecture* (2012) states that

$$\sum_{l,m\geq 0} q^{2l}\, a^{2m}\, t^{m^2}\, \mathfrak{w}(Hilb^{[l\leq l+m]})$$

is proportional to the Poincaré series of the HOMFLY-PT triply graded homology of the corresponding link. The connection with the perverse filtration of $\mathcal{J}_0$ is due to Maulik-Yun and Migliorini-Shende. The ORS series is a geometric variant of $Z(q,t,a)$.

The ORS-conjecture adds t to the formula we began with and the *Oblomkov-Shende conjecture*, extended by colors λ and proved by Maulik. The Cherednik-Danilenko conjecture was that the uncolored $\mathcal{H}(q,t,a)$ is proportional to the corresponding *reduced* KhR-polynomial. The passage from KhR-series to polynomials is an important step. This is manifest for the DAHA and motivic superpolynomials. Though it is a conjecture that the motivic ones depend on q polynomially.

5.3. **Quasi-rho-invariants.** The $\rho_{ab}-$ invariant is the von Neumann invariant defined for the abelianization representation $\pi_1(S^3\setminus K)\to\mathbb{Z}$. We will define a superpolynomial for a certain *integer* variant of ρ_{ab} for algebraic knots K. The superduality $q^\delta t^{2\delta}\mathbf{H}_K(q,\frac{1}{qt},a) = \mathbf{H}_K(q,t,a)$ and the connection conjecture for $a\to -1/q$ will be used. We set:

$$R_K(q,t,a) \stackrel{\textbf{def}}{=\!=} \bigl(\mathbf{H}_K(q,t,a) - t^\delta \mathbf{H}_K(q,t\!=\!1,a)\bigr)/\bigl((1-qt)(1-t)\bigr),$$
$$\rho_K(q,t) \stackrel{\textbf{def}}{=\!=} R_K(q,t,a\!=\!-1/q), \text{ where } \mathbf{H}_K(q,t,a\!=\!-1/q) = q^\delta.$$

Switching to L_{prncpl}, $\rho_K(q,t)$ is a sum of monic q^it^j (of multiplicity one); see the next section for the exact formula. The superduality becomes $q^{\delta-1}t^{2\delta-2}R_K(q,\frac{1}{qt},a)=R_K(q,t,a)$. The same symmetry holds for ρ_K.

Let us express $\rho_K(1,1)=\rho_K(q=1,t=1)$ in terms of $\Gamma=\nu_z(\mathcal{R})$. Let $G\stackrel{\textbf{def}}{=\!=}\mathbb{Z}_+\setminus\Gamma=\mathbb{Z}_+\cap S$ for $S=\cup_{i=1}^{\varpi}[g_i,g_i'+1]$, a disjoint union of segments, where $g_i\le g_i'\in G\not\ni g_i'+1$. Then $\delta=\sum_{i=1}^{\varpi}m_i$ for $m_i\stackrel{\textbf{def}}{=\!=}g_i'-g_i+1$. This is actually for any Gorenstein $\mathcal{R}\subset\mathbb{C}[[z]]$. Setting $\varsigma(x)=x$ for $x\in S$ and 0 otherwise,

$$\rho_K(1,1)=\textstyle\sum_{i=1}^{\varpi}m_i(g_i'+1-\frac{m_i}{2})-\frac{\delta^2}{2}=\int_0^\infty\varsigma(x)dx-\frac{\delta^2}{2}.$$

Similarly, $\rho_{ab}=\int_0^1\sigma_K(e^{2\pi ix})dx$ for the *Tristram-Levine signature* σ_K; see e.g. *"Signatures of iterated torus knots"* (Litherland, 1979). Our ς is some variant of σ_K; taking $\int_0^1$ makes ρ_{ab} "additive" (see below).

5.4. **Quasi-rho for cables.** For $r,s>0$ such that $\gcd(r,s)=1$, one has: $\rho_{r,s}(1,1)=\frac{(r^2-1)(s^2-1)}{24}$. The classical ρ_{ab} is $-\frac{1}{3}\frac{(r^2-1)(s^2-1)}{rs}$. Thus, we basically obtain the same formula up to some renormalization.

The classical ρ_{ab} for cables is known to be additive. For instance let $K=Cab(m,n)Cab(s,r)$. Then $\rho_{ab}=-\frac{1}{3}\big(\frac{(m^2-1)(n^2-1)}{mn}+\frac{(r^2-1)(s^2-1)}{rs}\big)$.

Let $\mathcal{R}=\mathbb{F}[[z^{\upsilon r},z^{\upsilon s}+z^{\upsilon s+p}]]$, where, $\gcd(r,s)=1$ as above, $\upsilon>1$ and $\gcd(\upsilon,p)=1$ for $p\ge 1$. Then $\Gamma=\langle\upsilon r,\,\upsilon s,\,\upsilon rs+p\rangle$, $2\delta=\upsilon^2rs-\upsilon(r+s)+(\upsilon-1)p+1$ and $K=Cab(m=\upsilon rs+p,\,n=\upsilon)\,Cab(s,r)$. One obtains: $\rho_K(1,1)=\frac{1}{24}\big((m^2-1)(n^2-1)+\upsilon^2(r^2-1)(s^2-1)\big)$.

More generally, $\rho_K(1,1)=\frac{1}{24}\sum_{i=1}^k\upsilon_i^2(a_i^2-1)(r_i^2-1)$ for the cable $K=Cab(a_k,r_k)\cdots Cab(a_2,r_2)Cab(a_1,r_1)$, where $1\le i\le k$, $\upsilon_i=r_k\cdots r_{i+1}$ and $\upsilon_k=1$. We will post the details elsewhere. Here $\upsilon_i=\gcd(u_1,\cdots,u_{i+1})$ for $\Gamma=\langle u_1,\cdots,u_{k+1}\rangle$, where $u_i<u_{i+1}$ and $\upsilon_{i+1}|\upsilon_i$; it is known that $\delta=\frac{1}{2}\sum_{i=1}^k\upsilon_i(a_i-1)(r_i-1)$.

We note the following natural embedding for $K'=Cab(a,r)K$ (both are algebraic knots as above). If δ is that for the ring $\mathcal{R}$ of K, then $\rho_K(q,t)$ is the sum of monomials q^it^j in $\rho_{K'}(q,t)$ such that $j<2\delta-1$.

The case of Cab(13,2)Cab(2,3). Here $\mathcal{R}=\mathbb{C}[z^4,z^6+z^7]]$, $r=3,s=2,\upsilon=2$, $\delta=8$. Then $\rho(1,1)=25$ and its refined version is $\rho(q,t)=1+qt+q^2t^2+q^3t^3+q^3t^4+q^4t^4+q^4t^5+q^5t^5+q^4t^6+q^5t^6+q^6t^6+q^5t^7+q^6t^7+q^7t^7+q^5t^8+q^6t^8+q^7t^8+q^6t^9+q^7t^9+q^6t^{10}+q^7t^{10}+q^7t^{11}+q^7t^{12}+q^7t^{13}+q^7t^{14}$.

RH holds for $\rho(q,t)$ when $q < q_{sup} \approx 0.802$. Presumably, $\lim_{p\to\infty} q_{sup} = 1$ for $Cab(2p+13,2)Cab(2,3)$; for instance, $q_{sup} \approx 0.996$ for $p=2000$.

Finally, $R(q,t,a)$ for $\mathbb{C}[z^4, z^6+z^7]]$ is: $1+t+qt+t^2+2qt^2+q^2t^2+t^3+2qt^3+3q^2t^3+q^3t^3+t^4+2qt^4+4q^2t^4+4q^3t^4+q^4t^4+t^5+2qt^5+4q^2t^5+6q^3t^5+4q^4t^5+q^5t^5+t^6+2qt^6+4q^2t^6+7q^3t^6+8q^4t^6+4q^5t^6+q^6t^6+t^7+2qt^7+4q^2t^7+7q^3t^7+10q^4t^7+8q^5t^7+4q^6t^7+q^7t^7+qt^8+2q^2t^8+4q^3t^8+7q^4t^8+8q^5t^8+4q^6t^8+q^7t^8+q^2t^9+2q^3t^9+4q^4t^9+6q^5t^9+4q^6t^9+q^7t^9+q^3t^{10}+2q^4t^{10}+4q^5t^{10}+4q^6t^{10}+q^7t^{10}+q^4t^{11}+2q^5t^{11}+3q^6t^{11}+q^7t^{11}+q^5t^{12}+2q^6t^{12}+q^7t^{12}+q^6t^{13}+q^7t^{13}+q^7t^{14}+a\Big(qt+qt^2+2q^2t^2+qt^3+3q^2t^3+3q^3t^3+qt^4+3q^2t^4+6q^3t^4+3q^4t^4+qt^5+3q^2t^5+7q^3t^5+9q^4t^5+3q^5t^5+qt^6+3q^2t^6+7q^3t^6+12q^4t^6+10q^5t^6+3q^6t^6+qt^7+3q^2t^7+7q^3t^7+13q^4t^7+17q^5t^7+10q^6t^7+3q^7t^7+q^2t^8+3q^3t^8+7q^4t^8+12q^5t^8+10q^6t^8+3q^7t^8+q^3t^9+3q^4t^9+7q^5t^9+9q^6t^9+3q^7t^9+q^4t^{10}+3q^5t^{10}+6q^6t^{10}+3q^7t^{10}+q^5t^{11}+3q^6t^{11}+3q^7t^{11}+q^6t^{12}+2q^7t^{12}+q^7t^{13}\Big)+a^2\Big(q^3t^3+q^3t^4+2q^4t^4+q^3t^5+3q^4t^5+3q^5t^5+q^3t^6+3q^4t^6+6q^5t^6+3q^6t^6+q^3t^7+3q^4t^7+7q^5t^7+8q^6t^7+3q^7t^7+q^4t^8+3q^5t^8+6q^6t^8+3q^7t^8+q^5t^9+3q^6t^9+3q^7t^9+q^6t^{10}+2q^7t^{10}+q^7t^{11}\Big)+a^3\Big(q^6t^6+q^6t^7+q^7t^7+q^7t^8\Big)$.

Recall that $R(q,t,a) \mapsto \rho(q,t)$ upon the substitution $a \mapsto -\frac{1}{q}$ in the parameters of $\mathbf{H}(q,t,a)$, which is generally the passage to the Heegaard-Floer homology and Alexander polynomials (when $q=1, a=-1$).

6. On physics connections

Generally, a challenge is to associate the Riemann and Lindelöf hypotheses with some physics phenomena in SCFT or similar theories. SCFT is connected with quite a few recent mathematical developments. DAHA can be considered as its part; their origin was in the Knizhnik-Zamolodchikov equations. DAHA superpolynomials can be interpreted as some physics partition functions.

The *p-adic strings* due to Witten and others must be mentioned; the starting point was an adelic product formula for the *Veneziano amplitude*. There is some relation to our expectations that L-functions of plane curve singularities over $\mathbb{F}_q$ can be related to some Dirac operators, which is schematically shown in the 2nd figure below.

The *Lee-Yang circle theorem* provide a different perspective. The Ising model with an external magnetic field is its main instance.

6.1. **Lee-Yang theorem.** For any lattice (of any dimension) with N vertices and the connected pairs of vertices denoted by $\langle n, n'\rangle$, let $\mathcal{Z} = \lim_{N\to\infty} \frac{\log(Z_N)}{N}$ for the partition function $Z_N = \sum_{\{\sigma_n\}} e^{-\beta\mathcal{H}}$, where the

Hamiltonian is $\mathcal{H} = -\sum_{\langle n,n'\rangle} J_{n,n'}\sigma_n\sigma_{n'} - H\sum_n \sigma_n$ and $\sigma = \pm 1$. This is the Ising model with an external magnetic field H. Here $\beta = (k_B T)^{-1}$ is the inverse temperature for the Boltzmann constant k_B. Assuming that $J_{n,n'} \geq 0$ (the ferromagnetic case) and $\beta > 0$, Lee-Yang proved that the zeros of Z_N in terms of the "complex fugacity" $\mu = e^{-2\beta H}$ belong to the unit circle $|\mu| = 1$; the corresponding symmetry of Z_N is simply $\sigma \mapsto -\sigma$. For the square lattice with $J = const > 0$, Z_N is a polynomial in terms of μ and $0 < u \stackrel{\mathbf{def}}{=\!=} e^{-4\beta J} < 1$. There is a q-version of this theorem and other physics-statistical variants.

The Lee-Yang-Fisher zeros are when u is considered as a free parameter for complex T. Numerical experiments showed that $|\mu| = 1$ for the μ-zeros can hold for some $u < 0$. The physics calculations are mostly when μ is fixed and the u-zeros are considered, but they can be used for the μ-zeros too. This phenomenon resembles the behavior of the t-zeros of our $\mathbf{H}(q, t, a)$. Actually, DAHA is directly related to the XXZ-model, which is somewhat similar to the Ising model with H as above, though all attempts to "integrate" the latter failed.

Only $\mathcal{Z}$ is physical; its phase transitions are positive *real* limits as $N \to \infty$ of (complex) μ-zeros of Z_N. Thus, these zeros can result in a phase transition only at $\mu = 1$ due to RH for Z_N, which point is the intersection of the unit μ-circle with $\mathbb{R}_+$. The relation between the failure of RH and "unwanted" phase transitions seems sufficiently general. Given $u < 0$, the μ-zeros of Z_N quickly become wild (near the real line), when N goes beyond N_u, which is the last N when RH still holds for this u. So do the points of phase transition for any $u < 0$. This can be clearly seen in the 1D Ising model.

One-dimensional case. The zeros of Z_N can be found explicitly in this case. One has: $-\beta\mathcal{H} = \beta\sum_{n=1}^{N} J\sigma_n\sigma_{n+1} + \frac{\beta}{2}H(\sigma_n + \sigma_{n+1})$, where the periodicity $\sigma_{N+1} = \sigma_1$ is assumed. Then $Z_N = \sum_{\{\sigma_n\}} e^{-\beta\mathcal{H}}$ can be calculated using the eigenvalues λ_1, λ_2 of the *transfer matrix* $\mathcal{T} = \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{pmatrix}$. Namely, $Z_N = \operatorname{tr}(\mathcal{T}^N) = \lambda_1^N + \lambda_2^N$ for $\lambda_{1,2} = e^{\beta J}\cosh(\beta H) \pm \sqrt{e^{2\beta J}\sinh^2(\beta H) + e^{-2\beta J}}$. Upon some algebraic manipulations, the μ-zeros of Z_N are for $H = \pm\imath\,\theta_n/\beta$, where $\cos(\theta_n) = \cos\frac{(2n-1)\pi}{2N}\sqrt{1-u}$ for $n = 1, \ldots, N$ and $u = e^{-4\beta J}$ as above. We obtain that RH, which is the condition $\theta_n \in \mathbb{R}$ for any n, holds if

$\cos\frac{\pi}{2N}\sqrt{1-u} \le 1$, i.e. for $u \ge -\tan^2\left(\frac{\pi}{2N}\right)$. Thus, u can be negative for RH, but the latter bound tends to 0 when $N\to\infty$. This example is not very helpful for our RH and clarifying its dependence on q, but provides some general physics insight.

6.2. **Landau-Ginzburg models.** Another physics approach to singularities is presented in paper *"Catastrophes and the classification of conformal theories"* (Vafa-Warner, 1989). The authors consider LGSM, Landau-Ginzburg Sigma Models, for *superpotentials* $W(x,y)$ corresponding to isolated singularities. They can be with several variables, and more than one superpotential W can be considered. Let us mention here two publications by Alexander Zamolodchikov in 1986. There are many other classes of superpotentials, for instance those for quiver varieties and KZ. The correspondence between SCFT and LGSM is one of the key in string theory.

A lot of information can be obtained directly in terms of $W(x,y)$ and the corresponding singularities. If only the topological class of singularity matters, then this is some "topological LGSM". We note that analytic parameters of singularities can be interpreted as *modes* in physics, but motivic theory captures (by now) only topological types.

For instance, the *Milnor number* $\mu = 2\delta$, which is $(r-1)(s-1)$ for $W_{r,s}(x,y)=x^s-y^r$, coincides with the number of (independent) chiral operators or superfields. It is the *Witten index* for plane curve singularities: the number of zero energy bosonic vacuum states minus the number of zero energy fermionic vacuum states. The dimensions of superfields for $W_{r,s}(x,y)$ are proportional to the corresponding quasi-homogeneity weights, which are $1/s$ for x, $1/r$ for y and so on.

Another example is the central charge, which is $c = 6\beta$ for $\beta = (\frac{1}{2}-\frac{1}{r})(\frac{1}{2}-\frac{1}{s})$ for $W_{r,s}(x,y)$. Generally, β is obtained from the asymptotic formula $\int W(x_1,\ldots,x_m)\prod_{i=1}^m \lambda^{1/2}dx_i \sim O(\lambda^\beta)$ for large λ. Also, the adjacency of singularities plays an important role in this approach.

6.3. **Refined Witten index.** Refined Witten and BPS indices were studied in the literature; see e.g. Gaiotto-Moore-Neitzke (Adv. Theor. Math. Phys. 2013). Generally, the challenge is to "split" the vacuum states counted by these indices.

As an example, we will split μ using $\mathcal{H}^{mot}(q,t,a == t/q)$, though actually we will use L_{prncpl} in the following calculation. Let us begin with $\delta_{q,t} \stackrel{\mathbf{def}}{=\!=} \frac{\mathcal{H}^{mot}(q,t,a=-t/q)-(qt)^\delta}{1-t} = \frac{L_{pncpl}(\frac{q}{t},t)-(qt)^\delta}{1-t}$. This formula is for any Gorenstein $\mathcal{R}\subset\mathbb{C}[[z]]$. The parameters q,t from $\mathcal{H}$ are used (not q from the definition of $\mathbf{H}$ and L).

As above: $G=\mathbb{Z}_+\backslash\Gamma=\cup_{i=1}^{\varpi}\{g_i\le x\le g_i'\}$, where $g_i'+1\in\Gamma$, and $m_i=g_i'-g_i+1$. Then $\delta_{q,t}=\frac{1-t^{g_1}}{1-t}+\sum_{i=1}^{\varpi-1}\frac{t^{g_i'+1}-t^{g_{i+1}}}{1-t}\big(\frac{q}{t}\big)^{m_1+\cdots+m_i}$, and $\delta_{1,1}=\delta$. This formula was actually used above in the definition of the refined quasi-rho invariant $\rho_K(q,t)$; also, see below.

Let $\mu_{q,t}\stackrel{\mathbf{def}}{=\!=}\delta_{q,t}+(qt)^{\delta-1}\delta_{t^{-1},q^{-1}}=\sum_{x=0}^{2\delta-1}t^{v(x)-1}q^{g(x)}$, where $v(x)=|\{\nu\in\Gamma\mid 0\le\nu\le x\}|$ and $g(x)=|\{g\in G\mid 0\le g<x\}|$ for G as above. Then $\mu_{1,1}=\mu$ and this definition ensures the superduality: $(qt)^{\delta-1}\mu(1/t,1/q)=\mu(q,t)$. Not all monomials are monic in $\mu(q,t)$: ϖ of them are with coefficient 2, which correspond to $x\in\Gamma\not\ni x+1$.

For example, $\mu_{q,t}=2+q+q^2+2q^3t+2q^4t^2+2q^5t^3+2q^6t^4+q^7t^5+q^7t^6+2q^7t^7$ for $\mathcal{R}=\mathbb{C}[[z^4,z^6+z^7]]$. Upon $q\mapsto qt$, it satisfies RH for $0<q<0.919090$. For $\mathcal{R}=\mathbb{C}[[z^6,z^9+z^{460}]]$, this range becomes $0<q<0.852561$. For $\mathcal{R}=\mathbb{C}[[z^6,z^8+z^{649}]]$, it is $0<q<0.846566$ and 0.848063 for z^8+z^{3003}.

Compare with the formula for $\varrho(q,t)\stackrel{\mathbf{def}}{=\!=}\frac{\mathcal{H}(q,t,a=-t/q)-q^\delta}{(1-t)(1-q)}=\rho(\frac{q}{t},t)$:
$\varrho(q,t)=\sum_{x\in G}q^{g(x)}\frac{1-t^{v(x)}}{1-t}=\sum_{G\ni x>y\in\Gamma}q^{g(x)}t^{v(y)-1}=(qt)^{\delta-1}\varrho(t^{-1},q^{-1})$. Note that we use the parameters q,t from $\mathcal{H}$ in this section.

Adding colors. The substitution $a\mapsto-\frac{t}{q}$ has remarkable properties for $\mathcal{H}^\lambda(q,t,a)$ for partitions λ more general than $\square$ (the uncolored case). Let n be the number of rows of λ and m the number of its columns.

For *hooks* λ, we expect that $\mathcal{H}^\lambda(q,t,a\mapsto-\frac{t}{q})=(qt^{n-1})^{\delta(n-1)}r^\lambda(q,t)$ for $r^\lambda(q,t)=r(q\mapsto t^{n-1}q^m,t\mapsto q^{m-1}t^n)$. Here $r(q,t)=\mathcal{H}(q,t,a\mapsto-\frac{t}{q})$ is for $\lambda=\square$, a polynomial with the constant term 1 considered above. For instance, $1-r^\lambda(q,t)$ is divisible by $(1-q^m)$ for $n=1$. This was checked for pure columns/rows and several hooks with $m=2$ or $n=2$. For example, $\big(1-r^{\square\square}(q,t)\big)/(1-q^2)=qt(1+q^2+q^4+q^7t+q^{10}t^2+q^{13}t^3+q^{16}t^4+q^{21}t^7)$ for $K=Cab(13,2)Cab(2,3)$.

This is far from being that simple beyond the hooks. For instance, for $T(3,2)$ and $\lambda=2\times2=\boxplus$: $1-r^{2\times2}_{3,2}(q,t)=q(1-qt)(1+q-q^2+t-q^2t+q^4t+q^3t^2-q^5t^2+q^2t^3-q^4t^3+q^5t^4+q^6t^4+q^4t^5-q^6t^5-q^6t^6+q^6t^7)$.

The definition of the $\varrho^\lambda(q,t)$ for *symmetric* λ can follow the uncolored case, but this is preliminary. For instance, one can set: $\varrho_K^{2\times2}(q,t)\stackrel{\mathbf{def}}{=\!=}$

$\big(r_K^{2\times 2}(q,t)-q^{4\delta}t^{4\delta}\big)/(1-qt)^2$. For the example above: $\varrho_{3,2}^{2\times 2}(q,t)=1-q-q^2+q^3+qt-q^2t+q^4t-q^5t+2q^2t^2-q^3t^2-q^4t^2+q^5t^2+2q^3t^3-q^4t^3+q^6t^3+2q^4t^4-q^5t^4-q^6t^4+q^5t^5-q^6t^5+q^6t^6$, which satisfies the superduality $q\leftrightarrow t^{-1}$ with the multiplier q^6t^6. Presumably, $\varrho_K^{2\times 2}(1,1)=4\varrho_K(1,1)$ for algebraic knots K. As above, $\mathcal{H}_K, \varrho_K$ are uncolored, i.e. for $\lambda=\Box$.

6.4. ***S*-duality.** The relation SCFT $\leftrightsquigarrow$ LGSM suggests that the S-*duality* in the former can be seen via the superpotential $W(x,y)$. The superduality of physics superpolynomials can be connected with that in M-*theory* and the symmetry $\epsilon_1\leftrightarrow\epsilon_2$ in *Nekrasov's instanton sums*. The general physics superduality (with λ) for superpolynomials is due to Gukov-Stosic (2012). For us, this correspondence is basically between DAHA, a "representative" of SCFT, and L-functions of singularities, which "represent" LGSM.

Namely, the DAHA superpolynomials, which are certain partition functions in SCFT, are conjectured to coincide with motivic superpolynomials, which are presumably some partition functions of properly defined *motivic topological* LCSM for superpotentials $W(x,y)$, where $W(x,y)=0$ is the equation of the corresponding plane curve singularities. Switching to the L-functions of the latter, we (conjecturally) *identify S from SCFT with the functional equation*.

This "identification" may be not too much surprising. The S-duality and mirror symmetry (CPT) are very universal in physics. The functional equation is certainly of the same calibre in mathematics. There are more than 20 different zeta-theories. The functional equation is almost always present, though RH does not always holds.

By "extrapolating" the Lee-Yang theorem, one can speculate that (topological) LGSM associated with plane curve singularities are "stable" when the "coupling constant" q is small enough to ensure RH for $\mathbf{H}(q,t,a)$. This is assuming that the failure of RH is somehow connected with the presence of unwanted phase transitions. Rephrasing, LGSM for plane curve singularities or for certain surface singularities can be "observed" only when the corresponding RH holds, which is granted for sufficiently small $q>0$.

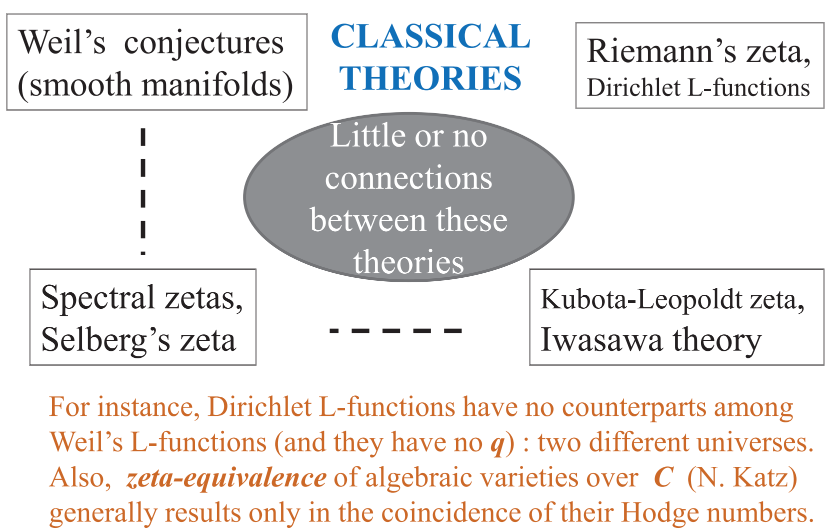
As such, $T(2,2p+1)$ can be "observed" for $a=0$ when $q>0$ is arbitrary. Setting $a=0$ matters; generally we can make a some constant or any quantity invariant under the functional equation. For instance, let

$a \mapsto -t/q$. Then we arrive at $\varrho_{2,2p+1}(q,t) = \frac{q^p - tq^{p+1} - t(qt)^p + (qt)^{p+1} + t - 1}{(q-1)(t-1)(qt-1)} = \sum_{0\le j\le i<p} q^i t^j$, and RH holds for $\rho_{2,2p+1}(q,t) = \varrho_{2,2p+1}(qt,t)$ only for sufficiently small q (not all).

Let me finish this section with little something on Manin's "*Mathematics as metaphor".* Yu.I. obviously expected number theory to play a major role in the alliance of physics and mathematics. If RH has something to do with the absence of unwanted phase transitions in physics theories or their stability of any kind, then number theory will not be just a "metaphor". Technically, DAHA accumulated quite a few integrable models and the fact that it appeared very "motivic" can be meaningful physically. I thank my friends-physicists for various talks on these matters (though they are not responsible for what I wrote).

7. Zeta-functions as invariants

7.1. **The first figure.** The recent progress in mathematics and physics was mainly in the fields related to geometry. Obviously, any new geometric understanding of the classical zeta and L-functions can open new avenues toward the justification of "Grand Conjectures".



In contrast to the Weil conjectures (proved by Deligne), the distribution of the zeros of Riemann's $\zeta(s)$ in the critical strip does not seem to reflect any "geometry". Riemann's zeta does occur in some geometric and physics considerations, but the interpretation of the Grand Conjectures geometrically or physically is missing. We note here that the zeros of Selberg's zeta functions contain a lot of geometric and analytic information. The Hasse-Weil zetas are very geometric.

Nicolas Katz proved that the zeta-equivalence of algebraic varieties X results in the coincidence of their virtual Hodge numbers. One needs to add the coefficients of the equations of X to the corresponding rings of functions and then consider the zeta function ζ_X of the resulting scheme over $\mathbb{Z}$. If $\zeta_X = \zeta_Y$, then we call X and Y zeta-equivalent. The coincidence of Hodge numbers is of course very far from the existence of any kind of isomorphism between X and Y. Let me mention here *motives* and Manin's *"Lectures on zeta functions and motives"* (1995).

Generally, *Kapranov's (motivic) zeta* and "true motivic superpolynomials" are with the coefficients in the Grothendieck ring $K_0(Var/\mathbb{F})$ of varieties over $\mathbb{F}$. The map $X \mapsto |X(\mathbb{F}_q)|$ is one of the *motivic measures.*

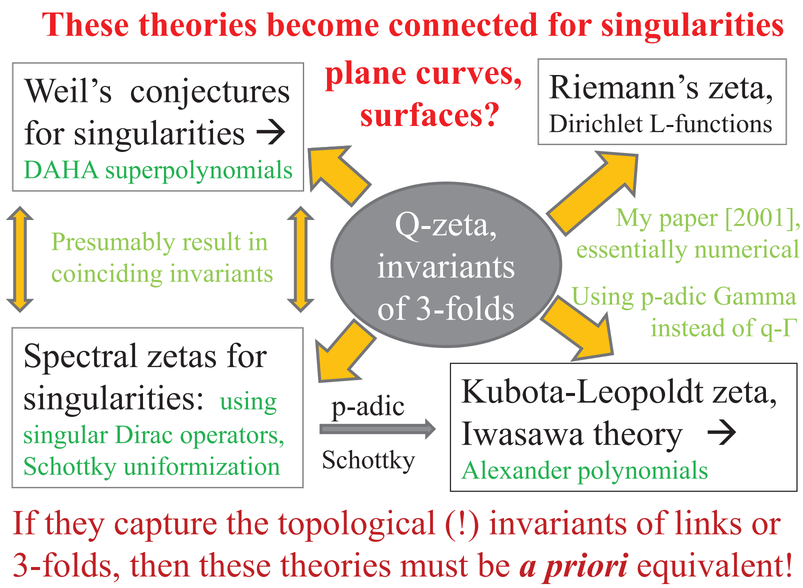
In the first figure, the Hasse-Weil zetas (over $\mathbb{F}_q$), the Selberg's zetas (via the Laplace operators) and the Kubota-Leopoldt ones are presented as disconnected blocks. There 3 theories are really different.

The last block (Riemann's $\zeta(s)$ and the Dirichlet L-functions) is obviously different from these three, but there are some (deep) relations. For instance, let us mention the connection (not reflected in this figure) of the Selberg's *"1/4 conjecture"* for arithmetic subgroups $\Gamma \subset SL(2,\mathbb{R})$ to the Riemann's zeta.

Concerning the upper-left block, the adelic products of Hasse-Weil zeta functions give those in the upper-right block. However, this does not help much to understand the zeros of the classical $\zeta(s), L(s)$. Any *direct* connections between these 2 blocks are unlikely simply because the classical $\zeta(s), L(s)$ do not contain q. What can be expected is *indirect*: via the refined invariants of 3-folds, q-L-functions, and the corresponding DAHA constructions. We try to outline this.

7.2. **The second figure.** Upon the switch to isolated singularities $\mathcal{X}$, the corresponding zetas can be expected to capture the topological type for some "good" $\mathcal{X}$, which holds for any plane curve singularities.

The "Hasse-Weil block" of the 2nd figure for plane curve singularities is discussed in this note. The block below is based on the covers of $\mathbb{P}^2$ ramified at $\mathcal{X}$, which is connected with *Schottky uniformization* in the variant due to Tate-Mumford of versal deformation of $\mathcal{X}$. It is expected



that the Selberg-type zetas of the corresponding Dirac operators lead to similar (if not the same) superpolynomials of singularities. Let us provide at least one reference: *"Zeta functions that hear the shape of a Riemann surface"* (Cornelissen-Marcolli, 2008).

The p-adic uniformization of modular curves associated with division algebras over totally real number fields, which was part of the ε-conjecture and Ribet's theorem, was my thesis. See e.g. recent *"On the p-adic uniformization of quaternionic Shimura curves"* (Boutot-Zink, 2022). This is somewhat beyond the figure, but is certainly related and demonstrates the potential of the p-adic uniformization.

The middle oval. It is mostly about direct adding q to the classical ζ and L-functions based on DAHA. If the corresponding deformed zeros become more regular at $0<q<1$, then the stochastic behavior of the classical ones can be due to $q\to 1$, which is not unusual in mathematics.

I defined such q-analogs in *"On q-analogues of Riemann's zeta function"* (2001) for A_1, following the q-Macdonald-Mehta formula. It is generally a formula for $\int \gamma_x\, \mu(x)dx$ in terms of $q-\Gamma$ functions. The measure function $\mu(x) = \prod_{\alpha, j\ge 0} \frac{(1-q^{(x,\alpha)+j})\,(1-q^{-(x,\alpha)+j+1})}{(1-q^{k+(x,\alpha)+j})(1-q^{k-(x,\alpha)+j+1})}$ is that making the Macdonald E-polynomials pairwise orthogonal, α are positive roots of a given reduced irreducible root system in $\mathbb{R}^n$ and $\gamma_x \stackrel{\mathbf{def}}{=\!=} q^{\pm x^2/2}$ (the Gaussian) is with "+" for the integration over $x \in \mathbb{R}^n$ and with "−" when $x \in \imath\mathbb{R}^n$ (the non-compact and compact cases). This is for the DAHA parameters $0<q<1$ and $t=q^k$ (k may depend on $|\alpha|$).

The link to the DAHA superpolynomials is due to the theorem that $\int f(x)\gamma_x\, \mu(x)dx$ is proportional to the *coinvariant* of $f(x)$, which is $f(x \mapsto -k\rho)$ for any Laurent series f in terms of $X = q^x$ provided the convergence of the corresponding integrals, real or imaginary.

The Stirling-Moak formula (Moak, 1984) gives that the (renormalized) limits $q \to 1_-$ of $\int \frac{1}{1-\gamma_x}\, \mu(x)dx$ for the corresponding choice of paths (real or imaginary) can be expressed in terms of Riemann's ζ and some products of Γ-functions. Then we perform the analytic continuation using "q-calculus", which is closely connected in the imaginary case with "picking up residues" (Arthur, Heckman-Opdam) in the theory of spherical Plancherel measure of affine Hecke algebras.

For A_1, we arrive at some q-analogs of (modified) $\zeta(s)$; the paper was only for A_1. Taking here $(1+\gamma_x)$ instead of $(1-\gamma_x)$, we obtain q-analogs (real and imaginary) of the simplest classical L-function $\zeta_+(s) = (1 - 2^{1-s})\zeta(s)$. The convergence of modified Dirichlet q-L-functions of this kind to the classical ones is for *any* non-singular $k \in \mathbb{C}$.

In the case of $(1-\gamma_x)$, the renormalized limit of $q - zeta$ for $k < 1/2$ is some interesting combination of Gamma-functions, which can be somehow connected to the so-called *Gram law* in the classical theory of $\zeta(s)$. The q-zeros approximate well the classical ones for q sufficiently close to 1 (but not too close!) and then slowly switch to those from the proper Gamma-functions.

The functional equation fails for the q-deformations, which is probably inevitable, but I confirmed numerically some q-RH for $s=k+\frac{1}{2}$.

Peter Sarnak noted once that many applications are based on the absence of zeta-zeros with $\frac{1}{2} < \Re s < 1$. This is basically what can be seen for $0 < q < 1$ and relatively small zeros in the case of imaginary integration. However, there was uncertainty when the neighboring zeros of the classical $\zeta(s)$ are getting "too close". Namely, the linear approximations of q- deformations of such "unusual" zeros of $\zeta(s)$ can be with $\Re s > 1/2$; though the linear approximations can become irrelevant for such zeros. Let me quote Harold Edwards: "the existence of nearly coincident zeros must give pause to even the most convinced believer" (his *"Riemann's Zeta Function"*).

The formulas for the linear approximations are those provided below for the *sharp q-zeta*, where the quantity $1-\bigl(\cdot\bigr)$ must be changed to $1+\bigl(\cdot\bigr)$, due to the passage from $\int_{\imath\mathbb{R}}$ to $\int_{\mathbb{R}}$. Sharp q-zetas and the corresponding $q-L$-functions are of real type. The corresponding sharp q-deformations of the classical zeros in proper horizontal strips (depending on $q<1$) are with $\Re k>0$ in our calculations.

7.3. **Sharp q-zeta.** We set $q=\exp(-1/\omega)$ for $\omega>0$ and $\varepsilon=\sqrt{\pi\omega/2}$. The integration path will be $\circlearrowleft_{\infty-\varepsilon i}^{\infty+\varepsilon i}$ around zero. For A_1 and the symmetric variant $\delta_k(x;q)\stackrel{\mathbf{def}}{=\!=}\prod_{j=0}^{\infty}\frac{(1-q^{j+2x})(1-q^{j-2x})}{(1-q^{j+k+2x})(1-q^{j+k-2x})}$ of μ, the function $\mathfrak{Z}_q^{\sqsubset}(k)\stackrel{\mathbf{def}}{=\!=}\frac{1}{2i}\oint_{\infty-\varepsilon i}^{\infty+\varepsilon i}\frac{\delta_k(x;q)}{1+q^{-x^2}}\,dx$ is analytic in the horizontal strip $K^\sharp=\{-2\varepsilon<\Im k<+2\varepsilon\}$ as $\Re k>-1/2$. Its meromorphic continuation to *all* $k\in\mathbb{C}$ via Cauchy's theorem, the *sharp q-zeta*, is:

$$\mathfrak{Z}_q^\sharp(k)=-\frac{\omega\pi}{2}\prod_{j=0}^{\infty}\frac{(1-q^{j+k})(1-q^{j-k})}{(1-q^{j+2k})(1-q^{j+1})}\times$$
$$\sum_{j=0}^{\infty}\frac{(1-q^{j+k})q^{-kj}}{(1-q^k)(q^{-\frac{(k+j)^2}{4}}+1)}\prod_{l=1}^{j}\frac{1-q^{l+2k-1}}{1-q^l}.$$

It has poles at $\{-\frac{1}{2}-\mathbb{Z}_+\}$ in $K^\sharp$. This strip is between the first zeros of $1+q^{-\frac{k^2}{4}}$. For all k apart from the poles, $\lim_{a\to\infty}(\frac{\omega}{4})^{k-1/2}\mathfrak{Z}_q^\sharp(k)=\sin(\pi k)(1-2^{\frac{1}{2}-k})\Gamma(k+\frac{1}{2})\zeta(k+\frac{1}{2})$.

Given a classical zero $k=z$ of $\zeta(1/2+k)$, let us assume that its $\sharp$-deformation $z^\sharp(\omega)$ exists and is differentiable with respect to $\varepsilon=1/\omega$. Then the formula for its linear approximation $\widetilde{z}^\sharp(\omega)$ is as follows: $\widetilde{z}^\sharp(\omega)=z\bigl(1-\frac{4(z+\frac{1}{2})\zeta_+(z+\frac{3}{2})-(z-1)\zeta_+(z-\frac{1}{2})}{12\omega\zeta'(z+\frac{1}{2})(1-2^{\frac{1}{2}-z})}\bigr)$. Thus, such "$\varepsilon$-deformable" zeros z are simple, an interesting reformulation of the classical conjecture that all z are simple. Similarly, $\mathfrak{Z}_q^\sharp(k;d)$ are for q^{-dx^2} instead of q^{-x^2} and *sharp L-functions* $\mathfrak{L}_q^\sharp(k;d)$ are for $\frac{q^{x^2/2}-q^{-x^2/2}}{q^{(d+1)x^2/2}-q^{-(d+1)x^2/2}}$.

Taking the classical $z=14.1347i$ and $\omega=750$ for $\mathfrak{Z}_q^\sharp(k;2)$:

$$z^\sharp=0.1304+14.1450i,\ \ \widetilde{z}^\sharp=0.1302+14.1465i.$$

Other zeros in $K^\sharp$ for $\omega = 750, d = 2$ are:

$zeta$	$sharp - zeta$	$linear\ approx.$
$21.0220i$	$0.3514 + 21.0702i$	$0.3504 + 21.0771i$
$25.0109i$	$0.5641 + 24.9586i$	$0.5745 + 24.9643i$
$30.4249i$	$0.9046 + 30.4014i$	$0.9134 + 30.4077i$
$32.9351i$	$1.1051 + 33.0341i$	$1.0998 + 33.0854i$
$37.5862i$	$1.6449 + 37.9660i$	$1.7675 + 38.1895i$
$40.9187i$	$1.9080 + 40.8119i$	$1.9141 + 40.7816i$
$43.3271i$	$2.2860 + 43.2485i$	$2.4497 + 43.3138i$
$48.0052i$	$2.9259 + 47.8424i$	$3.1103 + 47.5578i$.

There is a clear tendency for $z^\sharp$ to move to the right. Also, the zeros of the $\mathfrak{L}_q^\sharp(k;d) - \mathfrak{L}_q^\sharp(-k;d)$ we calculated at the end of this paper (not too many) were all satisfying the classical RH: they were such that $\Re k = 0$. This is within the corresponding strips. The convergence is very good here, but calculations when $\omega \gg 0$ and large $\Im(k)$ are involved.

Generally, the zeros of the q-deformed L-functions become more "regular" for $q < 1$ than the (nontrivial) zeros of the Riemann zeta function, conjecturally distributed like the eigenvalues of random Hermitian matrices (Dyson, Montgomery, Odlyzko,...).

It seems that the extension of these constructions from A_1 to the stable A_n theory is of importance. The relation between k and s will then depend on the rank as in the classical theory. The functional equation cannot be 'fixed", but the advantage is superduality. In the stable case, there is some connection between $\Im z \gg 0$ and $\Im z \sim 0$ for $q < 1$, but using it for RH is a long shot.

Concerning the "p-adic block", it is expected that there is a p-adic DAHA theory, where q-Gamma functions are replaced by their p-adic counterparts. This theory is doable, but it is not known how to go to the level of the p-adic zeta functions.

We note that the ρ_{ab}-invariants of knots are clearly of spectral nature. Accordingly, our quasi-rho invariants can be considered as some link between the "Weil block" and the spectral one (below it in this figure). Our $\rho(q,t)$ is given in terms of the motivic superpolynomials $\mathbf{H}^{mot}$.

Moreover, the integrality of the coefficients of $R(q,t,a)$ in Section 5.3 indicates that there can be their "triply-graded" categorification.

The classical ρ_{ab} is directly related to the Tristram-Levine signature. A different parametric deformation of the *Seifert matrix* (the one used for the Alexander polynomials) is presumably needed for our one. This would provide some link from superpolynomials to spectral invariants.

If this is true that zeta functions of singularities $\mathcal{X}$ and their corresponding a,q,t-versions provide strong topological invariants/moduli of some sort, then one can expect *a priori* links between the Hasse-Weil zetas, Selberg's zetas and p-adic zetas for such $\mathcal{X}$. This is mostly in the case of plane curve singularities, but the bottom 2 blocks of this figure are conditional even for such $\mathcal{X}$.

Toward 3-folds. A natural program is to extend the motivic approach from plane curve singularities to the *surface* singularities serving Seifert 3-folds and other *plumbed manifolds*. The corresponding invariants are expected to be a,q-deformations of some variants of the classical zeta and L-functions, satisfying the superduality. This would be the passage from superpolynomial to the super-series.

Another approach to the classical ζ and L-functions can be via the expansion of the theory of DAHA-superpolynomials from knots/links to 3-folds; the lens spaces are already very interesting. This is a more traditional direction. Here something is already known, and there is support in classical topology and physics.

7.4. **Strong polynomial count.** There are some restrictions for the types of singularities in the 2nd figure.

First, their topological types are expected to be the key. The definition of motivic zetas does require the rings $\mathcal{R}$, but they are of "discrete nature" (for singularities). For instance, some invariants of the isotopy classes of the links $\mathcal{X}\cap S^{2n-1}$ in $\mathbb{C}^n$ of isolated hypersurface singularities $0\in\mathcal{X}\subset\mathbb{C}^n$ can be hopefully obtained.

Second, we need to check that $\mathcal{X}$ can be defined over $\mathbb{Z}$ within its topological type and with good reductions for almost all prime p, which does not seem a real restriction.

Third, the varieties of modules (ideals) of finite colength in the corresponding local rings must be assumed of *strong polynomial count*: the number of their points over $\mathbb{F}_q$ must depend polynomially on q. This is quite a restriction. It is certainly true if they are paved by affine spaces.

We mention here that general affine Springer fibers can be *not* of strong polynomial count (of types $\neq A$). There is an example of Bernstein-Kazhdan, where zeta-functions of certain elliptic curves over $\mathbb{F}_q$ occur, which make this AFS *not* of polynomial count. See Appendix to *"Fixed point varieties on affine flag manifolds"* by Kazhdan-Lusztig (1988).

Under these conditions (there can be further restrictions), the corresponding $\zeta_{\mathcal{X}}(q,t,a)$ can be expected with all ifs and buts to be a powerful invariant of the singularity $\mathcal{X}$. This zeta-function can be presumably a strong topological invariant, which is conjectured for plane curve singularities and checked in many cases.

Hopefully, the surface singularities corresponding to lens spaces can be sufficient for reaching the Dirichlet L-functions, but this is speculative and anyway very far from understanding the zeros of the latter. It is not clear what can be counterparts of more general surface singularities in number theory.

Adding colors to DAHA superpolynomials is a similar challenge. This direction can potentially result in modular-type functions within knot theory (without 3-folds). We provided in Section 3.2 the superpolynomial of $T(2p+1,2)$ colored by "m-rows" (by $m\omega_1$).

Needless to say that isolated singularities are (and always were) among the key objects of algebraic geometry. Smooth projective manifolds proved to be very helpful in their study, but they are not really necessary for many aspects. We try to do as much as we can directly in terms of the singularity rings, when the theory of topological invariants becomes with strong "combinatorial" components.

Knörrer's periodicity. The theory of algebraic knots, related 3-folds and 5-folds make the usage of the corresponding zeta functions as geometric-topological invariants quite reasonable for (some) surface singularities. The Knörrer's periodicity for singularities is basically a connection between the plane curve singularities $W(x,y)=0$ and the ones given by the equations $u^2=W(x,y)$ is important here. Actually, 5-folds fit this picture too, for instance those for the singularities $uv=W(x,y)$ that naturally occur here; such Calabi-Yau threefolds were considered by Vafa-Dijkgraaf.

The passage from knots/links to 3-folds can shed light on the similarity of our considerations for knots and the classical number theory based on the periods of cusp forms, we began with. This is only an analogy so far; our superpolynomials are much simpler and much more algebraic than the zeta-polynomials in the beginning of this note.

Let me mention (again) that this note is very incomplete concerning the names and contributions. Only very few papers are mentioned. It is an introduction focused mostly on superpolynomials and some perspectives of their theory. We tried to outline some number theoretical aspects of this direction and possible physics connections. The exposition is sketchy and speculative in several places, especially in the last two sections. There are and can be various omissions; for instance, we do not discuss much the recent developments, even those directly related to the topics we touched upon.

To conclude, Manin's works and his vision of the role of number theory greatly influenced a lot of people, certainly all his students. We thank Yuri Tschinkel and Michael Finkelberg for help with this note.

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