RATIONAL POINTS OVER C_1 FIELDS

HÉLÈNE ESNAULT

1. Introduction

Early in his work, Yuri Manin established a conjectural relation between the geometry of certain smooth projective varieties and the existence of rational points on them on some specific fields. We shall focus on the first such example, published in 1966 in [Man66]. He was 29.

In his PhD thesis [Lan52], Lang defines the notion of a quasi algebraically closed field, a concept he attributes to E. Artin, see also the preface [LT65, p.x] to Artin's Collected Papers by Lang and Tate: this is a field k for which every hypersurface of degree $d \le n$ in $\mathbb{P}^n$ over k admits a rational point. He defines more generally a field to be C_i for some natural number i if every hypersurface of degree d with $d^i \le n$ in $\mathbb{P}^n$ over k admits a rational point. So quasi algebraically closed fields are precisely the C_1 fields. He lists fields which are C_1: finite fields, according to the theorem of Chevalley ([Che35, Théorème]) and its refinement by Warning (see [War35, Satz 1a] in which he shows a congruence modulo the characteristic p of k for the number of rational points); function fields over an algebraically closed field, according to the theorem of Tsen [Tse34, Satz 5] (who had also developed a variant of the C_i-notion). Lang proves in *loc. cit.* that complete discretely valued fields with an algebraically closed residue field are C_1. In equal characteristic, it includes the henselization at a finite place of a function field over an algebraically closed field (as then rational points are dense in rational points over the completion), in unequal characteristic it includes finite extensions of the maximal unramified extension of $\mathbb{Q}_p$ (as again rational points are dense in rational points over the completion).

Else there are conjectures. For example E. Artin conjectured that the maximal abelian extension $\mathbb{Q}^{\rm ab}$ of $\mathbb{Q}$ is C_1 (see the preface by Lang and Tate in *loc. cit.*). More recently, Fargues conjectured in [Far20, Conjecture 3.10] that the field of functions $\bar{\mathbb{Q}}_p(FF)$ of the Fargues-Fontaine curve is C_1, in analogy with Lang's conjecture on the function field $\mathbb{R}(C_0)$ of the conic, $C_\circ = \mathrm{Spec}\big(R[x,y]/(x^2+y^2+1)\big)$ over $\mathbb{R}$ without a point ([Lan53, p. 379]).

Manin in *loc. cit.* Theorem 3.12 b) proves that del Pezzo surfaces X over a finite field $\mathbb{F}_q$ have a rational point. He uses the Weil conjectures for del Pezzo surfaces: the second ℓ-adic cohomology $H^2(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ is spanned by algebraic cycles, which are all defined over a finite extension of $\mathbb{F}_q$, and the odd cohomology dies. So the Grothendieck-Lefschetz trace formula [Gro64, (25)] expresses the number of $\mathbb{F}_q$-points $|X(\mathbb{F}_q)|$ of X in the form

$$(\star) \quad |X(\mathbb{F}_q)| = 1 + q(\sum_{j=1}^{b_2} \xi_j) + q^2$$

where the ξ_i are roots of unity. This implies that $|X(\mathbb{F}_q)| \neq \emptyset$. In fact, more is true: as $(\sum_{i=1}^{b_2} \xi_j)$ is an algebraic integer, it implies that $|X(\mathbb{F}_q)|$ is congruent to 1 modulo q. Manin refers to Weil's article [Wei58, p.557] for the shape of the trace formula in ℓ-adic cohomology. Weil did not have ℓ-adic cohomology at disposal, he was conjecturing its existence and purity, proved later by Deligne in [Del74], and in [Del80]. In *loc. cit.* Weil mentions explicitly the formula $(\star)$ for rational surfaces as a consequence of his conjecture. He argues that on those, cohomological correspondences on $X \times_{\mathbb{F}_q} X$ are coming from algebraic cycles defined over some finite extension of $\mathbb{F}_q$. Even if he does not quote them, Manin likely had [Gro64] at disposal, and [Tat65] in which Tate formulates the Tate conjecture and details the relation between the zeta function and the cohomology as a Galois module.

Manin then formulates in *loc. cit.* Conjecture 4.1 to the effect that every rational surface X over a C_1 field k admits a rational point. He proves it by purely geometric methods for rational surfaces with an extra geometric property, for example when the surface is fibered $f : X \to C$ in conics. Then C itself is a conic, that is a curve of degree ≤ 2 in $\mathbb{P}^2$, so has a rational point by the C_1 definition, and by the same argument its fibre as well.

Manin's conjecture was really *programmatic*, which made arithmetic and geometric properties intertwined. It is interesting to mention that Deligne, to quote himself, coined the term *Arithmetic Geometry*, influenced by the title of the conference "Arithmetic Algebraic Geometry" held in Purdue in December 1963, in which Tate exposed his conjectures, but also after Manin's article *loc. cit.*. Of course, the Weil conjecture, the Tate conjecture etc. postulate properties shared by all varieties over certain fields, while Manin initiates the study of the relation between some specific fields and some specific varieties.

Campana in [Cam91] in complex geometry and Kollár, Miyaoka and Mori in [KMM92] more generality develop the notion of separably rationally connected varieties, a vast generalization of rational varieties on one hand, and of hypersurfaces of degree $d \leq n$ in $\mathbb{P}^n$ on the other. In dimension 1 and 2, rationally connected varieties are precisely rational varieties. In any dimension, a smooth projective connected variety X over a field k is separably rationally connected if over the algebraic closure $\bar{k}$ it admits a free rational curve, that is a morphism $\mathbb{P}^1 \to X_{\bar{k}}$ such that the pull-back of the tangent bundle is ample on the rational curve. If k has characteristic 0 this is equivalent to X being rationally chain connected, that is to the property that any two geometric points can be linked by a finite connected chain of rational curves. If k has characteristic $p > 0$, if X is separably rationally connected, it is rationally chain connected. The positivity of the pull-back of the tangent bundle forces those rational curves to be separable.

Kollár publishes his book [Kol96] in 1996 in which he studies general properties of rationally connected varieties. Manin's conjecture by this time has become the *Lang-Manin conjecture*: separably rationally connected varieties should have a rational point over a C_1 field. In the sequel we call it *the C_1-conjecture.* We mention also [Kol96, 6.1.1] in which Kollár poses the conjecture as a problem over a function field and [Kol96, 6.1.2] in which he asks for all fields over which separably rationally connected varieties have a rational point.

In this short note, we report on some progress on the C_1-conjecture. There are two main directions.

The first one relies on a classification of the varieties considered, see Section 3. The proof by Colliot-Thélène [CT86, Proposition 2] of Manin's initial conjecture, that is for rational surfaces, relies on the classification of rational surfaces over any field (by Iskovskikh [Isk80] for the last stone). They are fibered in conics, the case already essentially treated by Manin, or del Pezzo surfaces, see [Wit10, Théorème 1.11].

The second one consists in going through all the known C_1 fields one by one and try to apply methods specific to those fields. For example if k is a field of functions, do global geometry and degeneration on the special fibres, see Section 3.1. If k is the henselization at a finite place of a field of functions, reduce to the previous case, see Section 5.2. If k is finite, make sure that even if ℓ-adic cohomology is not spanned by algebraic cycles, the "level" of the cohomology is at least one so as to be able to apply q-divisibility in the Grothendieck-Lefschetz trace formula, see Section 4. If k is a complete discretely valued field with an algebraically closed residue field in unequal characteristic, we know very little, see Section 5.3 for a few modest facts on the index and on a "dynamical" result.

On the other hand, we do not know a complete list of C_1 fields. This shows that we are very far from understanding the conjecture in all generality. At the same time, there are classical varieties which are separably rationally connected and not defined by equations, but e.g. by a moduli functor, like in characteristic 0 the moduli of stable bundles of coprime rank and degree on a curve (see [Kau18] for first steps in this direction). It would be appealing if over a field from which one conjectures it is C_1, one could find a rational point on a separably rationally connected variety which is not defined by equations. Or perhaps no point: in the latter case, we would have to revise the hierarchy of problems!

This small text is by no means exhaustive and selects a few spots which are closer to the author's interests and understanding. For background and general studies, we refer to [Deb03], [Wit10] and [CT11].

Acknowledgements: We thank Pierre Deligne for a discussion on [Wei58] and on the origins of the terminology "Arithmetic Geometry" which is reflected in the Introduction. We thank Lars Hesselholt for discussions on $\bar{\mathbb{Q}}_p(FF)$, Olivier Benoist and Kay Rülling for discussions on $\mathbb{R}(C_\circ)$, Marco D'Addezio for discussions on $\mathbb{Q}^{\rm ab}$, which are reflected in Section 2.1, Olivier Wittenberg for general discussions on the topic when it came, János Kollár for precise references in his book, Akhil Mathew for his interest.

2. General Facts

2.1. **The Brauer group.** The Brauer group of a C_1 field is zero, see [Lan52, Corollary of Theorem 14], see [Wit10, Proposition1.15] for a self-contained proof, where the C_1 property is applied to the reduced norm of the skew field to its center.

In particular, to check whether a field k is C_1, one might first compute that its Brauer group $\mathrm{Br}(k)$ is equal to zero.

Witt proves in [Witt34, Satz 4] that the norm map is surjective on finite extensions of $\mathbb{R}(C_\circ)$, which shows that $\mathrm{Br}(\mathbb{R}(C_\circ)) = 0$ (see [Ser79, Proposition 11, 3)]).

For $\mathbb{Q}^{\rm ab}$, one proof is suggested in [Ser79, p. 162, d)]: $\mathbb{Q}^{\rm ab}$ contains the maximal cyclotomic extension of $\mathbb{Q}$ (in fact it is equal to it by the Kronecker-Weber theorem). The Brauer group of a p-adic field is killed by the maximal cyclotomic extension which is defined over $\mathbb{Q}$. The local-global principle applied to the tower of cyclotomic extensions of $\mathbb{Q}$ allows one to conclude.

Fargues proves $\mathrm{Br}(\mathbb{Q}_p(FF)) = 0$ in [Far20, Théorème 2.2, Corollaire 2.5].

2.2. **Severi-Brauer varieties.** As the Brauer group of a C_1 field k is trivial, Severi-Brauer varieties over k, that is smooth projective varieties over k which are isomorphic to $\mathbb{P}^d$ over an algebraic closure $\bar{k}$, have a rational point.

3. Geometry

In this section, we follow the Bourbaki talk by Debarre [Deb03].

3.1. **The theorem of Graber-Harris-Starr.** In [GHS03, Theorem 1.2] the authors prove the C_1 conjecture for $k = \mathbb{C}(B)$ the field of functions of a curve B over the field of complex numbers $\mathbb{C}$. This is equivalent to saying that if $f : X \to B$ is a projective morphism over a curve B over $\mathbb{C}$, with rationally connected generic fiber $X \times_B \mathbb{C}(B)$, then f has a section.

By replacing f by its Weil restriction with respect to a Noether normalization $B \to \mathbb{P}^1$, we may assume that $B = \mathbb{P}^1$.

A closed point of $X \times_{\mathbb{P}^1} \mathbb{C}(\mathbb{P}^1)$ corresponds to a diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & X \\ {\scriptstyle u}\downarrow & \swarrow{\scriptstyle f} & \\ \mathbb{P}^1 & & \end{array}$$

where C is a curve and $u : C \to \mathbb{P}^1$ is finite. If g, viewed as a point in the moduli space of maps to $\mathbb{P}^1$ with certain properties at the ramification of $C \to g(C)$, was smooth, then it would deform to a section of f ([Deb03, Theorem 4.1]). This is the first place where the authors use transcendental methods. The moduli $M(X, g_*[C])$ is the one of such maps for a fixed Betti class $g_*[C] \in H_2(X, \mathbb{Z})$. They need its compactification for the proof. Then they have to put themselves in this situation. To this aim, they use topological arguments combined with the abundance of rational curves to prove that once they have g, they can attach to C many rational curves so the union deforms well.

3.2. **The theorem of de Jong-Starr.** In [dJS03, Theorem] the authors prove the C_1 conjecture for $k = F(B)$ where F is any algebraically closed field. Strongly using the freeness of the rational curves, they deform directly algebraically the starting g bypassing the use of $M(X, g_*[C])$ and its compactification, and the topological arguments.

4. Motives and Cohomology

4.1. **Grothendieck-Lefschetz trace formula.** As we saw in the introduction, if the whole ℓ-adic cohomology of a smooth geometrically connected projective variety X is spanned by the cycle classes of algebraic cycles, then the Grothendieck-Lefschetz trace formula reads

$$|X(\mathbb{F}_q)| = 1 + \sum_{i=1}^{2d} (-1)^j \mathrm{Tr}\ \mathrm{Fr}|H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell) = 1 + \sum_{i=1}^{d} q^i (\sum_{j_i=1}^{b_{2i}} \xi_{j_i})$$

where the ξ_{j_i} are roots of unity, the b_{2i} are the Betti numbers of the even weighted ℓ-adic cohomology of $X_{\bar{\mathbb{F}}_p}$, Fr is the arithmetic Frobenius acting of ℓ-adic cohomology. (In addition the coefficient of q^d is equal to 1). As $(\sum_{j=1}^{b_{2i}} \xi_j)$ is an algebraic integer, from the formula follows the congruence

$$|X(\mathbb{F}_q)| \equiv 1 \bmod q.$$

More generally, if the trace formula reads

$$|X(\mathbb{F}_q)| = 1 + q\xi$$

where ξ is an algebraic integer, then we can conclude that $|X(\mathbb{F}_q)| \equiv 1 \bmod q$.

In [Esn03], we prove this property for rationally chain connected varieties, which in particular proves the C_1 conjecture (without the extra separability assumption) over finite fields.

We explain how we reach this congruence in the sequel. In *loc. cit.* we used crystalline cohomology. The same argument is transposed here on the ℓ-adic side.

4.2. **Integrality: Deligne's theorem.** In [Del73, Corollary 5.5.3] Deligne proves that the eigenvalues of Fr on $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ are algebraic integers. This is an important property in our context as it shows:

> the C_1 conjecture over finite fields follows from the statement that the eigenvalues of Fr on $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ for $i > 0$ are all divisible by q as algebraic integers.

4.3. **Motivic analogy.** In [Esn06, Introduction], we raise the question of

> the analogue in ℓ-adic cohomology of the Hodge level for Betti cohomology.

If X is a smooth complex projective variety, if $H^i(X, \mathbb{Q})$ is supported in codimension ≥ 1, then the Hodge level of $H^i(X, \mathbb{Q})$ is ≥ 1 as well. The converse is an example of Grothendieck's generalized Hodge conjecture.

Using purity, due to Gabber, we compute in [Esn03, Lemma 2.1] that if X is a smooth projective variety defined over $\mathbb{F}_q$, and if $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ is supported in codimension ≥ 1, then the eigenvalues of Fr on $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ are divisible by q as algebraic integers.

Thus for proving the C_1 conjecture over finite fields, we are reduced to proving that $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ is supported in codimension ≥ 1 for all $i \geq 1$.

In [BER12, Theorem 1.3] we relate the Hodge level in characteristic 0 to the analogue in crystalline cohomology of the q-divisibility of the eigenvalues of Fr in ℓ-adic cohomology: the former on a smooth projective variety in characteristic 0 implies that the slopes in rigid cohomology H^i of a specialization of a regular model all are ≥ 1 for $i \geq 1$. This makes in this context the philosophical analogy a concrete theorem.

4.4. **Trivial Chow group of 0-cycles after base change to any algebraic closed field.** By the very definition of rationally chain connected varieties, any two geometric points are linked by a finite connected chain of rational curves. In particular, the group $CH_0(X_K)$ of algebraic 0-cycles is equal to $\mathbb{Z}$ for any embedding $k \hookrightarrow K$ of k into an algebraically closed field K. Taking K to contain the generic point Spec($k(X)$), the resulting decomposition of the diagonal initiated by Bloch in his book [Blo80, Appendix to Lecture 1], a method which later became the basis of manifold developments, proves that $H^i(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ is supported in codimension ≥ 1 for all $i \geq 1$. This finishes the proof of C_1 conjecture over finite fields.

5. Remarks

5.1. **C_1 conjecture and minimal model program in characteristic 0.** A generalization of the study case by case of rational surfaces by Colliot-Thélène *loc. cit.* is performed in characteristic 0 by Pieropan in [Pie22]: over a given field of characteristic 0, the minimal model program implies that all rationally connected varieties of dimension $\leq d$ have a rational point if and only if $\mathbb{Q}$-factorial Fano varieties of dimension $\leq d$ and Picard rank 1 do. It is roughly a generalization of the reduction from all rational surfaces to the del Pezzo ones.

5.2. **Henselization of function fields at finite places.** In [CT11, Théorème 7.5], Colliot-Thélène proves that the C_1 conjecture holds true for k being the field of fractions of an henselian discrete valuation ring with algebraically closed residue field F in equal characteristic. Indeed, if X is defined over k, its rational points $X(k)$ are dense in $X(\widehat{k})$ for the topology defined by the discrete valuation, where $\widehat{k} = F((t))$ is the completion with respect to the valuation. If X was defined over $k(t)$, we could then apply directly the theorems Sections 3.1 and 3.2. In general X descends to X_A where $A = F[T_1, \ldots, T_n]/I \subset F[[t]]$ is an affine and smooth algebra over F, where the embedding is defined by a power series expansion in t of the T_i. For $n \in \mathbb{N}_{\geq 1}$ large enough, depending on the multiplicity of $\mathrm{Spec} F \in S = \mathrm{Spec}(A)$, the $F[[t]]/(t^n)$-point of S integrates to a finite type normal curve $C_n \to S$ over F, generically embedded in S, thus over which X_{C_n} has a C_n-point. In particular, $X(F[[t]]/(t^n)) \neq \emptyset$. As X is proper, a compactness argument allows one to conclude that $X(F[[t]]) \neq \emptyset$.

5.3. **Some remarks on complete discrete valuation rings $\mathcal{O}$ with algebraically closed residue field F in unequal characteristic.**

5.3.1. *A small dream.* We mention now where the proof from Section 5.2 can not be generalized to the unequal characteristic situation. By the same argument as in Section 3.1, we may assume that $\mathcal{O} = W(F)$, where F is the algebracially closed residue field of characteristic $p > 0$. So we have $A \subset W$, which is now defined by the p-adic expansion of the T_i, the resulting $W_n = W/p^n$ points of S, X_A. But we do not have the integration C_n as in the equal characteristic case. At the ICM 2014 in Seoul, we asked Scholze, which had just proved Deligne's weight-monodromy conjecture [Sch12, Theorem 1.14] for complete intersections by applying his tilting method, whether he would think there is a way to tilt X_W to say $Y_{F[[t]]}$, keeping the rational connectivity property for $Y_{F((t))}$, and if so if one could tilt back $Y_{F[[t]]}(F[[t]])$ constructed in Section 5.2 to $X_W(W)$. It is a difficult and badly posed question for many reasons, two immediate ones being that the tilt of the field of fractions of W contains also all the p-roots of t, and that over this large field, Y is not rationally connected. Nonetheless we can dream of a more evolved analogue of the classical proof in Section 5.2 in this context.

5.3.2. *Back to the Hodge level.* In [ELW15, Corollary 3] a weak form of the C_1 conjecture is proved over K the field of fractions of $\mathcal{O}$: the index of a separably rationally connected variety X, that is the g.c.d. of the degrees of its closed points, is equal to 1 if $\dim X + 1 < p$. It is really a very weak form. Indeed, the index 1 statement is true as soon as the Hodge level of X is ≥ 1. It is in the spirit of the theorem [BER12, Theorem 1.3] mentioned in Section 4.3

5.3.3. *Dynamic.* As explained previously, fixing a field K, complete with respect to a discrete valuation with algebraically closed residue field in unequal characteristic, the C_1 conjecture is not understood. In [DK17, Theorem 1.3], Duesler and Knecht prove a "dynamical" version of the conjecture which is close to the theorem of Ax-Knochen according to which, except for finitely many p, any hypersurface in $\mathbb{P}^n$ over $\mathbb{Q}_p$ of degree d with $d^2 \leq n$ has a rational point. They fix a Hilbert polynomial P and show using similar methods that, except for finitely many p, any rationally connected variety with Hilbert polynomial P and defined over the maximal unramified extension of $\mathbb{Q}_p$ has a rational point.

References

[BER12] Berthelot, P., Esnault, H., Rülling, K.: *Rational points over finite fields for regular models of algebraic varieties of Hodge type greater or equal to* 1, Annals of Mathematics **176** (2012), 1–96.

[Blo80] Bloch,S.: *Lectures on Algebraic cycles*, Duke University Mathematics Series **IV** (1980).

[Cam91] Campana, F.: *On twistor spaces of the class C*, J. Diff. Geom. **33** (1991), 541–549.

[Che35] Chevalley, C. it Démonstration d'une hypothèse de M. Artin, Hamburg Abh. **11** (1935), 73–75.

[CT86] Colliot-Thélène, J.-L.: *Arithmétique des variétés rationnelles et problèmes birationnels*, Proceedings ICM 1986, vol. 1, Berkeley, Amer. Math. Soc., (1987), 641–653.

[CT11] Colliot-Thélène, J.-L.: *Variétés presque rationnelles, leurs points rationnels et leurs dégénérescences*, In Arithmetic Geometry, Lecture Notes in Math. **2009** (2011), 1–44. Springer Verlag.

[Deb03] Debarre, O.: *Variétés rationnellement connexes*, Exposé **905**, Séminaire Bourbaki 2001/2002, Astérisque **290** (2003), 243–266.

[DK17] Duesler, B., Knecht, A.: *Rationally connected varieties over the maximally unramified extension of p-adic fields*, Rocky Mountain J. Mat. **47** (2017), (8), 2605–2617.

[dJS03] de Jong, J., Starr, J.: *Every rationally connected variety over the function field of a curve has a rational point*, Amer. J. Math. **125** (2003) (3), 567–580.

[Del73] Deligne, P.: *Théorème d'intégralité*, Appendix to "Le eniveau de la cohomologie des intersections complètes" by N. Katz, Expos'e XXI in SGA 7, Lecture Notes in Math. **340**, 363–400, Springer-Verlag, New York, 1973.

[Del74] Deligne, P.: *La conjecture de Weil I*, Publ. math. IHES **43** (1974), 273–308.

[Del80] Deligne, P.: *La conjecture de Weil II*, Publ. math. IHES **52** (1974), 137–252.

[Esn03] Esnault, H.: *Varieties over a finite field with trivial Chow group of* 0*-cycles have a rational point*, Invent. math. **151** (2003), 187–191.

[Esn06] Esnault, H.: *Deligne's integrality theorem in unequal characteristic and rational points over finite fields*, Annals of Mathematics **164** (2006), 715–730.

[ELW15] Esnault, H., Levine, M., Wittenberg, O.: *Index of varieties over Henselian fields and Euler characteristic of coherent sheaves*, J. Algebraic Geometry **24** (2015), 693–718.

[Far20] Fargues, L. : *G-torseurs en théorie de Hodge p-adique*, Compositio Math. **156** (2020), 2076–2110.

[GHS03] Graber, T., Harris, J., Starr, J.: *Families of ratonally connected varieties*, J. AMS **16** (2003), (1), 57–67.

[Gro64] Grothendieck, A.: *Formule de Lefschetz et rationalité des fonctions L*, Sém. Bourbaki **279**, 41–55, Soc. Math. France, Paris (1964/1965), 1995.

[Isk80] Iskovskikh, V.: *Minimal models of rational surfaces over arbitrary fields*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 19–43 (in russian), Math. USSR Izv. **14** (1980), no. 1. (in english).

[Kau18] Kaur, I.: *A pathological case of the C_1 conjecture in mixed characteristic*, In Mathematical Proceedings of the Cambridge Philosophical Society, **167** (2019), 61–64.

[KMM92] Kollár, J., Miyaokoa, Y., Mori, S.: *Rational connectedness and boundedness of Fano manifolds*, J. Diff. Geom. **36** (1992) 765–779.

[Kol96] Kollár, J.: *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete **32**, Springer Verlag (1996).

[Lan52] Lang, S.: *On quasi-algebraic closure*, Annals of Math. **55** (1952), 373–390.

[Lan53] Lang, S.: *The theory of real places*, Annals of Math. **57** (1957) (2), 378–391.

[LT65] Lang, S., Tate, J.: *The collected papers of Emil Artin*, Addison-Wesley Publishing Co. (1965), xvi+560 pp.

[Man66] Manin, Yu.: *Rational surfaces over perfect fields*, Publ. math. IHES **30** (1966), 55–113.

[Pie22] Pieropan, M. *On rationally connected varieties over C_1 fields of characteristic* 0, Algebra Number Theory bf 16 (2022), (8), 1811–1844.

[Sch12] Scholze, P.: *Perfectoid a spaces*, Publ. math. IHES **116** (2012), (1), 245–313.

[Ser79] Serre, J.-P.: *Local Fields*, Graduate Texts in Mathematics **67**, Springer Verlag (1979).

[Tat65] Tate, J.: *Algebraic cycles and poles of zeta functions*, in *Arithmetical Algebraic Geometry*, Proc. Conf. Purdue Univ., (1963) pp. 93–110 Harper and Row, New York (1965).

[Tse34] Tsen, Ch. C.: *Algebren über Funktionenkörpern*, Ph. D. dissertation, Göttingen University (1934).

[War35] Warning, E.: *Bemerkung zur vorstehender Arbeit von Herrn Chevalley*, Hamburg Abh. **11** (1935), 76–83.

[Wei58] Weil, A.: *Abstract versus classical algebraic geometry*, Matematika **4** Vol. 2, 59–66.

[Witt34] Witt, E.: *Zerlegung reeller algebraischer Funktionen in Quadrate. Schiefkörper über reellem Funktionenkörper*, J. Reine Angew. Math. **171** (1934), 4–11.

[Wit10] Wittenberg, O.: *La connexité rationnelle en arithmétique*, in *Variétés rationnellement connexes: aspects géométriques et arithmétiques*, Panorama et Synthèses **31**, Société mathématique de France (2010).

Freie Universität Berlin, Mathematik, Arnimallee 7, 14195 Berlin, Germany

Email address: esnault@math.fu-berlin.de