Spectral description of non-commutative local systems on surfaces and non-commutative cluster varieties

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To the memory of Yuri Ivanovich Manin

Contents

Abstract

Let R be a non-commutative field. We prove that generic triples of flags in an $m-$dimensional $R-$vector space are described by flat $R-$line bundles on the honeycomb graph with $\frac{(m-1)(m-2)}{2}$ holes.

Generalising this, we prove that the non-commutative stack $\mathcal{X}_{m,\mathbb{S}}$ of framed flat $R-$vector bundles of rank m on a decorated surface $\mathbb{S}$ contains open dense substacks, identified with stacks of flat line bundles on certain bipartite graphs Γ on $\mathbb{S}$.

We introduce non-commutative cluster Poisson varieties related to bipartite ribbon graphs. They carry a canonical non-commutative Poisson structure. The result above just means that the space $\mathcal{X}_{m,\mathbb{S}}$ has a structure of a non-commutative cluster Poisson variety, equivariant under the action of the mapping class group of $\mathbb{S}$.

For bipartite graphs on a torus, we get the non-commutative dimer cluster integrable system.

We develop a parallel dual story of non-commutative cluster $\mathcal{A}-$varieties related to bipartite ribbon graphs. They carry a canonical non-commutative 2-form. The dual non-commutative moduli space $\mathcal{A}_{m,\mathbb{S}}$ of twisted decorated local systems on $\mathbb{S}$ carries a cluster $\mathcal{A}-$variety structure, equivariant under the action of the mapping class group of $\mathbb{S}$. The non-commutative cluster $\mathcal{A}-$coordinates on the space $\mathcal{A}_{m,\mathbb{S}}$ are expressed as ratios of Gelfand-Retakh quasideterminants. In the case $m = 2$ this recovers the Berenstein-Retakh non-commutative cluster algebras related to surfaces.

For any split reductive group G with connected center, we prove that all stacks of framed G−Stokes data carry a cluster Poisson structure, equivariant under the wild mapping class group. Therefore these stacks can be equivariantly quantized.

The similar stacks of decorated G−Stokes data carry an equivariant cluster $\mathcal{A}-$variety structure.

We introduce admissible dg-sheaves, and define non-commutative stacks of Stokes data as stacks of admissible dg-sheaves of certain type.

1 Introduction

1.1 Spectral description of generic triples of flags

Let R be a *skew field*, i.e. a non-commutative division algebra. An m-dimensional $R-$vector space is a free rank m left R-module. If $m = 1$, we call it an $R-$line.

A *flag* $\mathcal{F}$ in an m-dimensional R-vector space V is a filtration by $R-$vector spaces

$$V = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \mathcal{F}^2 \supset \ldots \supset \mathcal{F}^{m-1} \supset \mathcal{F}^m = 0, \quad \text{codim } \mathcal{F}^i = i. \tag{1}$$

We associate to a flag $\mathcal{F}$ a colection of m lines $\text{gr}^i\mathcal{F} := \mathcal{F}^i/\mathcal{F}^{i+1}$, $i = 0, ..., m-1$.

The group of automorphisms of the R-vector space V acts transitively on the set of all flags in V.

In Section 1.1 we describe generic triples of flags in an $m-$dimensional $R-$vector space via flat line bundles on the bipartite graph Γ_m, shown on Figures 1, 2. It is a self-contained part of the paper, describing the simplest, and at the same time crucial result with all details. Let us start with generic pairs of flags.

1. Generic pairs of flags. A pair of flags $(\mathcal{A}, \mathcal{B})$ in an m-dimensional $R-$vector space V is *generic* if for any integers $a, b \geq 0$ such that $a + b = m$ the projections $V \longrightarrow V/\mathcal{A}^a$ and $V \longrightarrow V/\mathcal{B}^b$ induce an isomorphism

$$V \stackrel{\sim}{\longrightarrow} V/\mathcal{A}^a \oplus V/\mathcal{B}^b.$$

Given a generic pair of flags $(\mathcal{A}, \mathcal{B})$, let us set

$$\mathrm{L}^{b+1} := \mathcal{A}^a \cap \mathcal{B}^b, \quad a + b = m - 1, \quad a, b \geq 0.$$

The embeddings $\mathrm{L}^a \hookrightarrow V$, $a = 1, ..., m$, give rise to a canonical isomorphism

$$\mathrm{L}^1 \oplus \ldots \oplus \mathrm{L}^m \stackrel{\sim}{\longrightarrow} V.$$

Lemma 1.1. *Assigning to a generic pair of flags $(\mathcal{A}, \mathcal{B})$ in an m-dimensional R-vector space V the ordered collection of m lines $(\mathrm{L}^1, \ldots, \mathrm{L}^m)$ we get an equivalence of groupoids:*

$$\begin{gathered}\textit{Groupoid of generic pairs of flags in } m\textit{-dimensional spaces } (V, \mathcal{A}, \mathcal{B}) \stackrel{\sim}{\longrightarrow} \\ \textit{Groupoid of ordered collections of } m \textit{ lines } (\mathrm{L}^1, \ldots, \mathrm{L}^m).\end{gathered} \quad (2)$$

Proof. The inverse functor $(\mathrm{L}^1, \ldots, \mathrm{L}^m) \longrightarrow (V, \mathcal{A}, \mathcal{B})$ is given by

$$V := \mathrm{L}^1 \oplus \ldots \oplus \mathrm{L}^m, \quad \mathcal{A}^a := \mathrm{L}^1 \oplus \ldots \oplus \mathrm{L}^{m-a}, \quad \mathcal{B}^b := \mathrm{L}^{b+1} \oplus \ldots \oplus \mathrm{L}^m.$$

□

2. Spectral description of the groupoid of generic triples of flags. A triple of flags $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in an m-dimensional R-vector space V is *generic* if for any non-negative integers (a, b, c) with $a+b+c = m$, the following map is an isomorphism:[1]

$$V \longrightarrow V/\mathcal{A}^a \oplus V/\mathcal{B}^b \oplus V/\mathcal{C}^c, \qquad a + b + c = m. \quad (3)$$

Let us describe generic triples of flags in an m-dimensional R-vector space.

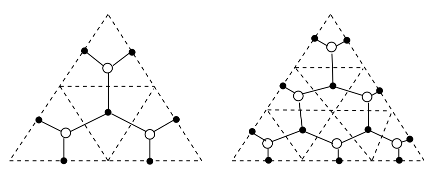


Figure 1: The bipartite graphs Γ_2 and Γ_3, shown by solid lines.

Let us assign a triple of flags $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ to the vertices of a triangle t, labeled by A, B, C.

We introduce a bipartite graph Γ_m, shown in Figures 1-2 for $m \leq 4$. Recall that a *bipartite graph* is a graph with vertices of two types, $\bullet$-vertices and $\circ$-vertices, such that each edge has a vertex of each type.

To define the graph Γ_m, take an m-triangulation of the triangle t, shown by punctured lines on Figure 1. It subdivides each side of the triangle into m equal segments, and tesselates the triangle into little triangles of two types: the $\bullet$-triangles, and the $\circ$-triangles. The bipartite graph Γ_m is the dual graph for this tesselation, with the extra $\bullet-$vertices added at the endpoints of the external edges.

Recall that a flat line bundle L on a graph Γ is given by the following data:

1. A line L_v for every vertex v of Γ.

[1]Note that a flag in V gives rise to the dual flag in V^*, but the dual to a generic triple of flags may not be generic.

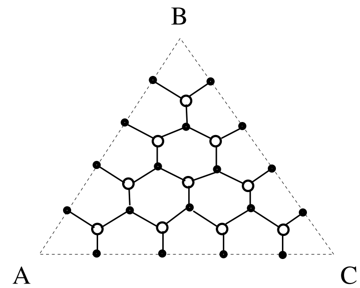


Figure 2: A bipartite graph Γ_4 for a generic triple of flags $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in a 4-dimensional space.

2. An isomorphism $t_{w,v} : L_v \to L_w$ for each edge of Γ with the ends v, w, with $t_{v,w} = t_{w,v}^{-1}$.

Theorem 1.2. *There is a canonical equivalence between the following two groupoids:*

1. *Groupoid of generic triples of flags in an m-dimensional R-vector space.*
2. *Groupoid of R-line bundles with connections on the bipartite graph Γ_m, see Figure 2.*

Proof. 1. *The functor $\mathcal{L} : \{$Generic triples of flags$\} \longrightarrow \{$Flat line bundles on the graph $\Gamma_m\}$.*
Given a generic triple of flags $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, let us define a flat R-line bundle $\mathcal{L}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ on the graph Γ_m.

1. The $\circ-$vertices of the graph Γ_m are parametrized by the triples of non-negative integers (a, b, c) such that $a + b + c = m - 1$. We assign to each $\circ-$vertex a one dimensional subspace:
$$\mathrm{L}^{\circ}_{a,b,c} := \mathcal{A}^a \cap \mathcal{B}^b \cap \mathcal{C}^c, \qquad a + b + c = m - 1. \tag{4}$$

2. The $\bullet-$vertices of the graph Γ_m are parametrized by the triples of non-negative integers (a, b, c) such that $a + b + c = m - 2$. We assign to each $\bullet-$vertex a two dimensional subspace:
$$\mathrm{P}_{a,b,c} := \mathcal{A}^a \cap \mathcal{B}^b \cap \mathcal{C}^c, \qquad a + b + c = m - 2. \tag{5}$$

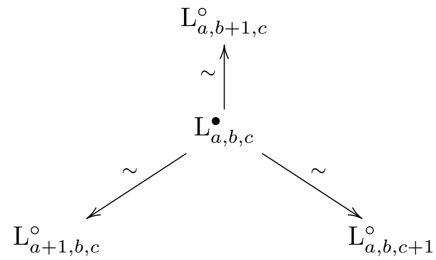
The plane $\mathrm{P}_{a,b,c}$ assigned to a $\bullet-$vertex b contains three lines assigned to the $\circ-$vertices incident to b:

$$\begin{array}{ccc} & \mathrm{L}^{\circ}_{a,b+1,c} & \\ & \downarrow & \\ & \mathrm{P}_{a,b,c} & \\ \nearrow & & \nwarrow \\ \mathrm{L}^{\circ}_{a+1,b,c} & & \mathrm{L}^{\circ}_{a,b,c+1} \end{array} \tag{6}$$

Indeed, the embedding $\mathcal{A}^{a+1} \hookrightarrow \mathcal{A}^a$ induces the embedding $\mathcal{A}^{a+1} \cap \mathcal{B}^b \cap \mathcal{C}^c \hookrightarrow \mathcal{A}^a \cap \mathcal{B}^b \cap \mathcal{C}^c$. Condition (3) just means that for each plane $\mathrm{P}_{a,b,c}$, the three lines (6) are disjoint. We define a line $\mathrm{L}^{\bullet}_{a,b,c}$ as the kernel of the natural map from the sum of the three lines in (6) to the plane:

$$\mathrm{L}^{\bullet}_{a,b,c} := \mathrm{Ker}\Big(\mathrm{L}_{a+1,b,c} \oplus \mathrm{L}_{a,b+1,c} \oplus \mathrm{L}_{a+1,b,c} \longrightarrow \mathrm{P}_{a,b,c}\Big). \tag{7}$$

Then there are three maps, induced by projections of the subspace $\mathrm{L}^{\bullet}_{a,b,c}$ onto the summands:



(8)

These three maps are isomorphisms if and only if the three lines in the plane $\mathrm{P}_{a,b,c}$ are disjoint.

So we get a flat line bundle on the graph Γ_m: its fibers at the vertices are given by the lines L° and $\mathrm{L}^{\bullet}$, and the parallel transport along an edge $\bullet \to \circ$ is given by a map $\mathrm{L}^{\bullet} \longrightarrow \mathrm{L}^{\circ}$ in (46).

Remark. If $R = K$ is commutative, the moduli space of flat line bundles on a graph Γ is identified with $H^1(\Gamma, K^*)$. So flat line bundles on the graph Γ_m are described by their monodromies around the $\binom{m-1}{2}$ holes in the graph. These are literally the canonical coordinates from [FG1], called the *triple ratios*.

If R is non-commutative, we no longer have canonical coordinates describing generic triples. For example, the isomorphism classes of flat R-line bundles on the graph Γ_3 are described by the elements of R^* considered modulo the conjugation. For $m > 3$ we need to pick a base point and paths to get a collection of elements of R^* defined up to a common conjugation.

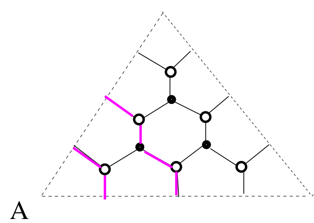


Figure 3: Vertices of the graph Γ_3 on the distance ≤ 2 from the vertex A are on the two (purple) zig-zags.

2. *The reconstruction functor* $\mathcal{R}$: $\{$*Flat line bundles on* $\Gamma_m\} \longrightarrow \{$*generic triples of flags*$\}$. Given a flat line bundle $\mathcal{L}$ on the graph Γ_m, we get a map from the direct sum over all internal $\bullet$−vertices $\{\mathbf{b}\}$ of the graph Γ_m to the direct sum over all $\circ$−vertices $\{\mathbf{w}\}$ of the graph Γ_m:

$$\mathrm{K}_{\mathcal{L}} : \bigoplus_{\text{internal } \bullet-\text{vertices } \mathbf{b}} \mathcal{L}_{\mathbf{b}} \longrightarrow \bigoplus_{\circ-\text{vertices } \mathbf{w}} \mathcal{L}_{\mathbf{w}}. \tag{9}$$

We define a vector space $V_{\mathcal{L}}$ as the cokernel of this map:

$$V_{\mathcal{L}} := \mathrm{Coker}(\mathrm{K}_{\mathcal{L}}). \tag{10}$$

Let us define three flags in this vector space, assigned to the vertices of the triangle t. Given a vertex A of the triangle, consider a function $d_{\mathrm{A}} : \{\text{vertices } v \text{ of the graph } \Gamma_m\} \longrightarrow \{1, ..., m\}$, given by the distance from a vertex v to the A, see Figure 3.[2] The function d_{A} induces a filtration on the complex (9).[3] So it induces a filtration on $\mathrm{Coker}(\mathrm{K}_{\mathcal{L}})$. We define the flag $\mathcal{A}_{\mathcal{L}}$ as the space (10) with this filtration. Explicitly:

$$\mathcal{A}^a_{\mathcal{L}} := \mathrm{Coker}\Big(\bigoplus_{d_{\mathrm{A}}(\mathbf{b}) \leq m-a} \mathcal{L}_{\mathbf{b}} \longrightarrow \bigoplus_{d_{\mathrm{A}}(\mathbf{w}) \leq m-a} \mathcal{L}_{\mathbf{w}}\Big), \qquad a = 0, ..., m-1.$$

[2] The distance d_{A} is the distance to A from the zig-zag strand on the graph Γ_m, parallel to the side BC, see Figure 3.
[3] That is a filtration on each of the spaces, preserved by the operator $K_{\mathcal{L}}$.

This way we get a functor

$$\mathcal{R} : \mathcal{L} \longrightarrow (\mathcal{A}_{\mathcal{L}}, \mathcal{B}_{\mathcal{L}}, \mathcal{C}_{\mathcal{L}}). \tag{11}$$

By the very definition, the two functors $\mathcal{L}$ and $\mathcal{R}$ provide the equivalence of categories. □

3. Reconstruction functor revisited. Here is a useful variant $\mathcal{R}'$ of the reconstruction functor $\mathcal{R}$.

A flat line bundle $\mathcal{L}$ on Γ_m provides three vector spaces assigned to the sides of the triangle. The space $V_{\rm AB}$ is the direct sum of the fibers of $\mathcal{L}$ at the vertices on the side AB, etc. Each of them carries two flags, assigned to the oriented sides. For example, let $v_1, ..., v_m$ be the vertices on the side AB, ordered $\rm A \to B$. Then the subspaces of the flag $\mathcal{F}^{\bullet}_{\rm AB}$ are

$$\mathcal{F}^a_{\rm AB} := \mathcal{L}_{v_1} \oplus \ldots \oplus \mathcal{L}_{v_{m-a}}. \tag{12}$$

For each vertex of the triangle, say A, let us define an isomorphism between the spaces assigned to the sides sharing the vertex. Namely, we consider A$-$*paths*, which are paths from a vertex on the one side to a vertex on the other side, which always go towards the other side. So given a vertex v_i on the side AB on the disctance i from A, and a vertex v'_j on the side AC on the distance j from A, there are $\binom{i-1}{j-1}$ A$-$paths $\gamma : v_i \longrightarrow v'_j$, see Figure 4. Then we define a linear map

$$\begin{aligned} &\varphi_{\rm A,B\to C} : V_{\rm AB} \longrightarrow V_{\rm AC}, \\ &\varphi_{\rm A,B\to C} := \sum_{\gamma} (-1)^{i-1} t_{\gamma}, \end{aligned} \tag{13}$$

where the sum is over all A$-$paths $\gamma : v_i \longrightarrow v'_j$ from AB to AC, and $t_\gamma : \mathcal{L}_{v_i} \longrightarrow \mathcal{L}_{v'_j}$ is the parallel transport in the local system $\mathcal{L}$ along the path γ.

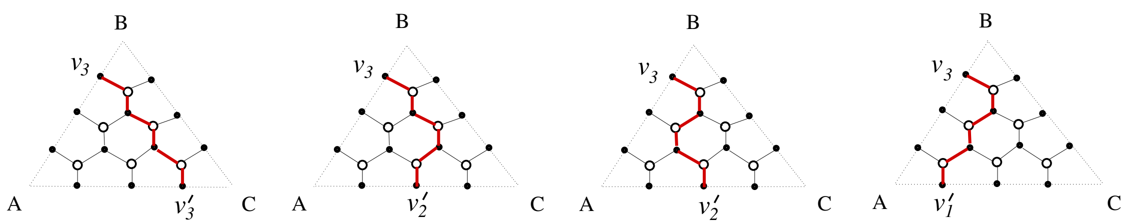


Figure 4: There are four A$-$paths from the vertex v_3 on the side AB to the side AC.

Evidently, this isomorphism preserves the flags assigned to the sides oriented out of the vertex A. It is easy to check that the two isomorphisms at the same vertex are mutually inverse, e.g.

$$\varphi_{\rm A,C\to B} \circ \varphi_{\rm A,B\to C} = {\rm Id}.$$

Furthermore, it is easy to check that the following composition is the identity map:

$$\varphi_{\rm B,C\to A} \circ \varphi_{\rm C,A\to B} \circ \varphi_{\rm A,B\to C} = {\rm Id}$$

Therefore using these isomorphism we identify canonically the three vector spaces, and transform the flags to one of them, getting a configuration of three flags. For example, we get a triple of flags in $V_{\rm AC}$:

$$(\mathcal{A},\ \varphi_{\rm A,B\to C}(\mathcal{B}),\ \mathcal{C})\ \subset\ V_{\rm AC}.$$

The flags are in generic position.

The linear map (13) seems to come out of the blue. Proposition 1.3 and Lemma 1.4 explain how to deduce it from the construction of Theorem 1.2.

Proposition 1.3. *The two reconstruction functors $\mathcal{R}$ and $\mathcal{R}'$ coincide.*

Proof. For each side of the triangle, say the side AB, there is a canonical map

$$\psi_{\rm AB}: V_{\rm AB} \longrightarrow V_{\mathcal{L}}. \tag{14}$$

It is induced by the embedding of the direct sum of the lines $\mathcal{L}_{\bf w}$ at the $\circ-$vertices closest to the side AB to the direct sum of all lines $\mathcal{L}_{\bf w}$.

Lemma 1.4. *The map $\psi_{\rm AB}$ is an isomorphism. One has*

$$\psi_{\rm AB}^{-1} \circ \psi_{\rm AB} = \varphi_{\rm A,B\to C}. \tag{15}$$

Proof. Denote by $V_{\mathcal{L}}^{\bullet}$ (respectively $V_{\mathcal{L}}^{\circ}$) the direct sum of the lines $\mathcal{L}_{\bf b}$ over all $\bullet-$vertices ${\bf b}$ (respectively the lines $\mathcal{L}_{\bf w}$ over the $\circ-$vertices) of the graph Γ_m. Denote also by $\widetilde V_{\mathcal{L}}^{\circ}$ the direct sum of all $\mathcal{L}_{\bf w}$ over all $\circ-$vertices but the ones closest to the side AB. Then we have

$$V_{\mathcal{L}}^{\circ} = V_{\rm AB} \oplus \widetilde V_{\mathcal{L}}^{\circ}.$$

There is a canonical isomorphism $K_{\mathcal{L},{\rm C}}: V_{\mathcal{L}}^{\bullet} \stackrel{\sim}{\longrightarrow} \widetilde V_{\mathcal{L}}^{\circ}$, given by the parallel transport from each $\bullet-$vertex ${\bf b}$ along the edge going towards the vertex C. □

□

Remark. The same arguments prove the analog of Theorem 1.2 for arbitrary polygons.

4. Spectral description of decorated triples of flags. A *decorated flag* is a flag $\mathcal{F}^\bullet$ as in (1) with a choice of a non-zero element $f_i \in {\rm gr}^i\mathcal{F} := \mathcal{F}^{i-1}/\mathcal{F}^i$ in each successive quotient.

A *zig-zag path*, or simply a *zig-zag*, on a ribbon graph Γ is a path turning at every vertex either left or right, so that the left and right turns alternate. Zig-zags on a bipartite ribbon graph are oriented so that they turn right at $\circ-$vertices. The graph Γ_m carries $3m$ oriented zig-zags, see Figure 5.

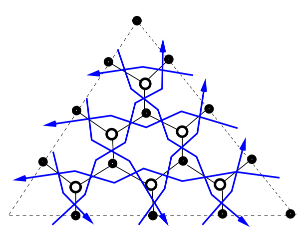


Figure 5: Nine zig-zag paths on the bipartite graph Γ_3. We push zig-zags a bit out of the vertices.

Theorem 1.5. *There is a canonical equivalence between the following two groupoids:*

1. *Groupoid of generic triples of decorated flags in an m-dimensional R-vector space.*
2. *Groupoid of R-line bundles with connections on the bipartite graph Γ_m, trivialised on every zig-zag.*

Theorem 1.5 is immediately deduced from Theorem 1.2 and the description of the reconstruction functor, which turns trivializations of the local system on zig-zags into decoration vectors of the flags.

The notable difference is that in the decorated setting there is a coordinate description. Indeed, for each edge E of the graph Γ_m there are two zig-zags γ_1, γ_2 containing E. They provide two trivialisations

s_{γ_1} and s_{γ_2} of the restriction of a flat line bundle to E. We assume that the oriented zig-zag γ_1 goes along the edge E in the direction $\circ \to \bullet$. We define an edge coordinate Δ_E as their ratio:

$$s_{\gamma_1} = \Delta_E \cdot s_{\gamma_2}, \qquad \Delta_E \in R^*. \tag{16}$$

Elements $\{\Delta_E\}$ satisfy the following monomial relations. For each internal vertex of Γ_m, the counterclockwise product of the elements Δ_E over the edges incident to the vertex is equal to 1, see Figure 6. There are no other relations between elements $\{\Delta_E\}$.

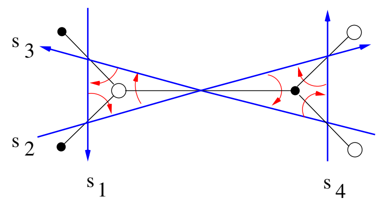


Figure 6: Writing $\Delta_E = s_1/s_2$ etc. where s_i are flat sections on zig-zags, the counterclockwise products are: $s_1/s_2 \cdot s_2/s_3 \cdot s_3/s_1 = 1$ near the $\circ-$vertex, and $s_2/s_3 \cdot s_3/s_4 \cdot s_4/s_2 = 1$ near the $\bullet-$vertex.

Proposition 1.6. *Given a generic triple of decorated flags, the invariants $\{\Delta_E\}$ at the edges of the graph Γ_m are ratios of two Gelfand-Retakh quasideterminants.*

See a detailed discussion of the simplest $m = 2$ case in Section 6.2.

In the commutative case, assuming that the space V carries a volume form, the invariants Δ_E are ratios of the $\mathcal{A}-$coordinates A_f [FG1], assigned to the faces of the graph Γ_m. Precisely, each edge E is shared by two faces f^+ and f^-, where f^+ is to the left of the $\circ \to \bullet$ oriented edge E, and $\Delta_E = A_{f^+}/A_{f^-}$.

1.2 Describing noncommutative local systems on surfaces

1. A generalization for surfaces glued from triangles. Theorem 1.2 has a generalization. Take a finite collection of triangles, glue them edge-to-edge in an arbitrary way, and delete the vertices of the glued triangles. The obtained "surface" $\mathcal{S}$ is singular along the edges where more then two triangles meet. It might be non-orientable. Its boundary is the union of the edges which are not glued to other edges. It has punctures at the deleted vertices. It comes with a triangulation $\mathcal{T}$.

Let us consider the following groupoid.

- Groupoid $\mathcal{G}_{\mathcal{S}}$ of local systems of m-dimensional R-vector spaces on $\mathcal{S}$, equipped with a flag in a fiber near each puncture, invariant under the monodromy around the puncture, such that for each triangle of $\mathcal{T}$ the triple of flags obtained by moving the flags near the vertices to one point is generic.

If $\mathcal{S}$ is just a triangle, $\mathcal{G}_{\mathcal{S}}$ is the groupoid of triples of flags in an m-dimensional R-vector space. Indeed, any local system on a triangle is trivial, so the triple of flags at the vertices is the only data left.

The graphs Γ_m are glued into a bipartite graph $\Gamma_m(\mathcal{T})$: gluing the triangles along the sides we glue the matching black vertices on the sides.

Theorem 1.7. *The groupoid $\mathcal{G}_{\mathcal{S}}$ on a possibly singular and possibly unorientable triangulated surface $\mathcal{S}$ is equivalent to the groupoid of 1-dimensional local systems on the related graph $\Gamma_m(\mathcal{T})$.*

2. Spectral description of local systems on decorated surfaces via $\mathrm{GL_m}-$*graphs***.** When $\mathcal{S}$ is smooth and orientable there are two more flexible variants of Theorem 1.7 described below.

In this case the surface glued from triangles is a decorated surface. Namely, a *decorated surface* $\mathbb{S}$ is a smooth oriented surface with a finite set of *punctures* inside and *special points* on the boundary, considered modulo isotopy. We assume that each boundary component has special points. A *marked point* is either a special point or a puncture.

Let us recall few more facts about bipartite ribbon graphs. We review them further in Section 2.1.

A bipartite ribbon graph Γ gives rise to a decorated surface $\mathbb{S}_\Gamma$ homotopy equivalent to Γ, obtained by gluing the ribbons. The graph Γ is embedded to the surface $\mathbb{S}_\Gamma$. A bipartite ribbon graph Γ gives rise to the conjugate bipartite ribbon graph Γ^*, obtained by altering the cyclic order of the edges at each $\bullet$-vertex. The surface Σ_Γ associated with the ribbon graph Γ^* is called the *spectral surface*.

The collection $\mathcal{Z}_\Gamma$ of oriented zig-zags of Γ, being pushed to $\mathbb{S}_\Gamma$, is well defined up to an isotopy. It allows to reconstruct the graph Γ.

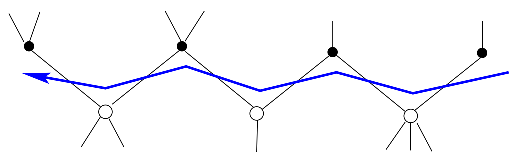


Figure 7: A zig-zag on a bipartite graph, naturally oriented, and pushed out of the vertices.

A special type of bipartite graphs on $\mathbb{S}$, called $\mathrm{GL_m}-$*graphs*, $m \geq 2$, was introduced in [G]. They are characterized the way their zig-zags relate to the marked points. Precisely, (on a finite cover of $\mathbb{S}$,) each zig-zag path goes around a single marked point on $\mathbb{S}$, and for each marked point s on $\mathbb{S}$ there are exactly m zig-zag paths going around s. There is also a technical condition: no parallel bigons, see Figure 14.

In particular, any *ideal triangulation* $\mathcal{T}$ of $\mathbb{S}$, i.e., a triangulation with vertices at the marked points, allows to glue $\mathbb{S}$ from triangles, and gives rise to a bipartite $\mathrm{GL_m}-$graph $\Gamma_m(\mathcal{T})$ on $\mathbb{S}$.

Note that a local system of R-vector spaces provides a local system of flags in its fibers.

Definition 1.8. *A framed R-local system on a decorated surface $\mathbb{S}$ is a local system of finite dimensional R-vector spaces on $\mathbb{S}$ with a framing, given by flat sections of the restriction of the associated local system of flags to the neighborhoods of the marked points.*

Theorem 1.9. *Let Γ be a $\mathrm{GL_m}-$graph on a decorated surface $\mathbb{S}$, homotopy equivalent to $\mathbb{S}$, and Σ_Γ the spectral surface. Consider the following two groupoids:*

1. *Groupoid $\mathcal{X}_{m,\mathbb{S}}$ of m-dimensional framed R-local systems on $\mathbb{S}$;*
2. *Groupoid $\mathrm{Loc}_1(\Sigma_\Gamma)$ of flat R-line bundles with connection on the spectral surface Σ_Γ.*

Then there is a canonical open embedding[4]

$$\varphi_\Gamma : \mathrm{Loc}_1(\Sigma_\Gamma) \hookrightarrow \mathcal{X}_{m,\mathbb{S}}. \tag{17}$$

The spectral surface Σ_Γ is homotopy equivalent to the graph Γ^*. So local systems on Σ_Γ are the same thing as local systems on the graph Γ^*. Note also that $\Gamma = \Gamma^*$. So there are canonical identifications

$$\mathrm{Loc}_1(\Sigma_\Gamma) = \mathrm{Loc}_1(\Gamma) = \mathrm{Loc}_1(\Gamma^*). \tag{18}$$

Theorem 1.9 is proved in Section 2, where we give a simple construction of a functor assigning to a generic local system on $\mathbb{S}$ a flat line bundle on the spectral surface Σ_Γ.

[4]In particular, they are birationally equivalent.

An entirely different proof of Theorem 1.9 is given in Section 9, and outlined in Section 1.3. It is based on the interpretation of each of the moduli spaces as a moduli space of *admissible dg-sheaves*.

Theorem 1.9 is closely related to the Gaiotto-Moore-Neitzke abelianization of vector bundles with flat connection on a Riemann surface via flat line bundle on the associated spectral curve [GMN].

Any two $\mathrm{GL_m}-$graphs on $\mathbb{S}$ are related by a sequence of moves, called *two by two moves* [G].

We show in Section 7.1 that any two by two move of arbitrary bipartite ribbon graphs $\mu : \Gamma \to \Gamma'$ gives rise to a non-commutative birational isomorphism μ intertwining φ_Γ and $\varphi_{\Gamma'}$:

$$\mu : \mathrm{Loc}_1(\Gamma) \xrightarrow{\sim} \mathrm{Loc}_1(\Gamma'). \tag{19}$$

Therefore the natural action of the mapping class group of $\mathbb{S}$ on the non-commutative moduli space $\mathcal{X}_{m,\mathbb{S}}$ can be presented by compositions of birational isomorphisms (19).

When $R = K$ is commutative and Γ is a bipartite ribbon graph, $\mathrm{Loc}_1(\Gamma)(K) = H^1(\Gamma, K^*)$. Any loop γ on Γ defines a function on $\mathrm{Loc}_1(\Gamma)$ given by the monodromy around the loop. Given a $\mathrm{GL_m}$-graph $\Gamma \subset \mathbb{S}$, the birational isomorphism (17) is nothing else but the cluster Poisson coordinate system on the space $\mathcal{X}_{\mathrm{GL_m},\mathbb{S}}$ of framed $\mathrm{GL_m}$-local systems on $\mathbb{S}$ [G]. The PGL_m-version of the story was done in [FG1].

Going back to a skew field R, the monodromy of a flat R-line bundle over a circle is described by an element of R^* modulo conjugation. So it does not give rise to a well defined rational function on non-commutative space. The analogs of cluster Poisson coordinate systems are non-commutative birational isomorphisms (17). The analogs of cluster Poisson transformations are non-commutative birational isomorphisms (19) intertwining the ones (17).

Let us introduce the non-commutative analog of the Poisson structure in a more general setup of non-commutative cluster Poisson varieties assigned to arbitrary bipartite ribbon graphs.

3. Non-commutative cluster Poisson varieties. In the commutative case, a cluster Poisson variety is defined by starting from a quiver, and considering a collection of quivers related by sequences of mutations. We assign to each quiver a cluster Poisson torus, and glue them by using birational isomorphisms between the tori assigned to mutations.

A special class of quivers and mutations is provided by bipartite ribbon graphs and two by two moves between them [GK]. A cluster Poisson torus assigned to a bipartite graph Γ is given by $\mathrm{H}^1(\Gamma, \mathbb{G}_m) = \mathrm{Loc}_1(\Gamma)$. It has the following Poisson structure. Recall that the graphs Γ and Γ^* coincide - only their ribbon structures are different. On the other hand, the graph Γ^* is homotopy equivalent to the spectral surface Σ_Γ. So the intersection pairing on $\mathrm{H}_1(\Sigma_\Gamma, \mathbb{Z})$ induces a Poisson structure on $\mathrm{H}^1(\Gamma, \mathbb{G}_m)$. Birational transformations (19) in the commutative case are cluster Poisson transformations.

In the non-commutative case, the analogs of quivers are bipartite ribbon graphs. The analogs of mutations are two by two moves. The non-commutative cluster Poisson torus assigned to a bipartite graph Γ is the moduli space $\mathrm{Loc}_1(\Gamma)$. Consider a collection of bipartite ribbon graphs $\{\Gamma'\}$ obtained by a sequence of two by two moves from the original graph Γ. Then groupoids $\mathrm{Loc}_1(\Gamma')$ are related by birational transformations (19). We show that they satisfy the pentagon relations. Therefore we get a non-commutative analog of a cluster Poisson variety $\mathcal{X}_{|\Gamma|}$ by gluing them according to these birational isomorphisms.

To introduce a non-commutative analog of the Poisson structure, consider the free abelian group $L(\Gamma)$ generated by the loops on the graph Γ. It has a Lie algebra structure, provided by the bipartite ribbon structure on Γ. Since the graphs Γ and Γ^* coincide, there is another Lie bracket on the same vector space, provided by the bipartite ribbon structure of Γ^*. We denote by $L(\Gamma^*)$ the latter Lie algebra.

The commutative algebra given by the symmetric algebra of $L(\Gamma^*)$ has a unique commutative Poisson algebra structure extending the Lie bracket on $L(\Gamma^*)$. It is the non-commutative analog of the Poisson structure on $H^1(\Gamma, \mathbb{G}_m)$. We prove that it is preserved by non-commutative birational isomorphisms (19).

We conclude that the moduli spaces of m-dimensional R-local systems on decorated surfaces carry a non-commutative cluster Poisson variety structure.

Another class of interesting examples are non-commutative Grassmannians, parametrising the m-dimensional subspaces in a standard N-dimensional R-vector space, and their strata.

Finally, a few words about "non-commutative spaces". Since we mostly concerned with their birational properties, one can consider their "points" with values in non-commutative K-algebras, e.g. the algebras of $N \times N$ matrices. A single non-commutative space determines a collection of classical moduli spaces parametrized by the integers N. For example, the non-commutative moduli space $\mathrm{Loc}_1(\Gamma)$ gives rise to a collection of classical moduli spaces parametrizing N-dimensional local systems.

4. An application: Non-commutative cluster integrable systems. In Section 8, in the special case when Γ is a bipartite graph on a torus, we define a non-commutative generalisation of the dimer models, and a non-commutative analog of the dimer cluster integrable system constructed in [GK]. After the definition of cluster Poisson varieties is given, the construction of the Hamiltonians and the proof that they Poisson commute follows literally the commutative case. In the simplest example we recover the non-commutative integrable system from [K].

Since the dimer cluster integrable systems is equivalent to the relativistic Toda systems studied by Fock and Marshakov [FM], this also gives a non-commutative analog of the latter for GL_m.

5. Spectral description of twisted local systems on decorated surfaces. Let $T'\mathbb{S}$ be the complement to the zero section of the tangent bundle on $\mathbb{S}$.

Definition 1.10. *A twisted local system on $\mathbb{S}$ is a local system on $T'\mathbb{S}$ with the monodromy -1 around a loop generating $\pi_1(T'_s\mathbb{S})$.*

We consider twisted framed local systems of R-modules of rank m on $\mathbb{S}$. Notice that the -1 monodromy around a loop generating $\pi_1(T'_s\mathbb{S})$ preserves the framing flags.

Theorem 1.11. *Let Γ be a $\mathrm{GL_m}-$graph on a decorated surface $\mathbb{S}$, homotopy equivalent to $\mathbb{S}$, and Σ_Γ the spectral surface. Consider the following two groupoids:*

1. *Groupoid of m-dimensional twisted framed R-local systems on $\mathbb{S}$;*
2. *Groupoid of flat twisted R-line bundles with connection on the spectral surface Σ_Γ.*

Then there is a canonical open embedding of the second groupoid to the first.

Theorem 1.9 is the "untwisted" version of Theorem 1.11, with the underlined words "twisted" deleted.

Note that we can identify untwisted and twisted local systems on Σ_Γ. Indeed, the obstruction lies in $H^2(\Sigma_\Gamma, \mathbb{Z}/2\mathbb{Z})$, which is zero.

6. Non-commutative cluster $\mathcal{A}-$varieties. A bipartite ribbon graph Γ gives rise to a non-commutative torus $\mathcal{A}_\Gamma$ - the moduli space of twisted flat $R-$line bundles on the associated spectral surface Σ_Γ, trivialized at the boundary. It comes with a canonical collection of $\mathcal{A}-$coordinates $\{\Delta_E\}$, parametrised by the edges of the graph Γ, and satisfying monomial relations parametrised by the vertices of Γ. [5]

For a two by two move $\mu : \Gamma_1 \longrightarrow \Gamma_2$, we define in Section 4 a non-commutative birational isomorphism

$$\mu : \mathcal{A}_{\Gamma_1} \longrightarrow \mathcal{A}_{\Gamma_2}. \tag{20}$$

[5] Note that boundary components of the spectral surface Σ_Γ are in natural bijection with zig-zags on the original surface $\mathbb{S}_\Gamma$. So this description match the one in subsection 3, except that now we can talk about the twisted flavor of the story.

It serves the role of a non-commutative cluster $\mathcal{A}-$transformation. Next, we show that transformations (20) satisfy the pentagon relations. Therefore we can glue them into a non-commutative space $\mathcal{A}_{|\Gamma|}$, which we call a non-commutative cluster $\mathcal{A}-$variety.

We prove that a non-commutative cluster $\mathcal{A}-$variety carries a canonical non-commutative $2-$form Ω. Precisely, given a bipartite ribbon graph Γ we introduce a non-commutative 2-form Ω_Γ on the torus $\mathcal{A}_\Gamma$. It has a simple expression in the cluster $\mathcal{A}-$coordinates. Birational isomorphisms (20) preserve them:

$$\mu^*\Omega_{\Gamma_2} = \Omega_{\Gamma_1}.$$

This feature of non-commutative cluster $\mathcal{A}-$varieties is quite non-trivial, see Section 11. In the commutative case, we recover the 2-form associated with cluster algebras/cluster $\mathcal{A}-$varieties [GSV], [FG2].

Just as in the commutative case, there is a canonical map of non-commutative cluster varieties

$$p : \mathcal{A} \longrightarrow \mathcal{X}.$$

It has Hamiltonian properties similar to the ones in the commutative case, where it is the projection along the null-foliation of the canonical 2-form on Ω on $\mathcal{A}$.

We stress that in sharp contrast with the commutative case, non-commutative cluster Poisson varieties do not carry any cluster coordinates. Therefore cluster Poisson considerations require new language.

On the other hand, the non-commutative cluster $\mathcal{A}-$varieties do come with a canonical collection of functions. Although we refer to them as cluster $\mathcal{A}-$coordinates, unlike in the commutative case, cluster coordinates in each cluster are not independent, and satisfy certain monomial relations.

7. Spectral description of the non-commutative decorated local systems on $\mathbb{S}$. In the commutative case there is the dual moduli space $\mathcal{A}_{\mathrm{SL_m},\mathbb{S}}$ parametrised twisted $\mathrm{SL_m}$-local systems on $\mathbb{S}$ equipped with a *decoration*: a choice of an invariant decorated flag near every marked point of $\mathbb{S}$. It has a cluster $\mathcal{A}$-variety structure [FG1], which is a geometric incarnation of the Fomin-Zelevinsky upper cluster algebras [FZI]. There are cluster $\mathcal{A}$-coordinate systems on $\mathcal{A}_{\mathrm{SL_m},\mathbb{S}}$ parametrised by ideal triangulations [FG1] or, more generally, $\mathrm{SL_m}$-graphs [G].

The noncommutative analog is the moduli space $\mathcal{A}_{m,\mathbb{S}}$ parametrised twisted decorated rank m R-local systems on $\mathbb{S}$. Note that there is no non-commutative analog of $\mathrm{SL_m}$. Its generic points parametrise twisted flat line bundles on the spectral surface Σ which are trivialized at the boundary of Σ. The trivilalisations allow to construct rational functions Δ_E on the moduli space $\mathcal{A}_{m,\mathbb{S}}$ parametrised by the edges E of the graph Γ. These functions are analogs of the cluster $\mathcal{A}$-coordinates. As we stressed above, the functions Δ_E do not form a coordinate system: they satisfy non-trivial monomial relations. There is no natural way to select a subset of the generators. A similar phenomenon occurs already for the commutative moduli space $\mathcal{A}_{\mathrm{GL_m},\mathbb{S}}$.

8. Berenstein-Retakh non-commutative cluster algebras related to surfaces. In Section 6 we show that in the case of two dimensional twisted decorated non-commutative local systems the obtained coordinate description recovers the non-commutative cluster functions decribed as ratios of 2×2 quasideterminants by A. Berenstein and V. Retakh in [BR]. In our approach formulas for these functions were obtained by geometric considerations, which made their properties evident.

In general the functions Δ_E on the moduli spaces $\mathcal{A}_{m,\mathbb{S}}$ are ratios of the Gelfand-Retakh quasideterminants [GR]. This way we get a geometric approach to quasideterminants.

1.3 Non-commutative cluster varieties and stacks of admissible dg-sheaves

Given a bipartite ribbon graph Γ, we take stacks $\mathrm{Loc}_1(\Gamma')$, where graphs Γ' are obtained from Γ by any sequences of two by two moves, and glue them via birational isomorphisms (19) into a space $\mathcal{X}_{|\Gamma|}$.

Then a question arises: can we assign to each bipartite ribbon graph Γ a finite stack $\mathcal{M}^{\circ}_{|\Gamma|}$, whose equivalence class is invariant under two by two moves, and which contains $\mathrm{Loc}_1(\Gamma)$ as a Zariski open substack? Equivalently, we want to have a Zariski open embedding of $\mathcal{X}_{|\Gamma|}$ into an an priori defined stack:

$$\mathcal{X}_{|\Gamma|} \subset \mathcal{M}^{\circ}_{|\Gamma|}. \tag{21}$$

To approach this question, we introduce stacks of admissible dg-sheaves on manifolds. This allows to answer the question in many interesting situations, discussed below, although not in full generality yet.

1. Admissible dg-sheaves. Given a cell complex $\mathcal{C}$, we define in Section 9.1 a dg-category of *dg-sheaves*, a dg-version of the derived category of complexes of sheaves constructible for the stratification of $\mathcal{C}$.

Recall that the microlocal support of a constructible complex of sheaves on a manifold X is a complex of sheaves on a conical Lagrangian in the cotangent bundle T^*X [KS]. A cooriented hypersurface in a manifold X gives rise to a Lagrangian in T^*X, and a Legendrian in its spherical bundle ST^*X. They are given, respectively, by the conormal vectors/rays in the direction specified by the coorientation.

Let $\mathcal{H}$ be a collection of <u>smooth</u> cooriented hypersurfaces in a manifold X with <u>disunct</u> Legendrians.

We consider a subcategory of $\mathcal{H}$-*supported* dg-sheaves. These are dg-sheaves whose microlocal support is given by complexes supported at the zero section and Lagrangians assigned to the hypersurfaces of $\mathcal{H}$.

Then we introduce a dg-subcategory of $\mathcal{H}$-*admissible* dg-sheaves. Its key features are the following.

1. Admissible dg-sheaves on a manifold X are concentrated in the degrees $[0, 1]$.

2. The microlocal support of an $\mathcal{H}$-admissible dg-sheaf is supported on the union of the zero section of T^*X and a Lagrangian $\mathrm{T}^*_{\mathcal{H}}X$, given by the union of conormal bundles to the cooriented hypersurfaces of $\mathcal{H}$. The restriction of the microlocal support to $\mathrm{T}^*_{\mathcal{H}}X$ is given by local systems on its components.

3. Stacks of $\mathcal{H}$-admissible dg-sheaves are invariant under deformations of $\mathcal{H}$ on X, called below *admissible isotopies*, which keep each component of $\mathcal{H}$ smooth, and their Legendrians disunct.

We demonstrate property 3) directly in Section 9.2 in the dimension 2. We stess that we allow in the process of deformation non-transversal intersections, e.g. three lines intersecting in a point. Note that a smooth Hamiltonian isotopy of a Legendrian curve can develop a cusp, which we do not allow.

Note that according to [GKS], the derived category of all constructible sheaves with a given microlocal support is invariant under Legendrian isotopies. However this does not imply that admissible isotopies preserve $\mathcal{H}$-admissibility, and not all Legendrian isotopies preserve $\mathcal{H}$-admissibility.

Since local systems on the conormal bundle to a cooriented hypersurface are pull backs of local systems on the hypersurface, we abuse terminology by referring to them as local systems on the hypersurface. Therefore below we refer to the microlocal support on $\mathrm{T}^*_{\mathcal{H}}X$ as the *Legendrian microlocal support*.

Since $\mathcal{H}$-admissible dg-sheaves are concentrated in degrees $[0, 1]$, not all complexes of constructible sheaves with Legendrian microlocal support in $\mathrm{T}^*_{\mathcal{H}}X$ are $\mathcal{H}$-admissible. Stacks of $\mathcal{H}-$admissible dg-sheaves are of finite type, while stacks of complexes of constructible sheaves with the Legendrian microlocal support in $\mathrm{T}^*_{\mathcal{H}}X$ are not: the latter contain local systems on X placed in arbitrary degrees.

Let Γ be a bipartite ribbon graph. Denote by $\mathcal{Z}_\Gamma$ the collection of oriented zig-zags on the decorated surface $\mathbb{S}_\Gamma$ associated to Γ, well defined up to isotopy. Since the surface $\mathbb{S}_\Gamma$ is oriented, zig-zags are cooriented. The complement $\mathbb{S}_\Gamma - \mathcal{Z}_\Gamma$ is a union of domains of three types, depending on the orientation types of their boundary segments, see Figure 8 & Section 9.3.

Lemma 1.12. *The stack* $\mathrm{Loc}_1(\Gamma)$ *of flat line bundles on a bipartite ribbon graph* Γ *is canonically equivalent to the stack of* $\mathcal{Z}_\Gamma$*-admissible dg-sheaves on* $\mathbb{S}_\Gamma$ *satisfying the following conditions:*

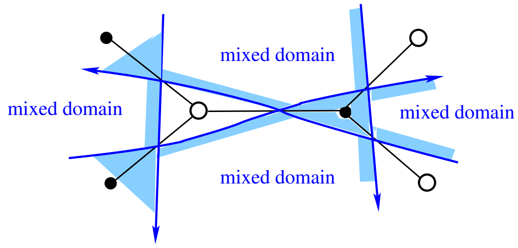


Figure 8: Zig-zag paths for a bipartite graph Γ on a surface $\mathbb{S}$ cut the surface into $\bullet$, $\circ$, and mixed domains. Each zig-zag path is coorientated by a shaded area to the right of the path.

1. *They are acyclic on mixed domains - note that this is an open condition.*[6]
2. *Their Legendrian microlocal support is given by flat $R-$line bundles at every curve of $\mathcal{Z}_\Gamma$.*

The dg-setting is crucial for Lemma 1.12: the connection on the line bundle on Γ is given by "homotopies between morphisms", absent in the classical description of the derived category.

Denote by $\mathcal{M}(\mathcal{H}, \mathbf{1})$ the stack of $\mathcal{H}-$admissible dg-sheaves on a surface satisfying property 2) of Lemma 1.12 for every curve of $\mathcal{H}$. It is a finite type stack. Then By Lemma 1.12 there is an embedding

$$\mathrm{Loc}_1(\Gamma) \subset \mathcal{M}(\mathcal{Z}_\Gamma, \mathbf{1}). \tag{22}$$

We define $\mathcal{M}^\circ_{|\Gamma|}$ as the component of $\mathcal{M}(\mathcal{Z}_\Gamma, \mathbf{1})$ containing $\mathrm{Loc}_1(\Gamma)$. We show that in many interesting cases we get this way Zarisky open embeddings (21).

2. Examples of stacks of $\mathcal{H}-$admissible dg-sheaves. They include the stacks of Stokes data, which are the Betti side of stacks of flat connections with irregular singularities on Riemann surfaces, see Sections 10.1 - 10.2. This way we get non-commutative stacks of Stokes data stacks as well.

Let us explain how to get the Betti stack of framed flat connections in the case of regular singularities. For each marked point s of $\mathbb{S}$, in a small punctured at s disc/half disc, consider m non intersecting and non selfintersecting circles/arcs ending on the boundary, cooriented out of s. Denote by $\mathcal{I}_{\mathbb{S},m}$ their union, see Figure 9. Its isotopy class is determined uniquely by the surface $\mathbb{S}$ and the integer m.

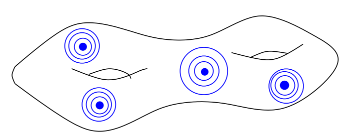


Figure 9: A collection of curves $\mathcal{I}_{\mathbb{S},3}$ on a genus two surface $\mathbb{S}$ with four punctures.

The following observation is an immediate corollary of the definition of $\mathcal{I}_{\mathbb{S},m}-$admissible dg-sheaves.

Lemma 1.13. *The stack $\mathcal{X}_{\mathbb{S},m}$ of m-dimensional R-local systems on a decorated surface $\mathbb{S}$ with complete filtrations near marked points is canonically equivalent to the stack of $\mathcal{I}_{\mathbb{S},m}-$admissible dg-sheaves on $\mathbb{S}$ satisfying the following two conditions:*

1. *They are acyclic near the marked points.*
2. *Their microlocal support is the union of the zero section and flat $R-$line bundles at every curve of $\mathcal{I}_{\mathbb{S},m}$.*

[6] The condition that a complex does not have cohomology in two neighboring degrees is an open condition. In particular the conditions that a complex is acyclic, or has a single non-trivial cohomology group, are open conditions. The latter are precisely the conditions we impose in Lemma 1.12.

Lemmas 1.12 - 1.13 provide a non-commutative cluster Poisson structure on the stack $\mathcal{X}_{\mathbb{S},m}$. Indeed, by Lemma 1.13, $\mathcal{X}_{\mathbb{S},m}$ is realized as the substack $\mathcal{M}_0(\mathcal{I}_{\mathbb{S},m}, \mathbf{1}) \subset \mathcal{M}(\mathcal{I}_{\mathbb{S},m}, \mathbf{1})$ given by condition 1):

$$\mathcal{X}_{\mathbb{S},m} = \mathcal{M}_0(\mathcal{I}_{\mathbb{S},m}, \mathbf{1}) \subset \mathcal{M}(\mathcal{I}_{\mathbb{S},m}, \mathbf{1}).$$

Consider a bipartite $\mathrm{GL_m}-$graph Γ on $\mathbb{S}$. By its very definition [G], there is an admissible isotopy of the collection $\mathcal{Z}_\Gamma$ of its zig-zags to the collection of curves $\mathcal{I}_{\mathbb{S},m}$, see Figure 10:

$$i_\Gamma : \mathcal{Z}_\Gamma \longrightarrow \mathcal{I}_{\mathbb{S},m}. \tag{23}$$

By property 3), isotopy (23) provides an equivalence of the related stacks of $\mathcal{H}$-admissible dg-sheaves

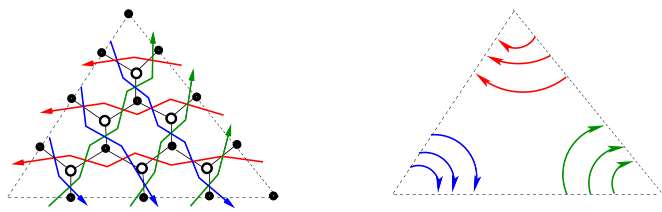


Figure 10: Zig-zag paths on the bipartite graph Γ_3 on the left can be admissibly deformed to concentric arcs around the vertices, shown on the right.

$$i_{\Gamma *} : \mathcal{M}(\mathcal{Z}_\Gamma, \mathbf{1}) \stackrel{\sim}{\longrightarrow} \mathcal{M}(\mathcal{I}_{\mathbb{S},m}, \mathbf{1}).$$

By Lemma 1.12, the stack $\mathrm{Loc}_1(\Gamma)$ is realized as the Zariski open substack $\mathcal{M}_{\mathrm{mix}=0}(\mathcal{Z}_\Gamma, \mathbf{1})$ of $\mathcal{M}(\mathcal{Z}_\Gamma, \mathbf{1})$ defined by condition 1):

$$\mathrm{Loc}_1(\Gamma) = \mathcal{M}_{\mathrm{mix}=0}(\mathcal{Z}_\Gamma, \mathbf{1}) \subset \mathcal{M}(\mathcal{Z}_\Gamma, \mathbf{1}).$$

Indeed, the acyclicity on mixed domains in Lemma 1.12 is an open condition. Acyclicity on mixed domains implies acyclicity near marked points since marked points are contained in the mixed domains, see Figure 10 when $\mathbb{S}$ is a triangle. The general case reduces to this since we can get a GL_m-graph on $\mathbb{S}$ by amalgamating the graphs Γ_m on triangles of an ideal triangulation of $\mathbb{S}$. Therefore $i_{\Gamma *}$ induces an open embedding $j_\Gamma : \mathcal{M}_{\mathrm{mix}=0}(\mathcal{Z}_\Gamma, \mathbf{1}) \subset \mathcal{M}_0(\mathcal{I}_{\mathbb{S},m}, \mathbf{1})$. Summarising, we get the commutative diagram

$$\begin{array}{ccc} \mathrm{Loc}_1(\Gamma) = \mathcal{M}_{\mathrm{mix}=0}(\mathcal{Z}_\Gamma, \mathbf{1}) & \stackrel{\subset}{\longrightarrow} & \mathcal{M}(\mathcal{Z}_\Gamma, \mathbf{1}) \\ {\scriptstyle j_\Gamma}\downarrow\cap & & {\scriptstyle i_{\Gamma *}}\downarrow\wr \\ \mathcal{X}_{\mathbb{S},m} = \mathcal{M}_0(\mathcal{I}_{\mathbb{S},m}, \mathbf{1}) & \stackrel{\subset}{\longrightarrow} & \mathcal{M}(\mathcal{I}_{\mathbb{S},m}, \mathbf{1}). \end{array} \tag{24}$$

In particular, for any GL_m-graph Γ on $\mathbb{S}$, we have, as was promised in (21),

$$\mathcal{M}^0_{|\Gamma|} = \mathcal{X}_{\mathbb{S},m}. \tag{25}$$

For example, the bipartite graph Γ_m on Figure 10 allows to recover the whole stack of all triples of flags as the stack $\mathcal{M}^0_{|\Gamma_m|}$, rather than describing its generic part, as Theorem 1.2 does.

The stack $\mathrm{Loc}_1(\Gamma)$ is a non-commutative Poisson torus. So open embeddings $j_\Gamma : \mathrm{Loc}_1(\Gamma) \subset \mathcal{X}_{\mathbb{S},m}$ for bipartite $\mathrm{GL_m}-$graphs Γ on $\mathbb{S}$ define a non-commutative cluster Poisson structure on the stack $\mathcal{X}_{\mathbb{S},m}$. The action of the mapping class group $\Gamma_\mathbb{S}$ of $\mathbb{S}$ preserves the class of $\mathrm{GL_m}-$graphs on $\mathbb{S}$. Any two of them are related by a sequence of two by two moves [G], providing admissible isotopies of their zig-zags. Therefore we get a $\Gamma_\mathbb{S}-$equivariant non-commutative cluster Poisson structure on the stack $\mathcal{X}_{\mathbb{S},m}$.

In Section 10.3 we describe stacks of framed Stokes data via admissible dg-sheaves, and show that

Stacks of (framed) Stokes data are shadows of non-commutative (framed) stacks of Stokes data: (26)

Theorem 1.14. *For any stack of framed Stokes data $\mathcal{S}$ on a decorated surface $\mathbb{S}$, there is a non-commutative stack $\mathcal{S}^{\rm nc}$ over* Spec($\mathbb{Z}$)*, whose restriction to commutative fields is the stack $\mathcal{S}$. The stack $\mathcal{S}^{\rm nc}$ carries a non-commutative cluster Poisson structure equivariant under the wild mapping class group.*

The argument is based on the observation that, similarly to Lemma 1.13, stacks of Stokes data can be realized as stacks of $\mathcal{L}-$admissible dg-sheaves on $\mathbb{S}$ vanishing near the punctures, where $\mathcal{L}$ is the union of certain local collection of curves $\mathcal{L}_p$ around the punctures p. When $\mathbb{S} = S^2 - \{0, \infty\}$, with one regular and one irregular point, these stacks are identified birationally with cluster Poisson varieties assigned in [FG3] to the cyclic closures of certain elements of the braid semigroup. Thanks to [FG3, Section 3.10], their quivers are described by bipartite graphs. The general case is reduced to this one plus the regular triangle case, except for single irregular points of rather degenerate type, see Section 10.3.

3. Stacks of framed G$-$Stokes data, their cluster Poisson structure & quantization. Stacks of G$-$Stokes data provide the Betti realization of stacks of G$-$bundles with meromorphic connections with arbitrary singularities on a Riemann surface Σ. We introduce stacks of *framed* G$-$Stokes data, and prove that they carry a cluster Poisson structure, equivariant under the wild mapping class group action.

Theorem 1.15. *Assume that $\mathbb{S} \neq S^2 - \{\infty\}$. For any split reductive group* G *over $\mathbb{Q}$ with connected center, the stack of* <u>*framed*</u> G$-$*Stokes data on $\mathbb{S}$ has a cluster Poisson structure, equivariant under the action of the wild mapping class group. Stacks of* <u>*plain*</u> G$-$*Stokes data carry compatible Poisson structures.*

The Poisson nature of the moduli spaces of Stokes data was studied by P. Boalch [Bo11].
Theorems 1.15 - 1.16 are valid for a much larger class of *ideal* Legendrian links $\mathcal{L}$, see Section 10.2.

Theorem 1.15 is proved in Section 10.4 using the following key ingredients.

1. If $\mathbb{S} = S^2 - \{0, \infty\}$, with one regular and one irregular point, the stacks of G$-$Stokes data are identified birationally with the cluster Poisson varieties assigned in [FG3] to the cyclic closures of certain elements of the braid semigroup $\mathrm{Br}^+_{\rm G}$ related to G.

2. When $\mathbb{S} = t$ is a triangle, the cluster Poisson structure of the related space $\mathcal{P}_{{\rm G},t}$ of configurations of triples of flags with pinnings was described in [GS19]. The key point is its cyclic invariance.

3. If $\mathbb{S} = S^2 - \{\infty\}$, the stacks of Stokes data are given by certain cyclically ordered configurations of flags. The cluster Poisson structure for sufficiently generic cyclic configurations follows from [GS19].

The general case is built from these ones, except for single irregular point - that is $\mathbb{S} = S^2 - \{\infty\}$ - of rather degenerate type, which requires more elaborate considerations, and thus is omitted.

Combining Theorem 1.15 with the cluster quantization machine [FG4], we get the following.

Theorem 1.16. *Under the same assumptions as in Theorem 1.15, the stacks of* G$-$*Stokes data, plain or framed, admit a cluster Poisson quantization, equivariant under the wild mapping class group action.*

Precisely, we apply the cluster quantization to the cluster Poisson space of framed G$-$Stokes data. Then we use the fact that forgetting the framing we get a map to the space of plane G$-$Stokes data, which is finite Galois map over a generic point. Its Galois group, given by a product of the Weyl groups of G, acts by cluster Poisson transformations. This allows us to pass to its invariants.

Theorem 1.16 completes the program of cluster quantization of the moduli spaces of flat connections on Riemann surfaces. In the regular case this was done in [FG1] for $\mathrm{G} = \mathrm{PGL}_m$, and in [GS19] in general.

4. Cluster $\mathcal{A}-$varieties via stacks of admissible dg-sheaves. The analogs of cluster $\mathcal{A}-$varieties in this set-up are given by stacks of (twisted) $\mathcal{H}-$admissible dg-sheaves with the following extra condition:

Local systems on all components of the microlocal support but the zero secion are trivialized. (27)

In particular, given a bipartite ribbon graph Γ we recover non-commutative cluster $\mathcal{A}-$varieties discussed in Section 1.2. Indeed, zig-zags on the surface $\mathbb{S}_\Gamma$ match boundary curves on the spectral surface

Σ_Γ. So the trivialization of the local systems on zig-zags on $\mathbb{S}_\Gamma$, describing the microlocal support of an admissible dg-sheaf, amounts to the trivialization of the local system on the spectral surface Σ_Γ on the boundary of Σ_Γ. The cluster $\mathcal{A}-$coordinates are assigned to the edges of Γ. They are obtained by comparing trivialisations of the local systems on the two boundary components by using the parallel transport along the canonical up to isotopy path connecting two boundary components.

Remark. The stacks of Stokes data are the geometric analogs of (families of) wildly ramified representations of the absolute Galois group, see [DMR]. So Theorem 1.16 should be useful in various flavors of the geometric quantum Langlands correspondence.

5. CY_3**-categories assigned to stacks of Stokes data.** Description of the cluster Poisson structure on stacks of Stokes data via webs of zig-zags for certain bipartite graphs on the surface S has another important application. Recall that any bipartite graph gives rise to a canonical quiver with potential, which determines a combinatorially defined 3d CY $A_\infty-$category. The categories assigned to bipartite graphs related by two by two moves are equivalant. Therefore we arrive to a combinatorially defined 3d Calabi-Yau $A_\infty-$category assigned to any topological Stokes data $(\mathcal{L}, d)$. This category should describe the Fukaya category of the family of open complex CY threefolds assigned to the same data $(\mathcal{L}, d)$. This story in the regular case is described in [G]. The $m = 2$ case is established in [BS], [S].

6. Comments. Sections 9 and 10.3 are very closely related to the series of works of Shende-Treumann-Williams-Zaslow [STZ14], [STWZ], [STZ16]. In particular, they studied extensively moduli spaces of rank one microlocal sheaves, including the ones related to bipartite surface graphs. Note that they work with all constructible complexes of sheaves with given microlocal support, while we introduce stacks of admissible ones, providing finite type stacks we need. Our exposition is entirely self-contained and elementary.

Our work also relates to some aspects of the Gao-Shen-Weng program [GSW]. In particular, non-commutative Plucker relations appear in their approach as well.

1.4 $\mathcal{Q}$-diagrams of surfaces and non-commutative Lagrangians in cluster symplectic varieties

1. Triple crossing diagrams. According to D. Thurston [Th], a *triple crossing diagram* $\mathcal{D}$ is a collection of smooth oriented curves on a surface S such that any intersection point is a triple transversal intersection, and any of the following three equivalent conditions holds, see Figure 11:

- The orientations of the curves at the intersection point alternate
- Following any curve, the orientations of the curves intersecting it alternate.
- The curve orientations induce consistent orientations of the complimentary domains.

If the surface S has boundary, we assume that the intersection points are not on the boundary.

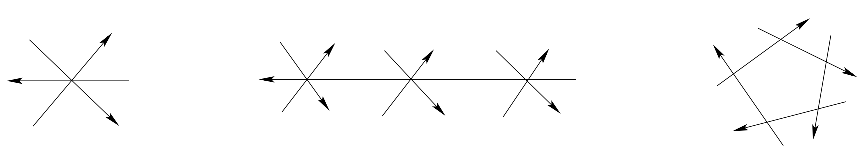


Figure 11: Triple crossing diagrams of oriented curves on surfaces.

A triple crossing diagram $\mathcal{D}$ on a surface S gives rise to a bipartite graph $\Gamma_\mathcal{D}$ on S as follows [GK]. Given a crossing point q, we move slightly the curves crossing at q in the direction of their conormals.

Then we get three types of the domains: white, blacks, and mixed. We shrink the white and black domains into $\circ-$ and $\bullet-$vertices, so that the crossing points provide the edges between these vertices.

Vice versa, take a bipartite graph Γ on a surface S. Let us assume it has only 3-valent vertices. we consider the collection of its zig-zag paths. moving slightly the zig-zag paths out of the vertices, we get a collection of curves whose intersection points are the centers of the edges. Then collapsing

Lemma 1.17. *A collection of smooth cooriented curves in a smooth surface S is a triple crossings diagram if all intersection points are triple transversal intersection points, and for any intersection point, the oriented conormals to the curves satisfy a linear relation with positive coefficients*

$$\alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3 = 0, \qquad \alpha_1, \alpha_2, \alpha_3 > 0. \tag{28}$$

Proof. Obvious. □

2. $\mathcal{Q}-$diagrams. Let M be a three-dimensional manifold with boundary ∂M. Consider a collection of smooth cooriented surfaces $\mathcal{H}$ in M. Below we call by surfaces components of the collection $\mathcal{H}$.

Definition 1.18. *A $\mathcal{Q}-$diagram*[7] *of surfaces in a threefold M is a collection of smooth cooriented surfaces $\mathcal{H}$ in M with disunct punctured conormal bundles, such that for any surface of the collection, the other surfaces induce on it a triple crossing diagram.*

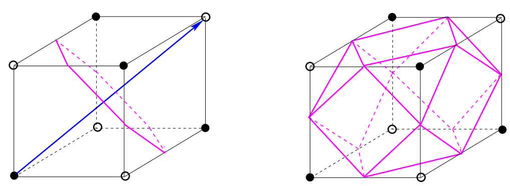


Figure 12: A central plane section perpendicular to a principal diagonal of a cube cuts out a hexagon. The four such planes cut out on the cube surface a polygon with 8 triangular and 6 square faces.

Lemma 1.19. *A collection of smooth cooriented surfaces in a threefold M is a $\mathcal{Q}-$diagram, if and only if the following conditions hold:*

1. *Intersection points are either double points, where two surfaces intersect transversally, or quadruple points, where four surfaces intersect.*
2. *For any quadruple intersection point, the oriented conormals $n_1, ..., n_4$ to the surfaces satisfy a unique up to a constant linear relation with positive coefficients*

$$\alpha_1 n_1 + ... + \alpha_4 n_4 = 0, \qquad \alpha_1, ..., \alpha_4 > 0. \tag{29}$$

Proof. Obvious. □

As we already discussed in Section 1.2.6, in Section 4 we assigned to a bipartite ribbon graph Γ a non-commutative torus $\mathcal{A}_\Gamma$ equipped with a 2-form Ω_Γ. A two by two move of bipartite ribbon graphs $\mu : \Gamma \longrightarrow \Gamma'$ gives rise to an elementary transformation (69) of $\mathcal{A}-$coordinates. By Theorem 5.7, it preserves the 2-form Ω_Γ. So we get a non-commutative birational isomorphism preserving the 2-forms:

$$\mu : (\mathcal{A}_\Gamma, \Omega_\Gamma) \longrightarrow (\mathcal{A}_{\Gamma'}, \Omega_{\Gamma'}).$$

[7] It can be also called a *quarduple crossings collection of surfaces.*

This just means that its graph is an isotropic subvariety for the form $(\Omega_\Gamma, -\Omega_{\Gamma'})$ on the product $\mathcal{A}_\Gamma \times \mathcal{A}_{\Gamma'}$.

In Section **??** we reveal the hidden A_4-symmetry of the transformation μ, and show that it describes a basic elementary A_4-symmetric Lagrangian subvariety in a non-commutative symplectic space. It is given by a system of equations in the cluster coordinates, which are equivalent to equations (69).

Furthermore, the A_4-symmetry of the basic Lagrangian allows to generalize it.

Namely, we introduce the notion of *quadruple intersection collection of cooriented surfaces* $\mathcal{Q}$ in a threefold M, referred to as a $\mathcal{Q}-$*diagram of surfaces*. Such a family induces a triple intersection diagram of curves $\mathcal{Q} \cap \partial M$ on the boundary ∂M. The latter determines uniquely up to an isotopy a bipartite graph $\Gamma_\mathcal{Q}$ on ∂M, such that the family of its zig-zag curves is isotopic to $\mathcal{Q} \cap \partial M$. The bipartite graph $\Gamma_\mathcal{Q}$ gives rise to a non-commutative $\mathcal{A}-$cluster variety $\mathcal{A}_\mathcal{Q}$. We show that the $\mathcal{Q}-$collection of surfaces $\mathcal{Q}$ gives rise to a non-commutative isotropic subvariety

$$\mathcal{L}_\mathcal{Q} \subset \mathcal{A}_\mathcal{Q}. \tag{30}$$

Taking the quotient by the action of the group $\mathcal{R}$ rescaling trivializations of the microsupport line bundles on zig-zag paths, we get a non-commutative symplectic variety $\mathcal{U}_\mathcal{Q} := \mathcal{A}_\mathcal{Q}/\mathcal{R}$ with the symplectic form induced by the 2-form Ω. The quotient $\mathcal{L}^\circ_\mathcal{Q} := \mathcal{L}_\mathcal{Q}/\mathcal{R}$ is a Lagrangian subvariety:

$$\mathcal{L}^\circ_\mathcal{Q} \subset \mathcal{U}_\mathcal{Q}. \tag{31}$$

The simplest example of a $\mathcal{Q}-$family of surfaces is obtained as follows. Take the vectors at the center of a tetrahedron pointed towards its vertices, and consider the planes through the center orthogonal to them. These planes are cooriented by the vectors. The Lagrangian corresponding to this quadruple of planes is the basic non-commutative cluster Lagrangian. It is manifestly symmetric under the action of the group A_4, acting by orientation preserving permutations of the planes, that is by even permutations.

7. Acknowledgments. We are grateful to A. Berenstein and V. Retakh for useful discussions during the Summer 2011 in IHES, when the project was started. The hospitality of IHES was crucial for the developing of the project.

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2 Spectral description of local systems

2.1 $\mathrm{GL_m}$-graphs and spectral surfaces

In Section 2.1 we recall some material from [GK] and [G].

A ribbon graph is a graph with a cyclic order of the edges at every vertex. We allow ribbon graphs with external, that is 1-valent, vertices. A ribbon graph Γ is the same thing as a graph on a decorated surface $\mathbb{S}$ which is homotopy equivalent to the graph.

A *face path* on Γ is a path turning right at every vertex. There is a bijection

$$\{\text{face paths on } \Gamma\} \longleftrightarrow \{\text{marked points on } \mathbb{S}\}. \tag{32}$$

Face intervals correspond to special points on $\mathbb{S}$. Face loops correspond to punctures of $\mathbb{S}$.

A *zig-zag path* on Γ is a path turning at every vertex either left or right, so that the left and right turns alternate. A zig-zag path on a bipartite ribbon graph Γ is oriented so that it turns right at the $\circ$-vertices, and turns left at the $\bullet$-vertices, see Figure 13.

The spectral surface Σ. Given a bipartite ribbon graph Γ, the *conjugate ribbon graph* Γ^* is obtained by changing the cyclic order of the edges at every $\bullet$-vertex. There is a bijection:

$$\{\text{zig-zag paths on } \Gamma\} \longleftrightarrow \{\text{face pathes on } \Gamma^*\}. \quad (33)$$

The conjugate ribbon graph Γ^* provides a topological surface Σ with holes, whose boundaries are the face pathes on Γ^*, described as follows. Let us add to the ribbon graph Γ^* its punctured faces F_{γ_i} bounding the face paths γ_i. Precisely, if γ_i is a face loop, then F_{γ_i} is a disc bounded by this loop, punctured inside. If γ_i is a face interval, then F_{γ_i} is a disc; the half of its boundary is identified with the path γ_i, and the rest becomes boundary of Σ. It is handy to puncture it at the marked point. We arrive at a topological surface, called the *spectral surface*:

$$\Sigma = \Gamma^* \cup \cup_{\gamma_i} F_{\gamma_i}. \quad (34)$$

Let $\mathbb{S}$ be a decorated surface with marked points $\{s_1, ..., s_n\}$. A *strand* on $\mathbb{S}$ is either an oriented loop, or an oriented path on $\mathbb{S} - \{s_1, ..., s_n\}$ connecting boundary points. Let γ be a strand such that $\mathbb{S} - \gamma$ has two connected components:

$$\mathbb{S} - \gamma = S_\gamma^\circ \cup S_\gamma^{\circ\prime}. \quad (35)$$

They inherit orientatations from $\mathbb{S}$. The domain S_γ° is the one whose orientation induces the original (clockwise) orientation of γ. We denote by S_γ the closure of S_γ°.

Let $\mathbb{S}$ be a decorated surface associated with a bipartite ribbon graph Γ, so that $\Gamma \subset \mathbb{S}$. Given a zig-zag path γ on Γ, the shape of the domain S_γ near a vertex v of Γ depends on the vertex type. Namely, the intersection $S_\gamma \cap U_v$ with a neighborhood U_v of v is a single sector if v is a $\circ$-vertex, and a union of $\text{val}(v) - 1$ sectors if v is a $\bullet$-vertex, see Figure 13.

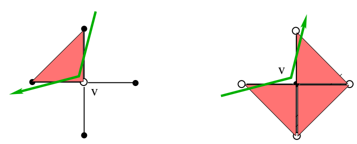


Figure 13: A (green) oriented zig-zag strand γ near a vertex v, deformed slightly off the vertex.

Pushing a bit zig-zag paths out of the vertices as on Figure 13, we get *zig-zag strands* on $\mathbb{S}$.
Let $\{\gamma\}$ be the collection of zig-zag strands on S. We assume that for each zig-zag strand γ

- $\mathbb{S} - \gamma$ has two connected components, and the domains S_γ are discs.

Using bijection (33), we construct the spectral surface Σ as follows. Denote by F_γ a copy of the disc S_γ. We attach the disc F_γ to every zig-zag path γ on Γ. Since the F_γ are discs, we get a surface homeomorphic to surface (34). Canonical maps $F_\gamma \longrightarrow S_\gamma$ provide a ramified map:

$$\pi : \Sigma \longrightarrow \mathbb{S}. \quad (36)$$

Definition 2.1. *The degree m of the map π is called the rank of the bipartite ribbon graph Γ.*

The map π has the following properties:

- The ramification points of π are the $\bullet$-vertices of Γ of valency ≥ 3: the ramification index of a $\bullet$-vertex equals its valency minus 2: this is clear from Figure 13.
- For every marked point $s \in \mathbb{S}$ there is a bijection

$$\varphi_s : \{1, ..., m\} \longrightarrow \pi^{-1}(s). \quad (37)$$

Definition 2.2. *Let $\Gamma \subset \mathbb{S}$ be a bipartite graph associated with a decorated surface $\mathbb{S}$.*

(i) A zig-zag strand γ on $\mathbb{S}$ is ideal if the domain S°_γ in (35) contains a single marked point s. We say that γ is associated to s.

(ii) Γ is a strict GL_m-graph if its rank is $m>1$, all zig-zag strands are ideal, and:

- *There are m zig-zag strands associated with each marked point.*
- *There are no parallel bigons and parallel half-bigons, see Figure 14.*

(iii) Γ is a GL_m-graph if for some finite cover $\rho : \widetilde{\mathbb{S}} \to \mathbb{S}$, $\rho^{-1}(\Gamma)$ is a strict GL_m-graph.

Figure 14: No parallel bigons (left) and half-bigons (right; the boundary of $\mathbb{S}$ is punctured).

An example. An ideal triangulation $\mathcal{T}$ of S determines a GL_m-graph Γ_m on $\mathbb{S}$ as follows. For each triangle of $\mathcal{T}$ we draw a bipartite graph shown on Figure 1, so that for each edge of $\mathcal{T}$ the external vertices of the two neighboring graphs match.

For a strict GL_m-graph Γ on $\mathbb{S}$, the *codistance* $\langle x, s\rangle$ from a point $x \in \mathbb{S}$ to a special point s is the number of zig-zag strands γ assocated with s such that x belongs to the open disc S°_γ. For example, the codistance to a special point s from a point x very close to s is m.

Lemma 2.3. *Let Γ be a strict GL_m-graph on $\mathbb{S}$. Then zig-zags associated with a marked point s form m concentric (half-) circles, going counterclockwise around s.*

For any point $x \in \mathbb{S} - \Gamma$, the sum of codistances from x to the marked points is equal to $\mathrm{rk}(\Gamma)$:

$$\sum_s \langle x, s\rangle = \mathrm{rk}(\Gamma). \tag{38}$$

Proof. Formula (38) is equivalent to the definition of rank as the degree of the covering π.

Alternatively, formula (38) is evident for a point x close to a marked point: in this case all codistances but one are 0, and the non-zero one is m. When x moves, crossing an edge E in an internal point, we alter codistances as follows: one of them increases by 1, another decreases by 1, the rest stay intact. □

2.2 Spectral description of non-commutative framed local systems

The monodromy of an m-dimensional framed R-local system $\mathcal{V}$ on $\mathbb{S}$ around a puncture p is an operator N in the fiber $\mathcal{V}_x$ over a nearby point x preserving the framing flag $\mathcal{F}$ in $\mathcal{V}_x$. The triple $(\mathcal{V}_x, \mathcal{F}, N)$ determines m operators, called the *monodromy eigenvalue operators* at p:

$$N_i : \mathrm{gr}^i\mathcal{F} \longrightarrow \mathrm{gr}^i\mathcal{F}, \quad i = 0, ..., m-1.$$

If R is commutative, these are the eigenvalues of N, which determine the semisimple part of N.

The points of Σ_Γ projecting to the marked points of $\mathbb{S}$ are the *marked points of* Σ_Γ.

Theorem 2.4. *Let Γ be a $\mathrm{GL_m}-$graph on a decorated surface $\mathbb{S}$, homotopy equivalent to $\mathbb{S}$. Then the following two groupoids are canonically birationally equivalent*

1. *Groupoid of m-dimensional framed R-local systems on $\mathbb{S}$;*

2. Groupoid of flat R-line bundles with connection on the spectral surface Σ_Γ.

The equivalence identifies the monodromy eigenvalue operators of the connection on $\mathcal{V}$ around a puncture p with the monodromy operators of $\mathcal{L}$ around the punctures on Σ_Γ over p, so that the eigenvalue operator N_i is the monodromy of $\mathcal{L}$ around the point $\varphi_p(i) \in \pi^{-1}(p)$, see (37).

Proof. Flat line bundles on the spectral surface Σ_Γ are the same thing as flat line bundles on the graph Γ. We follow the proof of Theorem 1.2, adjusted to the graph Γ. Let us define a functor

$$\mathcal{L} : \{\text{Generic } m\text{−dimensional framed local systems } \mathcal{V} \text{ on } \mathbb{S}\} \longrightarrow \{\text{Flat line bundles on } \Gamma\}. \tag{39}$$

Given a vertex v of Γ and a marked point s on $\mathbb{S}$ with the codistance $\langle v, s\rangle > 0$, there is a unique zig-zag γ containing v such that $s \in S_\gamma$. Since S_γ is a punctured disc, and the flag $\mathcal{F}^\bullet(s)$ near s is monodromy invariant, there is a canonical parallel transport of the flag $\mathcal{F}^\bullet(s)$ to a flag $\mathcal{F}^\bullet_{v,s}$ in the fiber $\mathcal{V}_v$ of $\mathcal{V}$ at v.

1. Let $\mathbf{w}$ be a $\circ-$vertex of the graph Γ. Denote by $a_1, ..., a_n$ the codistances from the $\mathbf{w}$ to the marked points $s_1, ..., s_n$ on $\mathbb{S}$. Then
$$a_1 + ... + a_n = m - 1, \qquad a_i \geq 0.$$
We assign to the $\circ-$vertex $\mathbf{w}$ a one dimensional subspace in the vector space $\mathcal{V}_{\mathbf{w}}$:
$$\mathrm{L}^\circ_{\mathbf{w}} := \mathcal{F}^{a_1}_{\mathbf{w},s_1} \cap \ldots \cap \mathcal{F}^{a_n}_{\mathbf{w},s_n}. \tag{40}$$

2. Let $\mathbf{b}$ be a $k-$valent $\bullet-$vertex of the graph Γ. Denote by $b_1, ..., b_n$ the codistances from the $\mathbf{b}$ to the marked points $s_1, ..., s_n$ on $\mathbb{S}$. Then
$$b_1 + ... + b_n = m - k + 1, \qquad b_i \geq 0.$$
We assign to the $k-$valent $\bullet-$vertex $\mathbf{b}$ a $(k-1)-$dimensional subspace in the vector space $\mathcal{V}_{\mathbf{b}}$:
$$\mathrm{P}_{\mathbf{b}} := \mathcal{F}^{b_1}_{\mathbf{b},s_1} \cap \ldots \cap \mathcal{F}^{b_n}_{\mathbf{b},s_n}. \tag{41}$$

The plane $\mathrm{P}_{\mathbf{b}}$ contains the k lines $\mathrm{L}^\circ_{\mathbf{w}}$ assigned to the $\circ-$vertices $\mathbf{w}$ incident to $\mathbf{b}$. Indeed, the codistances $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ from the vertices $\mathbf{w}$ and $\mathbf{b}$ are related as follows: $0 \leq a_i - b_i \leq 1$, and $a_j = b_j + 1$ for exactly k indices j. For any such an index j the embedding $\mathcal{F}^{b_j+1} \hookrightarrow \mathcal{F}^{b_j}$ induces the embedding $i_{\mathbf{w}\to\mathbf{b}} : \mathrm{L}^\circ_{\mathbf{w}} \longrightarrow \mathrm{P}_{\mathbf{b}}$.

We define a line $\mathrm{L}^\bullet_{\mathbf{b}}$ as the kernel of the sum of these maps over all $\circ-$vertices $\mathbf{w}$ incident to the $\mathbf{b}$:

$$\mathrm{L}^\bullet_{\mathbf{b}} := \mathrm{Ker}\Big(\bigoplus_{\mathbf{w}-\mathbf{b}} \mathrm{L}^\circ_{\mathbf{w}} \stackrel{i_{\mathbf{w}\to\mathbf{b}}}{\longrightarrow} \mathrm{P}_{\mathbf{b}}\Big). \tag{42}$$

Then there is a canonical map, induced by projections of the subspace $\mathrm{L}^\bullet_{\mathbf{b}}$ onto the summands:

$$\mathrm{L}^\bullet_{\mathbf{b}} \longrightarrow \bigoplus_{\mathbf{w}} \mathrm{L}^\circ_{\mathbf{w}}. \tag{43}$$

So we get a flat line bundle $\mathcal{L}_{\mathcal{V}}$ on the graph Γ. Its fibers at the vertices are given by the lines L° and $\mathrm{L}^\bullet$, and the parallel transport along an edge $\bullet \to \circ$ is given by a map $\mathrm{L}^\bullet \longrightarrow \mathrm{L}^\circ$ in (43).

Remark. If $R = K$ is commutative, the moduli space of flat line bundles on a graph Γ is identified with $H^1(\Gamma, K^*)$. The latter include the monodromies around the holes in the graph. These are the canonical coordinates from [G], generalizing the coordinates [FG1] to ideal bipartite graphs on $\mathbb{S}$.

If R is non-commutative, we no longer have canonical coordinates.

Let us show that the functor (39) is an equivalence of groupoids. Pick an ideal triangulation $\mathcal{T}$ of $\mathbb{S}$. Take an edge E of $\mathcal{T}$. The direct sum of the restrictions of the flat line bundle $\mathcal{L}$ to the $\circ-$vertices closest to E, and located from the one side of E, is an $m-$dimensional vector space, denoted by V_E. Take an edge F of $\mathcal{T}$ and a generic path π connecting midpoints of E and F. It intersects edges $E_1, E_2..., E_n$ of $\mathcal{T}$, ordered by the order of the intersection points $E_i \cap \pi$ on the path π. The functor $\mathcal{R}$ for the triangle provides isomorphisms

$$\mathrm{V}_E \xrightarrow{\mathcal{R}} \mathrm{V}_{E_1} \xrightarrow{\mathcal{R}} \ldots \xrightarrow{\mathcal{R}} \mathrm{V}_{E_n} \xrightarrow{\mathcal{R}} \mathrm{V}_F.$$

Their composition $\mathcal{R}_\pi : \mathrm{V}_E \longrightarrow \mathrm{V}_F$ does not change under deformations of the path π. So we get a local system of $m-$dimensional vector spaces on the surface $\mathbb{S}$, given by the vector space $\{\mathrm{V}_E\}$ and the isomorphisms $\mathcal{R}_\pi$ between them, providing the inverse functor. □

3 Spectral description twisted local systems

Convention. Dealing with <u>twisted</u> local systems on a decorated surface $\mathbb{S}$, we alter $\mathbb{S}$ by cutting out a little disc near each marked point m on $\mathbb{S}$. Abusing notation, we denote the obtained surface by $\mathbb{S}$. The tangent vectors to each boundary component of $\mathbb{S}$, positively oriented for the boundary orientation of $\mathbb{S}$, form a homotopy (semi)circle S^1_m in the bundle of non-zero tangent vectors to $\mathbb{S}$. We perform a similar procedure with the marked points on the spectral curve Σ. Talking about a flat section of a twisted local system near a marked point m of $\mathbb{S}$ or Σ, we mean a flat section of its restriction to the (semi)circle S^1_m.

The definition of a framed local system on a decorated surface in Definition 1.8 easily extends to the case of a twisted framed local system on $\mathbb{S}$. So by the monodromy of a twisted local system near a puncture we mean the monodromy of its restriction to the oriented homotopy circle near the puncture.

In Section 3.1 we consider the key example, when $\mathbb{S}$ is a triangle t. We replace t by an oriented circle S^1 with three marked points $\mathrm{A}, \mathrm{B}, \mathrm{C}$. Twisted framed local systems on $(S^1; \mathrm{A}, \mathrm{B}, \mathrm{C})$ are the same thing as the flat line bundle on S^1 with the monodromy -1, and flags $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in the fibers over the tangent vectors $v_\mathrm{A}, v_\mathrm{B}, v_\mathrm{C}$ to S^1 at the points $\mathrm{A}, \mathrm{B}, \mathrm{C}$, following the orientation of S^1. We call such data *twisted triples of flags.* There is a canonical one-dimensional local system ε on S^1 with the monodromy -1. Multiplying by ε, we identify the usual and twisted triples of flags. So there is almost no difference between the two notions. The distinction between the two will be important when we move to decorated twisted configurations of flags.

In Section 3.2 we consider the general case, describing twisted framed local systems on $\mathbb{S}$ via twisted framed flat line bundles on the spectral surface Σ of a bipartite graph on $\mathbb{S}$.

3.1 Spectral description of generic twisted triples of flags

Take a generic triple of flags $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in an m-dimensional R-vector space. We assign the flags to the vertices of a triangle t. The bipartite ribbon graph Γ_m on Figure 2 assigned to t gives rise to the spectral surface Σ_m. Namely, each zig-zag γ on Γ_m determines a smaller triangle t_γ, given by the component of $t - \gamma$ containing a single vertex of t. Take the disjoint union of all triangles t_γ. Then, for each edge E of the graph Γ_m, glue the triangles $t_{\gamma'}$ and $t_{\gamma''}$ assigned to the zig-zags γ' and γ'' containing E over the edge E. We get the surface Σ_m and the projection $\Sigma_m \longrightarrow t$, ramified over the $\bullet-$vertices. To avoid confusion, we denote by F_γ the triangle t_γ when it sits on the spectral surface.

Theorem 3.1. *There is a canonical equivalence between the following two groupoids:*

1. *Groupoid of generic triples of flags in an m-dimensional R-vector space.*

2. *Groupoid of flat <u>twisted</u> R-line bundles with connection on the spectral surface Σ_m of the graph Γ_m.*

Proof. $1 \longrightarrow 2$. A triple of flags $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in an m-dimensional R-vector space V gives rise to $3m$ lines

$$\mathrm{gr}^a\mathcal{A}, \quad \mathrm{gr}^b\mathcal{B}, \quad \mathrm{gr}^c\mathcal{C}, \quad a, b, c = 0, ..., m-1. \qquad (44)$$

Given a zig-zag γ on the graph Γ_m going around of the vertex A, we assign to the face F_γ of the spectral surface the line $\mathrm{gr}^{a_\gamma-1}\mathcal{A}$, where $a_\gamma \in \{1, ..., m\}$ is the codistance from γ to A. Abusing notation, we denote by $\mathrm{gr}^{a_\gamma-1}\mathcal{A}$ the twisted flat line bundle on F_γ whose restriction to oriented tangent vectors to the oriented side BC is given by the line $\mathrm{gr}^{a_\gamma-1}\mathcal{A}$. We do the same for the vertices B, C. Let us glue these twisted flat line bundles on the $3m$ faces F_γ to a twisted line bundle on the spectral surface.

Note that the edges of the graph Γ_m are parametrised by the following data:
i) A triple of non-negative integers (a, b, c) such that $a + b + c = m - 1$, plus
ii) A choice of two of the three numbers (a, b, c).

The faces of the graph Γ_m are labeled by the triples of non-negative integers (a, b, c) with $a+b+c = m$.

Given a data $(\underline{a}, \underline{b}, c)$ assigned to an edge E, where the chosen pair in ii) is the pair (a, b), the two faces $F_{\gamma'}$ and $F_{\gamma''}$ of the spectral surface sharing the edge E are labeled by the triples $(a, b + 1, c)$ and $(a + 1, b, c)$, accordantly. Then the faces $F_{\gamma'}$ and $F_{\gamma''}$ carry the line bundles

$$\text{Ł}_{\gamma'} := \mathrm{gr}^a\mathcal{A}, \qquad \text{Ł}_{\gamma''} := \mathrm{gr}^b\mathcal{B}.$$

We assign to the edge E the line

$$\text{Ł}_E := \mathcal{A}^a \cap \mathcal{B}^b \cap \mathcal{C}^c.$$

Then there are canonical isomorphisms, obtained by projecting along $\mathcal{A}^{a+1}$ and $\mathcal{B}^{b+1}$:

$$\mathrm{gr}^a\mathcal{A} \xleftarrow{\sim} \mathcal{A}^a \cap \mathcal{B}^b \cap \mathcal{C}^c \xrightarrow{\sim} \mathrm{gr}^b\mathcal{B}. \tag{45}$$

They allow to glue the twisted flat line bundles over the faces $F_{\gamma'}$ and $F_{\gamma''}$ along their common edge E.

Let us exten the obtained twisted flat line bundle on $\Sigma_m - \{\text{the vertices of } \Gamma^*_m\}$ to the spectral surface. In order to do this, we make the following observations.

For any triple (a, b, c) such that $a + b + c = m - 1$ there are three canonical isomorphisms:

$$\begin{array}{ccccc} & & \mathrm{gr}^b\mathcal{B} & & \\ & & \uparrow{\sim} & & \\ & & \mathcal{A}^a \cap \mathcal{B}^b \cap \mathcal{C}^c & & \\ & \swarrow{\sim} & & \searrow{\sim} & \\ \mathrm{gr}^a\mathcal{A} & & & & \mathrm{gr}^c\mathcal{C} \end{array} \tag{46}$$

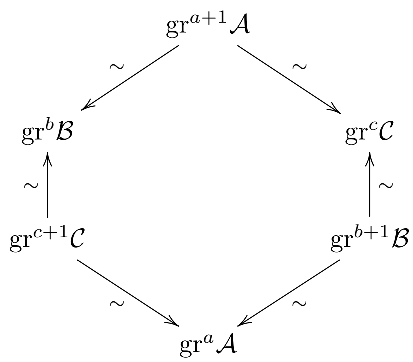
The first is the composition $\mathcal{A}^a \cap \mathcal{B}^b \cap \mathcal{C}^c \hookrightarrow \mathcal{A}^a \to \mathcal{A}^a/\mathcal{A}^{a+1}$. The others are similar.

Lemma 3.2. *There are the following relations between the isomorphisms (46):*

1. *For any integers $a, b, c \geq 0$ such that $a + b + c = m - 1$ there is a commutative triangle, i.e. the composition of the three isomorphism is the identity:*

$$\begin{array}{ccccc} & & \mathrm{gr}^b\mathcal{B} & & \\ & \nearrow & & \searrow & \\ \mathrm{gr}^a\mathcal{A} & & \longleftarrow & & \mathrm{gr}^c\mathcal{C} \end{array} \tag{47}$$

2. *For any integers $a, b, c \geq 0$ such that $a+b+c = m-2$ there is a sign-commutative hexagon, i.e. the composition of the following six isomorphisms/their inverces is equal to -1:*



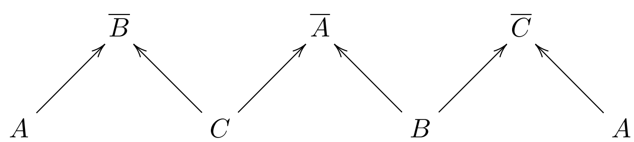
(48)

Proof. 1) Starting with a pair of isomorphisms $\mathrm{gr}^a\mathcal{A} \stackrel{\sim}{\longleftarrow} \mathcal{A}^a \cap \mathcal{B}^b \cap \mathcal{C}^c \stackrel{\sim}{\longrightarrow} \mathrm{gr}^b\mathcal{B}$, and composing the inverse of the first one with the second, we get an isomorphism $\mathrm{gr}^a\mathcal{A} \stackrel{\sim}{\longrightarrow} \mathrm{gr}^b\mathcal{B}$. The other two are similar. This makes the first claim is evident.

2) Intersecting with the plane $\mathcal{A}^a \cap \mathcal{B}^b \cap \mathcal{C}^c$, we reduce the claim to the following. Let A, B, C be three different lines in a plane P. Consider the quotient lines

$$\overline{A} := \mathrm{P}/A, \quad \overline{B} := \mathrm{P}/B, \quad \overline{C} := \mathrm{P}/C.$$

Projecting the line A to the quotient line $\overline{B}$ we get an isomorphism $A \longrightarrow \overline{B}$. There are six such line isomorphisms:



(49)

We compose them to a map $A \to A$, where going against an arrow means inverting the isomorphism. We claim that the composition is the minus identity map. Indeed, choose a triple of non-zero vectors $x \in A, y \in B, z \in C$ whose sum is zero:

$$x + y + z = 0. \tag{50}$$

Such a triple exists, and is defined uniquely up to a multiplication from the left by a non-zero scalar. Condition (50) just means that $x+z$ projects to zero in $\overline{B}$. So the map $A \longrightarrow C$ provided by the diagram $A \stackrel{\sim}{\longrightarrow} \overline{B} \stackrel{\sim}{\longleftarrow} C$ is given by $x \longmapsto -z$. Therefore the composition $A \longrightarrow C \longrightarrow B \longrightarrow A$ is given by $x \longmapsto -z \longmapsto y \longmapsto -x$. □

The part i) of Lemma 3.2 allows to extend the twisted flat line bundle to the $\circ-$vertices. Indeed, although the composition of the three maps on Figure 15 is the identity map, the tangent vectors are rotated by 2π, thus resulting the $-\mathrm{Id}$ map.

Similarly, the part ii) of of Lemma 3.2 allows to extend the twisted flat line bundle to the $\bullet-$vertices. Indeed, although the composition of the six maps (48) is $-\mathrm{Id}$, the tangent vectors are rotated by 4π, thus still resulting the $-\mathrm{Id}$ map.

$2 \longrightarrow 1$. *The functor $\mathcal{R}$: {Twisted flat line bundles on Σ_m} $\longrightarrow$ {generic twisted triples of flags}.* Take a twisted flat line bundle $\mathcal{L}$ on the spectral surface Σ_m of the bipartite ribbon graph Γ_m. It contains the graph Γ_m^*. For each vertex $v \in \Gamma_m^*$, restrict $\mathcal{L}$ to the positively oriented tangent vectors to an oriented circle $S_v^1 \subset T_v\Sigma - \{0\}$, getting a flat line bundle $\mathcal{L}_{S_v^1}$ on S_v^1 with the monodromy -1.

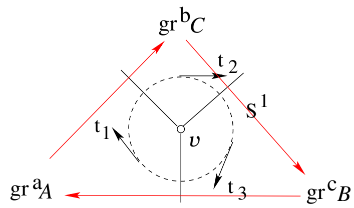


Figure 15: Let $a+b+c=m-1$, and the line $\mathrm{gr}^a\mathcal{A}$ sits over the tangent vector t_1, etc.. By (47), the composition $\mathrm{gr}^a\mathcal{A} \to \mathrm{gr}^b\mathcal{B} \to \mathrm{gr}^c\mathcal{C} \to \mathrm{gr}^a\mathcal{A}$ is the identity map. The tangent vector t_1 rotates by $t_1 \to t_2 \to t_3 \to t_1$ by 2π.

Then the flat connection on $\mathcal{L}$ provides a natural map

$$\mathrm{K}_{\mathcal{L}}: \bigoplus_{\text{internal }\bullet-\text{vertices }\mathbf{b}} \mathcal{L}_{S^1_{\mathbf{b}}} \longrightarrow \bigoplus_{\circ-\text{vertices }\mathbf{w}} \mathcal{L}_{S^1_{\mathbf{w}}}. \tag{51}$$

We identify all circles $S^1_{\mathbf{w}}$ with a circle S^1, and define the vector space $V_{\mathcal{L}}$ as the cokernel of this map:

$$V_{\mathcal{L}} := \mathrm{Coker}(\mathrm{K}_{\mathcal{L}}). \tag{52}$$

It is a flat bundle on the circle S^1 with the monodromy $-\mathrm{Id}$. Let us define three flags in the fibers of this local system over the points $\mathrm{A}, \mathrm{B}, \mathrm{C}$. The distance functions $d_{\mathrm{A}}, d_{\mathrm{B}}, d_{\mathrm{C}}$, see the proof of Theorem 1.2, induce three filtrations in the fibers of complex (51) over the points $\mathrm{A}, \mathrm{B}, \mathrm{C}$. They induce filtrations on $\mathrm{Coker}(\mathrm{K}_{\mathcal{L}})$, providing a twisted triple of flags. The functor $\mathcal{R}$ is the inverse to the one defined before. □

Remark. Let us compare constructions of the connection in the proofs of Theorems 1.2 and 3.1. The key point is that given three vectors x, y, z in a two dimensional vector space with $x+y+z=0$, we can define a connection either by $x \to y \to z \to x$, as we did proving Theorem 1.2, or by $x \to -y \to z \to -x$, getting a twisted connection.

3.2 Spectral description twisted local systems on surfaces

Below we use the notation Σ for the spectral surface Σ_Γ since the bipartite graph Γ is fixed.

Theorem 3.3. *Let Γ be a GL_m-graph on a decorated surface $\mathbb{S}$, and Σ the corresponding spectral surface. Then the following two groupoids are canonically birationally equivalent:*

1. *Groupoid of m-dimensional twisted framed R-vector bundles with flat connections $\mathcal{V}$ on $\mathbb{S}$.*
2. *Groupoid of flat twisted R-line bundles with connection $\mathcal{L}$ on the spectral surface Σ.*

The equivalence identifies the monodromy eigenvalue operators of the connection on $\mathcal{V}$ around a puncture p with the monodromy operators of $\mathcal{L}$ around the punctures on Σ over p just as in Theorem 2.4.

Proof. Let $\mathcal{V}$ be an m-dimensional twisted framed R-local system on $\mathbb{S}$. Let us construct a twisted flat line bundle $\mathcal{L}$ on Σ. Let $\mathcal{F}(s)$ be a flat section of the local system of flags in $\mathcal{V}$ defining a framing near a marked point $s \in \mathbb{S}$. For each zig-zag γ going around s, it provides a flat section over the punctured disc S_γ. Recall the punctured face F_γ of the spectral surface Σ assigned to γ. By the very construction of the spectral surface, there is a canonical identification:

$$F_\gamma = S_\gamma.$$

We use it to transport the flat section of the local system of flags on S_γ to F_γ, and denote it by $\mathcal{F}(s)$. We assign to a zig-zag γ a number $a_\gamma \in \{1, ..., m\}$ - the codistance from a point of the open disc S°_γ close to γ to the marked point associated with γ. Then there is a twisted flat line bundle on the face F_γ of Σ:

$$\mathcal{L}_{F_\gamma} := \mathrm{gr}^{a_\gamma - 1}\mathcal{F}(s) \tag{53}$$

Let us glue them into a twisted flat line bundle $\mathcal{L}$ on Σ.

Lemma 3.4. *Let $s_1, \ldots, s_n$ be the marked points on $\mathbb{S}$. Let y be a point of Σ, and $x := \pi(y) \in \mathbb{S}$.*
Let $0 \leq a_j \leq m$ be the codistance from x to s_j. Then we have:

1. *Let $y \in \Sigma - \Gamma^*$. Then*
$$a_1 + \ldots + a_n = m. \tag{54}$$

2. *Let $y \in \Gamma^* - \{\bullet-\text{vertices}\}$. Then*
$$a_1 + \ldots + a_n = m - 1. \tag{55}$$

3. *Let y be a $k-$valent $\bullet$-vertex of Γ^*. Then*
$$a_1 + \ldots + a_n = m - k + 1. \tag{56}$$

Proof. 1. When a point x is inside of a face, the claim follows from (38).

2. When a point x' moves and hits an inside point x of an edge E, just one of the codistances a_i decreases by 1. Namely, it is the one corresponding to the marked point s_γ surrounded by the zig-zag γ containing E, such that the open domain S°_γ containes x'. Same thing happens when x is a $\circ-$vertex. Indeed, take a small disc U_x containing x. Let $\gamma_1, \ldots, \gamma_k$ be the zig-zags containing x. The domains $S^\circ_\gamma \cap U_x$ are disjoint, and their closures cover U_x, as shown on the left in Figure 13.

3. When a point x' moves and hits a $\bullet-$vertex x, the $k-1$ of the codistances $\{a_i\}$ decrease by 1. They are the ones corresponding to the zig-zags γ containing the vertex x such that $x' \in S^\circ_\gamma$. □

Gluing over an edge E of Γ. Let γ_1 and γ_2 be the two zig-zags passing through E, and s_1 and s_2 the related marked points of $\mathbb{S}$. Faces F_{γ_1} and F_{γ_2} on Σ intersect along the edge (identified with) E. We denote by $\mathcal{F}^{a_i}$ the codimension a_i subspace of the flag $\mathcal{F}(s_i)$. Then, thanks to (55), $a_1 + a_2 + \ldots + a_n = m - 1$, and so there is a twisted flat line bundle near the edge E on $\mathbb{S}$:

$$\mathcal{L}_E := \mathcal{F}^{a_1} \cap \mathcal{F}^{a_2} \cap \mathcal{F}^{a_3} \cap \ldots \cap \mathcal{F}^{a_n}, \qquad a_1 + a_2 + \ldots + a_n = m - 1. \tag{57}$$

Note that there are canonical projections

$$\mathcal{F}^{a_1} \longrightarrow \mathrm{gr}^{a_1}\mathcal{F}(s_1), \qquad \mathcal{F}^{a_2} \longrightarrow \mathrm{gr}^{a_2}\mathcal{F}(s_2).$$

Restricting them to intersection (57) we get isomorphisms of twisted flat line bundles near the edge E:

$$\mathcal{L}_{F_{\gamma_1}} \stackrel{(53)}{=} \mathrm{gr}^{a_1}\mathcal{F}(s_1) \stackrel{\sim}{\longleftarrow} \mathcal{L}_E \stackrel{\sim}{\longrightarrow} \mathrm{gr}^{a_2}\mathcal{F}(s_2) \stackrel{(53)}{=} \mathcal{L}_{F_{\gamma_2}}. \tag{58}$$

Here the very left and right isomorphisms come from the fact that the codistances from a generic point of $S^\circ_{\gamma_1}$ close to the zig-zag γ_1 are given by $(a_1 + 1, a_2, \ldots, a_n)$, and for $S^\circ_{\gamma_2}$ they are $(a_1, a_2 + 1, \ldots, a_n)$. Using isomorphisms (58), we glue twisted flat line bundles (53) on faces F_γ into a twisted flat line bundle

$$\mathcal{L}^\times \text{on } \Sigma - \{\text{vertices of } \Gamma^*\}.$$

Proposition 3.5. *The twisted flat line bundle $\mathcal{L}^\times$ extends to a twisted flat line bundle $\mathcal{L}$ on Σ.*

Proof. Take a small circle S^1_v around a vertex $v \in \Gamma^*$ on the spectral surface Σ. Restrict the $\mathcal{L}^\times$ to a section of the bundle of non-zero tangent vectors on S^1_v. We get a flat line bundle $\mathcal{L}^\times_v$ over the circle S^1_v.

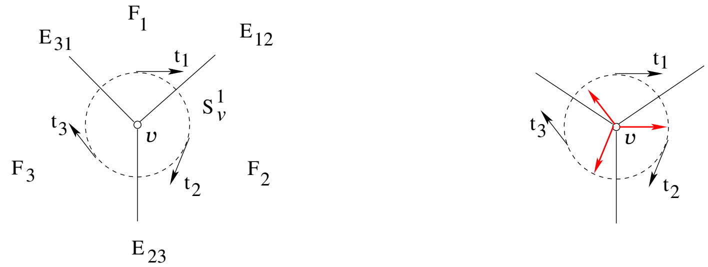


Figure 16: On the left: the edges $E_{i,i+1}$ and faces F_i of Γ^* near a $\circ$-vertex v.

Lemma 3.6. *The monodromy of the flat line bundle $\mathcal{L}_v^\times$ around every $\circ$-vertex v is equal to -1.*

Proof. Denote by $E_{i,i+1}$ the edge of Γ^* shared by the faces F_i and F_{i+1}, see Figure 16. Take a non-zero tangent vector t_i at a point of $S^1_v \cap F_i$ in the clockwise direction. Denote by $\mathcal{L}_{t_i}$ the fiber of the twisted local system $\mathcal{L}_{F_i}$ over t_i. To define the parallel transform $\mathcal{L}_{t_i} \to \mathcal{L}_{t_{i+1}}$, we move tangent vectors t_i and t_{i+1} towards each other to the tangent vector $t_{i,i+1}$ at the point $S^1_v \cap E_{i,i+1}$, and use isomorphisms (58). Denote by $\widetilde{t}_i$ the tangent vectors at the vertex v parallel to the tangent vectors t_i, see the red vectors on the right picture on Figure 16. We have canonical isomorphisms $\mathcal{L}_{t_i} = \mathcal{L}_{\widetilde{t}_i}$ provided by the parallel transform of tangent vectors $t_i \to \widetilde{t}_i$. After these identifications, the parallel transform of the vector t_i around the circle S^1_v amounts to the parallel transform $\mathcal{L}_{\widetilde{t}_i}$ for the rotation of the tangent vector $\widetilde{t}_i$ by 360°, resulting the -1 monodromy. □

Lemma 3.6 implies Proposition 3.5 a $\circ$-vertex.

Now let $\bullet$ be a $\bullet$-vertex of Γ. Let $v_1, \ldots, v_k$ be the $\circ$-vertices incident to $\bullet$, whose order is compatible with the cyclic structure at $\bullet$. A zig-zag $\ldots v_i \bullet v_{i+1} \ldots$, where $i \in \mathbb{Z}/k\mathbb{Z}$, passing via the vertex $\bullet$ determines a *$\bullet$-cosector*, see Figure 17:

$$C_{v_i \bullet v_{i+1}} := U_\bullet \cap S_\gamma.$$

The flat line bundle over this $\bullet$-cosector is denoted by $\mathcal{L}_{C_{v_i \bullet v_{i+1}}}$. Next, for each $\circ$-vertex v_i consider the domain C_{v_i} given by the intersection of U_{v_i} with an angle formed by the two edges at v_i on the left and on the right of the edge $\bullet v_i$, see Figure 17. Denote the twisted flat line bundle over this domain by $\mathcal{L}_{C_{v_i}}$.

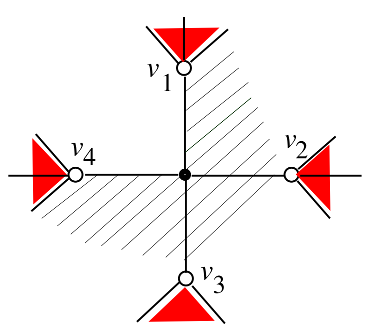


Figure 17: The shaded domain is the cosector $C_{v_4 \bullet v_1}$. The red ones are domains C_{v_i}.

For each $i = 1, ..., k$ the twisted flat structure of $\mathcal{L}$ provides a pair of isomorphisms

$$\mathcal{L}_{C_{v_i}} \longrightarrow \mathcal{L}_{C_{v_i \bullet v_{i+1}}} \longleftarrow \mathcal{L}_{C_{v_{i+1}}}.$$

They combine into a diagram of $2k$ line isomorphisms. For example, for $k = 3$ we get

$$\begin{array}{ccccccc} & \mathcal{L}_{C_{v_1\bullet v_2}} & & \mathcal{L}_{C_{v_2\bullet v_3}} & & \mathcal{L}_{C_{v_3\bullet v_1}} & \\ \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \searrow \\ \mathcal{L}_{C_{v_1}} & & \mathcal{L}_{C_{v_2}} & & \mathcal{L}_{C_{v_3}} & & \mathcal{L}_{C_{v_1}} \end{array} \qquad (59)$$

The same argument as in the proof of Lemma 3.2(ii) shows that the composition is $(-1)^k$.

Moving around the circle $S^1_\bullet$ on the spectral surface Σ we rotate the projection to $\mathbb{S}$ of the tangent vector to the circle by $(\text{val}(\bullet) - 1)2\pi$. The composition of $2\text{val}(\bullet)$ line isomorphisms in (59) is $(-1)^{\text{val}(\bullet)}$. Since $\text{val}(\bullet) - 1 + \text{val}(\bullet)$ is odd, the result is multiplied by -1. So we can extend the twisted flat line bundle $\mathcal{L}^\times$ across the $\bullet$-vertex. Proposition 3.5 is proved. □

To prove that the constructed map is an equivalence of groupoids, we proceed similarly to the proof of equivalence in Theorem 3.1. Theorem 3.3 is proved. □

4 Non-commutative cluster $\mathcal{A}-$varieties from bipartite ribbon graphs

Pick a vector field tangent to the boundary of each face of a decorated surface, following the boundary orientation, see Figure 18. A trivialization of a twisted flat line bundle $\mathcal{L}$ on the surface at the boundary is given by a choice of a non-zero section of the fiber of $\mathcal{L}$ over the tangent boundary vector field.

Figure 18: An edge E of a ribbon graph Γ determines two faces of $\mathbb{S}_\Gamma$, denoted f_E^+ and f_E^-. There is a canonical homotopy class of path p_E, connecting boundary components f_E^+ and f_E^-, shown by a blue arc.

Recall the surface $\mathbb{S}_\Gamma$ for a ribbon graph Γ, and the spectral surface Σ_Γ for a bipartite ribbon graph Γ.

Given a bipartite ribbon graph Γ, consider the groupoid, defined in either of the following two ways:

$$\begin{aligned} \mathcal{A}_\Gamma :=& \{\text{Twisted flat line bundles on the spectral surface } \Sigma_\Gamma, \text{ trivialized at the boundary}\}. \\ =& \{\text{Twisted flat line bundles on the surface } \mathbb{S}_\Gamma, \text{ trivialized at the zig-zag strands}\}. \end{aligned} \qquad (60)$$

These definitions are equivalent since there are canonical identifications:

$$\Gamma = \Gamma^*, \quad \Sigma_\Gamma = \mathbb{S}_{\Gamma^*}, \quad \{\text{zig-zags on } \mathbb{S}_\Gamma\} = \{\text{boundary components of } \Sigma_\Gamma\}.$$

Let us give a coordinate description of the non-commutative tori $\mathcal{A}_\Gamma$ in terms of the $\mathcal{A}-$coordinates.

1. $\mathcal{A}-$coordinates on ribbon graphs.

Definition 4.1. *Let Γ be a ribbon graph. The $\mathcal{A}-$*coordinates *on Γ are the elements $\{\Delta_E \in R^*\}$ assigned to the* <u>*oriented*</u> *edges E of Γ, satisfying the following monomial relations:*

- *Let E be an oriented edge, and $\overline{E}$ the same edge with the opposite orientation. Then*

$$\Delta_E \Delta_{\overline{E}} = -1. \qquad (61)$$

- *Let $E_1, ..., E_n$ be the edges incident to any vertex v of Γ, oriented out of the vertex v, whose order is compatible with their cyclic order. Then*

$$\Delta_{E_1}\Delta_{E_2}\dots\Delta_{E_n} = -1. \tag{62}$$

Calling the elements Δ_E "coordinates" is an abuse of terminology since they are not independent. We use the name $\mathcal{A}-$*decorated ribbon graph* for a ribbon graph with a collection of $\mathcal{A}-$coordinates.

Lemma 4.2. *Given a ribbon graph Γ, the $\mathcal{A}-$coordinates on Γ parametrise the isomorphism classes of twisted flat line bundles on the surface $\mathbb{S}_\Gamma$, trivialized at the boundary.*

Proof. We start from a twisted line bundle on the oriented decorated surface $\mathbb{S}_\Gamma$. An edge E of Γ determines two faces f_E^+ and f_E^- of $\mathbb{S}_\Gamma$. There is a canonical homotopy class of path p_E, connecting boundary components f_E^+ and f_E^-, see Figure 18. The parallel transport along the p_E acts on the trivializations $s_E^\pm$ at the boundary components $f_E^\pm$ as follows, defining elements $\Delta_E \in R^*$:

$$\text{par}_{p_E} : s_E^+ \longrightarrow \Delta_E s_E^-, \qquad \Delta_E \in R^*. \tag{63}$$

Going around any edge, or any vertex along the cyclic order at this vertex, as on Figure 18, we rotate the tangent vector by 2π, getting both relations (61) and (62). For example, for a 3-valent vertex we get

$$\begin{aligned} &s_{E_1}^+ \longrightarrow \Delta_{E_1}s_{E_1}^- = \Delta_{E_1}s_{E_2}^+ \longrightarrow \Delta_{E_1}\Delta_{E_2}s_{E_2}^- = \Delta_{E_1}\Delta_{E_2}s_{E_3}^+ \\ &\longrightarrow \Delta_{E_1}\Delta_{E_2}\Delta_{E_3}s_{E_3}^- = -\Delta_{E_1}\Delta_{E_2}\Delta_{E_3}s_{E_1}^+. \end{aligned} \tag{64}$$

Here the arrows are induced by the parallel transports (63) $\text{par}_{p_{E_i}}$, $i \in \mathbb{Z}/3\mathbb{Z}$. The $=$ signs mean canonical identifications provided by the parallel transport along the arcs α_i between E_i to E_{i+1}, see Figure 18, identifying $s_{E_i}^-$ with $s_{E_{i+1}}^+$. The $-$ sign in the end results from the fact that going around the circle amounts to the monodromy -1.

Conversely, picking a section s over a tangent vector to a boundary component, and using elements Δ_E satisfying the monomial relations (61) and (62), we recover the twisted flat line bundle on $\mathbb{S}_\Gamma$, trivialized at the boundary. Changing s we get an isomorphic object. □

2. Conventions on $\mathcal{A}-$coordinates on bipartite ribbon graphs. *Given a bipartite ribbon graph, its edges are orientated $\circ \longrightarrow \bullet$.*

Using this convention, the set of $\mathcal{A}-$coordinates on any bipartite ribbon graph can be given by a collection of elements $\{\Delta_E\}$ assigned to *nonoriented* edges E, assuming their default $\circ \longrightarrow \bullet$ orientation, subject to the following relations, where *antycyclic* means the order opposite to the cyclic order:

The cyclic product of the elements on the edges sharing any $\circ-$vertex is equal to -1.
The antycyclic product of the elements on the edges sharing any $\bullet-$vertex v is $(-1)^{\text{val}(v)-1}$. (65)

Indeed, given a $\bullet-$vertex v, we have, using $\Delta_{\overline{E}_i} = -\Delta_{E_i}^{-1}$:

$$-1 = \prod_{i=1}^{\text{val}(v)} \Delta_{\overline{E}_i} = \prod_{i=1}^{\text{val}(v)} (-\Delta_{E_i}^{-1}) = (-1)^{\text{val}(v)}\Big(\prod_{\text{val}(v)}^{i=1} \Delta_{E_i}\Big)^{-1}. \tag{66}$$

We apply this convention to the bipartite ribbon graph Γ^*, since the spectral surface where the twisted flat line bundles live, is $\mathbb{S}_{\Gamma^*}$.

Note that the bipartite ribbon graphs Γ and Γ^* are identical as graphs. Thinking about the graph Γ, e.g. drawing Figures, we depict the cyclic order on Γ as the counterclockwise one, and use below and in Section 6.2 a different convention:

The counterclockwise product of the elements on the edges sharing a $\circ-$vertex is equal to -1.
The counterclockwise product of the elements on the edges sharing a $\bullet-$vertex v is $(-1)^{\text{val}(v)-1}$. (67)

3. The action of the two by two moves on $\mathcal{A}-$coordinates on bipartite ribbon graphs. Recall the two by two moves of bipartite ribbon graphs, shown on Figure 19, see also Definition 7.1 below.

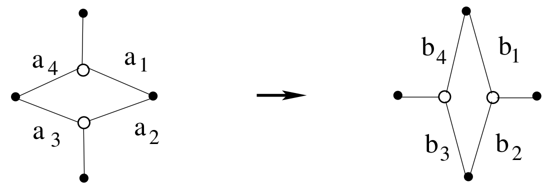


Figure 19: The relations between the coordinates a_i and b_i are $a_i a_{i+1} = (b_i b_{i+1})^{-1}$, $\forall i \in \mathbb{Z}/4\mathbb{Z}$. The cases $i = 1, 3$ are checked using Figure 20, the cases $i = 2, 4$ are similar.

We define a birational isomorphism $\mathcal{A}_\Gamma \longrightarrow \mathcal{A}_{\Gamma'}$ corresponding to a two by two move $\Gamma \longrightarrow \Gamma'$, serving the role of an elementary non-commutative $\mathcal{A}-$cluster transformation. Let us assign the coordinates $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ to the internal edges on the graphs on Figure 19. Let us set

$$A_i = a_i a_{i+1} a_{i+2} a_{i+3}, \qquad B_i = b_i b_{i+1} b_{i+2} b_{i+3}, \qquad i \in \mathbb{Z}/4\mathbb{Z}. \tag{68}$$

Consider the following transformation of the coordinates illustrated on Figure 19.

$$\begin{aligned} b_1 &= (1 + A_3^{-1}) a_3 &&= a_3 (1 + A_4^{-1}); \\ b_2 &= (1 + A_4)^{-1} a_4 &&= a_4 (1 + A_1)^{-1}; \\ b_3 &= (1 + A_1^{-1}) a_1 &&= a_1 (1 + A_2^{-1}). \\ b_4 &= (1 + A_2)^{-1} a_2 &&= a_2 (1 + A_3)^{-1}; \end{aligned} \tag{69}$$

To check the second equality in each line, note that $a_i a_{i+1} a_{i+2} a_{i+3} a_{i+4}$ can be written in two ways:

$$a_i A_{i+1} = A_i a_i, \qquad \forall i \in \mathbb{Z}/4\mathbb{Z}; \tag{70}$$

Lemma 4.3. *The square of the two by two transformation (69) is the identity transformation.*

Proof. This can be proved, by using the calculation of the coordinate transformations for the space $\mathrm{Conf}_4^{\mathrm{df}}(V_2)$ of four decorated flags in Section 6.2. Since the direct check is easy, let us elaborate it.

Going from b's to a's, the analog of the first formula in (69) is the following formula: $a_2 = b_4(1+B_1^{-1})$. Since $B_1^{-1} = A_3$, it is equivalent to $b_4 = a_2(1 + A_3)^{-1}$, which is indeed the case by (69).

Similarly, the analog of the second formula in (69) is the following: $a_3 = b_1(1+B_2)^{-1}$. Since $B_2^{-1} = A_4$, it is equivalent to $b_1 = a_3(1 + A_4^{-1})$, which is indeed the case by (69). □

Lemma 4.4. *We have for any $i \in \mathbb{Z}/4\mathbb{Z}$:*

$$\begin{aligned} &b_i b_{i+1} = (a_i a_{i+1})^{-1}, \\ &A_i = B_{i+2}^{-1}. \end{aligned} \tag{71}$$

Proof. We have

$$\begin{aligned} b_3 b_4 a_3 a_4 &= a_1(1 + A_2^{-1})(1 + A_2)^{-1} a_2 a_3 a_4 = a_1 A_2^{-1} a_2 a_3 a_4 = A_2^{-1} A_2 = 1. \\ b_4 b_1 a_4 a_1 &= a_2(1 + A_3)^{-1}(1 + A_3^{-1}) a_3 a_4 a_1 = a_2 A_3^{-1} a_3 a_4 a_1 = A_3^{-1} A_3 = 1. \end{aligned} \tag{72}$$

Since transformation formulas (69) are invariant under the cyclic shift $i \longmapsto i + 2$, the first formula in (71) follows. The second formula follows from the first. □

Figure 20: Using conventions (67), we have: $a_4 a_1 \alpha = -1$, $\alpha\beta = (-1)^{\mathrm{val}(v)-1}$, $b_4 b_1 \beta = (-1)^{\mathrm{val}(v)}$. Therefore $a_4 a_1 = (b_4 b_1)^{-1}$.

Figure 20 shows that formula (71) is consistent with the condition that the counterclockwise product of the coordinates around a $\circ-$vertex is 1, and around a $\bullet-$vertex is 1.

In Section 6.2 we define $\mathcal{A}-$coordinates for GL_2-graphs on a decorated surface $\mathbb{S}$, parametrising the moduli space $\mathcal{A}_{2,\mathbb{S}}$. The change of these $\mathcal{A}-$coordinates under a flip of a triangulation, studied in Section 6.2, motivated the definition of the $\mathcal{A}-$cluster coordinate transformations in (69).

Theorem 4.5. *The coordinate transformation (69) for a two by two move satisfies the pentagon relation.*

Proof. The proof can be obtained by a long calculation. Instead, we give a simple proof based on Theorem 6.5. Although we prove Theorem 6.5 later, its proof does not rely on Theorem 4.5.

Theorem 6.5 tells that for the configuration space $\mathrm{Conf}_4^{\mathrm{df}}(V_2)$ of four decorated flags in a two-dimensional $R-$vector space V_2, the non-commutative cluster $\mathcal{A}-$coordinates assigned in Section 6.2 to a bipartite GL_2-graphs corresponding to two different triangulations of the rectangle are related by the transformation (69). Let us elaborate on that.

Consider the cluster $\mathcal{A}-$coordinates on the configuration space $\mathrm{Conf}_5^{\mathrm{df}}(V_2)$ of five decorated flags in V_2, assigned to the bipartite GL_2-graphs corresponding to five different triangulations of the pentagon. By Theorem 6.5, each flip of a triangulation of the pentagon gives rise to the standard coordinate transformation (69) of the $\mathcal{A}-$coordinates for the corresponding bipartite ribbon graphs. Since, by the very construction of the coordinates, the composition of the five consecutive transformations tautologically describes the identity map on the configuration space, and since the cluster $\mathcal{A}-$coordinates for a given triangulation of the pentagon provide a birational isomorphism of the configuration space $\mathrm{Conf}_5^{\mathrm{df}}(V_2)$ with the cluster torus, this immediately implies, without any further calculations, that the pentagon relation for the cluster $\mathcal{A}-$coordinates holds. □

Theorem 4.5 allows to define the non-comutative cluster variety $\mathcal{A} = \mathcal{A}_{|\Gamma|}$ assigned to a bipartite ribbon graph Γ. Its clusters are parametrised by the bipartite ribbon graphs which can be obtained from Γ by two by two moves and the elementary transformations given by shrinking a two-valent vertex. We assign to each cluster the non-commutative torus $\mathcal{A}_\Gamma$, and glue them using the coordinate transformations (69) corresponding to the two by two moves. In Section 5 we will show that there is a canonical non-commutative 2-form $\Omega_\mathcal{A}$ on the cluster variety $\mathcal{A}$, preserved by the two by two moves of the graphs.

5 The canonical 2-form on a non-commutative cluster $\mathcal{A}-$variety.

1. The non-commutative analog of $d\log(a_1) \wedge \ldots \wedge d\log(a_n)$. Recall the cyclic envelope of the tensor algebra $\mathrm{T}(V)$ of a graded vector space V. It is spanned by the elements $(v_1 \otimes v_2 \otimes \ldots \otimes v_n)_\mathcal{C}$, which are cyclically invariant:

$$(v_1 \otimes v_2 \otimes \ldots \otimes v_n)_\mathcal{C} = (-1)^{|v_1|\cdot(|v_2|+\ldots+|v_n|)}(v_2 \otimes v_3 \otimes \ldots \otimes v_n \otimes v_1)_\mathcal{C}.$$

Below, if it can not lead to a misunderstanding, we use a shorthand

$$v_1 v_2 \ldots v_n := (v_1 \otimes v_2 \otimes \ldots \otimes v_n)_\mathcal{C}.$$

Definition 5.1. *The noncommutative analog of* $d\log(a_1) \wedge \ldots \wedge d\log(a_n)$ *is the cyclic product*

$$\{a_1, \ldots, a_n\} := da_1 \ldots da_n a_n^{-1} \ldots a_1^{-1}. \tag{73}$$

For example, the non-commutative 2-form generalizing $d\log(a) \wedge d\log(b)$ is given by:

$$\{a, b\} = da\ db\ b^{-1}a^{-1}. \tag{74}$$

Theorem 5.2. *For any elements* $a_0, \ldots, a_n$ *we have the Hochshild cocycle property:*

$$\{a_1, ..., a_n\} - \sum_{i=0}^{n-1}(-1)^i\{a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n\} + (-1)^n\{a_0, ..., a_{n-1}\} = 0. \tag{75}$$

For example:

$$\begin{aligned} &\{a\} - \{ab\} + \{b\} = 0, \\ &\{b, c\} - \{ab, c\} + \{a, bc\} - \{a, b\} = 0. \end{aligned} \tag{76}$$

Proof. Each of the following terms appears in the sum (75) twice, with the opposite signs

$$da_0 \ldots da_{i-1} da_i\ a_{i+1} da_{i+2} \ldots da_n a_n^{-1} ... a_0^{-1}.$$

□

Lemma 5.3. *i) One has*

$$\{a, b\} = -\{b^{-1}, a^{-1}\}. \tag{77}$$

ii) Suppose that $abc = 1$*. Then (78) implies that* $\{a, b\}$ *is cyclic invariant:*

$$\{a, b\} + \{b, c\} + \{c, a\} = 3 \cdot \{a, b\}. \tag{78}$$

iii) We have $\{x, 1 + x\} = \{1 + x, x\}$*.*

Proof. i) Recall the identity $c\ dc^{-1} = -dc\ c^{-1}$. Using this, we have

$$\{a, b\} = da\ db\ b^{-1}a^{-1} = a^{-1}da\ db\ b^{-1} = da^{-1}\ a\ b\ db^{-1} = -db^{-1}da^{-1}\ ab = -\{b^{-1}, a^{-1}\}.$$

ii) We have to show that

$$\{a, b\} + \{b, (ab)^{-1}\} + \{(ab)^{-1}, a\} = 3\{a, b\}. \tag{79}$$

Let us write the second identity in (76) for $c = (ab)^{-1}$:

$$\{a, b\} - \{a, a^{-1}\} + \{ab, (ab)^{-1}\} - \{b, (ab)^{-1}\} = 0. \tag{80}$$

By (77), we have $\{a, a^{-1}\} = -\{a, a^{-1}\}$. So $\{a, a^{-1}\} = 0$. So

$$\{a, b\} = \{b, (ab)^{-1}\}. \tag{81}$$

Using (77) we have: $\{a, b\} = -\{b^{-1}, a^{-1}\} \stackrel{(81)}{=} -\{a^{-1}, ab\} = \{(ab)^{-1}, a\}$. This and (81) imply the claim.

iii) One has $\{x, 1 + x\} = dxdx(1 + x)^{-1}x^{-1} = dxdxx^{-1}(1 + x)^{-1} = \{1 + x, x\}$. □

2. The non-commutative 2-form Ω_Γ for an $\mathcal{A}-$decorated bipartite ribbon graph Γ. Recall that an edge of a graph Γ is called an *external edge* if one of its vertices is univalent.

For simplicity, we do not consider graphs which have a component given by a single external edge.

We start from a trivalent ribbon graph Γ_v with a single-vertex v, decorated by a_1, a_2, a_3 in the order compatible with the cyclic structure at v, such that $a_1a_2a_3 = \pm 1$. Then we set, using notation (74):

$$\Omega_v := \{a_1, a_2\}. \tag{82}$$

It depends only on the cyclic order of the edges. Indeed, $\{a_1, a_2\} = \{a_2, a_3\} = \{a_3, a_1\}$ since $a_1a_2a_3 = \pm 1$.

Given a ribbon graph Γ_v with a single vertex v of valency > 3, we expand this vertex by adding two-valent vertices of the opposite color, as shown on Figure 21, producing a bipartite ribbon graph Γ'_v with trivalent vertices of the original color, and two-valent vertices of the opposite color. We set

$$\Omega_v := \sum_{x \in \Gamma'_v} \Omega_x. \tag{83}$$

The sum is over trivalent vertices of the graph Γ'_v. To see that it does not depend on the choice of Γ'_v, it is sufficient to check this for the graphs on Figure 21. This boils down to the cocycle condition for $\{a, b\}$:

$$\begin{aligned} &\{a_2, a_3\} + \{\alpha, a_4\} = \{a_1, a_2\} + \{\beta, a_3\} \longleftrightarrow \\ &\{a_2, a_3\} + \{a_1, a_2a_3\} = \{a_1, a_2\} + \{a_1a_2, a_3\}. \end{aligned} \tag{84}$$

Now we introduce the non-commutative 2-form assigned to an any decorated bipartite ribbon graph Γ.

Definition 5.4. *Given an $\mathcal{A}-$decorated bipartite ribbon graph Γ, we set*

$$\Omega_\Gamma := \sum_{w \in \Gamma} \Omega_w - \sum_{b \in \Gamma} \Omega_b. \tag{85}$$

Here the first sum is over all $\circ-$vertices of Γ, and the second over all $\bullet-$vertices.

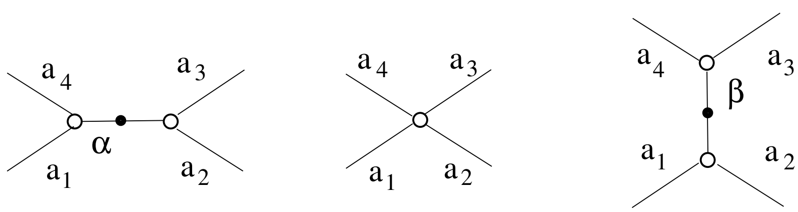


Figure 21: The cocycle condition guarantees that the 2-form Ω_Γ is well defined.

We assign to each edge E of an $\mathcal{A}-$decorated ribbon graph Γ, decorated by an element a_E, the following 3-form

$$\omega_E := (a_E^{-1} da_E)^3. \tag{86}$$

Theorem 5.5. *For an $\mathcal{A}-$decorated bipartite ribbon graph Γ, the $d\Omega_\Gamma$ is a sum is over all external edges E of Γ:*

$$d\Omega_\Gamma = \sum_E \omega_E. \tag{87}$$

The 2-form Ω_Γ assigned to an $\mathcal{A}-$decorated ribbon graph Γ without external edges is closed: $d\Omega_\Gamma = 0$.

Proof. Clearly the first claim implies the second.

Lemma 5.6. *Let* $abc = 1$. *Then*

$$3d\{a,b\} = -(a^{-1}da)^3 - (b^{-1}db)^3 - (c^{-1}dc)^3. \tag{88}$$

Proof. Straightforward calculation, using $c^{-1}dc = -d(c^{-1})c$ shows that the right hand side is

$$-(a^{-1}da)^3 - (b^{-1}db)^3 + \Big((ab)^{-1}d(ab)\Big)^3 = 3dadbd(b^{-1})a^{-1} + 3d(a^{-1})dadbb^{-1} = 3d\{a,b\}.$$

□

Therefore for each ∘−vertex w of Γ the differential of the 2-form Ω_w assigned to w is

$$d\Omega_w = -\sum_{w\in E_i}\omega_{E_i}. \tag{89}$$

Here the sum is over all edges E_i sharing the vertex w.

On the other hand, for each •−vertex b of Γ the differential of the 2-form Ω_b assigned to b is

$$d\Omega_b = \sum_{b\in E_i}\omega_{E_i}. \tag{90}$$

Here the sum is over all edges sharing the vertex b. Then evidently

$$d\Omega_\Gamma = \sum_{v\in\Gamma} d\Omega_v = -\sum_{E\subset\Gamma}\omega_E + \sum_{E\subset\Gamma}\omega_E = 0. \tag{91}$$

□

Theorem 5.7. *The 2-form Ω_Γ assigned to a decorated bipartite ribbon graph Γ is invariant under the two by two moves. This just means that the following identity holds, using the notation (69):*

$$\{a_4,a_1\} - \{a_1,a_2\} + \{a_2,a_3\} - \{a_3,a_4\} \ = \ \{b_1,b_2\} - \{b_2,b_3\} + \{b_3,b_4\} - \{b_4,b_1\}. \tag{92}$$

Proof. It is done by a tedious calculation, presented in Section 11. □

A variant. Take any ribbon graph Γ. Assign to its oriented edges $\mathbf{E}$ elements $a_{\mathbf{E}}$ such that denoting by $\overline{\mathbf{E}}$ the edge $\mathbf{E}$ with the opposite orientation, we have $a_{\mathbf{E}}a_{\overline{\mathbf{E}}} = 1$. For each vertex $v \in \Gamma$ we define Ω_v using the edges are oriented out of the vertex, and set take the sum over all vertices $v \in \Gamma$ of valency > 1:

$$\Omega_\Gamma := \sum_{v\in\Gamma}\Omega_v. \tag{93}$$

6 Spectral description of twisted decorated local systems

6.1 The spectral description

We use the same conventions about twisted local systems on $\mathbb{S}$ as in Section 3.2. We recall decorated flags, defined in Section 1.2.3.

Definition 6.1. *A* twisted decorated *R-local system on a decorated surface $\mathbb{S}$ is a twisted local system of finite dimensional $R-$vector spaces on $\mathbb{S}$ with a flat section of the associated local system of decorated flags over each marked boundary component.*

Let Γ be a $\mathrm{GL_m}$-graph on $\mathbb{S}$, homotopy equivalent to $\mathbb{S}$. Denote by Σ its spectral surface.

Theorem 6.2. *Given a* $\mathrm{GL_m}$*-graph* Γ *on* $\mathbb{S}$*, homotopy equivalent to* $\mathbb{S}$*, there is a birational equivalences of groupoids of non-commutative local systems:*

$$\begin{gathered}\{m\text{-dimensional } \underline{\text{twisted decorated}} \text{ local systems } \mathcal{V} \text{ on } \mathbb{S}\} \longleftrightarrow \\ \{\text{Flat twisted line bundles } \mathcal{L} \text{ on } \Sigma, \text{ trivialized on the boundary}\}.\end{gathered} \tag{94}$$

Proof. Equivalence (94) is obtained by applying the equivalence from Theorem 3.3 to twisted framed local systems on $\mathbb{S}$, equipped with an extra data: a decoration.[8] The description of the eigenvalue monodromy operators in Theorem 3.3 implies that the twisted flat line bundle $\mathcal{L}$ on Σ assigned to $\mathcal{V}$ has the following property:

- The restrictions of $\mathcal{L}$ to punctured discs around the marked points in $\pi^{-1}(s)$ are identified with the twisted flat line bundles $\mathrm{gr}^a\mathcal{A}$ associated to the decoration of $\mathcal{V}$ near s.

So a choice of a decoration on $\mathcal{V}$ near a marked point s is equivalent to a choice of trivializations of the restriction of $\mathcal{L}$ to the homotopy circles near the marked points $\pi^{-1}(s)$. ☐

A coordinate description. Recall that $\Gamma = \Gamma^*$ as graphs, and faces of Γ^* match zig-zags on Γ. By Theorem 4.2, the isomorphism classes of <u>twisted</u> flat line bundles on the spectral surface Σ, trivialised on the marked boundary, are described by a collection of invariants $\Delta_{\mathbf{E}} \in R^*$ at the oriented edges $\mathbf{E}$ of Γ, subject to the monomial relations (62).

Since the edges have canonical orientation $\circ \to \bullet$, given a GL_m-graph Γ on $\mathbb{S}$, we get a collection of functions $\{\Delta_E\}$ on the moduli space $\mathcal{A}_{m,\mathbb{S}}$ of twisted decorated local systems of $m-$dimensional vector spaces on $\mathbb{S}$, assigned to the edges E of Γ. They are non-commutative counterparts of the ratios of the cluster $\mathcal{A}$-coordinates [FG1], introduced in the context of GL_m-graphs Γ in [G].

Precisely, in the commutative case, for the twisted SL_m-local systems, the $\mathcal{A}$-coordinates $\{A_{\mathrm{F}}\}$ are assigned to the faces F of the graph Γ. Then $\Delta_E = A_{\mathrm{F}_+}/A_{\mathrm{F}_-}$, where F_+ and F_- are the two faces sharing the edge E, with the face F_+ on the right of the oriented $\circ \to \bullet$ edge. However the functions $\{\Delta_E\}$ do not form a coordinate system: they satisfy monomial relations (62).

In Section 6.2 we elaborate the $m = 2$ example.

6.2 A coordinate description of the non-commutative space $\mathcal{A}_{2,\mathbb{S}}$

Twisted triples of decorated flags. We define them as collections of decorated flags

$$\{\mathcal{F}_i\}, \; i \in \mathbb{Z}/6\mathbb{Z}, \; \text{ such that } \; \mathcal{F}_{i+3} = -\mathcal{F}_i \;\; \forall i \in \mathbb{Z}/6\mathbb{Z}.$$

Lemma 6.3. *A twisted triple of decorated flags* $\{\mathcal{F}_i\}$ *is the same thing as a twisted decorated local system on the triangle* t*.*

Proof. We consider t as a disc with three marked points x_1, x_2, x_3. Pick a non-zero tangent vector v_i at the point x_i, which follow the boundary orientation. There is the unique twisted local system $\mathcal{L}$ on the disc. We identify the fibers $\mathcal{L}_{v_i}$ of $\mathcal{L}$ at the vectors v_i by the parallel transport from v_1 to v_2 and to v_3 along the boundary circle, following the circle orientation, depicted as counterclockwise on the pictures.

Set $V := \mathcal{L}_{v_1}$. We assign to a triple of decorated flags $\{\mathcal{F}_i\}$ in V a pair $(\mathcal{L}, \alpha)$, where the decoration α is given by the flag $\mathcal{F}_i$ at the fibers $\mathcal{L}_{v_i}$ for $i = 1, 2, 3$, identified with V as above. ☐

A triple of decorated flags $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ gives rise to a twisted triple, setting

$$(\mathcal{A}, \mathcal{B}, \mathcal{C}) \longmapsto (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{A}', \mathcal{B}', \mathcal{C}'), \quad \mathcal{A}' = -\mathcal{A}, \; \mathcal{B}' = -\mathcal{B}, \; \mathcal{C}' = -\mathcal{C}. \tag{95}$$

[8] Note that a decorated local system is unipotent.

Describing twisted pairs of decorated flags in a two dimensional space. Let $\mathcal{A}$, $\mathcal{B}$ be two decorated flags in a two dimensional R-vector space V_2. Let

$$\begin{aligned} A &:= \mathcal{A}^1, & B &:= \mathcal{B}^1, \\ \overline{A} &:= \mathcal{A}/\mathcal{A}^1, & \overline{B} &:= \mathcal{B}/\mathcal{B}^1 \end{aligned} \tag{96}$$

be the pairs of associate graded lines of these flags. Each of the lines is equipped with a non-zero vector. We can describe decorated flags by these vectors, using the notation

$$\mathcal{A} = (a_1, a_2), \quad \mathcal{B} = (b_1, b_2). \tag{97}$$

Projecting the line A to the quotient line $\overline{B}$ we get an isomorphism of lines

$$\varphi_{a_1,b_2} : A \longrightarrow \overline{B}. \tag{98}$$

Since the lines are equipped with non-zero vectors, it is described by an element $\Delta(a_1, b_2) \in R^*$:

$$\varphi_{a_1,b_2} : a_1 \longmapsto \Delta(a_1, b_2) b_2. \tag{99}$$

The inverse map $\varphi_{b_2,a_1} : \overline{B} \longrightarrow$ A, $b_2 \longmapsto \Delta(b_2, a_1)a_1$ is described by the invariant

$$\Delta(b_2, a_1) := \Delta(a_1, b_2)^{-1}. \tag{100}$$

Generic pairs of flags $(\mathcal{A}, \mathcal{B})$ are described up to an isomorphism by the invariants $\{\Delta(a_1, b_2), \Delta(b_1, a_2)\}$.

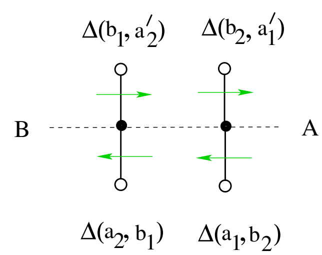


Figure 22: Invariants of a twisted pair of decorated flags $(\mathcal{A}, \mathcal{B}, -\mathcal{A}, -\mathcal{B})$ at the edges of a (disconnected) bipartite ribbon graph.

Let us consider now <u>twisted</u> pairs of decorated flags $(\mathcal{A}, \mathcal{B}, -\mathcal{A}, -\mathcal{B})$. Then the natural invariants are

$$\{\Delta(a_1, b_2), \Delta(b_1, a_2')\}, \qquad a_2' := -a_2.$$

They are invariant under the twisted cyclic shift $(\mathcal{A}, \mathcal{B}, -\mathcal{A}, -\mathcal{B}) \to (\mathcal{B}, -\mathcal{A}, -\mathcal{B}, -\mathcal{A})$.

Note that on the bipartite graph on Figure 22, consistently with (67), there are two more invariants:

$$\Delta(a_2, b_1)^{-1} = -\Delta(b_1, a_2'), \qquad \Delta(b_2, a_1') := -\Delta(a_1, b_2)^{-1}. \tag{101}$$

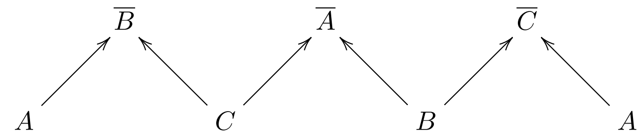
Describing twisted triples of decorated flags in a two dimensional space. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be three decorated flags in a two dimensional R-vector space V_2. Similar to (96), we set

$$\begin{aligned} A &:= \mathcal{A}^1, & B &:= \mathcal{B}^1, & C &:= \mathcal{C}^1, \\ \overline{A} &:= \mathcal{A}/\mathcal{A}^1, & \overline{B} &:= \mathcal{B}/\mathcal{B}^1, & \overline{C} &:= \mathcal{C}/\mathcal{C}^1. \end{aligned} \tag{102}$$

We describe decorated flags by the vectors in these lines, using the notation

$$\mathcal{A} = (a_1, a_2), \quad \mathcal{B} = (b_1, b_2), \quad \mathcal{C} = (c_1, c_2). \tag{103}$$

The three flags provide six such line isomorphisms, which we organize into a diagram:



(104)

We compose the six maps to a map $A \to A$, where going against an arrow means inverting the isomorphism. By Lemma 3.2 the composition is equal to $-\mathrm{Id}$. This just means that

$$\Delta(a_1, b_2)\Delta(c_1, b_2)^{-1}\Delta(c_1, a_2)\Delta(b_1, a_2)^{-1}\Delta(b_1, c_2)\Delta(a_1, c_2)^{-1} = -1. \tag{105}$$

Using the twisted triples of decorated flags (95), setting $a_1' := -a_1$, etc., and using (100), we write this as

$$\Delta(a_1, b_2)\Delta(b_2, c_1)\Delta(c_1, a_2')\Delta(a_2', b_1')\Delta(b_1', c_2')\Delta(c_2', a_1') = 1. \tag{106}$$

In fact this is just equivalent to

$$\Delta(a_1, b_2)\Delta(b_2, c_1)\Delta(c_1, a_2)\Delta(a_2, b_1)\Delta(b_1, c_2)\Delta(c_2, a_1') = 1. \tag{107}$$

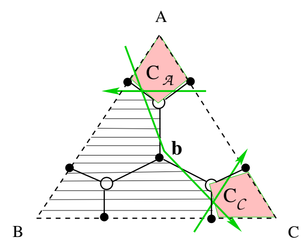


Figure 23: Sectors $C_{\mathcal{A}}$ and $C_{\mathcal{C}}$ (pink) and cosector $\widehat{C}_{\mathcal{B}}$ (shaded).

Let Γ be the GL_2-graph associated to the triangle t, see Figure 23. The lines A, B, C are assigned to the sectors $C_{\mathcal{A}}$, $C_{\mathcal{B}}$, $C_{\mathcal{C}}$. The lines $\overline{A}, \overline{B}, \overline{C}$ are assigned to the cosectors $\widehat{C}_{\mathcal{A}}$, $\widehat{C}_{\mathcal{B}}$, $\widehat{C}_{\mathcal{C}}$. The spectral surface of Γ is a hexagon h_t providing a $2:1$ cover of the triangle t, ramified at the central $\bullet$-vertex.

Using this and notation (100), we assign the coordinates to the edges of the bipartite GL_2-graph related to the counterclockwise oriented triangle on Figure 24. Then we assign coordinates to the edges of a bipartite GL_2-graph associated with an ideal triangulation of the surface, as shown on Figure 25.

The cyclic product of elements at the edges incident to a $\circ-$vertex is -1, and for the $\bullet-$vertices $(-1)^{\mathrm{val}(v)-1}$. For a $\bullet-$vertex this is the six-term identity (106), or, equivalently, (107).

The product of the two coordinates at the edges sharing a two-valent $\bullet-$vertex is -1. We shrink the edge containing a two-valent $\bullet-$vertex into a $\circ-$vertex. Then the cyclic product of the resulted coordinates at the edges of the new $\circ-$vertex is -1, as needed. The resulted coordinates for the two triangulations of a rectangle are shown on Figures 26 - 27.

In coordinates. Choose a basis (e_1, e_2) in the two dimensional R-vector space V_2. Then the three pairs of vectors $(a_1, a_2; b_1, b_2; c_1, c_2)$ are described by a 2×6 matrix with the columns given by decompositions

$$\begin{aligned} a_1 &= a_{11}e_1 + a_{12}e_2, \quad a_2 = a_{21}e_1 + a_{22}e_2; \\ b_1 &= b_{11}e_1 + b_{12}e_2, \quad b_2 = b_{21}e_1 + b_{22}e_2; \\ c_1 &= c_{11}e_1 + c_{12}e_2, \quad c_2 = c_{21}e_1 + c_{22}e_2. \end{aligned} \tag{108}$$

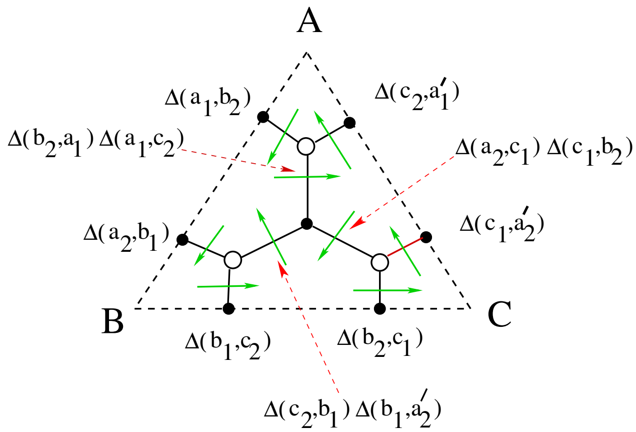


Figure 24: The $\mathcal{A}$-coordinates on the bipartite GL_2-graph assigned to a triangle. The coordinates at the side CA are assigned to the pair $(\mathcal{C}, \mathcal{A}')$ where $\mathcal{A}' := -\mathcal{A}$, and $(a_1', a_2') = (-a_1, -a_2)$. Identity (106) means that the counterclockwise product of the coordinates at the edges at the central $\bullet-$vertex is 1.

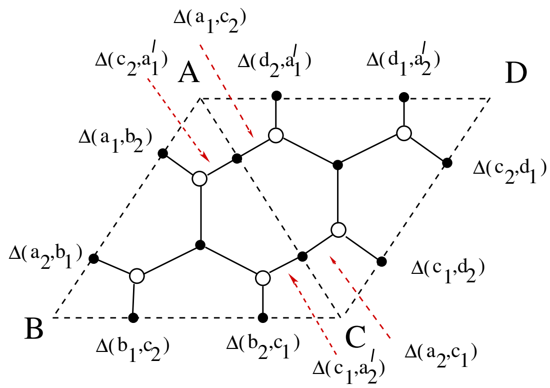


Figure 25: The $\mathcal{A}-$coordinates on the bipartite GL_2-graph associated with a triangulated rectangle.

The condition that $a_1 - \Delta(a_1, b_2)b_2 = \lambda b_1$ for some $\lambda \in R^*$ just means that

$$(a_{11}e_1 + a_{12}e_2) - \Delta(a_1, b_2)(b_{21}e_1 + b_{22}e_2) = \lambda(b_{11}e_1 + b_{12}e_2).$$

So we get a system of equations

$$\begin{cases} a_{11} - \Delta(a_1, b_2)b_{21} = \lambda b_{11}, \\ a_{12} - \Delta(a_1, b_2)b_{22} = \lambda b_{12}. \end{cases} \tag{109}$$

To solve the system, multiply the first equation from the right by b_{11}^{-1}, the second by b_{12}^{-1}, and subtract. We get the unique solution, which can be written in two slightly different ways:

$$\begin{aligned} \Delta(a_1, b_2) &= (a_{11} - a_{12}b_{12}^{-1}b_{11})(b_{21} - b_{22}b_{12}^{-1}b_{11})^{-1} = (a_1, b_1)_{11}(b_1, b_2)_{21}^{-1}; \\ \Delta(a_1, b_2) &= (a_{11}b_{11}^{-1}b_{12} - a_{12})(b_{21}b_{11}^{-1}b_{12} - b_{22})^{-1} = (a_1, b_1)_{12}(b_1, b_2)_{22}^{-1}. \end{aligned} \tag{110}$$

Here $(x_1, x_2)_{ij}$ is the quasideterminant [GR] assigned to the (ij) entry of a 2×2 matrix (x_{ij}):

$$\begin{aligned} (x_1, x_2)_{11} &:= x_{11} - x_{12}x_{22}^{-1}x_{21}, \quad (x_1, x_2)_{12} := x_{12} - x_{11}x_{21}^{-1}x_{22}. \\ (x_1, x_2)_{21} &:= x_{21} - x_{22}x_{12}^{-1}x_{11}, \quad (x_1, x_2)_{22} := x_{22} - x_{21}x_{11}^{-1}x_{12}. \end{aligned} \tag{111}$$

Remarks. 1. Equations (109) reflect the parallel transform in the twisted flat line bundle, see Remark in the end of Section 3.1.

2. If R is commutative, formulas (110) look as follows. Pick a symplectic form $\omega \in \det(V_2^*)$. Then for any two vectors $v, w \in V_2$ we have an invariant $\omega(v, w)$, and formula (110) can be written as:

$$\begin{aligned}\Delta(a_1, b_2) =& -\omega(a_1, b_1)\omega(b_1, b_2)^{-1} \\ =& \omega(b_1, a_1)\omega(b_1, b_2)^{-1}.\end{aligned} \tag{112}$$

Flips of triangulations and non-commutative Plücker relations. Given a basis (e_1, e_2) in the R-vector space V_2, the four pairs of vectors $(a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2)$ are described by a 2×8 matrix:

$$\begin{aligned}
a_1 &= a_{11}e_1 + a_{12}e_2, \quad a_2 = a_{21}e_1 + a_{22}e_2;\\
b_1 &= b_{11}e_1 + b_{12}e_2, \quad b_2 = b_{21}e_1 + b_{22}e_2;\\
c_1 &= c_{11}e_1 + c_{12}e_2, \quad c_2 = c_{21}e_1 + c_{22}e_2;\\
d_1 &= d_{11}e_1 + d_{12}e_2, \quad d_2 = d_{21}e_1 + d_{22}e_2.
\end{aligned} \tag{113}$$

Recall the Plücker identity for quasideterminants:

$$(a_1, c_1)_{11} = (a_1, b_1)_{11}(d_1, b_1)_{11}^{-1}(d_1, c_1)_{11} + (a_1, d_1)_{11}(b_1, d_1)_{11}^{-1}(b_1, c_1)_{11}. \tag{114}$$

Proposition 6.4. *There is a non-commutative Plücker relation for the elements $\Delta(*, *)$ from (110):*

$$\Delta(a_1, c_2) = \Delta(a_1, b_2)\Delta(d_1, b_2)^{-1}\Delta(d_1, c_2) + \Delta(a_1, d_2)\Delta(b_1, d_2)^{-1}\Delta(b_1, c_2). \tag{115}$$

It is equivalent to the Plücker identity for quasideterminants.

Proof. We use formula $\Delta(a_1, b_2) = (a_1, b_1)_{11}(b_1, b_2)_{21}^{-1}$ from (110). Then we have to check that

$$\begin{aligned}
(a_1, c_1)_{11}(c_1, c_2)_{21}^{-1} \overset{?}{=}& (a_1, b_1)_{11}(b_1, b_2)_{21}^{-1} \cdot (b_1, b_2)_{21}(d_1, b_1)_{11}^{-1} \cdot (d_1, c_1)_{11}(c_1, c_2)_{21}^{-1} +\\
& (a_1, d_1)_{11}(d_1, d_2)_{21}^{-1} \cdot (d_1, d_2)_{21}(b_1, d_1)_{11}^{-1} \cdot (b_1, c_1)_{11}(c_1, c_2)_{21}^{-1}\\
=& (a_1, b_1)_{11}(d_1, b_1)_{11}^{-1}(d_1, c_1)_{11}(c_1, c_2)_{21}^{-1} +\\
& (a_1, d_1)_{11}(b_1, d_1)_{11}^{-1}(b_1, c_1)_{11}(c_1, c_2)_{21}^{-1}.
\end{aligned} \tag{116}$$

Multiplying this from the right by $(c_1, c_2)_{21}$ we get precisely (114). □

Theorem 6.5. *The flip $AC \to BD$ of the triangulated rectangle $ABCD$ amounts to the following transformation of the $\mathcal{A}-$coordinates, depicted on Figures 26-27:*

$$\begin{aligned}
\overline{x} &= (1 + (zwxy)^{-1})z = z(1 + (wxyz)^{-1});\\
\overline{y} &= (1 + wxyz)^{-1}w = w(1 + xyzw)^{-1};\\
\overline{z} &= (1 + (xyzw)^{-1})x = x(1 + (yzwx)^{-1});\\
\overline{w} &= (1 + yzwx)^{-1}y = y(1 + zwxy)^{-1}.
\end{aligned} \tag{117}$$

Each of these formulas is equivalent to the non-commutative Plücker identity (115).

Proof. The clockwise products of the rhombus edge coordinates on Figure 26, starting from a $\bullet-$vertex, are given by:

$$\begin{aligned}
yzwx &= -\Delta(d_2, c_1)\Delta(c_1, b_2)\Delta(b_2, a_1)\Delta(a_1, d_2),\\
wxyz &= -\Delta(b_2, a_1)\Delta(a_1, d_2)\Delta(d_2, c_1)\Delta(c_1, b_2).
\end{aligned} \tag{118}$$

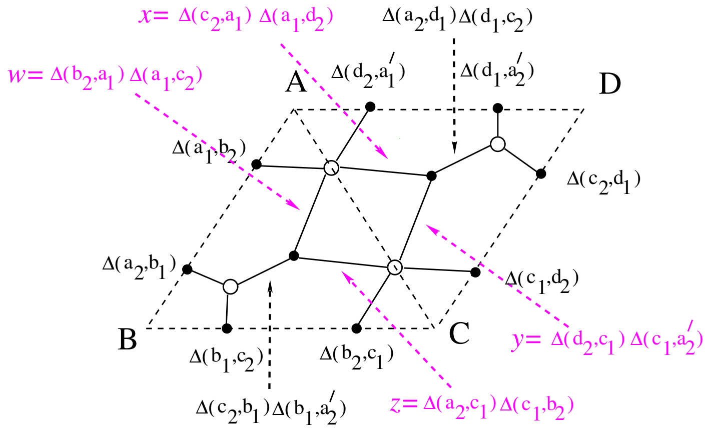


Figure 26: The $\mathcal{A}-$coordinates for the triangulation AC of counterclockwise oriented rectangle $ABCD$.

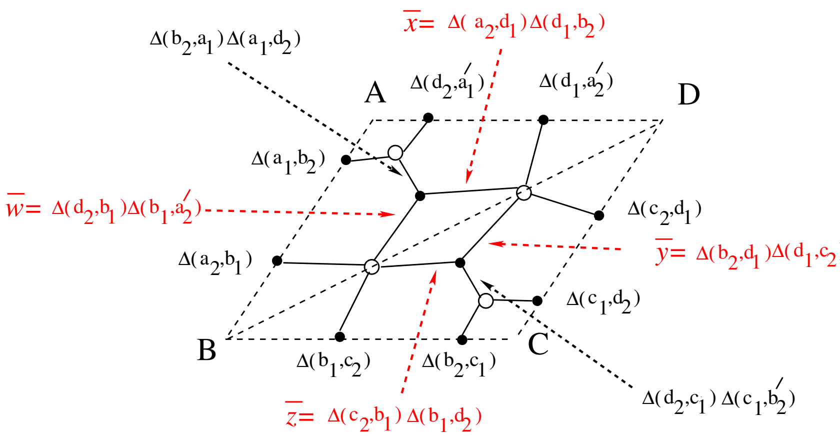


Figure 27: The $\mathcal{A}-$coordinates for the triangulation BD of counterclockwise oriented rectangle $ABCD$. We start with a configuration of 4 decorated flags $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. It induces triples $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $(\mathcal{B}, \mathcal{C}, \mathcal{D})$, which we promote to twisted configurations, and introduce the coordinates as on Figure 24.

Clockwise products of rhombus edge coordinates on Figure 27, starting from a $\bullet-$vertex, are given by:

$$\begin{aligned}\overline{x}\ \overline{y}\ \overline{z}\ \overline{w} &= -\Delta(a_2, d_1)\Delta(d_1, c_2)\Delta(c_2, b_1)\Delta(b_1, a_2),\\ \overline{z}\ \overline{w}\ \overline{x}\ \overline{y} &= -\Delta(c_2, b_1)\Delta(b_1, a_2)\Delta(a_2, d_1)\Delta(d_1, c_2).\end{aligned} \tag{119}$$

Now the proof is deduced easily from the following Lemma.

Lemma 6.6. *We have:*

$$\begin{aligned}y\overline{w}^{-1} &= 1 + \Delta(d_2, c_1)\Delta(c_1, b_2)\Delta(b_2, a_1)\Delta(a_1, d_2) = 1 + yzwx;\\ w\overline{y}^{-1} &= 1 + \Delta(b_2, a_1)\Delta(a_1, d_2)\Delta(d_2, c_1)\Delta(c_1, b_2) = 1 + wxyz;\\ z^{-1}\overline{x} &= 1 + \Delta(b_2, c_1)\Delta(c_1, d_2)\Delta(d_2, a_1)\Delta(a_1, b_2) = 1 + (wxyz)^{-1};\\ x^{-1}\overline{z} &= 1 + \Delta(d_2, a_1)\Delta(a_1, b_2)\Delta(b_2, c_1)\Delta(c_1, d_2) = 1 + (yzwx)^{-1}.\end{aligned} \tag{120}$$

Proof. It is easy to see that we have the following:

$$\begin{aligned} w\overline{y}^{-1} &= \Delta(b_2, a_1)\Delta(a_1, c_2)\Delta(c_2, d_1)\Delta(d_1, b_2), \\ z^{-1}\overline{x} &= \Delta(b_2, c_1)\Delta(c_1, a_2)\Delta(a_2, d_1)\Delta(d_1, b_2), \end{aligned} \tag{121}$$

and similar two identities obtained by the cyclic shift by two: $(a, b, c, d) \longmapsto (c, d, a, b)$.

Substituting the non-commutative Plücker identity (115) to formula (121) for $w\overline{y}^{-1}$ we get

$$\begin{aligned} w\overline{y}^{-1} &= \Delta(b_2, a_1)\Big(\Delta(a_1, b_2)\Delta(b_2, d_1)\Delta(d_1, c_2) + \Delta(a_1, d_2)\Delta(d_2, b_1)\Delta(b_1, c_2)\Big)\Delta(c_2, d_1)\Delta(d_1, b_2) \\ &= 1 + \Delta(b_2, a_1)\Delta(a_1, d_2)\Delta(d_2, b_1) \cdot \Delta(b_1, c_2)\Delta(c_2, d_1)\Delta(d_1, b_2) \\ &= 1 + \Delta(b_2, a_1)\Delta(a_1, d_2)\Delta(d_2, b_1) \cdot \Delta(b_1, d_2)\Delta(d_2, c_1)\Delta(c_1, b_2) \\ &= 1 - \Delta(b_2, a_1)\Delta(a_1, d_2)\Delta(d_2, c_1)\Delta(c_1, b_2) \\ &= 1 + wxyz. \end{aligned} \tag{122}$$

Here to get the third equality we use the following identity, equivalent to (105), combined with (100):

$$\Delta(b_1, c_2)\Delta(c_2, d_1)\Delta(d_1, b_2) = -\Delta(b_1, d_2)\Delta(d_2, c_1)\Delta(c_1, b_2). \tag{123}$$

The identity for $y\overline{w}^{-1}$ is obtained from this by the cyclic shift by two.

Similarly, to get formula formula (121) for $z^{-1}\overline{x}$ we need the non-commutative Plücker identity

$$\Delta(c_1, a_2) = \Delta(c_1, d_2)\Delta(d_2, b_1)\Delta(b_1, a_2) + \Delta(c_1, b_2)\Delta(b_2, d_1)\Delta(d_1, a_2). \tag{124}$$

Substituting identity (124) to formula (121) for $z^{-1}\overline{x}$, we get

$$\begin{aligned} z^{-1}\overline{x} &= \Delta(b_2, c_1)\Delta(c_1, a_2)\Delta(a_2, d_1)\Delta(d_1, b_2) \\ &\overset{(124)}{=} \Delta(b_2, c_1)\Big(\Delta(c_1, d_2)\Delta(d_2, b_1)\Delta(b_1, a_2) + \Delta(c_1, b_2)\Delta(b_2, d_1)\Delta(d_1, a_2)\Big)\Delta(a_2, d_1)\Delta(d_1, b_2) \\ &= \Delta(b_2, c_1)\Delta(c_1, d_2)\Delta(d_2, b_1) \cdot \Delta(b_1, a_2)\Delta(a_2, d_1)\Delta(d_1, b_2) + 1 \\ &= 1 - \Delta(b_2, c_1)\Delta(c_1, d_2)\Delta(d_2, b_1) \cdot \Delta(b_1, d_2)\Delta(d_2, a_1)\Delta(a_1, b_2) \\ &= 1 - \Delta(b_2, c_1)\Delta(c_1, d_2)\Delta(d_2, a_1)\Delta(a_1, b_2) \\ &\overset{(118)}{=} 1 + (wxyz)^{-1}. \end{aligned} \tag{125}$$

Here to get the fourth equality we use the identity similar to (123). □

□

Remark. Formulas (117) imply that the monodromies starting at the •−vertex are related as follows:

$$\overline{x}\ \overline{y}\ \overline{z}\ \overline{w} = (zwxy)^{-1}. \tag{126}$$

Indeed, the left hand side is equal to

$$\begin{aligned} &z(1 + (wxyz)^{-1})(1 + wxyz)^{-1}wx(1 + (yzwx)^{-1})(1 + yzwx)^{-1}y \\ =&z \cdot (wxyz)^{-1} \cdot wx \cdot (yzwx)^{-1} \cdot y = (zwxy)^{-1}. \end{aligned} \tag{127}$$

It is also clear from Figures 26 - 27 that each side of (126) is equal to $\Delta(a_2, d_1)\Delta(d_1, c_2)\Delta(c_2, b_1)\Delta(b_1, a_2)$.

6.3 Gluing spectral surfaces for ideal bipartite GL_2−graphs from hexagons

1. The spectrtal surface for the triangle. Let Γ be the GL_2-graph associated to the triangle t, see Figure 23. The spectral surface of Γ is a hexagon h_t providing a $2:1$ cover of the triangle t, ramified at the central •-vertex. The map $h_t \to t$ can be realized by the map $z \to z^2$ sending a regular hexagon in the complex plane z with the vertices at the sixth roots of unity onto a triangle with vertices at the cubic roots of unity. The vertices of the hexagon h_t are labeled by elements of the set $\{1, 2\}$, so that going around the hexagon the labels alternate. For each vertex v of t there is a bijection

$$\{\text{The vertices of the hexagon } h_t \text{ over } v\} \stackrel{\sim}{\longrightarrow} \{1, 2\}.$$

The sides of the hexagon inherit a "$1 \to 2$" orientation.

2. Gluing the spectral surface from hexagons. Let $\mathcal{T}$ be an ideal triangulation of $\mathbb{S}$. For each triangle t of $\mathcal{T}$ choose a hexagon h_t over t with the vertices labeled as above. Let us glue the hexagons h_t into a complete smooth topological surface $\mathbf{\Sigma}$. Given an edge E of $\mathcal{T}$, there are two triangles t and t' sharing E. Let h and h' be the hexagons covering these triangles. We identify the vertices of the hexagons with the same labels, projecting onto the same vertex of $\mathcal{T}$. Then we glue two edges whose vertices are identified. For example, gluing two hexagons assigned to a triangulation of a rectangle we get an annulus. Denote by $\mathbf{S}$ a complete surface obtained by filling the punctures on $\mathbb{S}$. Gluing hexagons h_t we get a surface $\mathbf{\Sigma}$ with the following properties:

- There is a $2:1$ map $\pi : \mathbf{\Sigma} \to \mathbf{S}$ with one ramification point in every triangle of $\mathcal{T}$.
- For every marked point $s \in \mathbf{S}$ there is a bijection $\pi^{-1}(s) \longrightarrow \{1, 2\}$. Moving around a triangle of $\mathcal{T}$ we get a permutation $\{1, 2\} \longrightarrow \{2, 1\}$.

The spectral surface Σ is obtained by deleting the vertices of the triangles from $\mathbf{\Sigma}$.

3. The twisted decorated flat line bundle on Σ. Let $\mathcal{V}$ be a twisted decorated local system of two dimensional R-vector spaces over $\mathbb{S}$. So there is an invariant decorated flag $\mathcal{A}_s$ near each marked point s. For every ideal triangle t there is a triple of decorated flags $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ assigned to the vertices of t. We assign the six trivialised lines (102) to the vertices of the hexagon h_t so that going around the hexagon we get the lines $(A, \overline{B}, C, \overline{A}, B, \overline{C})$. The map $h_t \to t$ sends the vertices labeled by $A, \overline{A}$ to the vertex a, and so on, see Figure 28. We get a twisted flat line bundle $\mathcal{L}_{h_t}$ over the hexagon, trivialised at the tangent vectors to the boundary near the vertices: the parallel transports along the oriented edges are defined by the maps (98) / their inverses. Given an ideal triangulation of $\mathbb{S}$, and gluing the hexagons h_t into the

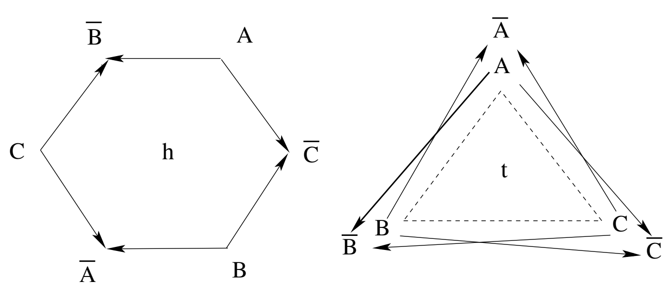


Figure 28: The twisted flat line bundle on a hexagon assigned to a triple of flags.

surface Σ, the twisted flat line bundles $\mathcal{L}_{h_t}$ are glued into a twisted flat line bundle $\mathcal{L}$ on Σ, trivialized near the marked points.

We construct $\mathcal{A}-$coordinates by assigning to each edge E of the hexagonal tesselation of Σ a function Δ_E on $\mathcal{A}_{2,\mathbb{S}}$. Let A and $\overline{B}$ be the lines over the vertices of E, trivialized by non-zero vectors $a_1 \in A$ and $b_2 \in \overline{B}$. The parallel transport $\varphi_E : A \to \overline{B}$ along the E acts by $\varphi_E(a_1) = \Delta_E b_2$. For every hexagon h_t, the alternated product of the six functions Δ_E assigned to its consecutive edges is equal to -1. There are no more relations between the functions $\{\Delta_E\}$.

7 Non-commutative cluster Poisson varieties from bipartite ribbon graphs

Definition 7.2 for the action of two by two moves on flat line bundles is dictated by the example provided by configurations of four lines in a two dimensional space given in Section 7.1.

7.1 Two by two moves for non-commutative flat line bundles

1. Defining two by two moves.

Definition 7.1. *Let Γ be a bipartite ribbon graph containing a subgraph α on the left of Figure 29. A* two by two move $\mu_\alpha : \Gamma \to \Gamma'$ at α *produces a new bipartite graph Γ', obtained by replacing α by the subgraph α' on the right of Figure 29.*

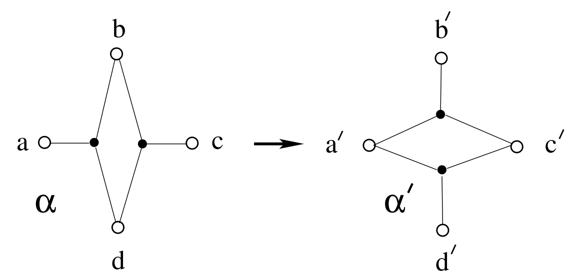


Figure 29: A two by two move $\alpha \to \alpha'$.

Given a two by two move $\mu_\alpha : \Gamma \to \Gamma'$, let us define a birational isomorphism of the moduli spaces of non-commutative flat line bundles on the graphs:

$$\mu_\alpha : \mathrm{Loc}_1(\Gamma) \longrightarrow \mathrm{Loc}_1(\Gamma'). \tag{128}$$

Consider a two by two move at α shown on Figure 29:

$$\mu_\alpha : \Gamma \to \Gamma'.$$

Let α_0 be the graph α without the $\circ$-vertices, and similarly α_0'. By the definition, $\Gamma - \alpha_0 = \Gamma' - \alpha_0'$. Denote by $b \bullet a$ the arc t on the graph α going from the $\circ$-vertex a to the $\bullet$-vertex, and then to the $\circ$-vertex b - see Figure 30. We use a similar notation for the graph α'.

Starting from a flat line bundle L on Γ, we produce a new flat line bundle L' on Γ' as follows:

Let $M_v : L_v \to L_v$ be the minus of the operator of counterclockwise monodromy around the simple loop on L_α, acting at the fiber at v, see Figure 31.

- The restriction of L' to $\Gamma' - \alpha_0'$ is defined as the restriction of L to $\Gamma - \alpha_0$.
- The restriction of L' to α', denoted by $L_{\alpha'}$, is defined in Definition 7.2.

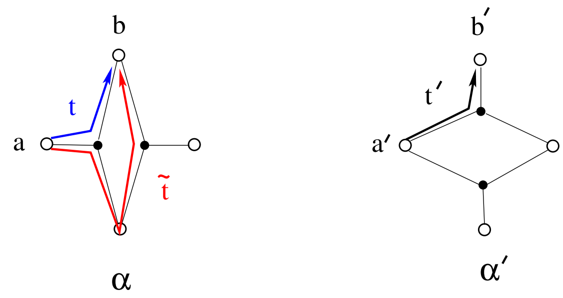


Figure 30: The two by two move action on the parallel transform: $t' := t + \widetilde{t} = (1 + M_b)t$.

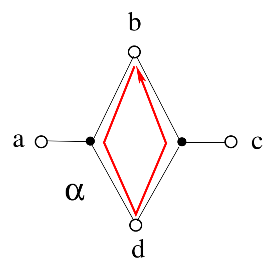


Figure 31: M_b is the negative of the counterclockwise monodromy around the loop α.

Definition 7.2. *Given a flat line bundle L_α on the graph α, we define a flat line bundle $L_{\alpha'}$ on the graph α' by setting the parallel transports along the arcs between the $\circ$-vertices on α' to be:*

$$\begin{aligned} t_{b'\bullet a'} &:= (1 + M_b)t_{b\bullet a}, \quad t_{c'\bullet b'} := t_{c\bullet b}(1 + M_b)^{-1}, \\ t_{d'\bullet c'} &:= (1 + M_d)t_{d\bullet c}, \quad t_{a'\bullet d'} := t_{a\bullet d}(1 + M_d)^{-1}. \end{aligned} \tag{129}$$

We glue the flat line bundle $L_{\alpha'}$ to the restriction of the flat line bundle L to $\Gamma - \alpha_0$.

Let $\widetilde{t}_{b,a}$ be "the other way around α" parallel transport from a to b, see Figure 30. Then

$$t_{b'\bullet a'} \stackrel{(129)}{=} (1 + M_b)t_{b\bullet a} = t_{b\bullet a}(1 + M_a) = t_{b\bullet a} + \widetilde{t}_{b,a}. \tag{130}$$

It is a sum of the two different simple paths from a to b on α.

Lemma 7.3. *Monodromies of the local systems L_α and $L_{\alpha'}$ at each of the four $\circ$-points coinside:*

$$M_a = M'_a, \quad M_b = M'_b, \quad M_c = M'_c, \quad M_d = M'_d.$$

Proof. We have

$$\begin{aligned} &M'_a = t_{a'\bullet d'}t_{d'\bullet c'}t_{c'\bullet b'}t_{b'\bullet a'} = \\ &t_{a\bullet d}(1 + M_d)^{-1}(1 + M_d)t_{d\bullet c}t_{c\bullet b}(1 + M_b)^{-1}(1 + M_b)t_{b\bullet a} = \\ &t_{a\bullet d}t_{d\bullet c}t_{c\bullet b}t_{b\bullet a} = M_a. \end{aligned} \tag{131}$$

Next, using $M_a = M'_a$ we have

$$M'_b = t_{b'\bullet a'}M'_a t^{-1}_{b'\bullet a'} \stackrel{(129)}{=} t_{b\bullet a}(1 + M_a)M_a t^{-1}_{b'\bullet a'} = t_{b\bullet a}M_a(1 + M_a)t^{-1}_{b'\bullet a'}. \tag{132}$$

The first two equalities in (130) imply:

$$t^{-1}_{b'\bullet a'} \stackrel{(130)}{=} (t_{b\bullet a}(1 + M_a))^{-1} = (1 + M_a)^{-1}t^{-1}_{b\bullet a}.$$

Substituting this to (132), we get:

$$M_{b'} \overset{(132)}{=} t_{b\bullet a} M_a (1+M_a)(1+M_a)^{-1} t^{-1}_{b\bullet a} = t_{b\bullet a} M_a t^{-1}_{b\bullet a} = M_b.$$

The last two equalities follow from the cyclic shift by two symmetry of the picture.

Corollary 7.4. *i) We can rewrite Definition 7.2 as follows:*

$$\begin{aligned} t_{b'\bullet a'} &:= (1+M_b) t_{b\bullet a}, \quad (1+M_{c'}) t_{c'\bullet b'} := t_{c\bullet b}, \\ t_{d'\bullet c'} &:= (1+M_d) t_{d\bullet c}, \quad (1+M_{a'}) t_{a'\bullet d'} := t_{a\bullet d}. \end{aligned} \tag{133}$$

ii) One has $\mu_{\alpha'} \circ \mu_\alpha = \mathrm{Id}$.

Proof. i) Indeed, using Lemma 7.3,

$$t_{c'\bullet b'} = t_{c\bullet b}(1+M_b)^{-1} = t_{c\bullet b}(1+M_{b'})^{-1} \quad \Longrightarrow \quad t_{c'\bullet b'}(1+M_{b'}) = t_{c\bullet b} \quad \Longrightarrow \quad (1+M_{c'}) t_{c'\bullet b'} = t_{c\bullet b}.$$

The identity $(1+M_{a'}) t_{a'\bullet d'} := t_{a\bullet d}$ follows by the cyclic shift by two symmetry.

ii) Since the cyclic shift by one of α' is isomorphic to α, this implies $\mu_{\alpha'} \circ \mu_\alpha = \mathrm{Id}$. □

2. Configurations of lines and two by two moves. To prove that transformations (128) satisfy the pentagon relations we need the following comparison result. Our construction from Section 3.1 provides a birational equivalence of groupoids

$$\begin{gathered} \{\text{Quadruples of lines in a 2-dimensional } R\text{-vector space } V, \text{ at the } \circ\text{-vertices of } \alpha\} \\ \longleftrightarrow \{\text{Flat } R\text{-line bundles on the graph } \alpha\}. \end{gathered} \tag{134}$$

Proposition 7.5. *The transformation in Definition 7.2 is characterised by the condition that it commutes with equivalence (134), providing a commutative diagram*

$$\begin{array}{ccc} \{\textit{Line bundles } \mathcal{L} \textit{ on } \alpha\} & \xrightarrow{\textit{Definition 7.2}} & \{\textit{Line bundle } \mathcal{L}' \textit{ on } \alpha'\} \\ & {\scriptstyle (134)}\searrow \quad \swarrow{\scriptstyle (134)} & \\ & (A,B,C,D) \subset V & \end{array} \tag{135}$$

Proof. We start with a configuration of four generic lines (A,B,C,D) in V. We assign them to the $\circ$-vertices (a,b,c,d) of the graph α, and to the $\circ$-vertices (a',b',c',d') of the graph α'.

We abuse notation by denoting by the same letter a a vector in the line A, etc.

The two $\bullet$-vertices provide us five vectors in the lines (A,B,C,D), defined uniquely up to a common left R^*-factor by the condition that the sum of the three vectors at the $\circ$-vertices incident to any $\bullet$-vertex is zero. These are the vectors illustrated on the left of Figure 32:

$$(a,b,b_1,c,d), \qquad \text{So } b_1 = Xb, \quad X \in R^*.$$

The counterclockwise monodromy around α acts on any of them by the left multiplication by X. Indeed,

$$t_{a\bullet b}(b) = a, \quad t_{d\bullet a}(a) = d, \quad t_{c\bullet d}(d) = c, \quad t_{b_1\bullet c}(c) = b_1 = Xb. \tag{136}$$

There are similar five vectors in V related to the graph α', illustrated on the right in Figure 32:

$$(a',b',b'_1,c',d'), \quad \text{So } b'_1 = Yb', \quad Y \in R^*.$$

The counterclockwise monodromy around α' acts on them as the left multiplication by Y.

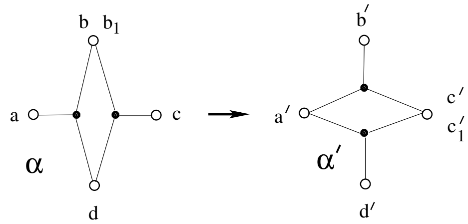


Figure 32: Calculating the two by two move for twisted configurations of lines in V_2.

One summarises definition of these vectors by the following relations, one per each $\bullet$-vertex:

$$a+b+d=0, \quad d+c+b_1=0; \qquad a'+b'+c'=0, \quad a'+d'+c'_1=0. \tag{137}$$

Notice that $b_1 = Xb$ and $c'_1 = Yc'$. So the equations read now as

$$a+b+d=0, \quad d+c+Xb=0; \qquad a'+b'+c'=0, \quad a'+d'+Yc'=0.$$

These vectors are related as follows.

Lemma 7.6. *One has*

$$a' = a, \qquad b' = (1-X)b, \qquad c' = -c, \qquad d' = (1-X^{-1})d. \tag{138}$$

Proof. Each pair of the vectors $(a', a), \ldots, (d', d)$ is collinear, so $a' = \lambda_a a$, $b' = \lambda_b b$, $c' = \lambda_c c$, $d' = \lambda_d d$. Let us solve these equations. First, let us express c, d via a, b, and respectively c', d' via a', b':

$$d = -a - b, \quad c = (1-X)b + a, \quad c' = -a' - b', \quad d' = -(1-Y)a' + Yb'.$$

Next, the identities $\lambda_c c = c'$ and $\lambda_d d = d'$ give:

$$\lambda_c((1-X)b + a) = -\lambda_a a - \lambda_b b, \qquad \lambda_d(a+b) = (1-Y)a - Y\lambda_b b.$$

Set $\lambda_a = 1$. Then we get the following equations:

$$\lambda_b = 1-X, \quad \lambda_c = -1, \quad \lambda_d = 1-Y, \quad -Y\lambda_b = \lambda_d.$$

From the first and last equations we see that $Y = X^{-1}$, and therefore we finally get

$$\lambda_a = 1, \quad \lambda_b = 1-X, \quad \lambda_c = -1, \quad \lambda_d = 1-X^{-1}.$$

□

Lemma 7.6 together with (136) implies

$$\begin{aligned} t_{b'\bullet a'} &= (1-X)t_{b\bullet a}, \quad t_{c'\bullet b'} = (1-X^{-1})^{-1}t_{c\bullet b_1}, \\ t_{d'\bullet c'_1} &= (1-X)t_{d\bullet c}, \quad t_{a'\bullet d'} = (1-X^{-1})^{-1}t_{a\bullet d}. \end{aligned} \tag{139}$$

Recall that M_a is the *negative* of the counterclockwise monodromy around α.

Lemma 7.7. *Formulas (139) are equivalent to formulas (129).*

Proof. The first line in (129) is equivalent to the second one by the cyclic shift by two argument. The right formula in each line in (129) is equivalent to the left one, applied to the two by two move $\mu_{\alpha'} : \Gamma' \to \Gamma$. So we need to check only one of the four formulas in (129).

In the top left formula in (139) the variable X, by its definition, is the action of the counterclockwise monodromy along α on the vector b. So it can be written as $t_{b'\bullet a'} = (1+M_b)t_{b\bullet a}$, which is just the top left formula in (129). □

Proposition 7.5 is proved □

3. The pentagon relation.

Theorem 7.8. *The fifth power of "two by two move followed by the non-trivial symmetry of the triangulated pentagon" for the two by two moves on Figure 33 acts as the identity on flat line bundles.*

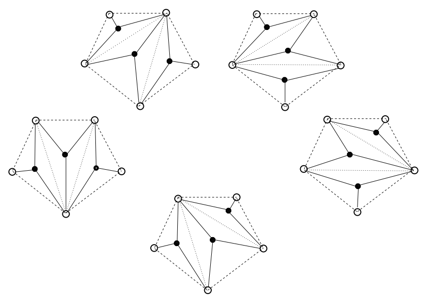


Figure 33: The pentagon relation.

Proof. Let $\{\Gamma_i\}$, where $i \in \mathbb{Z}/5\mathbb{Z}$, be the sequence of bipartite graphs on Figure 33. They are assigned to triangulations of the pentagon. The two by two move $\Gamma_i \to \Gamma_{i+1}$ corresponds to a flip of the triangulation. For each of the graphs Γ_i there is an equivalence of groupoids

$$\{\text{Generic flat line bundles on the graph } \Gamma_i\} \longleftrightarrow \tag{140}$$

$$\{\text{Generic 5-tuples of lines in a two dimensional space } V, \text{ assigned to } \circ\text{-vertices of a pentagon.}\}$$

Thanks to Lemma 7.7 the mutation of flat line bundles corresponding to a two by two move intertwines these equivalences. Indeed, a two by two move corresponding to a flip of a triangulation of the pentagon keeps intact one of the triangles of the triangulation, and does not affect the restriction of the flat line bundle to the graph inside of that triangle. □

7.2 Poisson algebra related to a bipartite ribbon graph

In Section 7.2 we define a non-commutative analog of the Poisson algebra assigned to a bipartite ribbon graph in [GK]. The output is still a *commutative* Poisson algebra. We prove that two by two moves give rise to homomorphisms of Poisson algebras.

1. The algebra $\mathcal{O}_\Gamma$. Let Γ be a graph. An oriented loop α on Γ is a continuous map $\alpha : S^1 \to \Gamma$ of an oriented circle. A vertex of the loop α is a point $v \in S^1$ which is mapped to a vertex of Γ.

Denote by l_Γ the set of all oriented loops on Γ, considered modulo homotopy. Let

$$L_\Gamma := \mathbb{Z}[l_\Gamma]$$

be the free abelian group generated by the set l_Γ. Let $\mathcal{O}_\Gamma$ be its symmetric algebra:

$$\mathcal{O}_\Gamma := S^*(L_\Gamma). \tag{141}$$

Let $\mathrm{Loc}_{N,\mathbb{C}}(\Gamma)$ be the moduli space of all N-dimensional complex vector bundles with connections on Γ. Denote by $\mathcal{O}(\mathrm{Loc}_{N,\mathbb{C}}(\Gamma))$ the algebra of regular functions on this space. There is a canonical map of commutative algebras

$$r : \mathcal{O}_\Gamma \longrightarrow \mathcal{O}(\mathrm{Loc}_{N,\mathbb{C}}(\Gamma)). \tag{142}$$

Namely, a loop α on Γ give rise to a function M_α on $\mathrm{Loc}_{N,\mathbb{C}}(\Gamma)$ given by the $1/N$ times the trace of the monodromy along the loop α. Homotopic loops give the same function. Since $\mathcal{O}_\Gamma$ is the free commutative algebra generated by the set l_Γ, the map $\alpha \longmapsto M_\alpha$ extends uniquely to a homomorphism (142). The trivial loop maps to the unit.

2. The Goldman bracket for a ribbon graph Γ. Let us define a Lie bracket $\{*,*\}_{\rm G}$ on L_Γ.

A valency m vertex v of a ribbon graph Γ gives rise to a free abelian group $\mathbb{A}_v$ of rank $m-1$, given by $\mathbb{Z}$-linear combinations of the oriented out of v edges $\overrightarrow{E_i}$ at v, of total degree zero:

$$\mathbb{A}_v = \{\sum n_i \overrightarrow{E_i} \ \mid \ \sum n_i = 0, \ n_i \in \mathbb{Z}\}.$$

It is generated by the "oriented paths" $\overrightarrow{E_i} - \overrightarrow{E_j}$, where $-\overrightarrow{E_j}$ is the edge E_j oriented towards v.

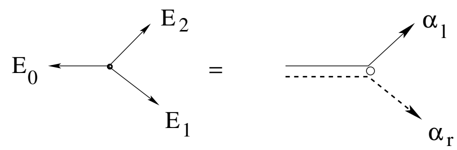


Figure 34: One has $\delta_v(\alpha_r, \alpha_l) = \frac{1}{2}$.

Lemma 7.9. *Given a vertex v of Γ, there is a unique skew symmetric bilinear form*

$$\delta_v : \mathbb{A}_v \wedge \mathbb{A}_v \longrightarrow \frac{1}{2}\mathbb{Z}$$

such that for any triple E_0, E_1, E_2 of edges as in Figure 34 one has

$$\delta_v(\alpha_r \wedge \alpha_l) = \frac{1}{2}, \qquad \alpha_r := \overrightarrow{E_1} - \overrightarrow{E_0}, \quad \alpha_l := \overrightarrow{E_2} - \overrightarrow{E_0} \ . \tag{143}$$

Take two loops α and β on a ribbon graph Γ. Let $\alpha_v \in \mathbb{A}_v$ be the element induced by the loop α at the vertex v. Set $\delta_v(\alpha, \beta) := \delta_v(\alpha_v, \beta_v)$. Given a pair (v, w), where v is a vertex of α, w is a vertex of β, and $\alpha(v) = \beta(w)$, consider a new loop $\alpha \circ_{v=w} \beta$ obtained by starting at the vertex $\beta(w)$, going around the loop β, and then around the loop α. We define a bracket $\{\alpha, \beta\}_{\rm G}$ by taking the sum over all such pairs of vertices (v, w):

$$\{\alpha, \beta\}_{\rm G} := \sum_{v=w} \delta_v(\alpha, \beta) \cdot \alpha \circ_{v=w} \beta.$$

This is a ribbon graph version of the Goldman bracket. It is easy to check the following.

Theorem 7.10. *The bracket $\{*,*\}_{\rm G}$ provides L_Γ with a Lie algebra structure. So it extends via the Leibniz rule to a Poisson bracket $\{*,*\}_{\rm G}$ on the commutative algebra $\mathcal{O}_\Gamma$. The map (142) is a homomorphism of Poisson algebras.*

3. The Lie bracket $\{*,*\}$ on the space L_Γ for a bipartite ribbon graph Γ. Let Γ be a bipartite ribbon graph. We apply the previous construction to the conjugate graph Γ^*.

Definition 7.11. *i) The Lie bracket $\{*,*\}$ on L_Γ is induced, via the isomorphism $L_\Gamma = L_{\Gamma^*}$, by the Lie bracket on L_{Γ^*}, for the conjugate graph Γ^*.*

ii) It induces, via the Leibniz rule, a Poisson algebra structure $\{*,*\}$ on $\mathcal{O}_\Gamma$.

Let α, β be two oriented loops on a bipartite surface graph Γ. Then

$$\{\alpha, \beta\} = \sum_v {\rm sgn}(v) \delta_v(\alpha, \beta) \cdot \alpha \circ_v \beta.$$

The sum is over all vertices $v \in \alpha \cap \beta$, and ${\rm sgn}(v) = 1$ for the $\circ$-vertex, and -1 for the $\bullet$-vertex.

To calculate the bracket $\{\alpha, \beta\}$ between two loops α, β on Γ we have the following two cases:

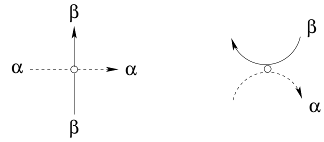


Figure 35: On the left: $\delta_v(\alpha,\beta)=1$. On the right: $\delta_v(\alpha,\beta)=0$.

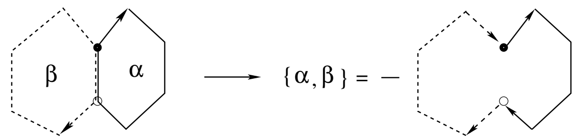


Figure 36: One has $\{\alpha,\beta\}=-\alpha\circ\beta$, where $\alpha\circ\beta$ is the loop on the right.

- A vertex $v\in\alpha\cap\beta$ is an isolated intersection point. Then its contribution is $\pm1,0$ depending on the geometry of the intersection, as depicted on Figure 35.
- A vertex $v\in\alpha\cap\beta$ lies on an edge $vw\subset\alpha\cap\beta$. Then its contribution is $\pm1,0$ depending on the geometry of the intersection, as depicted on Figures 36, 37.

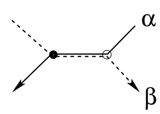


Figure 37: The total contribution of the $\circ$ and $\bullet$-vertices to $\{\alpha,\beta\}$ is zero.

Changing the orientation of the surface amounts to changing the sign of the Poisson bracket. Interchanging black and white we change the sign of the Poisson bracket.

Theorem 7.12. *Let $\Gamma\to\Gamma'$ be a two by two move of bipartite ribbon graphs. Then the map induced by Definition 7.2 is a Lie algebra homomorphism of completed Lie algebras:*

$$\mu_{\Gamma\to\Gamma'}:\widehat{L}_\Gamma\longrightarrow\widehat{L}_{\Gamma'}.$$

Proof. Denote by $\gamma_{b\bullet a}$ a loop on Γ containing the segment $t_{b\bullet a}$, etc. Let us check the following:

$$\begin{aligned}&\{\gamma_{b'\bullet a'},\gamma_{d'\bullet c'}\}=\{\gamma_{b\bullet a},\gamma_{d\bullet c}\},\\&\{\gamma_{b'\bullet a'},(1+M_{c'})\gamma_{c'\bullet b'}\}=\{\gamma_{b\bullet a},(1+M_c)\gamma_{c\bullet b}\},\\&\{\gamma_{b'\bullet a'},\gamma_{c'\bullet b'}\}=\{\gamma_{b\bullet a},\gamma_{c\bullet b}\}.\end{aligned}\tag{144}$$

i) We have

$$\{\gamma_{b'\bullet a'},\gamma_{d'\bullet c'}\}=\{(1+M_b)\gamma_{b\bullet a},(1+M_d)\gamma_{d\bullet c}\}\overset{\text{Fig }38}{=}\{\gamma_{b\bullet a},\gamma_{d\bullet c}\}.$$

ii) The second boils down to

$$\begin{aligned}&\{(1+M_b)\gamma_{b\bullet a},\gamma_{c\bullet b}\}\overset{\text{Fig }39}{=}\{\gamma_{b\bullet a},(1+M_c)\gamma_{c\bullet b}\}.\\&\{(1+M_b)\gamma_{b\bullet a},(1+M_c)^{-1}\gamma_{c\bullet b}\}\overset{\text{Fig }39}{=}\{\gamma_{b\bullet a},\gamma_{c\bullet b}\}.\end{aligned}\tag{145}$$

□

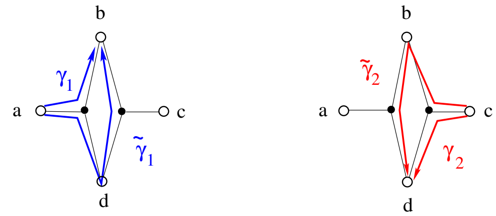


Figure 38: The contributions of the edge $\bullet b$ to $\{\gamma_1, \widetilde{\gamma}_2\}$ and $\{\widetilde{\gamma}_1, \gamma_2\}$ are cancelled. Therefore $\{\gamma_1, \widetilde{\gamma}_2\} + \{\widetilde{\gamma}_1, \gamma_2\} = 0$, and thus $\{\gamma_1 + \widetilde{\gamma}_1, \gamma_2 + \widetilde{\gamma}_2\} = \{\gamma_1, \gamma_2\}$.

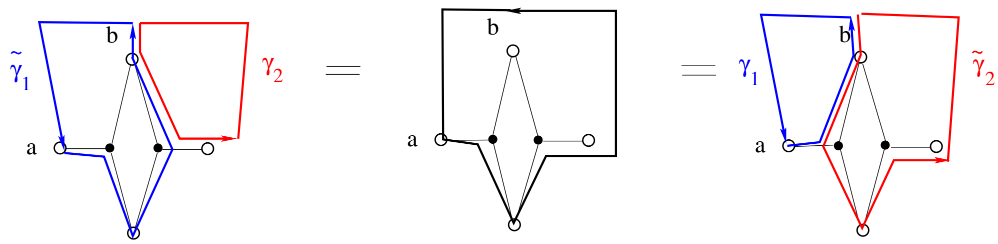


Figure 39: $\{\widetilde{\gamma}_1, \gamma_2\} =$ the middle loop $= \{\gamma_1, \widetilde{\gamma}_2\}$.

□

Corollary 7.13. *A two by two move gives rise to a birational isomorphism of Poisson algebras:*

$$\mu_{\Gamma \to \Gamma'} : \widehat{\mathcal{O}}_\Gamma \longrightarrow \widehat{\mathcal{O}}_{\Gamma'}.$$

Conclusion. *Groupoids* $\mathrm{Loc}_1(\Gamma)$ *assigned to bipartite graphs* Γ *and related by birational transformations (128) form a non-commutative analog of a cluster Poisson variety, with the Poisson bracket on the algebra (141) given by Definition 7.11.*

8 Dimers and non-commutative cluster integrable systems

1. Dimer covers. A *dimer cover* of a bipartite graph Γ is a subset of edges of Γ such that each vertex is the endpoint of a unique edge in the dimer cover. Given a dimer cover M, we assign to each edge E of Γ the weight $\omega_M(E) \in \{0, 1\}$ such that $\omega_M(E) = 1$ if and only if the edge E belongs to M. Let $[E]$ be the edge E oriented as $\bullet \to \circ$. Then we get a 1-chain

$$[M] := \sum_E \omega_M(E)[E].$$

Here the sum is over all edges. We define $\mathrm{sgn}(v) := -1$ for the $\bullet$-vertex v, and $\mathrm{sgn}(v) := 1$ for the $\circ$-vertex. The condition that M is a dimer cover just means that

$$d[M] = \sum_v \mathrm{sgn}(v)[v].$$

Here the sum is over all vertices v of Γ. So if M_1, M_2 are dimer covers, $[M_1] - [M_2]$ is a 1-cycle.

2. Zig-zag paths revisited. Recall that a zig-zag γ on a bipartite ribbon graph Γ is oriented so that it turns right at the $\circ$-vertices, and turns left at the $\bullet$-vertices. For every edge there are exactly two zig-zag paths containing the edge. They traverse the edge the opposite ways.

3. A reference dimer cover Φ for a bipartite graph on a torus [GK]. We consider from now bipartite graphs Γ on a torus T. A bipartite graph Γ on a torus is *minimal* if it does not have parallel bigons, see the left picture on Figure 14. Homology classes of oriented zig-zag paths on a bipartite graph $\Gamma \subset \mathrm{T}$ form a collection of vectors $e_1, ..., e_n$ of the rank two lattice $H_1(\mathrm{T}, \mathbb{Z})$. We order them cyclically by the angle they make with a given direction. Their sum is zero. So they are oriented sides of a convex polygon $N \subset H_1(\mathrm{T}, \mathbb{Z})$, called the *Newton polygon* of Γ. The set $\{e_1, \ldots, e_n\}$ is identified with the set of primitive edges of the boundary ∂N of N. Alternatively, N is the convex hull of 0, e_1, $e_1 + e_2$, ..., $e_1 + ... + e_{n-1}$.

Let X be the set of circular-order-preserving maps

$$\alpha : \{e_1, \ldots, e_n\} \to \mathbb{R}/\mathbb{Z}. \tag{146}$$

Let E be an edge of Γ. Denote by z_r (respectively z_l) the zig-zag path containing E in which the $\circ$-vertex of E precedes (respectively follows) the $\bullet$-vertex. Then a map α provides a function

$$\varphi_\alpha = \varphi : \{\text{Edges of } \Gamma\} \longrightarrow \mathbb{R}, \qquad \varphi(E) := \alpha_r - \alpha_l. \tag{147}$$

Here we use the identification of zig-zag loops with their homology classes e_i. Then α_r (respectively α_l) is the evaluation of the map α on the zig-zag path z_r (respectively z_l) crossing the edge E, and $\alpha_r - \alpha_l$ denotes the length of the counterclockwise arc on $S^1 = \mathbb{R}/\mathbb{Z}$ from α_l to α_r.

Theorem 8.1. *[GK, Theorem 3.3] For any map $\alpha \in X$, the function φ_α satisfies*

$$d\varphi_\alpha = \sum_v \operatorname{sgn}(v)[v].$$

Take a map α which sends all e_i to 0 and assigns 1 to the arc between e_i and e_{i+1}, and 0 to all other arcs. Then φ_α is a dimer cover of Γ. We denote it by Φ. It corresponds to a vertex of the Newton polygone, perhaps a degenerate one, i.e. located on a side.

4. The commuting Hamiltonians. Since M and Φ are perfect matchings, the 1-cycle $[M] - [\Phi]$ is a union of non-intersecting and nonselfintersecting loops. Considered modulo homotopy on the graph Γ, it defines an element

$$(M) := [M] - [\Phi] \in L(\Gamma).$$

Definition 8.2. *Let Γ be a bipartite graph on a torus. The non-commutative partition function $\mathcal{P}_\Phi$ is an element of $L(\Gamma)$ given as the sum over all dimer covers M of Γ*

$$\mathcal{P}_\Phi := \sum_M \operatorname{sgn}(M)(M). \tag{148}$$

The element $[M] - [\Phi]$ defines a class in $H_1(\mathrm{T}, \mathbb{Z})$, denoted by $h_\mathrm{T}(M)$. The set of homology classes for different dimer covers M of Γ coincides with the set of internal points of a shift of the Newton polygon N. Take the sum over the dimer covers M with the same homology class a:

$$H_a := \sum_{h_\mathrm{T}(M) = a} (M) \in L(\Gamma). \tag{149}$$

Furthermore, for any nonzero integer n, we define a new element

$$H_{na} \in L(\Gamma).$$

Namely, if $n > 0$, it is obtained by travelling n times each loop of H_a. If $n < 0$, we reverse the orientations of the loops of H_a, and then travel each obtained loop $|n|$ times.

Definition 8.3. *The elements H_{na} are the Hamiltonians of the dimer system.*

Remark. A map α related to another vertex of the Newton polygon leads to another collection of Hamiltonians which differ from $\{H_a\}$ by a sum of zig-zag loops, which lies in the center of the Poisson algebra $\mathcal{O}(\Gamma)$. Therefore the Hamiltonian flows do not depend on the choice of α, and on each symplectic leaf any two collections of Hamiltonians differ by a common factor.

Theorem 8.4. *Let Γ be a minimal bipartite graph on a torus $\mathbb{T}$. Then the Hamiltonians H_{na} commute under the Lie bracket on $L(\Gamma)$. Therefore they Poisson commute in $\mathcal{O}(\Gamma)$.*

Proof. We prove that $\{H_a, H_b\} = 0$. The proof almost literally follows the proof of [GK, Theorem 3.7]. For the convenience of the reader we indicate the main steps. The claim that $\{H_{na}, H_{mb}\} = 0$ follows from this.

Take a pair of matchings (M_1, M_2) on Γ. Let us assign to them another pair of matchings $(\widetilde{M_1}, \widetilde{M_2})$ on Γ. Since $[M_1] - [M_2]$ is a 1-cycle, and M_1, M_2 are dimer covers, it is a disjoint union of non-itersecting and non-selfintersecting loops. So $[M_1] - [M_2]$ is a disjoint union of:

1. homologically trivial on the torus loops,
2. homologically non-trivial loops,
3. edges shared by both matchings.

For every edge E of each homologically trivial loop we switch the label of E: if E belongs to a matching M_1 (respectively M_2), we declare that it will belong to a matching $\widetilde{M_2}$ (respectively $\widetilde{M_1}$). For all other edges we keep their labels intact. Switching the labels of a homologically trivial loop we do not change the homology classes $[M_1] - [\Phi]$, $[M_2] - [\Phi]$.[9] Clearly $\widetilde{\widetilde{M}}_i = M_i$.

Lemma 8.5. *[GK, Lemma 3.8] One has*

$$\{(M_1), (M_2)\} + \{(\widetilde{M_1}), (\widetilde{M_2})\} = 0. \tag{150}$$

Proof. Let us write (150) as a sum of the local contributions corresponding to the vertices v of Γ, and break it in three pieces as follows:

$$\varepsilon(\mu_1, \mu_2) + \varepsilon(\widetilde{\mu}_1, \widetilde{\mu}_2) = (\sum_{v \in \mathcal{E}_1} + \sum_{v \in \mathcal{E}_2} + \sum_{v \in \mathcal{E}_3})\Big(\delta_v(\mu_1, \mu_2) + \delta_v(\widetilde{\mu}_1, \widetilde{\mu}_2)\Big).$$

Here $\mathcal{E}_1$ (respectively $\mathcal{E}_2$ and $\mathcal{E}_3$) is the set of vertices which belongs to the homologically trivial loops (respectively homologically non-trivial loops, double edges).

For any $v \in \mathcal{E}_1$ we have $\delta_v(\mu_1, \mu_2) + \delta_v(\widetilde{\mu}_1, \widetilde{\mu}_2) = 0$ since the elements $l_v(\mu_1) \in \mathbb{A}_v$ provided by μ_i coincides with the $l_v(\widetilde{\mu}_2)$, and similarly $l_v(\mu_2) = l_v(\widetilde{\mu}_1)$. So the first sum is a sum of zeros.

The third sum is also a sum of zeros by the skew symmetry of the bracket.

It remains to prove that

$$\sum_{v \in \mathcal{E}_2} \Big(\delta_v(\mu_1, \mu_2) + \delta_v(\widetilde{\mu}_1, \widetilde{\mu}_2)\Big) = 0.$$

Let γ be an oriented loop on Γ. The *bending* $b_v(\gamma; \varphi)$ of the function φ at a vertex v of γ is

$$b_v(\gamma; \varphi) = \sum_{E \in R_v} \varphi(E) - \sum_{E \in L_v} \varphi(E) \in \mathbb{R}.$$

Here R_v (respectively L_v) is the set of all edges sharing the vertex v which are on the right (respectively left) of the oriented path γ. Lemma 8.5 now follows from Lemma 8.6 below. □

[9] Contrary to this, switching the labels of homologically non-trivial loops in $[M_1] - [M_2]$, we do change the homology classes $[M_i] - [\Phi]$.

Lemma 8.6. *[GK, Lemma 3.9] For any simple topologically nontrivial loop γ, and for any $\alpha \in X$, see (146), the corresponding function φ_α satisfies,*

$$\sum_{v\in\gamma} b_v(\gamma;\varphi_\alpha) = 0.$$

□

9 Admissible dg-sheaves on surfaces and non-commutative cluster varieties

9.1 Admissible dg-sheaves

DG-category of constructible dg-sheaves on a cell complex. Let X be a stratified topological space, such that the closure of each stratum is a ball. The strata form a partially ordered set. We write $\sigma_1 < \sigma_2$ if $\sigma_1 \subset \overline{\sigma}_2$ and $\sigma_1 \neq \sigma_2$. It determines a quiver Q, whose vertices are the strata σ, and vertices σ_1, σ_2 are related by an arrow $\sigma_1 \to \sigma_2$ if and only if $\sigma_1 < \sigma_2$.

The quiver Q determines a nonunital A_∞-category $\mathcal{Q}$ whose objects are the strata σ, and

$$\mathrm{Hom}_{\mathcal{Q}}(\sigma_1,\sigma_2) := \begin{cases} \mathbb{Z} & \text{if } \sigma_1 < \sigma_2, \\ 0 & \text{otherwise.} \end{cases} \tag{151}$$

The composition of Hom's is defined by the composition of arrows.

Definition 9.1. *A constructible dg-sheaf on X is an A_∞-functor from the A_∞-category $\mathcal{Q}$ to the dg-category* Vect$^\bullet$ *of complexes of vector spaces. Constructible dg-sheaves form a dg-category*

$$\mathcal{C}_X := \mathrm{Fun}_{A_\infty}(\mathcal{Q}, \mathrm{Vect}^\bullet).$$

Elaborating this definition, an A_∞-functor $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}_X)$ is given by the following data:

- For each nested collection of strata $\sigma_0 < \sigma_1 < \ldots < \sigma_n$, a map

$$f_{\sigma_0,\ldots,\sigma_n} : \mathcal{F}(\sigma_0) \otimes \mathrm{Hom}_{\mathcal{Q}}(\sigma_0,\sigma_1) \otimes \ldots \otimes \mathrm{Hom}_{\mathcal{Q}}(\sigma_{n-1},\sigma_n) \longrightarrow \mathcal{F}(\sigma_n)[1-n]. \tag{152}$$

  These maps satisfy the well known quadratic equations.

Since the spaces $\mathrm{Hom}_{\mathcal{Q}}(\sigma_0,\sigma_1)$ are canonically identified with $\mathbb{Z}$, one can write (152) just as

$$f_{\sigma_0,\ldots,\sigma_n} : \mathcal{F}(\sigma_0) \longrightarrow \mathcal{F}(\sigma_n)[1-n]. \tag{153}$$

One pictures a map (152) by a collection of points on the half-line, with the objects σ_i at the intervals between the points, and the object $\mathcal{F}$ at the end point, see Figure 40. Then the terms of the quadratic equations match all the ways to collapse a collection of consecutive points.

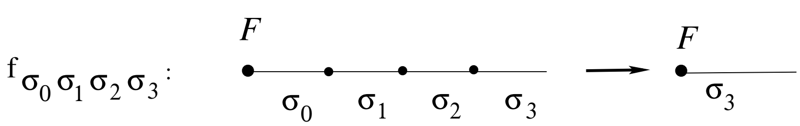


Figure 40: A map $\mathcal{F}(\sigma_0) \otimes \mathrm{Hom}(\sigma_0,\sigma_1) \otimes \mathrm{Hom}(\sigma_1,\sigma_2) \otimes \mathrm{Hom}(\sigma_2,\sigma_3) \longrightarrow \mathcal{F}(\sigma_3)[-2]$.

Elaborating further, an object of the category $\mathcal{C}_X$ is given by the following data:
A complex C_σ assigned to each cell σ.

For each pair of strata $\sigma_0 < \sigma_1$, a map of complexes $f_{\sigma_0,\sigma_1} : C_{\sigma_0} \longrightarrow C_{\sigma_1}$.
For each triple of strata $\sigma_0 < \sigma_1 < \sigma_2$, a homotopy $f_{\sigma_0,\sigma_1,\sigma_2} : C_{\sigma_0} \to C_{\sigma_2}[-1]$:

$$[d, f_{\sigma_0,\sigma_1,\sigma_2}] = f_{\sigma_1,\sigma_2} \circ f_{\sigma_0,\sigma_1} - f_{\sigma_0,\sigma_2}.$$

And so on.

Given two A_∞-functors $\mathcal{F}$ and $\mathcal{G}$, one defines a graded vector space

$$\begin{gathered}\mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G}) := \\ \prod_{n\geq 0}\prod_{\sigma_0<\ldots<\sigma_n} \mathrm{Hom}^\bullet\Big(\mathcal{F}(\sigma_0)\otimes \mathrm{Hom}_{\mathcal{Q}}(\sigma_0,\sigma_1)\otimes\ldots\otimes \mathrm{Hom}_{\mathcal{Q}}(\sigma_{n-1},\sigma_n), \mathcal{G}(\sigma_n)\Big)[-n] = \\ \prod_{n\geq 0}\prod_{\sigma_0<\ldots<\sigma_n} \mathrm{Hom}^\bullet\Big(\mathcal{F}(\sigma_0), \mathcal{G}(\sigma_n)\Big)[-n].\end{gathered} \tag{154}$$

One equips it with a natural differential, getting a complex $\mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G})$.

$\mathcal{H}$-supported complexes of constructible sheaves on a manifold. Let X be an oriented n-dimensional manifold. A *coorientation* of a hypersurface H in X is a choice of a connected component of the conormal bundle to H minus the zero section. This component is called the conormal bundle to the cooriented hypersurface H.

Recall that the microlocal support of a complex of constructible sheaves $\mathcal{F}$ on X is a closed conical Lagrangian subvariety of T^*X, defined as follows. Take a non-zero covector $\eta \in T^*_xX$, and a little ball passing through the point x and tangent to the hyperplane $\eta = 0$ in T_xX. Move it slightly, getting a ball B_η containing x; move it the other way, getting a ball B'_η which does not contain x. Let $B''_\eta := B'_\eta \cap B_\eta$. Consider the cone of the restriction map from B_η to B''_η:

$$\mathrm{Cone}\Big(\mathcal{F}(B_\eta) \stackrel{\mathrm{Res}}{\longrightarrow} \mathcal{F}(B''_\eta)\Big). \tag{155}$$

Then η does not enter the microlocal support of $\mathcal{F}$ if and only if the complex (155) is acyclic. Finally, if an open neighborhood U of x is contained in the support of $\mathcal{F}$, then the zero section of T^*U belongs to the microlocal support of $\mathcal{F}$.

Definition 9.2. *Let $\mathcal{H}$ be a collection of cooriented hypersurfaces in a manifold X. A complex of constructible sheaves on X is $\mathcal{H}$-supported if its microlocal support is contained in the union of the zero section of T^*X and the conormal bundles to the cooriented hypersurfaces in $\mathcal{H}$.*

Generic cooriented stratification $\mathcal{H}$ of a manifold. Take a collection of cooriented hypersurfaces in generic position, that is locally isomorphic to a collection of coordinate hyperplanes. It provides a stratification $\mathcal{H}$ of X. The 0-cells are the intersection points of n hypersurfaces. The 1-cells are components of the intersections of $n-1$ hypersurfaces minus 0-cells, and so on. Let S be a codimesion k stratum of the stratification $\mathcal{H}$. We can identify an open neighbourhood of S with $S\times\mathbb{R}^k$, where $\mathbb{R}^k$ is the vector space with coordinates $\{x_i\}$, so that the codimension 1 strata containing S are given by equations $x_i = 0$, cooriented by $dx_i > 0$. The conormal bundle T^*_SX contains 2^k octants. Using an isomorphism $T^*_SX = \mathbb{R}^k \times S$, they are given by inequalities $\pm x_1 \geq 0, \ldots, \pm x_k \geq 0$. The octant $x_1 \geq 0, \ldots, x_k \geq 0$ is called the *positive octant*, and denoted by T^+_SX. The dual stratification is described by an oriented cube $\mathcal{K}_S$. Its vertices match the octants in $\mathbb{R}^k$. The oriented edges match the cooriented codinension 1 strata, etc. The cubes $\mathcal{K}_S$ for different strata S are assembled into a cubical complex $\mathcal{K}$ dual to the stratification $\mathcal{H}$, see Figure 41.

In particular, for each face K of $\mathcal{K}$ there is a vertex $v_-(K)$ assigned to the negative octant, and the opposite vertex $v_+(K)$ assigned to the positive octant. They coincide if $\dim(K) = 0$.

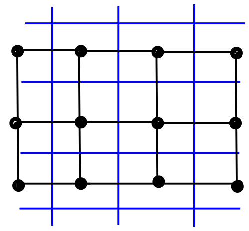


Figure 41: A cubical complex dual to a generic stratification.

Lemma 9.3. *The microlocal support of an $\mathcal{H}$-supported constructible complex of sheaves $\mathcal{F}$ on X is contained in the union of the positive octants $T^+_S X$, where S are the strata of $\mathcal{H}$.*

Proof. The microlocal support of $\mathcal{F}$ is a union of certain octants in $T^*_S X$. Indeed, varying the convector η inside of one octant we do not change the cohomology groups of complex (155). If the microlocal support intersects one of the non-positive open octants, the closure of this octant intersects a negative component of the conormal bundle to one of the codimension 1 strata, which contradicts to the condition that $\mathcal{F}$ is $\mathcal{H}$-supported. □

$\mathcal{H}$-supported dg-sheaves on a manifold. A dg-enhancement of the derived category of $\mathcal{H}$-supported complexes of constructible sheaves is given by the $\mathcal{H}$-supported dg-sheaves. Let us define a more economic model for it for stratifications by generic cooriented hypersurfaces.

Proposition 9.4. *Let $\mathcal{H}$ be a stratification of a manifold X by generic cooriented hypersurfaces. Then $\mathcal{H}$-supported dg-sheaves can be described as follows, using the dual cubical complex $\mathcal{K}$:*

Data:

- *A graded vector space $C^\bullet_v$ assigned to each vertex v of $\mathcal{K}$.*
- *A map $\delta_K : C^\bullet_{v_-(K)} \longrightarrow C^\bullet_{v_+(K)}[1-k]$ for each k-dimensional face K of $\mathcal{K}$, where $k \geq 0$.*

Condition: *For each face K of $\mathcal{K}$, take the direct sum of the graded spaces $C^\bullet_v$ at the vertices v of K, shifted by the distance d_v from v to the vertex $v_-(K)$:*[10]

$$C^\bullet_K := \oplus_{v\in K} C^\bullet_v[-d_v].$$

- *Then the signed sum d_K of the maps δ_F assigned to the faces F of K in $\mathcal{K}$ satisfies $d_K^2 = 0$.*

Example. Let us elaborate the cubical description of an $\mathcal{H}$-supported dg-sheaf in the dimension 3:
For each vertex v we get a complex $(C^\bullet_v, \delta_v)$.
For each oriented edge $E = (0, 1)$ we get a morphism of complexes $\delta_E : C^\bullet_0 \longrightarrow C^\bullet_1$.
For each square K with the vertices $(00, 01, 10, 11)$ we get a homotopy

$$h_K : C^\bullet_{00} \longrightarrow C^\bullet_{11}[-1], \qquad \delta_{11} h_K + h_K \delta_{00} = \delta_{01\to 11} \circ \delta_{00\to 01} - \delta_{10\to 11} \circ \delta_{00\to 10}.$$

For each cube C with vertices $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ we get a higher homotopy $H_{\mathrm{C}} : C^\bullet_{000} \longrightarrow C^\bullet_{111}[-2]$:

$$\begin{aligned} \delta H_{\mathrm{C}} - H_{\mathrm{C}}\delta &= \delta_{110\to 111} \circ h_{000\to 110} + \delta_{101\to 111} \circ h_{000\to 101} + \delta_{011\to 111} \circ h_{000\to 011} \\ &- h_{100\to 111} \circ \delta_{000\to 100} - h_{010\to 111} \circ \delta_{000\to 010} - h_{001\to 111} \circ \delta_{000\to 001}. \end{aligned} \tag{156}$$

[10] If one parametrizes the vertices of a cube K by $(\varepsilon_1, \ldots, \varepsilon_k)$, where $\varepsilon_i \in \{0, 1\}$, then the distance is $\varepsilon_1 + \ldots + \varepsilon_k$.

Proof. Let us show how a data $(C_v^\bullet, \delta_K)$ determines a dg-sheaf on X. We start with a simple observation. Let H be a component of a cooriented hypersurface strata. Denote by $\mathcal{D}^+$ and $\mathcal{D}^-$ the connected domains of $X - \mathcal{H}$ sharing H, so that H is cooriented towards $\mathcal{D}^+$. Let $\mathcal{F}$ be an $\mathcal{H}$-supported dg-sheaf on X. Then the natural map

$$\mathcal{F}_{|H} \longrightarrow \mathcal{F}_{|\mathcal{D}^-} \tag{157}$$

is a quasi-isomorphism. Indeed, its cone detects the microlocal support of $\mathcal{F}$ in the direction opposite to the coorientation of H, and hence is acyclic. We can alter the dg-sheaf data on the component H so that the map (157) is an isomorphism, getting a quasi-isomorphic dg-sheaf.

Generalising this, let us construct an A_∞-functor $\mathcal{F}$. Given a stratum σ of $\mathcal{H}$, let us define $\mathcal{F}(\sigma)$. Let us move a bit the point $p \in \sigma$ inside of the unique n-dimensional stratum S_p in the negative direction from p, whose closure contains p. The stratum S_p is encoded by a vertex v of $\mathcal{K}$. We set $\mathcal{F}(\sigma) := C_v^\bullet$. The maps $f_{\sigma_1,\ldots,\sigma_n}$ are defined naturally.

Let us describe a dg-sheaf $\mathcal{C}^\bullet$ near the intersection point v of two cooriented lines in $\mathbb{R}^2$ on Figure 42. The cubical description of the dg-sheaf $\mathcal{C}^\bullet$ near v is given by the following data, where $h : C_{00}^\bullet \longrightarrow C_{11}^\bullet[-1]$, see also Figure 42:

$$\begin{array}{ccc} C_{01}^\bullet & \xrightarrow{g_1} & C_{11}^\bullet \\ {\scriptstyle f_0}\uparrow & \nearrow^{h} & \uparrow{\scriptstyle f_1} \\ C_{00}^\bullet & \xrightarrow[g_0]{} & C_{10}^\bullet \end{array} \tag{158}$$

The restriction of $\mathcal{C}^\bullet$ to a ball $\mathcal{B}$ around v is given, by the definition, by the complex $C_{00}^\bullet$. The restriction of $\mathcal{C}^\bullet$ to the domain $\mathcal{D}$ on Figure 42 is given by the following complex, where C_{ij}^0 is in the degree $i+j-1$:

$$\begin{array}{ccc} C_{01}^\bullet & \xrightarrow{g_1} & C_{11}^\bullet \\ & & \uparrow{\scriptstyle f_1} \\ & & C_{10}^\bullet \end{array} \tag{159}$$

Indeed, to calculate $\mathcal{C}_{|\mathcal{D}}^\bullet$ we cover $\mathcal{D}$ by open subsets $\mathcal{U}$ and $\mathcal{V}$, where $\mathcal{U}$ is disunct with the horizontal line, and $\mathcal{V}$ with the vertical. Then $\mathcal{C}_{|\mathcal{U}}^\bullet = C_{01}^\bullet$, $\mathcal{C}_{|\mathcal{V}}^\bullet = C_{10}^\bullet$. Their restrictions to $\mathcal{V} \cap \mathcal{U}$ are given by the maps g_1, f_1.

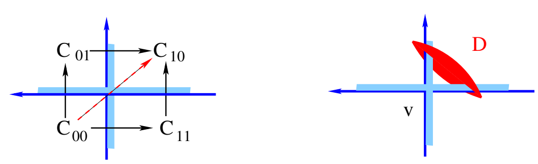


Figure 42: The domain $\mathcal{D}$ intersects the two lines, but does not contain their intersection v.

Assume that $\mathcal{D} \subset \mathcal{B}$. Then, using (159), the restriction of $\mathcal{C}_{|\mathcal{B}}^\bullet$ to $\mathcal{D}$ is given by the map of complexes:

$$(f_0, g_0, h) : C_{00}^\bullet \longrightarrow \mathrm{Cone}\Big(C_{01}^\bullet \oplus C_{10}^\bullet \stackrel{(g_1,f_1)}{\longrightarrow} C_{11}^\bullet\Big)[-1]. \tag{160}$$

The shifted cone of this map is the total complex associated with (158). $\square$

Lemma 9.5. *Given an $\mathcal{H}$-supported complex of sheaves $\mathcal{F}$ on X, the complex (155) for any convector η inside the positive octant T_S^+X is quasiisomorphic to the complex $(C_K^\bullet, d_K)$, where K is the face of $\mathcal{K}$ dual to the strata S.*

Proof. Let us elaborate the microlocal support at the vertex v on Figure 42. Then the map (155) is given by the map (160). So the claim boils down to the last sentence of the proof of Proposition 9.4. $\square$

$\mathcal{H}$-admissible dg-sheaves. Among all $\mathcal{H}$-supported dg-sheaves we distinguish a crucial for us subcategory of *$\mathcal{H}$-admissible dg-sheaves.*

Definition 9.6. *Given a stratification $\mathcal{H}$ provided by a collection of cooriented hypersufaces in a manifold X whose Legendrians are disunct, an $\mathcal{H}$-supported dg-sheaf as in Proposition 9.4 is $\mathcal{H}$-*admissible *if*

- *The complexes $(C^\bullet_K, d_K)$ satisfy the following conditions:*

$$\begin{aligned} &i)\ \ \dim \mathrm{K}=0: \quad H^p(C^\bullet_K)=0 \quad \text{if } p\notin\{0,1\},\\ &ii)\ \ \dim \mathrm{K}=1: \quad H^p(C^\bullet_K)=0 \quad \text{if } p\notin\{0\},\\ &iii)\ \ \dim \mathrm{K}>1: \quad H^p(C^\bullet_K)=0 \quad \text{for all } p. \end{aligned} \tag{161}$$

Lemma 9.7. *The restriction of an $\mathcal{H}$-admissible dg-sheaf to each hypersurface of $\mathcal{H}$ is a local system.*

Proof. Same as the proof of Lemma 9.10 which we spelled in detail. □

9.2 The bigon and triangle moves

We consider moves of the hypersurfaces supporting the microlocal support of an admissible dg-sheaf such that the corresponding Legendrians remain disunct. We call them admissible moves. Focusing on the dimension 2, we prove that admissible moves preserve the category of $\mathcal{H}$-admissible dg-sheaves.

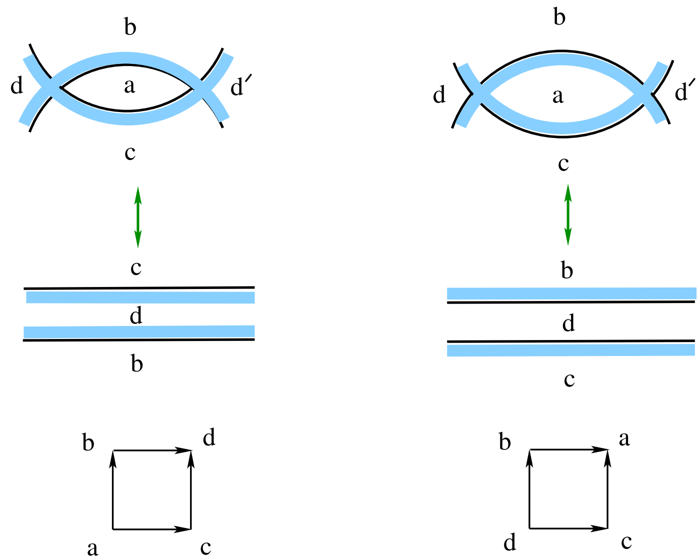


Figure 43: There are two different admissible bigon moves.

The bigon moves. There are two different admissible moves of a pair of cooriented lines in the plane, called bigon moves and shown on Figure 43. Complexes assigned to the faces and arrows between them corresponding to the cooriented lines separating them are described by the vertices and oriented edges of a square, shown at the bottom of each move.

Recall that any object of the triangulated category of representations of a Dynkin quiver is a direct sum of the shifts $R_i[n_i]$, $n_i \in \mathbb{Z}$ of indecomposable representations R_i. The isomorphism classes of the latter are parametrized by positive roots of the root system.

The two arrows from the bottom left vertex of the square form an A_3-quiver. It has six indecomposable representations. Taking into account a $\mathbb{Z}/2\mathbb{Z}-$symmetry of the A_3-quiver, there are just four of them to consider, depicted on Figure 44. Each of them determines uniquely an acyclic homotopy square, as shown on Figure 44.

The quiver representation obtained by forgetting the vertex a of the square and the arrows from this vertex describe a dg-sheaf obtained by the elementary move of the original dg-sheaf.

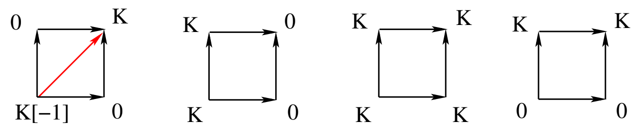


Figure 44: The squares for bigon moves. The non-trivial homotopy is shown by a red arrow. Each arrow $K \to K$ is an isomorphism.

Theorem 9.8. *The bigon moves transform admissible dg-sheaf to admissible ones.*

Proof. This it is clear from Figure 44. □

The triangle moves. There are two moves of a triple of cooriented lines in the plane, called IIa and IIb moves, or just triangle moves, shown on Figure 45. The complexes assigned to the faces and the arrows between them are described by the quivers shown on the bottom.

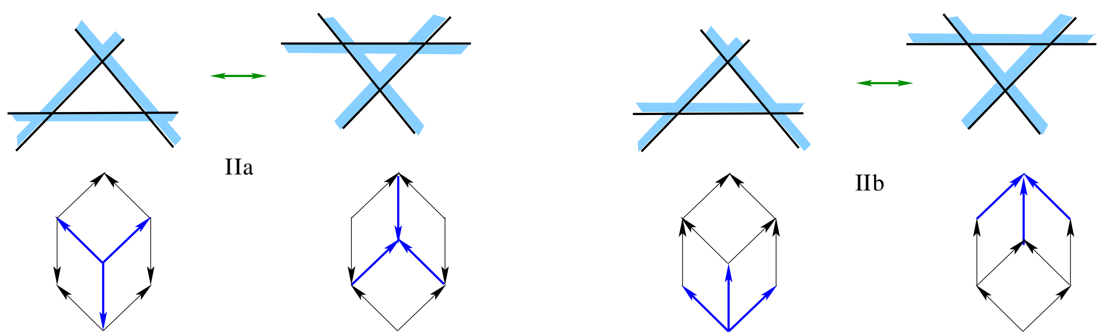


Figure 45: The triangle moves. The D_4-quivers are shown by blue arrows.

The three arrows from the central vertex form an D_4-quiver. A D_4-quiver has 12 indecomposable representations. Pick an orientation of the edges of the D_4-quiver such that the three arrows go out of the central vertex. The symmetry group of this quiver is the group S_3. Modulo the S_3-symmety, there are 6 indecomposable representations. They are depicted on Figure 46 for the D_4-quiver whose central vertex is the bottom left vertex of the cube.

A triangle move of three cooriented lines in a 2d space is described in the 3d space-time by the three cooriented coordinate planes. The $\mathcal{H}$-supported dg-sheaves for the union of cooriented coordinate planes are described by data on a cube shown on Figure 46, and their shifts.

Forgetting the top right vertex of the cube diagram we describe the original admissible dg-sheaf. Forgetting the bottom left vertex we describe the resulting one. This shows how indecomposable dg-sheaves transforms under type IIa moves. Type IIb moves are treated similarly, with the comment that this time we remove bottom right vertex instead of the top right.

Theorem 9.9. *Triangle moves transform admissible dg-sheafs to admissible ones.*

Proof. This is clear from Figure 46. Indeed, the dg-sheaf in $\mathbb{R}^3$ assigned to the cube is admissible, so forgetting any vertex we still get an admissible dg-sheaf. □

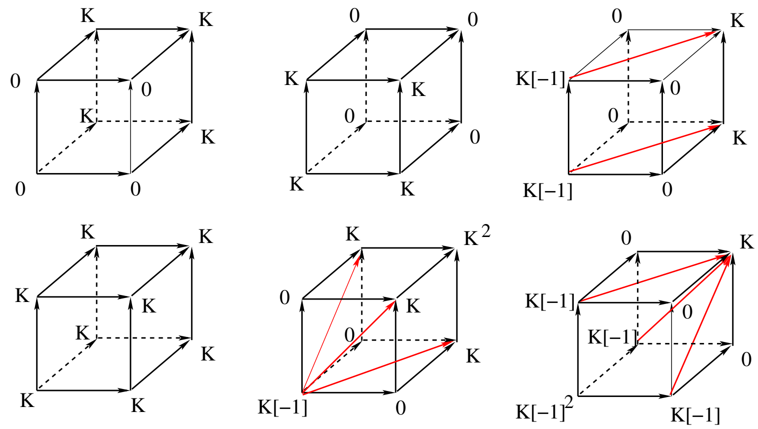


Figure 46: The cubes for elementary moves IIa and IIb. Homotopies are shown by red arrows.

Remark. Triangle moves provide a bijection between the isomorphism classes of indecomposable objects for two differently oriented A_3-quivers: the one oriented out of the central vertex to the one oriented towards the central vertex for the IIa move, and to the one with two edges oriented towards the central vertex for the IIb move.

9.3 Local systems on ideal bipartite graphs and decorated surfaces

1. Bipartite ribbon graphs and zig-zag paths. A bipartite graph Γ gives rise to a collection of zig-zag paths on the decorated surface $\mathbb{S}$ associated with Γ. Denote by $\mathcal{Z}$ their union. The complement $\mathbb{S} - \mathcal{Z}$ is a disjoint union of connected domains of three types: $\bullet$, $\circ$, and mixed domains, see Figure 47. The $\bullet-$domains are the ones containing a single $\bullet-$vertex of Γ. The $\circ-$domains contain a single $\circ-$vertex of Γ. The rest are mixed domains. The $\bullet$ and $\circ-$domains are contractible.

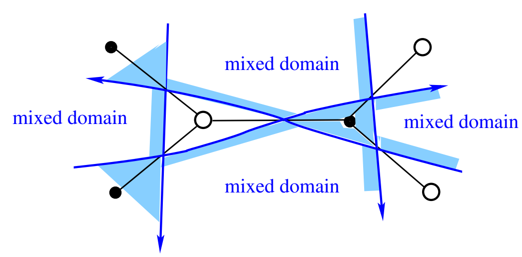


Figure 47: Zig-zag paths for a bipartite graph Γ on a surface $\mathbb{S}$ cut the surface into $\bullet$, $\circ$, and mixed domains. Each zig-zag path is coorientated by a shaded area to the right of the path.

Each zig-zag path is orientated so that the $\bullet-$vertices are on the right. This plus the surface orientation provide coorientation of zig-zag paths: moving along a zig-zag path the coorientation points to the right. All sides of a $\bullet-$domain are cooriented inside, all sides of a $\circ-$domain are cooriented outside, and coorintations of the sides of a mixed domain alternate.

So the union of zig-zag paths give rise to a cooriented stratification $\mathcal{Z}$ of $\mathbb{S}$. Its zero stratum is the set $\mathcal{Z}^0$ of crossing points. So there is a category of $\mathcal{Z}$-admissible dg-sheaves on $\mathbb{S}$.

2. $\mathcal{Z}$-admissible dg-sheaves. A $\mathcal{Z}$-admissible dg-sheaf on $\mathbb{S}$ is given by the following data:

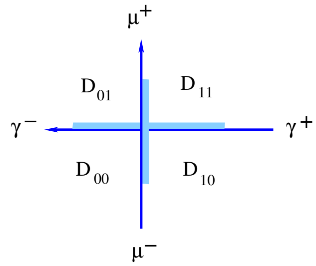


Figure 48: The stratification near a crossing point of zig-zag paths γ and μ has one 0-dimensional stratum, four 1-dimensional strata $\gamma^{\pm}, \mu^{\pm}$ and four 2-dimensional strata $\mathcal{D}_{**}$.

- A complex of local systems $\mathcal{F}^{\bullet}_{\mathcal{D}}$ on each domain $\mathcal{D}$ of $\mathbb{S} - \mathcal{Z}$, concentrated in degrees $[0, 1]$:
- A map of complexes $\varphi_{\gamma} : \mathcal{F}^{\bullet}_{\mathcal{D}^-} \to \mathcal{F}^{\bullet}_{\mathcal{D}^+}$ for each component γ of $\mathcal{Z} - \mathcal{Z}^0$, such that

$$H^i \mathrm{Cone}(\varphi_{\gamma}) = 0 \ \text{ unless } i = 0. \tag{162}$$

- For each crossing point of zig-zags γ and μ, a homotopy - see Figure 48:

$$h : \mathcal{F}^{\bullet}_{\mathcal{D}_{00}} \longrightarrow \mathcal{F}^{\bullet}_{\mathcal{D}_{11}}[-1], \qquad dh + hd = \varphi_{\mu^+} \circ \varphi_{\gamma^-} - \varphi_{\gamma^+} \circ \varphi_{\mu^-}.$$

$$\begin{array}{ccc} \mathcal{F}^{\bullet}_{\mathcal{D}_{01}} & \xrightarrow{\varphi_{\mu^+}} & \mathcal{F}^{\bullet}_{\mathcal{D}_{11}} \\ {\scriptstyle \varphi_{\gamma^-}} \uparrow & \overset{h}{\nearrow} & \uparrow {\scriptstyle \varphi_{\gamma^+}} \\ \mathcal{F}^{\bullet}_{\mathcal{D}_{00}} & \xrightarrow[\varphi_{\mu^-}]{} & \mathcal{F}^{\bullet}_{\mathcal{D}_{10}} \end{array} \tag{163}$$

- Let D be the signed sum of the differentials on $\mathcal{F}^{\bullet}_{\mathcal{D}_{ij}}$, the maps $\varphi_{\mu^{\pm}}$, $\varphi_{\gamma^{\pm}}$, and h. The data above satisfies the condition that the following complex with the differential D is acyclic:

$$\mathcal{F}^{\bullet}_{\mathcal{D}_{00}} \oplus \mathcal{F}^{\bullet}_{\mathcal{D}_{01}}[-1] \oplus \mathcal{F}^{\bullet}_{\mathcal{D}_{10}}[-1] \oplus \mathcal{F}^{\bullet}_{\mathcal{D}_{11}}[-2]. \tag{164}$$

The last condition is equivalent to either of the following conditions:
1. The natural map of complexes $\mathrm{Cone}(\varphi_{\gamma^-}) \longrightarrow \mathrm{Cone}(\varphi_{\gamma^+})$ is a quasi-isomorphism
2. The natural map of complexes $\mathrm{Cone}(\varphi_{\mu^-}) \longrightarrow \mathrm{Cone}(\varphi_{\mu^+})$ is a quasi-isomorphism.

a) $\mathcal{Z}$-admissible dg-sheaves on $\mathbb{S} \longrightarrow$ local systems on zig-zag paths. Condition (162) just means that on each component γ of $\mathcal{Z} - \mathcal{Z}^0$ there is a local system $\mathrm{Cone}(\varphi_{\gamma})$.

Lemma 9.10. *Given a zig-zag path γ, local systems* $\mathrm{Cone}(\varphi_{\gamma})$ *on* $\gamma - \{\text{crossing points}\}$ *extend to a local system on γ, denoted by* $\mathrm{Cone}(\varphi_{\gamma})$.

Proof. Condition (164) implies that for each crossing point on γ separating two germs of zig-zags γ^- and γ^+, there is an isomorphism $\mathrm{Cone}(\varphi_{\gamma^-}) \xrightarrow{\sim} \mathrm{Cone}(\varphi_{\gamma^+})$. It allows to extend local systems on components of $\gamma - \{\text{crossing points}\}$ to a local system on γ. □

b) Local systems on $\Gamma \longrightarrow \mathcal{Z}$-admissible dg-sheaves on $\mathbb{S}$, acyclic on mixed strata. A local system of vector spaces on a graph Γ is given by a collection of vector spaces V_v at the vertices v of Γ and linear maps $\varphi_{(v,w)} : V_v \to V_w$ for each edge (v, w) of Γ, so that $\varphi_{(w,v)} = \varphi^{-1}_{(v,w)}$.

Given a local system $\mathcal{V}$ on a bipartite graph Γ on $\mathbb{S}$, let us define a $\mathcal{Z}$-admissible dg-sheaf $\mathcal{F}_{\mathcal{V}}$ on $\mathbb{S}$. Since the $\bullet$ and $\circ$ domains of $\mathbb{S} - \mathcal{Z}$ are contractible, we can talk about the fibers of a complex of $\mathcal{Z}$-constructible sheaves over them. Then we define

- The fiber of $\mathcal{F}_{\mathcal{V}}$ at the $\circ-$domain with a vertex $\circ$ of Γ is the shifted fiber $\mathcal{V}_\circ[-1]$ of $\mathcal{V}$ at $\circ$.
- The fiber of $\mathcal{F}_{\mathcal{V}}$ at the $\bullet-$domain containing a vertex $\bullet$ of Γ is the fiber $\mathcal{V}_\bullet$ of $\mathcal{V}$.
- The fibers of $\mathcal{F}_{\mathcal{V}}$ over mixed domains are zero. So the maps φ_γ for zig-zags γ are zero.
- The homotopy map for each edge $\circ - \bullet$ of Γ, as illustrated on Figure 49:

$$h : \mathcal{V}_\circ[-1] \longrightarrow \mathcal{V}_\bullet[-1]. \tag{165}$$

  This map is the shift of the parallel transport map along the edge $\circ - \bullet$.

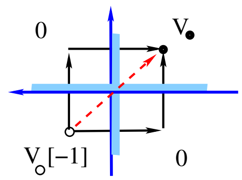


Figure 49: The admissible dg-sheaf $\mathcal{F}_{\mathcal{V}}$ on $\mathbb{S}$ assigned to a local system $\mathcal{V}$ on Γ.

So we get the following diagram near each crossing of two zig-zag paths:

$$\begin{array}{ccc} 0 & \longrightarrow & \mathcal{V}_\bullet \\ \uparrow & \overset{h}{\nearrow} & \uparrow \\ \mathcal{V}_\circ[-1] & \longrightarrow & 0 \end{array} \tag{166}$$

c) $\mathcal{Z}$-admissible dg-sheaves on $\mathbb{S}$, acyclic on mixed strata $\longrightarrow$ local systems on Γ. Construction b) can be inverted. Namely, let $\mathcal{F}$ be a $\mathcal{Z}$-admissible dg-sheaf $\mathcal{F}$ on $\mathbb{S}$, acyclic on mixed strata. Then $\mathcal{Z}$-admissibility implies that

$$H^i(\mathcal{F}_{\mathcal{D}_\circ}) = \ \ 0 \quad \text{if } i \neq 1, \qquad H^i(\mathcal{F}_{\mathcal{D}_\bullet}) = \ \ 0 \quad \text{if } i \neq 0. \tag{167}$$

Furthermore, the homotopy h provides an isomorphism

$$h : H^1(\mathcal{F}_{\mathcal{D}_\circ}) \stackrel{=}{\longrightarrow} H^0(\mathcal{F}_{\mathcal{D}_\bullet})[-1].$$

So we get a local system $\mathcal{L}_{\mathcal{F}}$ on the bipartite graph Γ by assigning the spaces $H^1(\mathcal{F}_{\mathcal{D}_\circ})[1]$ to the $\circ-$vertices $\circ$ of Γ, the ones $H^0(\mathcal{F}_{\mathcal{D}_\bullet})$ to the $\bullet-$vertices $\bullet$ of Γ, and using the isomorphism h as the parallel transport along the edge $\circ \to \bullet$.

d) Framed local systems on $\mathbb{S}$ as $\mathcal{Z}_m$-admissible dg-sheaves. Given a decorated surface $\mathbb{S}$, consider a collection of m nested simple clockwise loops around each puncture, and a collection of m simple clockwise arcs around each special point, which do not intersect and do not self-intersect. They are cooriented "inside the surface", see Figure 50. Denote by $\mathcal{Z}_m$ their union.

Lemma 9.11. *A $\mathcal{Z}_m$-admissible dg-sheaf on a decorated surface $\mathbb{S}$, acyclic near the punctures and special points, is equivalent to an m-dimensional local system on $\mathbb{S}$ with a framing.*

Proof. Take a punctured disc $\mathcal{D}^*$ with m concentric circles going clockwise around the puncture, see Figure 50. Take a radius of the disc and restrict to it a $\mathcal{Z}_m$-admissible complex of sheaves on $\mathcal{D}^*$. The

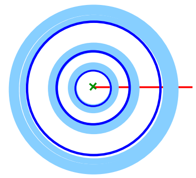


Figure 50: An admssible dg-sheaf on a punctured disc stratified by m concentric outward cooriented circles, which is zero near the puncture, is a filtered vector space.

restriction is described by complexes $V_0^\bullet, \ldots V_m^\bullet$ and maps $\varphi_{i,i+1} : V_i^\bullet \longrightarrow V_{i+1}^\bullet$ between them. To proceed further we need the following simple observation.

Let $f : W_1^\bullet \longrightarrow W_2^\bullet$ be a map of complexes of vector spaces with the following properties:

1. Cone(f) is quasi isomorphic to a one dimensional vector space in the degree 0.
2. $H^i(W_1^\bullet) = 0$ if $i \neq 0$.

Then $H^i(W_2^\bullet) = 0$ if $i \neq 0$, and the induced map $H^0(W_1^\bullet) \longrightarrow H^0(W_2^\bullet)$ is injective.

We apply this to the maps $\varphi_{i,i+1}$. Since $H^\bullet(V_0^\bullet) = 0$, we get a flag of vector spaces:

$$V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow \ldots \hookrightarrow V_m, \qquad \mathrm{dim} V_i = i, \qquad V_i := H^0(V_i^\bullet).$$

The monodromy around the puncture provides an operator in the vector space V_m preserving the flag. This just means that we get a framed m-dimensional local system of vector spaces over the punctured disc $\mathcal{D}^*$. Applying this argument at all punctures and special points we get an m-dimensional local system on the surface $\mathbb{S}$ equipped with an invariant flag near each puncture and each special point, that is a framed local system on the decorated surface $\mathbb{S}$. □

Framed local systems on $\mathbb{S} \longleftrightarrow$ flat line bundles on an ideal bipartite graph Γ. Let $\mathcal{L}$ be a one dimensional local system on a GL_m-bipartite graph Γ on $\mathbb{S}$. It determines a $\mathcal{Z}$-admissible dg-sheaf $\mathcal{F}_\mathcal{L}$ on $\mathbb{S}$ via Construction b). By the very definition, the zig-zag paths of Γ can be deformed using the elementary moves to a collection $\mathcal{Z}_m$, formed by m nested loops/arcs near each marked points. By Lemma 9.11, the resulting $\mathcal{Z}_m$-admissible dg-sheaf is just a framed m-dimensional local system on $\mathbb{S}$. The elementary moves provide birational equivalences between the corresponding categories of $\mathcal{Z}$-admissible dg-sheaves.

Conversely, let $\mathcal{L}$ be a framed m-dimensional local system on $\mathbb{S}$. Construction d) assigns to it a $\mathcal{Z}_m$-admissible dg-sheaf on $\mathbb{S}$. Deforming back using the elementary moves the collection $\mathcal{Z}_m$ of loops/arcs around the marked points to the collection $\mathcal{Z}_\Gamma$ of zig-zag paths of the ideal bipartite graph Γ, we get a $\mathcal{Z}_\Gamma$-admissible dg-sheaf $\mathcal{E}_\mathcal{L}$ on $\mathbb{S}$, which is zero near the marked points. As Figure 51 shows, this admissible

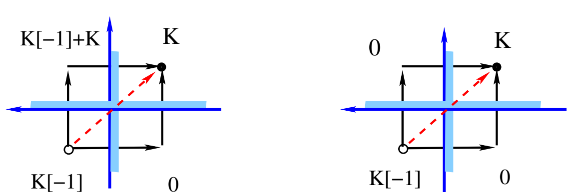


Figure 51: Two admissible dg-sheaves near a crossing point. Each provides a rank one local system on the microlocal support.

dg-sheaf may not be acyclic on all mixed domain. However for a generic $\mathcal{Z}_m$-admissible dg-sheaf on $\mathbb{S}$ we get an admissible dg-sheaf acyclic on the mixed domain. The microlocal support of the obtained

$\mathcal{Z}_\Gamma$-admissible dg-sheaf is described by flat line bundles on zig-zag paths γ of Γ. They are the same line bundles we had on γ before the deformation. Construction c) gives us a flat line bundle on the graph Γ.

So we get a regular injective map from the moduli space $\mathrm{Loc}_1(\Gamma)$ of flat line bundles on the graph Γ to the moduli space $\mathcal{X}_{\mathrm{GL_m},\mathbb{S}}$ of framed m-dimensional local systems on $\mathbb{S}$, called the reconstruction map:

$$R_\Gamma : \mathrm{Loc}_1(\Gamma) \hookrightarrow \mathcal{X}_{\mathrm{GL}_m,\mathbb{S}}. \tag{168}$$

It provides an open domain in the moduli space space $\mathcal{X}_{\mathrm{GL}_m,\mathbb{S}}$, which is a cluster Poisson torus. This is achieved by the following chain of constructions:

$$\begin{aligned}
&\text{Flat line bundles on a rank } m \text{ ideal bipartite graph } \Gamma \text{ on } \mathbb{S} \stackrel{b)+c)}{\longleftrightarrow} \\
&\mathcal{Z}_\Gamma\text{-admissible dg-sheaves on } \mathbb{S}, \text{ acyclic on mixed domains } \stackrel{\text{Sec. } 9.2}{\subset} \\
&\mathcal{Z}_m\text{-admissible dg-sheaves, vanishing near the marked points } \stackrel{d)}{\longleftrightarrow} \\
&\text{Framed } m\text{-dimensional local systems on } \mathbb{S}.
\end{aligned} \tag{169}$$

The middle map in (169) is an open embedding. The others are isomorphisms of moduli spaces.

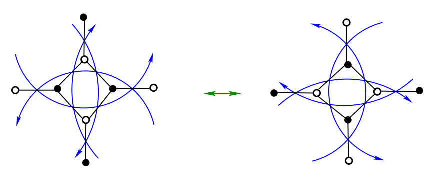


Figure 52: A two by two move of bipartite graphs.

A Lagrangian surface in $\mathcal{L}_\Gamma \subset T^*\mathbb{S}$. Given a bipartite graph Γ, note the following:

1. Each $\bullet$ or $\circ-$domain $\mathcal{D} \subset \mathbb{S}$ gives rise to a domain $\mathcal{D} \subset T^*\mathbb{S}$ - the zero section on $\mathcal{D}$.
2. Each cooriented zig-zag path γ gives rise to the conormal bundle $T^*_\gamma\mathbb{S} \subset T^*\mathbb{S}$.

Definition 9.12. *The surface $\mathcal{L}_\Gamma \subset T^*\mathbb{S}$ is the union of all these domains in $T^*\mathbb{S}$:*

$$\mathcal{L}_\Gamma := \cup_\bullet \mathcal{D}_\bullet \cup_\circ \mathcal{D}_\circ \cup_\gamma T^*_\gamma \tag{170}$$

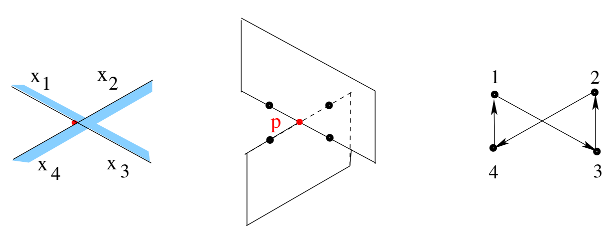


Figure 53: The link of the surface $\mathcal{L}_\Gamma$ near a crossing point p is a circle $1 \to 3 \to 2 \to 4 \to 1$.

Lemma 9.13. *The surface $\mathcal{L}_\Gamma$ is homeomorphic to a smooth oriented surface, identified with the oriented spectral surface Σ_Γ.*

Proof. The surface $\mathcal{L}_\Gamma$ is evidently homeomorphic to a smooth surface everywhere except the crossing point of the conormal bundles $p := T^*_{\gamma_1}\mathbb{S} \cap T^*_{\gamma_2}\mathbb{S}$ given by the zero cotangent vector at the crossing point

$\gamma_1 \cap \gamma_2$ of two zig-zag paths. To prove that it is homeomorphic to a smooth surface at p, we calculate the intersection of a small sphere at p with $T^*_{\gamma_1}\mathbb{S} \cup T^*_{\gamma_2}\mathbb{S}$. We claim that it is homeomorphic to a circle. This is proved on Figure 53, where we showed the link of a crossing point, observing at the same time that the projection $\mathcal{L}_\Gamma \longrightarrow \mathbb{S}$ reversing the orientation on one of the triangles. (Indeed, both arrows $3 \to 2$ and $4 \to 1$ look up on Figure 53). □

Two by two moves of bipartite graphs. Recall the two by two move $\Gamma_1 \to \Gamma_2$ of bipartite graphs on Figure 52. It is decomposed into a sequence of bigon and triangle moves on Figures 54 - 55. Note that the collections of cooriented lines which we get in the process are no longer associated with bipartite graphs.

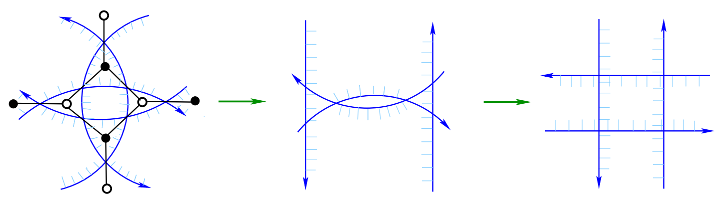


Figure 54: From the left to the right: two type II triangle moves, followed by a bigon move.

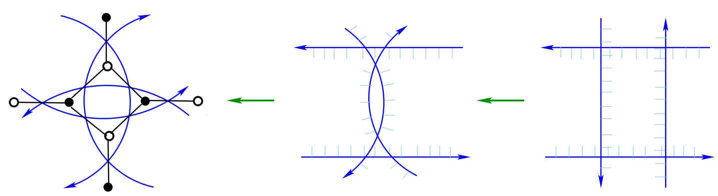


Figure 55: From the right to the left: a bigon move, followed by two triangle moves.

Starting from a local system $\mathcal{L}_1$ on the graph Γ_1, we interpret it as an admissible dg-sheaf, and transform accordantly along the sequence of admissible moves above. In the process we get admissible dg-sheaf not associated with any bipartite graphs. However, assuming that we started from a generic local system $\mathcal{L}_1$ on the graph Γ_1, the resulting admissible dg-sheaf in the end does correspond to a local system $\mathcal{L}_1$ on the graph Γ_2. So we get a birational isomorphism of stacks of local systems on the graphs Γ_1 and Γ_2. In particular, we get a birational isomorphism between the stacks of flat line bundles:

$$\mathrm{Loc}_1(\Gamma_1) \longrightarrow \mathrm{Loc}_1(\Gamma_2).$$

Our next goal is to get an explicit expression for this map for any skew field R.

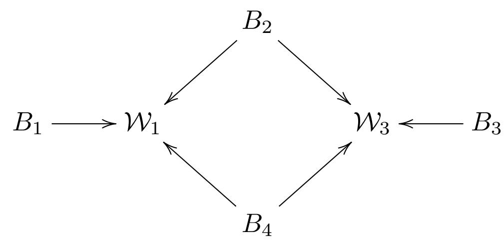
Let B_1, B_2, B_3, B_4 be the fibers of the local system over the $\bullet-$domains and W_1, W_3 the fibers over the $\circ-$domains on the left on Figure 54. Then in the domain on the Figure the local system and hence the corresponding dg-sheaf are described by the following diagram of isomorphisms:

$$\begin{array}{ccccc} & & B_2 & & \\ & \nearrow^{\sim} & & \nwarrow^{\sim} & \\ B_1 \xleftarrow{\sim} W_1 & & & & W_3 \xrightarrow{\sim} B_3 \\ & \searrow_{\sim} & & \swarrow_{\sim} & \\ & & B_4 & & \end{array} \tag{171}$$

Let us set

$$\mathcal{W}_1 := \frac{B_1 \oplus B_2 \oplus B_4}{W_1}, \qquad \mathcal{W}_3 := \frac{B_2 \oplus B_3 \oplus B_4}{W_3}.$$

Here we identify W_1 with its image under the map $W_1 \hookrightarrow B_1 \oplus B_2 \oplus B_4$, and similarly for W_3. Then after the triangle moves at the two $\circ-$vertices we get a dg-sheaf with the microlocal support shown in the middle picture on Figure 54. Since an admissible dg-sheaf related to a bipartite graph has zero fibers over the mixed domains, the only relevant picture describing a single type IIa triangle move is the fifth picture on Figure 46. It follows that the resulting dg-sheaf is described by the following diagram:

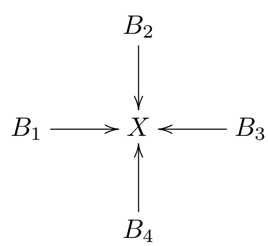


(172)

These maps are no longer isomorphisms since $\mathrm{rk}(\mathcal{W}_1) = \mathrm{rk}(\mathcal{W}_3) = 2\mathrm{rk}(B_i)$, and the diagram is no longer assigned to a bipartite graph. There are canonical isomorphisms

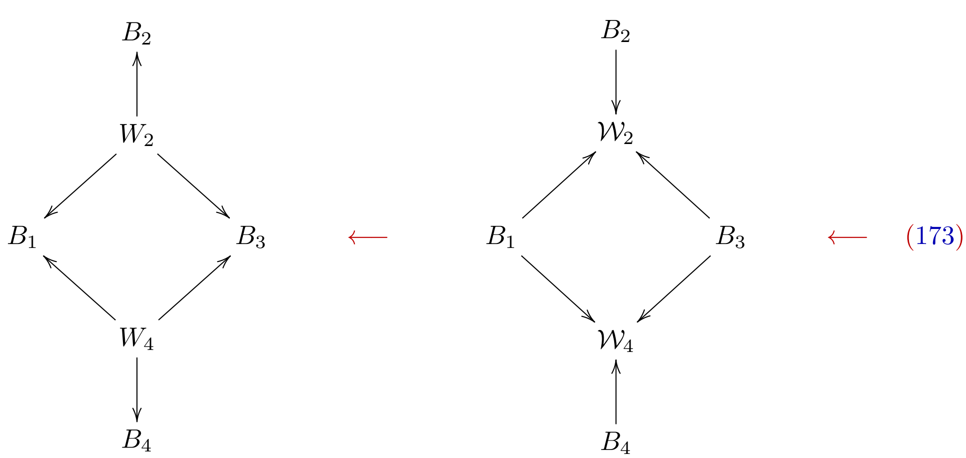
$$\mathcal{W}_1 \stackrel{\sim}{\longleftarrow} B_2 \oplus B_4 \stackrel{\sim}{\longrightarrow} \mathcal{W}_3.$$

The left one is the embedding $B_2 \oplus B_4 \hookrightarrow B_1 \oplus B_2 \oplus B_4$, followed by the projection to $\mathcal{W}_1$. The right one is similar. Combining them we get an isomorphism $f : \mathcal{W}_1 \stackrel{\sim}{\longrightarrow} \mathcal{W}_3$. Setting $X := \mathcal{W}_1 \stackrel{f}{=} \mathcal{W}_3$, we get a diagram



(173)

It describes the dg-sheaf on the right of Figure 54. The transition (171) $\to$ (173) describes the composition of the three moves on Figure 54. Then we apply the three moves on Figure 55. The corresponding transformations of quiver representations are recorded on diagram (174).



(174)

As the result we altered the dg-sheaf in the domain $\mathcal{D}$ inside of the four $\bullet-$vertices, but did not alter it outside. Let us calculate how the corresponding local system inside of the domain $\mathcal{D}$ changes. Consider the following diagram on the left, where all maps are isomorphisms:

$$\begin{array}{ccccc} & & \mathrm{L}_2 & & \\ & & \uparrow{\scriptstyle i_2} & & \\ \mathrm{L}_1 & \xleftarrow{i_1} & \mathrm{L} & \xrightarrow{i_3} & \mathrm{L}_3 \end{array} \qquad \begin{array}{ccccc} & & \mathrm{L}_2 & & \\ & & \downarrow & & \\ \mathrm{L}_1 & \longrightarrow & \mathcal{L} & \longleftarrow & \mathrm{L}_3 \end{array} \tag{175}$$

Setting $\mathcal{L} := (\mathrm{L}_1 \oplus \mathrm{L}_2 \oplus \mathrm{L}_3)/\mathrm{L}$, it determines the diagram on the right. So there are three subspaces in $\mathcal{L}$. Then the third subspace L_3 provides a linear isomorphism

$$\varphi_{\mathrm{L}_1} : \mathrm{L}_2 \longrightarrow \mathrm{L}_3.$$

We claim that $\varphi_{\mathrm{L}_1} = -i_3 \circ i_2^{-1}$. Indeed, for any $l \in \mathrm{L}$, let $l_k := i_k(l)$. Then there are three vectors in $\mathcal{L}$ which sum to zero:

$$(l_1, 0, 0) + (0, l_2, 0) + (0, 0, l_3) = 0 \quad \text{in} \quad \mathcal{L}.$$

Now return to diagram (173), providing four subspaces in X. Then there are two isomorphisms:

$$\varphi_{\mathrm{B}_3} : \mathrm{B}_1 \longrightarrow \mathrm{B}_2, \qquad \varphi_{\mathrm{B}_4} : \mathrm{B}_1 \longrightarrow \mathrm{B}_2.$$

Let $M_{\mathrm{B}_2} : \mathrm{B}_2 \longrightarrow \mathrm{B}_2$ be the minus counterclockwise monodromy in diagram (171), given by

$$\mathrm{B}_2 \longrightarrow \mathrm{W}_1 \longrightarrow \mathrm{B}_4 \longrightarrow \mathrm{W}_3 \longrightarrow \mathrm{B}_2.$$

Lemma 9.14. *One has*

$$\varphi_{\mathrm{B}_4} = (1 + M_{\mathrm{B}_2})\varphi_{\mathrm{B}_3}.$$

Proof. If our local system is a rank one flat $R-$line bundle, it is proved by the calculation done in proof of Proposition 7.5. □

Changing $\{\bullet-\text{vertices}\} \longleftrightarrow \{\circ-\text{vertices}\}$ and reversing the arrows amounts to dualisation of the diagrams. Precisely, let W_1, W_2, W_3, W_4 be the fibers of the local system over the black domains, and B_1, B_3 the fibers over the white domains. Then the local system is described by the following diagram where all maps are isomorphisms:

$$\begin{array}{ccccccc} & & & W_2 & & & \\ & & \swarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\searrow & & \\ W_1 & \xrightarrow{\sim} & B_1 & & B_3 & \xleftarrow{\sim} & W_3 \\ & & \nwarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\nearrow & & \\ & & & W_4 & & & \end{array} \tag{176}$$

Let us set

$$\mathcal{B}_1 := \mathrm{Ker}\Big(W_1 \oplus W_2 \oplus W_4 \longrightarrow B_1\Big), \qquad \mathcal{B}_3 := \mathrm{Ker}\Big(W_2 \oplus W_3 \oplus W_4 \longrightarrow B_3\Big).$$

After the triangle moves at the two $\bullet-$vertices we get a dg-sheaf described by the diagram

$$\begin{array}{ccccccc} & & & W_2 & & & \\ & & \nearrow & & \nwarrow & & \\ W_1 & \longleftarrow & \mathcal{B}_1 & & \mathcal{B}_3 & \longrightarrow & W_3 \\ & & \searrow & & \swarrow & & \\ & & & W_4 & & & \end{array} \tag{177}$$

The composition of canonical isomorphisms $\mathcal{B}_1 \stackrel{\sim}{\longrightarrow} W_2 \oplus W_4 \stackrel{\sim}{\longleftarrow} \mathcal{B}_3$ provides an isomorphism $g : \mathcal{B}_1 \stackrel{\sim}{\longrightarrow} \mathcal{B}_3$. Setting $Y := \mathcal{B}_1 = \mathcal{B}_3$, we get a diagram

$$\begin{array}{ccccc} & & W_2 & & \\ & & \uparrow & & \\ W_1 & \longleftarrow & Y & \longrightarrow & W_3 \\ & & \downarrow & & \\ & & W_4 & & \end{array} \tag{178}$$

Conclusion. Section 9.3 gives an alternative approach to main definitions & results of Section 7.1.

Twisted and untwisted dg-sheaves. Given a sheaf of categories $\mathcal{C}$ on a space X, and a cocycle c_2 representing a class in $H^2(X, \mathrm{Cent}(\mathcal{C})^*)$, where $\mathrm{Cent}(\mathcal{C})^*$ is given by the invertible elements of $\mathcal{C}$, one can make a new sheaf of categories $\widetilde{\mathcal{C}}$, by twisting $\mathcal{C}$ by the 2-cocyle c_2. Precisely, take an open cover $\{U_i\}$ of X. Then an object of $\widetilde{\mathcal{C}}$ over X is given by objects A_i of categories $\mathcal{C}(U_i)$, related by isomorphisms $f_{ij} : A_i \longrightarrow A_j$ on $U_i \cap U_j$ which satisfy the following relations on the triple intersections:

$$f_{jk} \circ f_{ij} = c_2(U_i, U_j, U_k) f_{ik}$$

Below we use twisting by the second Stiefel-Whitney class $\mathrm{sw}_2(\mathbb{S}) \in H^2(\mathbb{S}, \mathbb{Z}/2\mathbb{Z})$. We can twist both the Fukaya category and the category of constructible or dg-sheafs. The Fukaya category of objects with the microlocal support on $T^*_Z\mathbb{S}$ is naturally equivalent to the twisted constructible category on $\mathbb{S}$, and vice versa. Note that $\mathrm{sw}_2(\mathbb{S}) = 0$ unless $\mathbb{S}$ is a compact non-oriented surface. So in our case the twisted category is equivalent to the original one. Yet the twisted version is somewhat better. It is the one providing positivity phenomenon in all formulas.

10 Non commutative stacks of framed Stokes data

10.1 Stokes data: local and global pictures

1. Admissible sheaves and local Stokes data. We start with two topological definitions.

Definition 10.1. *A collection $\mathcal{L}$ of smooth oriented loops $L_1, \ldots, L_n$ in the punctured disc $\mathrm{D}^* \subset \mathbb{C}^*$ is called* ideal *if it enjoys the following properties:*

1. *The (self)intersection $L_i \cap L_j$ of any two loops is transversal.*
2. *Each loop L_i rotates to the right, that is the argument function φ on D^* strictly decreases as we go along the loop following its orientation. We coorient the loops out of the puncture.*

Cooriented loops L_i are the connected components of the Legendrian link in $\mathrm{ST}(\mathrm{D}^*)$ provided by $\mathcal{L}$. We refer to $\mathcal{L}$ as an *ideal Legendrian link*, although it is only a projection of a Legendrian.

Lemma 10.2. *An ideal Legendrian link on the punctured disc D^* is the same thing as an element of the cyclic envelope $[\mathrm{Br}_n^+]$ of the braid semigroup, $n \geq 0$.*

Proof. Cut the collection of curves $\mathcal{L}$ by a generic ray r from the origin. Scanning the $\mathcal{L}$ by rotating the ray to the right we get a sequence of permutations, and hence an element of the braid semigroup Br_n^+, given by their product. The $n = 0$ case is the empty link. This construction can evidently be inverted. Scanning stating from a different ray produces the same element of the cyclic braid semigroup. □

Consider the subcategory $\mathcal{C}_0(\mathrm{D}^*, \mathcal{L})$ of the category $\mathcal{C}(\mathrm{D}^*, \mathcal{L})$ of admissible dg-sheaves on D^* for an ideal Legendrian $\mathcal{L}$ consisting of admissible dg-sheaves which are acyclic near the puncture. By Proposition 10.5, they are automatically concentrated in the degree zero, so we refer to them just as admissible sheaves. We denote by $[\mathcal{L}]$ the homotopy class of an ideal Legendrian $\mathcal{L}$ on D^* under Legendrian isotopies keeping Legendrians ideal and disjunct, called admissible isotopies. Categories assigned to admissibly isotopic Legendrians $\mathcal{L}$ are equivalent by Theorems 9.8-9.9. So the category $\mathcal{C}_0(\mathrm{D}^*, \mathcal{L})$ is determined by the admissible isotopy class $[\mathcal{L}]$.

Definition 10.3. *An ideal Legendrian $\mathcal{L}$ in the punctured disc $\mathrm{D}^* \subset \mathbb{C}^*$ is of* Stokes type *if it*

- *The intersection $L_i \cap L_j$ of any two different components of $\mathcal{L}$ is non-empty.*

Definition 10.4. *A local* Stokes data *is a local system $\mathcal{V}$ of vector spaces on the circle of rays in the tangent space at zero, with an increasing filtration $\mathcal{F}_\bullet$ everywhere except the fibers over a finite number of rays, called Stokes rays, related near them as follows.*

Denote by $\mathcal{F}^{\pm}_\bullet(\mathcal{V})$ filtrations to the right and to the left of a Stokes ray r. Then $\mathcal{F}^+_i(\mathcal{V}) = \mathcal{F}^-_i(\mathcal{V})$ if $i \neq a$, and the subquotient $\mathrm{gr}_{[a+1,a]}\mathcal{F}(\mathcal{V})$ is the direct sum of its subspaces $\mathrm{gr}_a\mathcal{F}^{\pm}(\mathcal{V})$:

$$\mathrm{gr}_{[a+1,a]}\mathcal{F}(\mathcal{V}) := \mathcal{F}_{a+1}(\mathcal{V})/\mathcal{F}_{a-1}(\mathcal{V}) = \mathrm{gr}_a\mathcal{F}^+(\mathcal{V}) \oplus \mathrm{gr}_a\mathcal{F}^-(\mathcal{V}). \tag{179}$$

Proposition 10.5. *The following categories are canonically equivalent:*

1. *The category of local Stokes data from Definition 10.4.*
2. *The direct sum of categories $\bigoplus_{[\mathcal{L}]} \mathcal{C}_0(\mathrm{D}^*, \mathcal{L})$ over all homotopy types $[\mathcal{L}]$ of Stokes Legendrians.*

Proof. Before we proceed to the proof, let us state the following simple Lemma.

Lemma 10.6. *An admissible sheaf near a simple crossing point is determined by a commutative diagram*

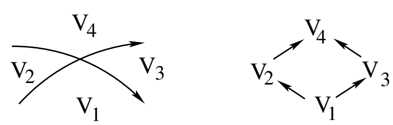


Figure 56: An admissible sheaf near a crossing point.

of four vector spaces V_i on the right of Figure 56, where all arrows are injective, and the map

$$V_2/V_1 \oplus V_3/V_1 \xrightarrow{\sim} V_4 \tag{180}$$

is an isomorphism.

Now let us prove the Proposition.

2) $\longrightarrow$ 1). The circle S^1 of rays minus the Stokes rays is a union of a finite number of intervals:

$$S^1 - \{\text{Stokes rays}\} = \mathrm{I}_1 \cup \ldots \cup \mathrm{I}_m.$$

Since the sheaf is zero near the puncture, the admissibility just means that on each of the intervals I_k we get a local system of filtered vector spaces. It follows from Lemma 10.6 that the admissibility implies that this local system extends across the Stokes rays, while the filtration is altered: the filtration in $\mathrm{gr}_{[a+1,a]}$ flips, and the rest of the filtration data is preserved.

1) $\longrightarrow$ 2). We start from a local Stokes data on S^1. For each ray $r \in \mathrm{I}_k$, we assign the associate graded quotients $\mathrm{gr}_i\mathcal{F}^+(\mathcal{V})$ to some points of the ray so that the index i grows when we go along the ray, and when the ray moves in the interval I_k, we get a collection of curves together with local systems of

vector spaces on them for each sector between the Stokes rays. We erase the curves with the zero local systems on them.

Let us glue these curves with local systems, which we have so far in the sectors of the punctured disc determined by the Stokes rays. For each $i \neq a, a+1$, at a Stokes ray r, we glue the curves which carry the local systems gr_i, and then glue the local systems on them using the local system structure on $\mathcal{V}$ near the ray. Finally, we glue the curve which carries gr_a on the one side of r with the one with gr_{a+1} on the other side, and glue local systems on them via the isomorphisms provided by condition (179):

$$\mathrm{gr}_a\mathcal{F}^-(\mathcal{V}) \stackrel{\sim}{\longrightarrow} \mathrm{gr}_{a+1}\mathcal{F}^+(\mathcal{V}), \qquad \mathrm{gr}_{a+1}\mathcal{F}^-(\mathcal{V}) \stackrel{\sim}{\longrightarrow} \mathrm{gr}_a\mathcal{F}^+(\mathcal{V}). \tag{181}$$

We get a collection $\mathcal{L}$ of loops on the disc, which never turn back, with simple crossings at the Stokes rays. Each of the loops carries a local system of vector spaces.

The claim that we get an admissible sheaf follows from Lemma 10.6. Proposition 10.5 is proved. □

Remark. One can relax condition 1) in Definition 10.3 allowing $m > 2$ curves intersect at a point, with different tangents. Let us describe the corresponding analog of condition (179) in Definition 10.4. Recall that a pair of length m flags $\mathcal{F}_\bullet$ and $\mathcal{G}_\bullet$ in a vector space V is complimentary if $V = \mathcal{F}_a \oplus \mathcal{G}_b$ if $a+b = m$.

Lemma 10.7. *An admissible sheaf near an $m-$fold crossing point with different tangents, see Figure 57, is determined by a pair of vector spaces $V_0 \subset V$ and two length m complimentary flags in V/V_0.*

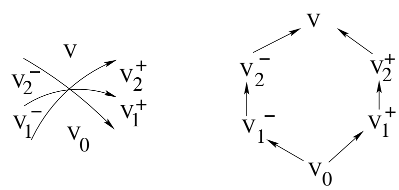


Figure 57: An admissible sheaf near a triple crossing point: the following two flags in V/V_0 are complimentary to each other: $V_1^-/V_0 \subset V_2^-/V_0$ and $V_1^+/V_0 \subset V_2^+/V_0$.

Proof. Given such an admissible sheaf, going from the bottom to the top on the left of the crossing points gives one flag, and going from the right gives the other flag. □

Then there is an analog of Proposition 10.5 if instead of (179) we have the following condition: *The left and right flags in the subquotient associated with the crossing point are opposite to each other.*

Our next goal is to explain how Stokes Legendrian links $\mathcal{L}$ and the categories $\mathcal{C}_0(\mathrm{D}^*, \mathcal{L})$ arise in complex geometry, as topological invariants of holomorphic vector bundles with connections on a punctured disc with any, possibly irregular, singularity at zero. For this we need to recall classification of connections on a formal punctured disc Spec $\mathbb{C}((z))$, due to Hukuhara, Turrittin, Levelt [M2].

2. Formal classification of irregular singularities. A Puiseux series is a Laurent series in $z^{1/d}$, $d \in \mathbb{Z}_{\geq 1}$. The field of all Puiseux series is the algebraic closure $\overline{\mathbb{C}((z))}$ of the field $\mathbb{C}((z))$ of Laurent series in z, with the Galois group $\widehat{\mathbb{Z}}$. The integral closure of the ring of Taylor series $\mathbb{C}[[z]]$ is the local ring $\mathcal{O}_{\overline{\mathbb{C}((z))}}$ of this field. It is given by Taylor series in $z^{1/d}$, $d \geq 1$.

Consider the quotient of the field of Puiseux series by its local ring:

$$\overline{\mathbb{C}((z))} \ / \ \mathcal{O}_{\overline{\mathbb{C}((z))}} = \lim_{d\geq 1} \ \mathbb{C}((z^{1/d})) \ / \ \mathbb{C}[[z^{1/d}]]. \tag{182}$$

Definition 10.8. *The space $\mathcal{G}$ is the space of orbits of the Galois action on the elements $f(z)$ in (182).*

So an element $f(z) \in \mathcal{G}$ is presented as a Puiseux polynomial in $z^{-1/d}$ for some positive integer d:

$$f(z) = \sum_{k \geq 1} \lambda_k z^{-k/d}, \qquad \lambda_k \in \mathbb{C}, \tag{183}$$

where we identify $\{f(z)\} \sim \{f(e^{2\pi i/d}z)\}$, and assume d is the smallest such integer.

Consider the $d:1$ cover of the formal punctured disc:

$$\pi_d : \text{Spec } \mathbb{C}((w)) \longrightarrow \text{Spec } \mathbb{C}((z)), \qquad z = w^d, \quad d \geq 1.$$

Take a holonomic $\mathcal{D}-$module on this cover, given by the tensor product of the $\mathcal{D}-$module generated by $e^{f(z)}$, and a local system $\mathcal{V}_w$ on Spec $\mathbb{C}((w))$:

$$e^{f(z)} \otimes \mathcal{V}_w.$$

The $\mathcal{V}_w$ is determined by a local system on the circle of the rays in the tangent space to 0 in the $w-$plane. We consider the push-forward of this $\mathcal{D}-$module under the map π_d:

$$\pi_{d*}(e^{f(z)} \otimes \mathcal{V}_w).$$

Theorem 10.9. *The category of holonomic $\mathcal{D}-$modules on the formal punctured disc is decomposed into blocks, parametrised by the elements of $\mathcal{G}$. So any holonomic $\mathcal{D}-$modules on* Spec $\mathbb{C}((z))$ *is isomorphic to a direct sum of the standard $\mathcal{D}-$modules*

$$\bigoplus_{j=1}^{m} \pi_{d_j *}(e^{f_j(z)} \otimes \mathcal{V}_{w_j}), \qquad f_j(z) \in \mathcal{G}. \tag{184}$$

3. Legendrian links $\mathcal{L}$ of Stokes type, and local Stokes data. Take a flat connection on a disc with a single singular point. By Theorem 10.9, its restriction to the formal punctured disc is a direct sum (184). So it determines a collection of different Puiseux polynomials with negative exponents:[11]

$$f_j(z) = \sum_{k \in \mathbb{Z}_{\geq 1}} \lambda_{j,k} z^{-k/d_j}, \qquad \lambda_{j,k} \in \mathbb{C}, \quad d_j \in \mathbb{Z}_{\geq 1}, \qquad j = 1, ..., m. \tag{185}$$

They define multivalued functions $f_j(z)$. Restricting them to a certain concrete but sufficiently small radius ε circle $\{\varepsilon e^{i\theta}\}$ centered at 0, we get a collection of curves $\{L_j\}$ given parametrically by

$$Z_j(\theta) := R_j(\theta) \cdot e^{i\theta}, \qquad R_j(\theta) := e^{\text{Re} f_j(\varepsilon e^{i\theta})} \qquad j = 1, ..., m. \tag{186}$$

Taking the 1-jet of the radial part $R_j(\theta)$ of the function $Z_j(\theta)$, we get a Legendrian knot L_j in the space $\mathcal{J}^1(S^1)$ of 1-jets of real functions on the circle. Explicitly,

$$\theta \longmapsto (\theta, R_j(\theta), dR_j(\theta)).$$

Their union is a Legendrian link $\mathcal{L}$ in $\mathcal{J}^1(S^1)$. We call it a Legendrian link of *Stokes stype.* For a generic small ε it fits Definition 10.3.

The push-forward of the local systems $\mathcal{V}_{w_j}$ provides a local system $\mathcal{V}_j$ of vector spaces on the loop L_j.

The pairs $(L_j, \mathcal{V}_j)$ describe the microlocal support of an admissible sheaf.

The filtration by the order of growth provides the filtration at all but Stokes rays.

Therefore we get a local Stokes data from Definition 10.4.

[11] An irreducible analytic flat connection on a punctured disc, restricted to the formal punctured disc, can be a sum of several $\mathcal{D}-$modules from different blocks.

Legendrian links $\mathcal{L}$ assigned to different sufficiently small ε are isotopic, although considered as collections of curves they may look differently. Therefore the corresponding categories are canonically equivalent by Theorem 9.9.

One can consider, as Deligne and Malgrange do [M1], [M2], the limiting case when $\varepsilon \to 0$. Then the corresponding Legendrian link may have non-transversal crossings, with different tangents. Note that Deligne-Malgrange Stokes rays are determined by the singularity, while our Stokes rays depend also on the choice of a small radius ε.

4. Example. Given an $n \times n$ matrix A with distinct eigenvalues $\{\lambda_j\}$, consider a differential equation

$$df = \frac{A}{z^2} f(z)dz. \tag{187}$$

Its solutions in the eigenbasis of A are linear combinations of functions $f_j(z) = e^{-\lambda_j/z}$. To draw the curve $\mathcal{L}$, we project orthogonally points $\{\lambda_j\}$ onto the oriented line obtained by rotating the real axis by an angle θ, exponentiate their coordinates on the line, and put them onto the positive ray in the direction θ. Rotating θ from 0 to 2π, these points move along the curve $\mathcal{L}$. Rotating $\theta \to \theta + \pi$, we change the order of the points on the ray to the opposite one, see Figure 58. So any two of these loops intersect. Therefore we get a collection of loops as in Definition 10.3.

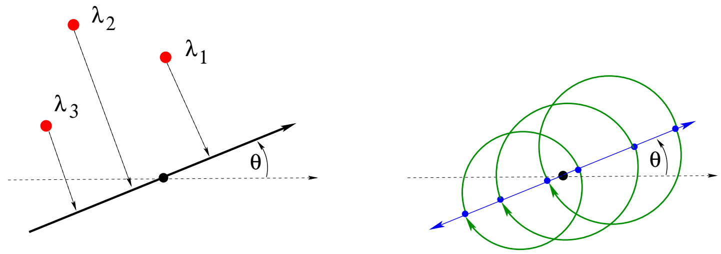


Figure 58: On the left: projecting red complex points $\lambda_1, \lambda_2, \lambda_3$ onto a line. On the right: rotating their exponentials - blue points on the axis. The curve $\mathcal{L}$ is the union of three green loops obtained this way.

The Stokes rays are the rays $\mathrm{Re}(\lambda_i - \lambda_j) = 0$ in $\mathbb{C}$. The local Stokes data from Definition 10.4 is given by filtration of the space of solutions in a sector around a non-Stokes ray r by the growth of a solution. It satisfies the tranversality assumption (179) if no three λ_i's are on the same line.

In a more general situation, solutions of type $f_j(z) = z^\alpha e^{-\lambda_j/z}$ describe one dimensional local systems on these circles with the monodromies $e^{2\pi i \alpha}$.

5. Legendrian links of analytic Stokes type. Only some Stokes Legendrians describe the singular type of a connection on a disc. We call them Legendrian links of *analytic Stokes* type.

One can describe Legendrian links of analytic Stokes type as follows. Take a radius ε sphere $S^3 \subset \mathbb{C}^2$, and delete from it the circle in the $x = 0$ plane. The obtained contact threefold $S^3_{(0)}$ is identified with $\mathcal{J}^1(S^1)$. Let us consider a germ of an algebraic curve C in $\mathbb{C}^2$, reduced, but possibly reducible, with a singularity at $(0, 0)$. So it is the set of zeros of a product of different irreducible polynomials in two variables. Intersecting the curve C with $S^3_{(0)} \subset \mathbb{C}^2$ we get a Legendrian link. All Legendrian links of analytic Stokes type are obtained this way.

We are now return back to arbitrary Stokes Legendrinans, and describe the corresponding category of admissible sheaves in terms of representations of a certain quiver.

6. Quiver description. Consider an oriented quiver $Q_{\mathcal{L}}$ with the vertices v_i assigned to the oriented loops L_i of $\mathcal{L}$, and the following arrows assigned to the loops and crossing points of $\mathcal{L}$, see Figure 59:

1. For each vertex v, there is a simple loop $l_v : v \longrightarrow v$ at this vertex.
2. For each crossing $q \in L_i \cap L_j$, we have an arrow $\alpha_q : v_i \to v_j$ if L_i is above L_j to the right of q.

Theorem 10.10. *1. The category $\mathcal{C}(\mathrm{D}^*, \mathcal{L})$ is abelian. It is equivalent to the category $\mathcal{R}(Q^*_{\mathcal{L}})$of representations of the quiver $Q_{\mathcal{L}}$, with invertible arrows l_v.*

2. If a Legendrian link $\mathcal{L}$ is of analytic Stokes type, that is arises from a singular point of a holomorphic vector bundle with connection on D^*, possibly irregular, then the stack $\mathcal{M}_{\mathrm{D}^*}(\mathcal{L})$ is equivalent to the Betti stack of given irregular type, also known as the stack of Stokes data.

Remark. Since the category of representations of the quiver $Q_{\mathcal{L}}$ is abelian, the category $\mathcal{C}_{\mathrm{D}^*}(\mathcal{L})$ is abelian. The category $\mathcal{C}(\mathrm{D}^*, \mathcal{L})$ assigned to a general ideal Legendrian is not abelian. Indeed, if $\mathcal{L}$ is isotopic to a collection of concentric circles, we get the category of filtered vector spaces.

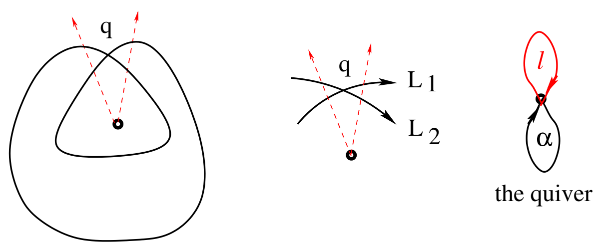


Figure 59: At the crossing point q, the local arc L_1 is above the one L_2, producing a loop α of the quiver. The loop l on the diagram, shown red, corresponds to the loop itself

Proof. i) Pick a reference point p_i at each loop L_i of $\mathcal{L}$.

For each intersection/selfintersection point $q \in L_i \cap L_j$, let κ_1, κ_2 be little intervals on the loops L_i, L_j containing q. They are ordered so that the interval κ_1 goes above κ_2 to the right of q, see Figure 59. Pick paths in the positive direction connecting the point q with reference points p_i and p_j.

Let us assign to a representation $\mathcal{R}$ of the quiver $Q_{\mathcal{L}}$ a sheaf $\mathcal{S}_{\mathcal{R}}$ on the punctured disc D^*.

First, we define a local system $\mathcal{V}_i$ on each loop L_i. Recall the vertex v of the quiver assigned to the loop L_i. Put the vector space $\mathcal{R}(v)$ to the reference point p_i on L_i. Then the operator $\mathcal{R}(l_v) : \mathcal{R}(v) \longrightarrow \mathcal{R}(v)$ for the loop l_v desrcibes the monodromy of the local system on the loop L_i in the positive direction.

Next, we define the fiber of the sheaf $\mathcal{S}_{\mathcal{R}}$ at a point $d \in \mathrm{D}^* - \{\text{Stokes rays}\}$. Denote by $\gamma_1, \ldots, \gamma_k$ the little intervals on the curve $\mathcal{L}$ containing the intersection points of the segment connecting 0 and d with $\mathcal{L}$. Then fiber of the sheaf $\mathcal{S}_{\mathcal{R}}$ at d is the direct sum of the fibers of the local systems $\mathcal{V}_i$ at the intervals:

$$\mathcal{S}_{\mathcal{R}}(d) := \oplus_{i=1}^{k} \mathcal{V}_i(\gamma_i).$$

We order the intervals away from 0, and filter the fiber by subspaces $\oplus_{i=1}^{p} \mathcal{V}_i(\gamma_i)$.

We extend the sheaf to $\mathrm{D}^* - \{\text{crossing points}\}$ in the obvious way. Note that the filtration above is defined only away from the Stokes rays. Let us use the maps $\mathcal{R}(\alpha_q)$ assigned to the arrows α_q of the quiver to extend the sheaf to the crossing points.

Denote by $\mathcal{V}_{\kappa_i}$ the fiber of the local system on $\mathcal{L}$ over the interval κ_i, $i = 1, 2$. The stalk $\mathcal{S}_{\mathcal{R}}(q)$ of the sheaf at a crossing point q is mapped by the restriction map to the stalks $\mathcal{S}_{\mathcal{R}}(q_{\pm})$ on the right/left of q.

These maps are isomorphisms, since the sheaf does not have microlocal support in those directions. So an extension of the sheaf to the point q provides isomorphisms

$$\mathcal{V}_{<q} \oplus \mathcal{V}_{\kappa_1} \oplus \mathcal{V}_{\kappa_2} =: \mathcal{S}_{\mathcal{R}}(q_-) \stackrel{\sim}{\longleftarrow} \mathcal{S}_{\mathcal{R}}(q) \stackrel{\sim}{\longrightarrow} \mathcal{S}_{\mathcal{R}}(q_+) := \mathcal{V}_{<q} \oplus \mathcal{V}_{\kappa_1} \oplus \mathcal{V}_{\kappa_2}.$$

So to extend the sheaf to the point q one just needs to define a linear automorphism of $\mathcal{V}_{<q} \oplus \mathcal{V}_{\kappa_1} \oplus \mathcal{V}_{\kappa_2}$, which is the identitity on $\mathcal{V}_{<q}$, and preserves $\mathcal{V}_{\kappa_1} \oplus \mathcal{V}_{\kappa_2}$. We define the latter by setting

$$(v_1, v_2) \longrightarrow (v_1, \mathcal{R}(\alpha_q)(v_1) + v_2). \qquad \mathcal{R}(\alpha_q) : V_{\kappa_1} \longrightarrow V_{\kappa_2}. \tag{188}$$

Then it also preserving the filtration $V_{\kappa_1} \subset V_{\kappa_1} \oplus V_{\kappa_2}$, as required in our situation.

This functor is an equivalence. This is a non-trivial result, and we do not present here a proof.

ii) This is a reformulation of the known description [DMR], [M2] of the analytic Stokes data in terms of the filtration of the space of solutions given by the order of growth, and the way it changes when we cross Stokes rays. □

7. Global Stokes data. Below S is an oriented topological surface, possibly with punctures.

Definition 10.11. *A Legendrian $\mathcal{L}$ on S is* ideal / Stokes type*, if it admissibly isotopic to a disjoint union of Legendrians $\mathcal{L}_p$ from Definition 10.1 / 10.3 near punctures p on S:*

$$\mathcal{L} = \cup_p \mathcal{L}_p, \qquad \mathcal{L}_p \cap \mathcal{L}_q = \emptyset.$$

Definition 10.12. *A* Stokes data *on S is a local system $\mathcal{V}$ of vector spaces on S, plus a local Stokes data at the specialization of $\mathcal{V}$ to each puncture.*

The following Proposition is an immediate consequence of Proposition 10.5.

Proposition 10.13. *Given an ideal Legendrian $\mathcal{L}$ on S, the category of Stokes data on S of the topological type $\mathcal{L}$ is equivalent to the category $\mathcal{C}_0(S; \mathcal{L})$ of admissible sheaves on S with the microlocal support on $\mathcal{L}$ and the zero section of T^*S, which vanish near the marked points.*

8. Non-commutative stacks of Stokes data. Denote by $\mathcal{S}(\mathcal{L}, d)$ the stack of Stokes data of a given topological type $(\mathcal{L}, d)$ on S. The topological types are described by the pairs $(\mathcal{L}, d)$, where $\mathcal{L}$ is an ideal Legendrian links on $\mathrm{ST}(S)$ from Definition 10.11, modulo admissible Legendrian isotopies, equipped with local systems of vector spaces on the connected components $\mathcal{L}_\alpha$ of the Legendrian $\mathcal{L}$. Dimensions of local systems are encoded by the dimension vector $d = \{d_\alpha\}$. When $\mathcal{L}$ is empty, we recover the stack $\mathrm{Loc}_m(S)$.

Definition 10.14. *The non-commutative stack of Stokes data $\mathcal{S}^{\mathrm{nc}}(\mathcal{L}, d)$ of the topological type $(\mathcal{L}, d)$ is the stack $\mathcal{M}_0(\mathcal{L}, d)$ of $\mathcal{L}-$admissible dg-sheaves of $R-$vector spaces on S vanishing near the punctures of S, whose microlocal support on $\mathcal{L}_\alpha$ is a local system on the loop $\mathcal{L}_\alpha$ of dimension d_α:*

$$\mathcal{S}^{\mathrm{nc}}(\mathcal{L}, d) := \mathcal{M}_0(\mathcal{L}, d).$$

10.2 The Riemann-Hilbert correspondence and the wild mapping class group

1. The classical Riemann-Hilbert correspondence. Given a Riemann surface Σ, recall the de Rham stack $\mathcal{M}^{\mathrm{reg}}_{\mathrm{DR}}(m, \Sigma)$ of rank m vector bundles with meromorphic connections with regular singularities on Σ. Its Betti variant is the stack $\mathrm{Loc}_m(S)$ of local systems of $m-$dimensional vector spaces on the topological surface S underlying Σ.

The classical Riemann-Hilbert correspondence [D] identifies the associated complex analytic stacks:

$$\mathcal{RH}^{\mathrm{reg}}_\Sigma : \mathcal{M}^{\mathrm{reg}}_{\mathrm{DR}}(m, \Sigma)(\mathbb{C}) \stackrel{\sim}{\longrightarrow} \mathrm{Loc}_m(S)(\mathbb{C}).$$

However the algebraic structures of these stacks are different. The algebraic structure of $\mathcal{M}_{\mathrm{DR}}^{\mathrm{reg}}(m,\Sigma)$ depends on the complex structure on Σ. When Σ varies, we get the algebraic de Rham stack $\mathcal{M}_{\mathrm{DR}}^{\mathrm{reg}}(m)$ over the moduli space $\mathcal{M}_{g,n}$. Contrary to this, the algebraic structure of the Betti stack depends only on the topology of Σ. When Σ varies, we get a local system of the Betti stacks over $\mathcal{M}_{g,n}$. Equivalently, the mapping class group $\Gamma_S = \pi_1(\mathcal{M}_{g,n})$ of S acts by automorphisms of the stack $\mathrm{Loc}_m(S)$.

Summarising, we get the canonical universal Riemann-Hilbert equivalence:

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{DR}}^{\mathrm{reg}}(m)(\mathbb{C}) & \xrightarrow{\mathcal{RH}} & \mathcal{M}_{\mathrm{B}}^{\mathrm{reg}}(m)(\mathbb{C}) \\ & \searrow \quad \swarrow & \\ & \mathcal{M}_{g,n} & \end{array} \tag{189}$$

To generalize it to irregular singularities, we need the Betti side, given by moduli spaces of Stokes data.

2. The Riemann-Hilbert correspondence. We start with a pair $(S,\mathcal{L})$, where $\mathcal{L} = \cup\mathcal{L}_p$ is a Legendrian link of analytic Stokes type on a surface S, considered up to admissible isotopies.

Definition 10.15. *The moduli space $\mathcal{P}_{S,\mathcal{L}}$ parametrises data $(\Sigma,\{f_j\})$, where Σ is a Riemann surface of the topological type S, and $\{f_j \in \mathcal{G}_p\}$ is a collection of distinct Puiseux polynomials*[12] *from Definition 10.8 near the punctures p on Σ, with a given admissible isotopy class of the corresponding Legendrian $\mathcal{L}$.*

Forgetting Puiseux polynomilas, we get a projection, where S is a genus g surface with n punctures:

$$\mathcal{P}_{S,\mathcal{L}} \longrightarrow \mathcal{M}_{g,n}. \tag{190}$$

Given a Legendrian link $\mathcal{L}$ of Stokes type on S, there are the following categories and stacks:

1. The category of admissible sheaves $\mathcal{C}(S,\mathcal{L})$, and the stack of its objects $\mathcal{M}(S,\mathcal{L})$.

2. The DeRham stack $\mathcal{M}_{\mathrm{DR}}(S,\mathcal{L})$ of holomorphic vector bundles with connection over a Riemann surface Σ of topological type S, with the given homotopy class of the Legendrian link $\mathcal{L}$ assigned to its formal equivalence class. It is fibered over the stack $\mathcal{P}_{S,\mathcal{L}}$:

$$\mathcal{M}_{\mathrm{DR}}(S,\mathcal{L}) \longrightarrow \mathcal{P}_{S,\mathcal{L}}. \tag{191}$$

3. The Betti stack $\mathcal{M}_{\mathrm{B}}(S,\mathcal{L})$, which is a local system of stacks $\mathcal{M}_{\mathrm{B}}(S,\mathcal{L})$ over $\mathcal{P}_{S,\mathcal{L}}$.

Its fiber over a point $(\Sigma,\{f_j\}) \in \mathcal{P}_{S,\mathcal{L}}$ is the stack $\mathcal{M}(S,\mathcal{L})$ for the Legendrian link $\mathcal{L}$ of $\{f_j\}$.

The fact that it is indeed a well defined local system of stacks, e.g. does not depend on the choice of radius ε in the assignment $\{f_j\} \longmapsto \mathcal{L}$, and locally constant over the base, is guaranteed by Theorem 9.9.

The wild Riemann-Hilbert correspondence [M2] is a complex analytic equivalence of stacks over $\mathcal{P}_{S,\mathcal{L}}$:

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{DR}}(S,\mathcal{L}) & \xrightarrow{\mathcal{RH}} & \mathcal{M}_{\mathrm{B}}(S,\mathcal{L}) \\ & \searrow \quad \swarrow & \\ & \mathcal{P}_{S,\mathcal{L}} & \end{array} \tag{192}$$

The algebraic structure of the fibers of the De Rham stack over $\mathcal{P}_{S,\mathcal{L}}$ varies, while for the Betti stack it does not. The Betti stack form a local system of stacks over $\mathcal{P}_{S,\mathcal{L}}$. In particular, we can define the analog of isomonodromic deformation for irregular holomorphic connections.

[12] The space $\mathcal{G}_p$ is defined via the local ring $\mathcal{O}_p$ of functions at a point $p \in \Sigma$. It does not depend on a local parameter z.

3. The wild mapping class group. Comparing the classical Riemann-Hilbert correspondence (189) with the one (192), we see that the right analog of $\mathcal{M}_{g,n}$ is the moduli space $\mathcal{P}_{S,\mathcal{L}}$, and arrive at

Definition 10.16. *The* wild mapping class group *is the fundamental group* $\Gamma_{S,\mathcal{L}} := \pi_1(\mathcal{P}_{S,\mathcal{L}})$.

The wild mapping class group $\Gamma_{S,\mathcal{L}}$ acts by automorphisms of the stack $\mathcal{M}(S,\mathcal{L})$.

The map (190) provides a surjection $\Gamma_{S,\mathcal{L}} \longrightarrow \Gamma_S$. The group $\Gamma_{S,\mathcal{L}}$ is an extension of the mapping class group Γ_S by the product over the punctures p of S of the local wild mapping class groups $\Gamma_{\mathrm{D}^*,\mathcal{L}_p}$:

$$1 \longrightarrow \prod_p \Gamma_{\mathrm{D}^*,\mathcal{L}_p} \longrightarrow \Gamma_{S,\mathcal{L}} \longrightarrow \Gamma_S \longrightarrow 1. \tag{193}$$

Example. In example (187) in Section 10.1, the space $\mathcal{P}_{\mathrm{D}^*,\mathcal{L}}$ for the punctured disc is given by collections of distinct complex numbers $\{\lambda_j\}$. The wild mapping class group $\Gamma_{S,\mathcal{L}}$ is the braid group Br_n.

Below we will discuss topological ramifications of this definition.

4. Three flavors of Legendrian mapping class groups. There are three flavors of the space of Legendrian links $\mathcal{L} \subset \mathrm{ST}^*(S)$ and Hamiltonian isotopies betwen them. Therefore there are three flavors of Legendrian mapping class group, defined as the fundamental group of the connected component of $\mathcal{L}$.

1. The space of all Legendrians $\mathcal{L} \subset \mathrm{ST}^*(S)$ and Hamiltonian isotopies. The *Legendrian mapping class group* $\Gamma_{\mathrm{leg},S,\mathcal{L}}$ is the fundamental group of the connected component containing $\mathcal{L}$. By [GKS], it acts by autoequivelences of the triangulated/dg category of constructible sheaves on S.

2. Legendrians given by smooth loops in S, and admissible Hamiltonian isotopies between them. The *admissible Legendrian mapping class group* $\Gamma_{\mathrm{adm},S,\mathcal{L}}$ is the fundamental group of the component of $\mathcal{L}$. By Theorems 9.8 - 9.9, it acts by autoequivelences of the category $\mathcal{C}(S,\mathcal{L})$ of admissible dg-sheaves.

3. The ideal Legendrians $\mathcal{L}$ which are, by the definition, admissibly isotopic to a union of local ideal Legendrians $\mathcal{L}_p$ near the punctures p of S, modulo admissible Hamiltonian isotopies. The *wild mapping class group* $\Gamma_{S,\mathcal{L}}$ is the fundamental group of the connected component of $\mathcal{L}$. It acts by preserving cluster structures of the moduli spaces of Stokes data, see Section 10.3.

These spaces are nested: (3) $\subset$ (2) $\subset$ (1). Therefore there are maps of the corresponding groups:

$$\Gamma_{S,\mathcal{L}} \longrightarrow \Gamma_{\mathrm{adm},S,\mathcal{L}} \longrightarrow \Gamma_{\mathrm{leg},S,\mathcal{L}}. \tag{194}$$

5. A combinatorial description of the wild mapping class group. Given a Weyl group W, consider the following stratified space $\mathcal{C}_W$. Its strata $\mathcal{S}_\beta$ are given by collections $\beta = \{w_1, ..., w_n\}$ of the cyclically ordered elements of W assigned to distinct points on a circle, so that the cyclic order of the points on the circle coincides with the one of β. So $\mathcal{S}_\beta$ is homeomorphic to $S^1 \times \mathbb{R}^{n-1}$.

If $l(w_iw_{i+1}) = l(w_i)+l(w_{i+1})$, the stratum $\mathcal{S}_{w_1,...,w_iw_{i+1},...,w_n}$ lies on the boundary of $\mathcal{S}_{w_1,...,w_i,w_{i+1},...w_n}$. Denote by $\mathcal{C}_\beta$ the connected component of the cell complex $\mathcal{C}_W$ containing the strata $\mathcal{S}_w$. The space $\mathcal{C}_\beta$ is determined by the image $[\beta]$ of the element β in the cyclic envelope of the braid semigroup Br^+_W assigned to W, discussed in Section 10.4.

Let $W = S_N$. Then $[\beta]$ determines a cyclic braid. Denote the corresponding Legendrian by $\mathcal{L}_{[\beta]}$.

Proposition 10.17. *The local wild mapping class group* $\Gamma_{\mathcal{L}_{[\beta]}}$ *is the fundamental group of* $\mathcal{C}_\beta$*:*

$$\Gamma_{\mathcal{L}_{[\beta]}} := \pi_1(\mathcal{C}_\beta).$$

10.3 Non-commutative cluster structure of stacks of Stokes data

1. Non-commutative stacks of framed Stokes data. Just like in the regular case, to quantize the stack of Stokes data $\mathcal{S}(\mathcal{L}, d)$, we introduce a larger stack $\mathcal{X}(\mathcal{L}, d)$ of *framed* Stokes data. Namely, we add a framing, defined as a complete filtration of the local system $\mathcal{L}_\alpha$ on each loop α of the Legendrian $\mathcal{L}$, invariant under the monodromy around the loop.

We interpret a filtration on a d_j-dimensional local system on the loop α_j of the Legendrian $\mathcal{L}$ as follows. We replace the loop α_j by a collection of d_j loops $\alpha_j^{(1)}, \ldots, \alpha_j^{(d_j)}$ obtained by expanding the loop α_j out of the puncture by $\varepsilon, 2\varepsilon, ..., d_j\varepsilon$, see Figure 60. We equip each loop $\alpha_j^{(p)}$ with the local system given by gr^p of the filtration on the loop α_j. So get a new link $\mathcal{L}_d$ supporting one dimensional local systems.

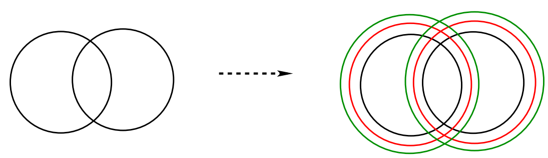


Figure 60: Inflating a Legendrian link.

Set $\mathbf{1} := (1, ..., 1)$. Consider the stack $\mathcal{M}_0(\mathcal{L}_d, \mathbf{1})$ of $\mathcal{L}_d-$admissible dg-sheaves on S, vanishing near the punctures, whose microlocal support is described by the union of the zero section and arbitrary 1-dimensional local systems on the Legendrian $\mathcal{L}_d$. The next observation generalizes Lemma 1.13.

Lemma 10.18. *The non-commutative stack $\mathcal{M}_0(\mathcal{L}_d, \mathbf{1})$ is canonically equivalent to the one of framed Stokes data of type $(\mathcal{L}, d)$:*

$$\mathcal{M}_0(\mathcal{L}_d, \mathbf{1}) = \mathcal{X}(\mathcal{L}, d). \tag{195}$$

There is a projection given by assigning the microlocal support local system on the Legendrian $\mathcal{L}_d$:

$$\mu : \mathcal{M}_0(\mathcal{L}_d, \mathbf{1}) \longrightarrow \mathrm{Loc}_1(S^1)^{|d|}, \qquad |d| := \sum_\alpha d_\alpha. \tag{196}$$

The product of the symmetric groups $\prod_\alpha S_{d_\alpha}$ acts by birational automorphisms of $\mathcal{X}(\mathcal{L}, d)$. To define it, note that given a generic $p-$dimensional $R-$local system on a circle, there are $p!$ complete filtrations of the local system. Indeed, a generic linear automorphism L of an $p-$dimensional vector space V_p is decomposed uniquely into a direct sum of $L-$invariant one-dimensional subspaces. The filtrations are parametrised by orderings of these subspaces. Therefore the symmetric group S_p acts on the fltrations.

This just means that the canonical projection, obtained by forgetting the filtrations

$$p : \mathcal{X}(\mathcal{L}, d) \longrightarrow \mathrm{Loc}(\mathcal{L}, d) \tag{197}$$

is a Galois cover at the generic point with the Galois group $\prod_\alpha S_{d_\alpha}$.

Theorem 10.19. *For any decorated surface $\mathbb{S}$, and any topological data $(\mathcal{L}, d)$ where $\mathcal{L}$ is an ideal Legendrian on $\mathbb{S}$, the stack $\mathcal{X}(\mathcal{L}, d)$ of framed non-commutative Stokes data has a non-commutative cluster Poisson structure equivariant under the wild cluster mapping class group $\Gamma_{\mathcal{L},d}$.*

In the commutative case, the subalgebra $\mu^*\mathcal{O}(\mathbb{G}_m^{|d|})$, see (196), is the Poisson center of $\mathcal{O}(\mathcal{X}(\mathcal{L}, d))$.

We prove Theorem 10.19 assuming that in the case when $\mathbb{S} = S^2 - \{\infty\}$ the Legendrian $\mathcal{L}$ is not very degenerate, see Theorem 10.21 for the precise statement.

2. Idea of the proof of Theorem 10.19. Lemma 10.18 tells that any stack of framed Stokes data has a canonical realization as a stack of admissible sheaves with the following two features:

i) The admissible sheaves vanish near the punctures.
ii) The microlocal support local systems on the Legendrian $\mathcal{L}_d$ are <u>one-dimensional</u>.

We show that the Legendrian link $\mathcal{L}_d$ has an admissible deformation to (a slight modification of) the web $\mathcal{Z}_\Gamma$ of zig-zag strands for a certain bipartite graph Γ on S. By the invariance of the stack of admissible dg-shaves under admissible deformations, $\mathcal{M}(\mathcal{L}_d, \mathbf{1}) = \mathcal{M}(\mathcal{Z}_\Gamma, \mathbf{1})$. By Lemma 1.12, the stack $\mathrm{Loc}_1(\Gamma)$ of 1-dimensional local systems on Γ is realised as the open substack defined by the condition of vanishing on mixed domains: $\mathrm{Loc}_1(\Gamma) \subset \mathcal{M}(\mathcal{Z}_\Gamma, \mathbf{1})$. Since vanishing on mixed domains implies vanishing near the punctures, we get an open embedding

$$\mathrm{Loc}_1(\Gamma) \subset \mathcal{M}_0(\mathcal{L}_d, \mathbf{1}) = \mathcal{X}(\mathcal{L}, d).$$

The last equivalence is given by Lemma 1.12. This way we get an open cluster Poisson torus assigned to the graph Γ. The tori assigned to different graphs Γ provide the cluster atlas. The elements of the cluster wild mapping class group transform such a graph Γ to another one Γ', related to Γ by two by two moves. Therefore we get a cluster Poisson atlas equivariant under the wild mapping class group.

Before we start the proof, let us recall the following.

3. Bipartite graphs and triple crossing diagrams. Let us recall triple crossing diagrams [Th].

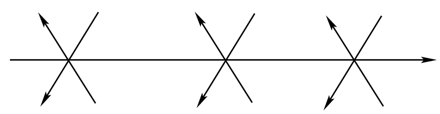


Figure 61: A triple crossing diagram.

Definition 10.20. *A triple crossing diagram is a collection $\mathcal{I}$ of smooth oriented strands on an oriented decorated surface $\mathbb{S}$ such that:*
1. *Each intersection point of $\mathcal{I}$ is a triple point.*
2. *Strand orientations are consistent with an orientation of each component of $\mathbb{S} - \mathcal{I}$.*
3. *Each strand is either a loop, or ends on the boundary of $\mathbb{S}$. Endpoinds of strands are disjunct.*

See counterexamples to condition 2) on Figure 62). Note that condition 2) implies the following:
• For each intersection point, passing strands are oriented as on Figure 61.
• Going along a strand of $\mathcal{I}$, the orientations of the intersecting it strands alternate, as on Figure 61.

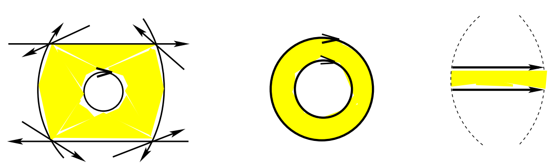


Figure 62: Strand orientations are not consistent with any orientations of internal (yellow) components.

Any bipartite graph can be transformed to a bipartite graph with 3-valent $\bullet-$vertices obtained by splitting the $\bullet-$vertices of valency > 3 by adding edges, and introducing 2-valent $\circ-$vertices in the middle of these edges. A bipartite ribbon graph Γ with 3-valent $\bullet-$vertices gives rise to a triple crossing diagram on the associated surface S_Γ, provided by its zig-zag strands, [GK, Section 2.6]. Namely, we first push zig-zag strands out of the graph, and then shrink the triangles near the $\bullet-$vertices to points, see Figure 47. Then all original double crossings collide into triple crossings corresponding to the $\bullet-$vertices.

This construction can be inverted. Any triple crossing diagram $\mathcal{I}$ on $\mathbb{S}$ has a canonical up to isotopy resolution defined as follows [GK, Section 2.3]. For each crossing, move slightly one of the strands, inflating the crossing into a little triangle, whose strands are oriented consistently with the orientation of $\mathbb{S}$ - counterclockwise - getting a web $\mathcal{I}'$. There are three types of components of $\mathbb{S} - \mathcal{I}'$, see Figure 47:

i) *Black* components, whose boundary strands are oriented clockwise.
ii) *White* components, whose boundary strands are oriented counterclockwise.
iii) *Mixed* components, whose boundary strands orientations alternate.

We shrink all black (respectively white) components to black (respectively white) vertices, and connect two vertices by an edge if they are separated by a crossing of the web $\mathcal{I}'$. We get a bipartite graph $\Gamma_{\mathcal{I}}$.

4. Ideal Legendrians → triple crossing diagrams → non-commutative cluster Poisson varieties. Take an ideal Legendrian $\mathcal{L}$ on a punctured surface S, whose local components $\mathcal{L}_p$ are ideal Legendrians near punctures p. So each $\mathcal{L}_p$ satisfies conditions of Definition 10.1, different $\mathcal{L}_p$ are disjunct, and $\mathcal{L} = \cup_p \mathcal{L}_p$. The number of crossings of a generic ray r from p with $\mathcal{L}_p$ is the *local rank* of $\mathcal{L}_p$ at p. An ideal Legendrian $\mathcal{L}$ is of *rank* m if for each local Legendrian $\mathcal{L}_p$ its local rank at p is equal to m.

Theorem 10.21. *Any rank m ideal Legendrian $\mathcal{L}$ on a punctured surface S different from $S^2 - \{\infty\}$, or a rank m ideal Legendrian on $S^2 - \{\infty\}$ corresponding to a cyclic closure of the braid, $w_0 x w_0 y$, for any $x, y \in \mathrm{Br}_m$, can be admissibly isotoped to a triple crossing diagram with the following property:*

The black and white components are either contractible, or homeomorphic to an annulus.
The union of all annuli is a disjoint union of punctured discs around (some of the) punctures.

Before we proceed to the proof of Theorem 10.21, let us discuss its main implications.

The triple crossing diagrams $\mathcal{L}$ which appear in our story have black and white components which are either contractible, or annuli. Furthermore, let us assume that the union of all annuli is given by a union of discs around the punctures. Assuming these conditions, consider the category $\mathcal{C}(S; \mathcal{L}, \mathbf{1})$ of $\mathcal{L}-$admissible dg-sheaves such that the local systems of the microlocal support supported on the components of $\mathcal{L}$ are 1-dimensional. Denote by $\mathcal{M}_0(S; \mathcal{L}, \mathbf{1})$ the substack of the stack of objects $\mathcal{M}(S; \mathcal{L}, \mathbf{1})$ of the category $\mathcal{C}(S; \mathcal{L}, \mathbf{1})$ given by $\mathcal{L}-$admissible dg-sheaves which are zero on the mixed components.

Lemma 10.22. *The stack $\mathcal{M}_0(S; \mathcal{L}, \mathbf{1})$ is a torus.*

Proof. According to our assumption, the annuli surrounding a given puncture □

Corollary 10.23. *Any rank m ideal Legendrian $\mathcal{L}$ on S gives rise to a Zariski open torus $\mathcal{M}_0(S; \mathcal{L}, \mathbf{1})$ in the non-commutative moduli stack $\mathcal{M}(S; \mathcal{L}, \mathbf{1})$ of $\mathcal{L}-$admissible dg-sheaves with one-dimensional local systems on the components of $\mathcal{L}$.*

We will proceed to the proof of Theorem 10.21, after discussing two examples.

5. The simplest example. The stack of $m-$dimensional local systems on $S^2 - \{0, \infty\}$ with complete filtrations near the punctures is just the stack of admissible sheaves vanishing near the punctures for the oriented web $\mathcal{L}_m$, illustrated for $m = 3$ on the left of Figure 63. An admissible deformation of $\mathcal{L}_m$ shown on the right describes the following constructible sheaves. Denote by $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_{2m+1}$ the open components of $S^2 - \mathcal{L}_m$, counted from 0. Then the sheaves are zero on $\mathcal{A}_{2k+1}$, and are 1-dimensional local systems on $\overline{\mathcal{A}}_{2k}$. Their moduli space is $(\mathbb{G}_m)^m$, parametrising the monodromies.

Figure 63: Left: the link $\mathcal{L}_3$ describing the stack of rank 3 local systems on $\mathbb{C}^*$ with complete filtrations near the punctures. Right: admissible deformed link providing an open locus identified with $\mathbb{G}_m^3$.

Figure 64: Left: a triple crossing diagram from two intersecting clockwise oriented loops around the puncture. Right: the corresponding bipartite graph.

6. Another simple example. Let $S^2 - \{0, \infty\}$ be a sphere a regular puncture at ∞, and an irregular one at 0. The Legendrian $\mathcal{L}_0$ goes clockwise around 0. It bounds a domain $\mathcal{D}_{\mathcal{L}_0}$. Precisely, take a point $x \in \mathrm{D}^*$. Let r_x be the ray from p to x. Then $x \in \mathcal{D}_{\mathcal{L}_0}$ if and only if it is in the convex hull of $\{r_x \cap \mathcal{L}_0\}$.

Now we follow [FG3]. Call the selfintersection points of $\mathcal{L}_0$ *vertices*. Take a component $\mathcal{V}$ of $\mathcal{D}_{\mathcal{L}_0} - \mathcal{L}_0$. Connect the rightmost vertex on the boundary of $\mathcal{V}$ with the leftmost one by a counterclockwise path $\gamma_{\mathcal{V}} \subset \mathcal{V}$, transversal to $\mathcal{L}_0$. Take the union of these paths, a counterclockwise arc β near ∞, and $\mathcal{L}_0$, see Figure 64:

$$\mathcal{I} := \cup_{\mathcal{V}} \gamma_{\mathcal{V}} \cup \beta \cup \mathcal{L}_0. \tag{198}$$

It is a triple point diagram. Its number of counterclockwise arcs is the same as the local rank of $\mathcal{L}_0$ at 0.

7. Proof of Theorem 10.21. Take a triangulation $\mathcal{T}$ of S with vertices at the punctures. Inflate each side of $\mathcal{T}$ to a pair of thin *bigons*, see Figure 65. Then S - {the bigons} = a union of *internal triangles*. Performing admissible isotopies, we can assume that $\mathcal{L}$ crosses internal triangles t and bigons b as follows.

1. For each vertex v of t: the $\mathcal{L}$ induces m non-intersecting, clockwise (red) arcs around v.

2. For each bigon b: near one of its angles, called the *red angle*, $\mathcal{L}$ induces m non-intersecting clockwise arcs near its vertex. We put no conditions on the restriction of $\mathcal{L}$ to the other one, called the *blue angle*.

3. For each vertex v of each edge of $\mathcal{T}$: just one of the bigons at this edge has a red angle at v.

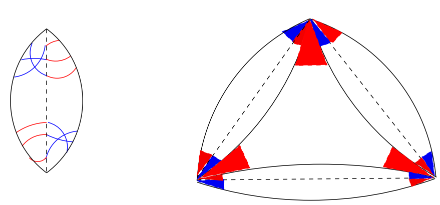


Figure 65: Inflating an edge to a pair of bigons. Inflating edges of a triangle to pairs of colored bigons.

Let b be a bigon. Denote by $\mathcal{L}_b$ the Legendrian in the bigon.

Lemma 10.24. *One can admissibly isotope the Legendrian $\mathcal{L}_b$ to a Legendrian $\mathcal{L}'_b$ so that*

1. *On each side of the bigon b, the endpoints of the red and blue strands of $\mathcal{L}'_b$ alternate.*

2. The strands of $\mathcal{L}'_b$ form a triple crossing diagram.

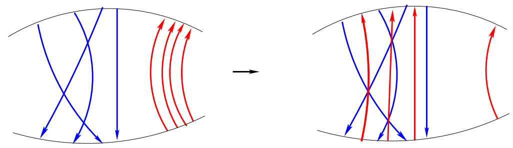


Figure 66: From concentric red and blue arcs near the vertices of a bigon, to a triple crossing diagram.

Proof. Move the red arcs towards the blue angle of b so that condition 1) holds. Then there is a unique up to isotopy way to deform them to a triple crossing diagram, with the minimal number of crossings.

Namely, forget the red arcs. Then for each component of the complement to the blue arcs pick the leftmost vertex/boundary point, and the similar rightmost one, and connect them by the unique up to isotopy red arc inside the component, see Figure 66. We get a triple crossing diagram. □

Lemma 10.24 provides triple crossing diagrams in the bigons. For each internal triangle t, see Figure 65, there is a standard triple crossing diagram provided by the $m-$trianulation of the triangle, which induces on the boundary the given alternating sequence of intersection points of $\mathcal{L}'_b$ provided by Lemma 10.24. Indeed, take the $m-$trianulation of a triangle, see Figure 67, and move each of its strands towards the parallel to it side of the triangle, so that the centers of looking down small triangles become the triple crossing points. Then the endpoints alternate, and we can match them with the ones on the bigons $\mathcal{L}'_b$. We denote this triple crossing diagram by $\mathcal{L}'_t$. Take the union of these triple point diagrams:

$$\mathcal{L}_{\mathcal{T}} := \cup_t \mathcal{L}'_t \bigcup \cup_b \mathcal{L}'_b. \tag{199}$$

Theorem 10.21 is proved.

Figure 67: A triple crossing diagram assigned to an m-triangulation of the triangle t, for $m = 1, 2$.

10.4 Cluster structure of stacks of $\mathrm{G}-$Stokes data on surfaces

1. The cyclic envelope $\mathrm{Br}^+_{\mathfrak{g}}$ of the braid semigroup. Let G be a split semi-simple algebraic group over $\mathbb{Q}$. Denote by I the set of positive simple roots, and by C_{ij} the Cartan matrix of the root system of the Lie algebra $\mathfrak{g}$ of G. Let $\mathrm{Br}_{\mathfrak{g}}$ (respectively $\mathrm{Br}^+_{\mathfrak{g}}$) be the braid group (respectively semigroup) of the root system of $\mathfrak{g}$. It is generated by the elements $s_i, i \in \mathrm{I}$, subject to the relations

$$\begin{array}{lll} s_is_j = s_js_i & \text{if} & C_{ij} = C_{ji} = 0, \\ s_is_js_i = s_js_is_j & \text{if} & C_{ij} = C_{ji} = -1, \\ s_is_js_is_j = s_js_is_js_i & \text{if} & C_{ij} = 2C_{ji} = -2, \\ s_is_js_is_js_is_j = s_js_is_js_is_js_i & \text{if} & C_{ij} = 3C_{ji} = -3. \end{array} \tag{200}$$

The Weyl group W is the quotient of the group $\mathrm{Br}_{\mathfrak{g}}$ by the relations $s_i^2 = 1$. Denote by $l(*)$ the length function on W. There is a set theoretic section $\mu : W \to \mathrm{Br}^+_{\mathfrak{g}}$, such that $\mu(s_i) = s_i$ and $\mu(ss') = \mu(s)\mu(s')$ if $l(ss') = l(s) + l(s')$. Abusing notation, we denote by s_i the elements $\mu(s_i) \in \mathrm{Br}^+_{\mathfrak{g}}$.

Denote by $[\mathrm{Br}^+_{\mathfrak{g}}]$ the set of coinvariants of the cyclic shift $s_{j_1}\dots s_{j_{n-1}}s_{j_n} \longmapsto s_{j_n}s_{j_1}\dots s_{j_{n-1}}$ on $\mathrm{Br}^+_{\mathfrak{g}}$, called the *cyclic envelope* of the braid semigroup.

We denote by $[s_{j_1}\dots s_{j_n}] \in [\mathrm{Br}^+_{\mathfrak{g}}]$ the cyclic element corresponding to $s_{j_1}\dots s_{j_n} \in \mathrm{Br}^+_{\mathfrak{g}}$.

Given a pair of flags B, B', we define $w(\mathrm{B}, \mathrm{B}') \in W$ via the isomorphism $\mathrm{G}\backslash(\mathcal{B}\times\mathcal{B}) = W$.

Take a defining relation $s_{l_1}s_{l_2}...s_{l_m} = s_{r_1}s_{r_2}...s_{r_m}$ in (200). For any flags $(\mathrm{B}_0, ..., \mathrm{B}_m)$ with $w(\mathrm{B}_{a-1}, \mathrm{B}_a) = s_{j_a}$, where $a = 1, ..., m$, there is a unique collection of flags $(\mathrm{B}'_0, ..., \mathrm{B}'_m)$ with $w(\mathrm{B}'_{a-1}, \mathrm{B}'_a) = s_{r_a}$.

Given a reduced decomposition $s_{j_1}s_{j_2}...s_{j_k}$ of a cyclic element $\beta \in [\mathrm{Br}^+_{\mathfrak{g}}]$ there is the moduli space of $\mathrm{G}-$orbits of flags $(\mathrm{B}_1, ..., \mathrm{B}_k)$ such that $w(\mathrm{B}_{i-1}, \mathrm{B}_i) = s_{j_i}$, where $i \in \mathbb{Z}/k\mathbb{Z}$. The moduli spaces assigned to different reduced decompositions of β are canonically isomorphic.

The moduli space $\mathcal{X}(\mathrm{G}, \beta; S)$. Let $\mathcal{B}$ be the flag variety of G. A $\mathrm{G}-$local system $\mathcal{L}$ on S gives rise to the associated local system of flags $\mathcal{L}_{\mathcal{B}} := \mathcal{L}\times_{\mathrm{G}}\mathcal{B}$. Denote by S^1_p the circle of rays at T_pS. Given a reduced decomposition $s_{j_1}\dots s_{j_k}$ of the cyclic word $\beta_p \in [\mathrm{Br}^+_{\mathfrak{g}}]$, take distinct points $x_1, ..., x_k \in S^1_p$ going cyclically around the circle, and put the elements s_{j_i} at the points x_i.

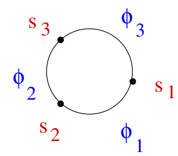


Figure 68: Cyclic framing of type $[s_1s_2s_3]$.

Definition 10.25. *Let $\beta := \{\beta_p\}$ be a collection of elements of $[\mathrm{Br}^+_{\mathfrak{g}}]$ assigned to the punctures $\{p\}$ of a surface S. A β_p-framing at a puncture p on a G-local system $\mathcal{L}$ on S is given by flat sections $\varphi = (\varphi_1, \dots, \varphi_k)$ of the local system of flags $\mathcal{L}_{\mathcal{B}}$ over intervals x_1x_2, ..., x_kx_1 on S^1_p so that $w(\varphi_i, \varphi_{i+1}) = s_{\alpha_i}$:*

$$\varphi_1 \xrightarrow{s_{j_1}} \varphi_2 \xrightarrow{s_{j_2}} \dots \varphi_k \xrightarrow{s_{j_k}} \varphi_1.$$

A $\beta-$framing on $\mathcal{L}$ is a collection of local framings $\{\beta_p\}$ at the punctures of S.

The moduli space $\mathcal{X}(\mathrm{G}, \beta; S)$ parametrises $\beta-$framed G-local systems $\mathcal{L}$ on S.

The stack $\mathcal{X}(\mathrm{G}, \beta; S)$ depends on the elements $\beta_p \in [\mathrm{Br}^+_{\mathfrak{g}}]$, rather than their reduced decompositions.

Examples of stacks $\mathcal{X}(\mathrm{G}, \beta; S)$.

1. If $\mathrm{G} = \mathrm{GL}_m$, a local framed Stokes data is just a $\beta-$framing at the puncture on an $m-$dimensional local system of vector spaces on the punctured disc for an element $\beta \in [\mathrm{Br}^+_{\mathfrak{gl}_m}]$.

2. If $\beta_p = e$ is the unit, then a local β_p-framing at the puncture p is just a monodromy invariant flag near the puncture.

3. If all $\beta_p = e$ are the unit elements, then $\mathcal{X}(\mathrm{G}, \beta; S)$ is the stack $\mathcal{X}(\mathrm{G}, S)$ [FG1] parametrising G-local system $\mathcal{L}$ on S with a flat section of the local system of flags $\mathcal{L}_{\mathcal{B}}$ near each puncture of S.

4. Let $\mathbb{S}$ be a decorated surface with k boundary components, with $b_1, ..., b_k$ special points on them.

   If $\beta_p = [\mu(w_0)\cdot\ldots\cdot\mu(w_0)] \in [\mathrm{Br}^+_{\mathfrak{g}}]$ is the cyclic product of b_p elements $\mu(w_0)$ assigned to the longest element $w_0 \in W$, then $\mathcal{X}(\mathrm{G}, \beta; S)$ is the stack $\mathcal{X}_{\mathrm{G},\mathbb{S}}$ for from [FG1], as we show in Section 10.5.

   The cluster Poisson structure on moduli spaces 1) and 2) was defined for $\mathrm{G} = \mathrm{PGL}_m$ in [FG1], for classical groups and G_2 by case-by case study in [Le1]-[Le2], and in the general case in [GS19].

5. If $S = S^2 - \{0, \infty\}$ is a punctured disc, and $\beta_0 = e$, $\beta_\infty = \beta$, then the stack $\mathcal{X}(\mathrm{G}, \{e, \beta\}; \mathrm{D}^*)$ carries a cluster Poisson variety structure assigned to the cyclic braid semigroup element β in [FG4].

6. If $S = S^2 - \{0, \infty\}$, then the stack $\mathcal{X}(\mathrm{G}, \beta_0, \beta_p; \mathrm{D}^*)$ contains as a Zariski open part the double Bott-Samelson variety [SW] assigned to a pair cyclic braid semigroup elements $\beta_0, \beta_\infty \in [\mathrm{Br}^+_{\mathfrak{g}}]$.

Theorem 10.26. *Any stack of framed* G*-Stokes data on a surface* S *is the stack* $\mathcal{X}(\mathrm{G}, \beta; S)$ *for some* β.

Proof. This can be deduced from the formal classification in the G-case just as in Section 10.1. □

Theorem 10.27. *Let* S *be a punctured surface, and* $S \neq S^2 - \{\infty\}$. *Then the stack* $\mathcal{X}(\mathrm{G}, \beta; S)$ *has a cluster Poisson structure, equivariant under the wild mapping clsass group. Therefore the stack* $\mathcal{X}(\mathrm{G}, \beta; S)$ *admits a cluster quantization, equivariant under the wild mapping class group.*

For each puncture p, *the braid group* $\mathrm{Br}_{\mathfrak{g}}$ *acts by automorphisms of the stack* $\mathcal{X}(\mathrm{G}, \beta; S)$, *preserving the cluster Poisson structure.*

Proof. Assume first that $S \neq S^2 - \{0, \infty\}$. Then there is an ideal triangulation $\mathcal{T}$ of S.

Take an ideal triangulation $\mathcal{T}$ of S, and inflate each of its edges E to a pair of thin bigons b_E, as we did in Section 10.3. For each puncture p, put all the points $x_1, ..., x_{b_p}$ inside of the arc on the circle S^1_p located inside of the left bigon at one of the edges sharing p. Then the moduli space $\mathcal{X}(\mathrm{G}, \beta; S)$ is birationally equivalent to the result of amalgamation of the moduli spaces $\mathcal{P}_{\mathrm{G},t}$ from [GS19] assigned to the triangles t of the triangulation, and the moduli spaces $\mathcal{P}_{\mathrm{G},r_E}$ assigned to the rectangles corresponding to the bigons, that is the edges of $\mathcal{T}$. The independence of the choice of $\mathcal{T}$ follows from [FG4] & [GS19].

Now let $S = S^2 - \{0, \infty\}$ be a cylinder. Recall the decorated flag variety $\mathcal{A} := \mathrm{G}/\mathrm{U}$. Recall the elementary moduli space $\mathcal{P}_{\mathrm{G},s}$ paramterising triples $(\mathrm{B}_1, \mathrm{A}_2, \mathrm{A}_3) \in \mathrm{G}\backslash(\mathcal{B} \times \mathcal{A} \times \mathcal{A})$, such that the pairs $(\mathrm{B}_1, \mathrm{A}_2)$ and $(\mathrm{B}_1, \mathrm{A}_2)$ are generic, and $w(\mathrm{B}_2, \mathrm{B}_3) = s$ for the pair of flags $(\mathrm{B}_2, \mathrm{B}_3)$ underlying the decorated flags $(\mathrm{A}_2, \mathrm{A}_3)$. We picture such a triple by a wedge with the flag B_1 at the vertex, and two arrows pointing out to the decorated flags at the vertices on the short edge. The space $\mathcal{P}_{\mathrm{G},s}$ carries a cluster Poisson structure [GS19, Section 6]. The same space with the opposite Poisson structure is denoted by $\mathcal{P}_{\mathrm{G},\overline{s}}$.

Take reduced decompositions for the braid semigroup elements $\beta_0 = s_{i_1}...s_{i_a}$ and $\beta_\infty = \overline{s}_{j_1}...\overline{s}_{j_b}$. Amalgamate the corresponding moduli spaces $\mathcal{P}_{\mathrm{G},s_i}$ and $\mathcal{P}_{\mathrm{G},\overline{s}_j}$ so that the order of the spaces $\mathcal{P}_{\mathrm{G},s_i}$ (respectively $\mathcal{P}_{\mathrm{G},\overline{s}_j}$) follows a reduced decomposition β_0 (respectively β_∞), see Figure 69. Then glue the left and the right pairs (A, B) for the resulting moduli spaces if they oriented the same way. Otherwise alter one of them by $(\mathrm{B}_1, \mathrm{A}_2) \longrightarrow \overline{w}_0(\mathrm{B}_1, \mathrm{A}_2)$ where $\overline{w}_0$ is the canonical lift of w_0 to G, and then glue.

This way we get a Zariski open part of $\mathcal{X}(\mathrm{G}, \beta; S)$. Therefore the result follows from [GS19].

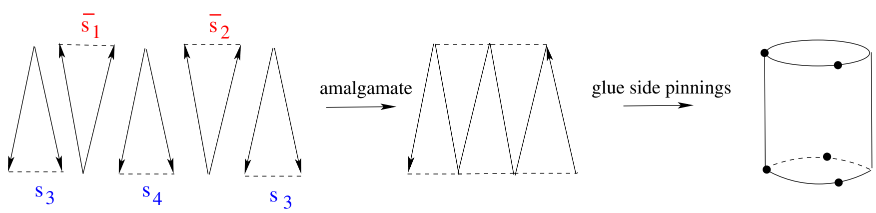


Figure 69: Getting a cluster chart with $\beta_0 = s_1 s_2$, $\beta_\infty = s_3 s_4 s_3$ on a cylinder by the amalgamation.

The second claim follows from this by applying the cluster Poisson quantization machine [FG4].

Let us define the action a_p of the braid group $\mathrm{Br}_{\mathfrak{g}}$ on the stack $\mathcal{X}(\mathrm{G}, \beta; S)$. Pick a generator s_i of the the braid group $\mathrm{Br}_{\mathfrak{g}}$. Take a cyclic word $\mathbf{b} = [s_{j_1} \cdot \ldots \cdot s_{j_n}]$ assigned to the cyclic collection of points $x_1, ..., x_n$ on the circle $S^1_p = T_pS - \{0\}/\mathbb{R}_{>0}$. We write this schematically as $[s_{j_1}|x_1, \cdot \ldots \cdot s_{j_n}|x_n]$. Then we find its representative of $[\mathbf{b}]$ starting from s_1, and shift the cyclic word by one step to the right:

$$[s_1|x_1, s_{j_2}|x_2, \cdot \ldots \cdot s_{j_n}|x_n] \longmapsto [s_1|x_2, s_{j_2}|x_3, \cdot \ldots \cdot s_{j_n}|x_1].$$

which starts from s_i. $\square$

7. Cluster quantization of $\mathcal{M}_{\rm DR}({\rm G}, \beta; S)$. To quantize the De Rham stack $\mathcal{M}_{\rm DR}({\rm G}, \beta; S)(\mathbb{C})$, we apply the G−analog of the universal Riemann-Hilbert equivalence (192), and then develope the $\Gamma_{{\rm G},\beta;S}-$equivariant quantization of the Betti stack $\mathcal{M}_{\rm B}({\rm G}, \beta; S)$.

To do this, we quantize first the bigger stack $\mathcal{X}({\rm G}, \beta; S)$ of framed Stokes data. There is a finite group $W_{{\rm G},S,\beta}$ which acts birationally on $\mathcal{X}({\rm G}, \beta; S)$ by altering filtrations. Forgetting the filtrations we get a map, which is a finite Galois cover at the generic point:

$$p : \mathcal{X}({\rm G}, \beta; S) \longrightarrow \mathcal{M}_{\rm B}([S, \mathcal{L}]).$$

The space $\mathcal{X}({\rm G}, \beta; S)$ has a $\Gamma_{{\rm G},\beta;S} \times W_{{\rm G},\beta;S}-$equivariant Poisson structure. Therefore [FG4] gives its $\Gamma_{{\rm G},\beta;S} \times W_{{\rm G},\beta;S}-$equivariant cluster Poisson quantization, given by:

1. A $q-$deformed algebra of functions $\mathcal{O}_q(\mathcal{X}({\rm G}, \beta; S))$.
2. A unitary projective representation of the group $\Gamma_{{\rm G},\beta;S} \times W_{{\rm G},\beta;S}$ in a Hilbert space $\mathcal{H}$.
3. A representation of the $*-$algebra given by the modular double

$$\mathcal{O}_q(\mathcal{X}({\rm G}, \beta; S)) \otimes \mathcal{O}_{q^\vee}(\mathcal{X}({\rm G}, \beta; S)), \qquad q = e^{i\pi\hbar}, \quad q^\vee = e^{i\pi/\hbar}. \tag{201}$$

by unbounded operators in the Hilbert space $\mathcal{H}$, intertwining the action of the group $\Gamma_{{\rm G},\beta;S} \times W_{{\rm G},\beta;S}$ on the $*-$algebra (201) with its projective unitary action in $\mathcal{H}$ from 2). Here $\hbar \in \mathbb{R}_{>0}$ or $|\hbar| = 1$.

The group $W_{{\rm G},\beta;S}$ acts by cluster Poisson transformations (the regular case is done in [GS19]). Set

$$\mathcal{O}_q(\mathcal{M}_{\rm B}({\rm G}, \beta; S)) := \mathcal{O}_q(\mathcal{X}({\rm G}, \beta; S))^{W_{{\rm G},\beta;S}}.$$

We consider the representation of the modular double of this subalgebra in the Hilbert space $\mathcal{H}$. It commutes with the unitary action of the group $W_{{\rm G},\beta;S}$ in $\mathcal{H}$.

10.5 More examples

1. Non-commutative stacks $\mathcal{X}_m(\mathbb{S})$ and $\mathcal{A}_m(\mathbb{S})$ as stacks of Stokes data Below $\mathbb{S}$ is a decorated surface with boundary. We assume that the local systems on the strands of the webs, provided by the microlocal supports of admissible sheaves, are one dimensional.

We assign to each boundary component π of $\mathbb{S}$, with b_π special points, an ideal Legendrian link $\mathcal{L}_{\pi,m}$ in a little cylinder $C_\pi \subset \mathbb{S}$ containing π, obtained by concatenating b_π copies of the Legendrian with m strands on a rectangle realising the longest permutation $w_0 : (1, 2, ..., m) \longmapsto (m, ..., 2, 1)$, see Figure 70.

Let us assign to each puncture p on $\mathbb{S}$ a collection $\mathcal{L}_{p,m}$ of m small disjoint loops around the puncture.

Denote by $\mathcal{L}_{\mathbb{S},m}$ the union of these Legendrians:

$$\mathcal{L}_{\mathbb{S},m} := \bigcup_{\pi \in \pi_0(\partial\mathbb{S})} \mathcal{L}_{\pi,m} \ \cup \bigcup_{\text{punctures } p} \mathcal{L}_{p,m}. \tag{202}$$

Definition 10.28. *The non-commutative stack $\mathcal{M}^{(1)}_{\mathcal{X}}(\mathbb{S}, \mathcal{L}_{\mathbb{S},m})$ parametrises admissible sheaves on $\mathbb{S}$ from $\mathcal{C}(\mathbb{S}, \mathcal{L}_{\mathbb{S},m})$, vanishing near marked points, whose microlocal support restricted to the Legendrian $\mathcal{L}_{\mathbb{S},m}$ is given by one dimensional non-commutative local systems.*

The stack $\mathcal{M}^{(1)}_{\mathcal{A}}(\mathbb{S}, \mathcal{L}_{\mathbb{S},m})$ parametrises similar data with trivialized local systems on the Legendrian.

Theorem 10.29. *The stack $\mathcal{M}^{(1)}_{\mathcal{X}}(\mathbb{S}, \mathcal{L}_{\mathbb{S},m})$ is birationally isomorphic to the stack $\mathcal{X}_m(\mathbb{S})$. It has a non-commutative $\Gamma_{\mathbb{S},\mathcal{L}_{\mathbb{S},m}}-$equivariant cluster Poisson structure. The stack $\mathcal{M}^{(1)}_{\mathcal{A}}(\mathbb{S}, \mathcal{L}_{\mathbb{S},m})$ is birationally isomorphic to the stack $\mathcal{A}_m(\mathbb{S})$. It has a $\Gamma_{\mathbb{S},\mathcal{L}_{\mathbb{S},m}}-$equivariant non-commutative cluster $\mathcal{A}-$structure.*

Proof. The complement $\mathbb{S} - \mathcal{L}_{\mathbb{S},m}$ contains the unique internal component. It is homotopic to $\mathbb{S}$. Its boundary is a union of circles $\{S^1_p\}$ and $\{S^1_\pi\}$ parametrised by punctures p and boundary components π.

Restricting an admissible sheaf $\mathcal{S} \in \mathcal{M}^{(1)}(\mathbb{S}, \mathcal{L}_{\mathbb{S},m})$ to the internal component of $\mathbb{S} - \mathcal{L}_{\mathbb{S},m}$ we get an $m-$dimensional local system $\mathcal{V}$. For each puncture p, its restriction to the circle S^1_p has a complete filtration, provided by the Legendrian $\mathcal{L}_p$.

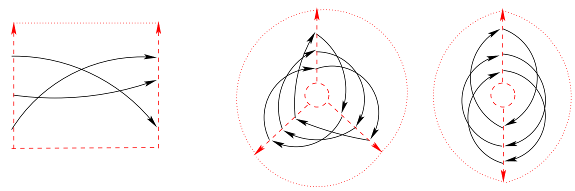


Figure 70: A Legendrian realising permutation $(1, 2, 3) \to (3, 2, 1)$, and examples of concatenation.

Given a boundary component π, consider b_π generic normal rays cutting the cylinder C_π into b_π rectangles, see Figure 70, so that the link $\mathcal{L}_{\pi,m}$ is a concatenation of Legendrians in these rectangles realizing the longest permutation w_0. For each of the rays r, the admissible sheaf $\mathcal{S}$ provides a flag $\mathcal{F}_{\pi,r}$ in the fiber of $\mathcal{V}$ over $S^1_\pi \cap r$. The local system $\mathcal{V}$ with the flags on the circles $\{S^1_\pi\}$ and filtrations on the circles $\{S^1_p\}$ is exactly the data which determines a framed local system on $\mathbb{S}$ of dimension m.

Evidently, this construction can be inverted starting with a framed local systems with the flags $\mathcal{F}_{\pi,r}$ in generic position. So we get a birational isomorphism. The $\mathcal{A}-$case is completely similar.

The rest follows from the already established results about the spaces $\mathcal{X}_m(\mathbb{S})$ and $\mathcal{A}_m(\mathbb{S})$.

Alternatively, we can deform a Legendrian $\mathcal{L}_{\mathbb{S},m}$ to a web of zig-zags of a bipartite graph of type $\mathbb{S}$, say by using an ideal triangulation of $\mathbb{S}$ for a start. So we get a cluster torus of each of the two flavors. □

2. Non-commutative Grassmannians as stacks of Stokes data Consider a web $\mathcal{W}_{k,n}$ given by a collection of oriented arcs on a disc, cooriented away from the boundary, as shown on Figure 71. We can describe it as follows. Take n special points on the boundary, and consider arcs which encircle $k - 1$ subsequent points, so that for each point there is just one arc starting a bit to the right of a special points, and ending a bit to the left of another special point.

Consider the stack $\mathrm{C}^\circ_{\mathcal{X}}(\mathcal{W}_{k,n})$ of admissible dg-sheaves with the following two properties:

i) They vanish in the middle of each interval between the special points.

ii) The local system on each arc describing the microlocal support is one dimensional.

In fact these dg-sheaves are sheaves: they are concentrated in the degree zero.

The $\mathcal{A}-$flavor of this space is the stack $\mathrm{C}^\circ_{\mathcal{A}}(\mathcal{W}_{k,n})$ of admissible sheaves as above such that the one dimensional local systems on each curve are trivialized.

Lemma 10.30. *i) The stack $\mathrm{C}^\circ_{\mathcal{X}}(\mathcal{W}_{k,n})$ describes cyclically ordered collections of n one dimensional subspaces in a $k-$dimensional $R-$vector space such that any subsequent k of them generate the space. It has a non-commutative cluster Poisson variety structure.*

ii) The stack $\mathrm{C}^\circ_{\mathcal{A}}(\mathcal{W}_{k,n})$ describes cyclic collections of n vectors in a $k-$dimensional $R-$vector space such that any subsequent k of them generate the space. It has a cluster $\mathcal{A}-$variety structure.

Proof. The ambient vector space is reconstructed as the fiber of an admissible sheaf in the center of the disc. The one-dimensional subspaces are the fibers of the admissible sheaf near special boundary points. The non-zero vectors generating these one dimensional subspaces are given by trivializations of the local systems sitting on the arcs of the web. The admissibility at a crossing point is given by Lemma 10.6. This implies that the admissibility at the crossing points is equivalent to the condition that each k subsequent subspaces generate the space.

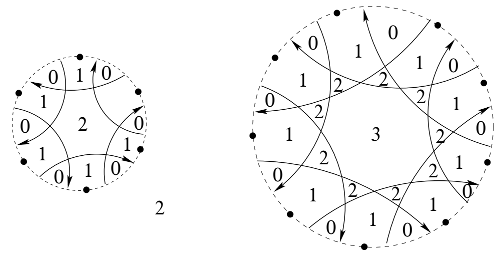


Figure 71: The webs $\mathcal{W}_{2,5}$ and $\mathcal{W}_{3,9}$ on a disc describe configurations of 5 and 9 lines/vectors in non-commutative vector spaces of dimension 2 and 3 repectively, as stacks of admissible sheaves. The numbers tell dimensions of the fibers of admissible sheaves in each component of the complement to the web.

To get a cluster description, it is sufficient to find a bipartite graph whose zig-zag web is admissibly isotopic to a web $\mathcal{W}_{k,n}$. Then our general result implies that the torus $\mathrm{Loc}_1(\Gamma)$ sits inside of the stack we consider. This kind of zig-zag webs are the ones considered by Postnikov [P]. □

11 Proof of Theorem 5.7

Recall the transformation formulas, where $A_1 = a_1a_2a_3a_4$ etc.:

$$\begin{aligned} b_1 &= (1 + A_3^{-1})a_3 = a_3(1 + A_4^{-1}), \\ b_2 &= (1 + A_4)^{-1}a_4 = a_4(1 + A_1)^{-1}, \\ b_3 &= (1 + A_1^{-1})a_1 = a_1(1 + A_2^{-1}), \\ b_4 &= (1 + A_2)^{-1}a_2 = a_2(1 + A_3)^{-1}. \end{aligned} \tag{203}$$

Recall the identities

$$\begin{aligned} &\{a, b\} = -\{b^{-1}, a^{-1}\}, \\ &\{b, c\} - \{ab, c\} + \{a, bc\} - \{a, b\} = 0, \\ &\{xy, y^{-1}\} = -\{x, y\}, \qquad \{x^{-1}, xy\} = -\{x, y\}. \end{aligned} \tag{204}$$

Lemma 11.1.

$$\{b_2, b_3\} = \{a_2a_3a_4, (1 + A_1)^{-1}\} + \{a_2a_3, a_4\} + \{1 + A_1, a_4^{-1}\}. \tag{205}$$

Proof. By the definition (117), we have

$$\{b_2, b_3\} = \{a_4(1 + A_1)^{-1}, (1 + A_1^{-1})a_1\}. \tag{206}$$

By the cocycle identity in (204) for the triple $(a_4, (1 + A_1)^{-1}, (1 + A_1^{-1})a_1)$,

$$\{b_2, b_3\} = \{(1 + A_1)^{-1}, (1 + A_1^{-1})a_1\} + \{a_4, A_1^{-1}a_1\} - \{a_4, (1 + A_1)^{-1}\}. \tag{207}$$

By the cocycle identity for the triple $((1 + A_1)^{-1}, 1 + A_1^{-1}, a_1)$ we can rewrite the first term, and get

$$\{b_2, b_3\} = \{A_1^{-1}, a_1\} - \{1 + A_1^{-1}, a_1\} + \{(1 + A_1)^{-1}, 1 + A_1^{-1}\} + \{a_4, A_1^{-1}a_1\} - \{a_4, (1 + A_1)^{-1}\}. \tag{208}$$

Writing the middle term as $-\{1 + A_1, A_1^{-1}\}$ by using the last identiy in (204), applying the cocycle identity for the triple $(1 + A_1, A_1^{-1}, a_1)$, and observing that $\{a_4, A_1^{-1}a_1\} = \{a_2a_3, a_4\}$ and $\{a_3, A_4^{-1}\} = -\{A_4, a_3^{-1}\} = \{a_4a_1a_2, a_3\}$ using the last line in (204), we get the claim. □

Lemma 11.2.

$$-\{b_1, b_2\} =\{a_3A_4^{-1}, (1 + A_4)\} - \{a_4, a_1a_2\} - \{a_4^{-1}, 1 + A_4\}. \tag{209}$$

Proof. By the definition (117), we have

$$\{b_2^{-1}, b_1^{-1}\} =\{a_4^{-1}(1 + A_4), (1 + A_4^{-1})^{-1}a_3^{-1}\}. \tag{210}$$

Using the cocycle identity for the triple $(a_4^{-1}(1 + A_4), (1 + A_4^{-1})^{-1}, a_3^{-1})$ we write this as

$$\begin{aligned}-\{b_1, b_2\} =& - \{(1 + A_4^{-1})^{-1}, a_3^{-1}\} + \{a_4^{-1}A_4, a_3^{-1}\} + \{a_4^{-1}(1 + A_4), (1 + A_4^{-1})^{-1}\}\\ =&\{a_3, 1 + A_4^{-1}\} - \{a_1a_2, a_3\} - \{1 + A_4^{-1}, (1 + A_4)^{-1}a_4\}.\end{aligned} \tag{211}$$

The cocycle identity in (204) allows to write the last term as

$$\{1 + A_4, (1 + A_4^{-1})^{-1}\} - \{a_4^{-1}, A_4\} - \{a_4^{-1}, 1 + A_4\}. \tag{212}$$

Since the first term here is equal to $\{A_4^{-1}, 1 + A_4\}$ and $\{a_3, A_4^{-1}a_4\} = \{a_1a_2, a_3\}$, we get

$$-\{b_1, b_2\} =\{a_3, A_4^{-1}(1 + A_4)\} - \{a_1a_2, a_3\} + \{A_4^{-1}, (1 + A_4)\} - \{a_4, a_1a_2a_3\} - \{a_4^{-1}, 1 + A_4\}. \tag{213}$$

Using the cocycle relation for the triple $(a_3, A_4^{-1}, 1 + A_4)$, and then the cocycle relation for (a_4, a_1a_2, a_3) we get the claim. ☐

Denote by Cycl_2 the operator on functions in $a_1, ..., a_4$ given by the identity + the cyclic shift by two:

$$\text{Cycl}_2\text{F}(\text{a}_1, \text{a}_2, \text{a}_3, \text{a}_4) = \text{F}(\text{a}_1, \text{a}_2, \text{a}_3, \text{a}_4) + \text{F}(\text{a}_3, \text{a}_4, \text{a}_1, \text{a}_2).$$

We split the sum to calculate as A) + B), where:

A) It is the total contribution of the middle terms in the Lemmas is:

$$\begin{aligned}&\text{Cycle}_2\Big(-\{a_4, a_1a_2\} + \{a_2a_3, a_4\}\Big)\\ =& - \{a_4, a_1a_2\} + \{a_2a_3, a_4\} - \{a_2, a_3a_4\} + \{a_4a_1, a_2\}\\ =&\{a_1, a_2\} - \{a_2, a_3\} + \{a_3, a_4\} - \{a_4, a_1\}.\end{aligned} \tag{214}$$

The last equality uses cocycle identities for (a_4, a_1, a_2) and (a_2, a_3, a_4). We get exactly what we wanted.

B) It is the rest:

$$\text{Cycle}_2\Big(\{1 + A_1, a_4^{-1}\} + \{a_2a_3a_4, (1 + A_1)^{-1}\} - \{a_4^{-1}, 1 + A_4\} - \{(1 + A_4)^{-1}, a_4a_1a_2\}. \tag{215}$$

This can be rewritten as

$$\begin{aligned}&\text{Cycle}_2\Big(\Big(-dA_1a_4^{-1}da_4 + dA_1(a_2a_3a_4)^{-1}d(a_2a_3a_4)\Big)(1 + A_1)^{-1}+\\ &\Big(da_4a_4^{-1}dA_4 - d(a_4a_1a_2)(a_4a_1a_2)^{-1}dA_4\Big)(1 + A_4)^{-1}\Big).\end{aligned} \tag{216}$$

Expanding this, we get

$$\begin{aligned}&d(a_1a_2a_3a_4)(a_2a_3a_4)^{-1}d(a_2a_3)a_4(1 + A_1)^{-1}\\ &+ d(a_3a_4a_1a_2)(a_4a_1a_2)^{-1}d(a_4a_1)a_2(1 + A_3)^{-1}\\ &- a_4d(a_1a_2)(a_4a_1a_2)^{-1}dA_4(1 + A_4)^{-1}\\ &- a_2d(a_3a_4)(a_2a_3a_4)^{-1}dA_2(1 + A_2)^{-1}.\end{aligned} \tag{217}$$

Interchanging lines 2 and 3, writing $1+A_4 = a_4(1+A_1)a_4^{-1}$, and similarly for $1+A_3$ and $1+A_2$, we get:

$$\begin{aligned} & d(a_1a_2a_3a_4)(a_2a_3a_4)^{-1}d(a_2a_3)a_4(1+A_1)^{-1} \\ & - d(a_1a_2)(a_4a_1a_2)^{-1}d(a_4a_1a_2a_3)a_4(1+A_1)^{-1} \\ & + (a_3a_4)^{-1}d(a_3a_4a_1a_2)(a_4a_1a_2)^{-1}d(a_4a_1)a_2a_3a_4(1+A_1)^{-1} \\ & - (a_3a_4)^{-1}d(a_3a_4)(a_2a_3a_4)^{-1}d(a_2a_3a_4a_1)a_2a_3a_4(1+A_1)^{-1}. \end{aligned} \tag{218}$$

Lemma 11.3. *The expresssion (218) is equal to zero.*

Proof. Let us write the first two lines as

$$\begin{aligned} & \Big(d(a_1a_2)a_2^{-1}d(a_2a_3)a_4 + a_1a_2d(a_3a_4)(a_2a_3a_4)^{-1}d(a_2a_3)a_4 \\ & - d(a_1a_2)(a_4a_1a_2)^{-1}d(a_4a_1)a_2a_3a_4 - d(a_1a_2)a_2^{-1}d(a_2a_3)a_4\Big)(1+A_1). \end{aligned} \tag{219}$$

The terms 1 and 4 cancel, so we get

$$(219) = \Big(a_1a_2d(a_3a_4)(a_2a_3a_4)^{-1}d(a_2a_3)a_4 - d(a_1a_2)(a_4a_1a_2)^{-1}d(a_4a_1)a_2a_3a_4\Big)(1+A_1). \tag{220}$$

A similar calculation with the lines 3 and 4 in (218) gives

$$\begin{aligned} & \Big((a_3a_4)^{-1}d(a_3a_4)a_4^{-1}d(a_4a_1) + d(a_1a_2)(a_4a_1a_2)^{-1}d(a_4a_1) \\ & - (a_3a_4)^{-1}d(a_3a_4)(a_2a_3a_4)^{-1}d(a_2a_3)a_4a_1 - (a_3a_4)^{-1}d(a_3a_4)a_4^{-1}d(a_4a_1)\Big)a_2a_3a_4(1+A_1). \end{aligned} \tag{221}$$

The terms 1 and 4 cancel, so we get

$$(221) = \Big(d(a_1a_2)(a_4a_1a_2)^{-1}d(a_4a_1) - (a_3a_4)^{-1}d(a_3a_4)(a_2a_3a_4)^{-1}d(a_2a_3)a_4a_1\Big)a_2a_3a_4(1+A_1). \tag{222}$$

Therefore (219) + (221) is equal to

$$\begin{aligned} & \Big(a_1a_2d(a_3a_4)(a_2a_3a_4)^{-1}d(a_2a_3)a_4 \\ & - d(a_1a_2)(a_4a_1a_2)^{-1}d(a_4a_1)a_2a_3a_4 \\ & + d(a_1a_2)(a_4a_1a_2)^{-1}d(a_4a_1)a_2a_3a_4 \\ & - (a_3a_4)^{-1}d(a_3a_4)(a_2a_3a_4)^{-1}d(a_2a_3)a_4A_1\Big)(1+A_1). \end{aligned} \tag{223}$$

Terms 2 and 3 evidently cancel. Terms 1 and 4 cancel if we permute A_1 and $1+A_1$ in the term 4. □

Therefore the sum B) is equal to zero. Theorem 5.7 is proved.

An identity between non-commutative 2-forms. Note that we have

$$\{x, y^{-1}\} = -x^{-1}dxy^{-1}dy, \quad \{x^{-1}, y\} = -dxx^{-1}dyy^{-1}.$$

Theorem 11.4. *Let us introduce the notation*

$$\langle a|b\rangle := \{1+ab, b^{-1}\} - (b^{-1}, 1+ba\}. \tag{224}$$

Then one has

$$\mathrm{Cycle}_2\Big(\langle a_1a_2a_3|a_4\rangle - \langle a_1|a_2a_3a_4\rangle\Big) = 0. \tag{225}$$

Proof. We can write (215) as

$$\mathrm{Cycle}_2\Big(\{1+A_1, a_4^{-1}\} - \{1+A_1, (a_2a_3a_4)^{-1}\} - \{a_4^{-1}, 1+A_4\} + \{(a_4a_1a_2)^{-1}, 1+A_4\}\Big). \tag{226}$$

This coincides with (225). But we already proved that the sum (215) is equal to zero. □

Remark. If variables a_i commute, then $\langle a|b\rangle = -2 \cdot d\log(1+ab) \wedge d\log b$, and identity (225) reads as

$$d\log(1+a_1a_2a_3a_4) \wedge d\log(a_1a_2a_3a_4) = 0.$$

However for a general matrix X we have $d\log(1+X) \wedge d\log X \neq 0$. The identity (225) is the analog of the Steinberg relation $d\log(1+z) \wedge d\log(z) = 0$ for the non-commutative 2-forms. It is interesting that it involves four non-commutative variables a_1, a_2, a_3, a_4 rather than one.

It would be interesting to find a similar non-commutative generalization of the de Rham realization of the defining relation in the Milnor K-group: $d\log(1+z) \wedge d\log(z) \wedge d\log(z_1) \wedge \ldots \wedge d\log(z_n) = 0$.

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