Manin's work in birational geometry

Vyacheslav V. Shokurov

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Rational surfaces. One of the main Manin contributions to birational geometry of algebraic surfaces was given in two subsequent papers in late 60's about rational surfaces over a perfect field [Manin66] [Manin67]. In both papers Manin uses very classical methods but with a new cohomological style and also add some fresh ideas which are inherited in modern birational geometry. [Both publications were in Russian.] Unfortunately, in the first of them there are a lot of misprints because it was published in France in lack of a Russian editor. I hope that its English translation is better. I try to present the most important results of these two papers from modern perspective and with slightly updated terminology.

Both papers are about algebraic surfaces F over a perfect field k which are *rational* over an algebraic closure $\overline{k}$ of k. Such surfaces are called *rational*. This terminology is still used in modern birational geometry. In Grothendieck stile it would be better to say that F is *geometrically rational*, that is, $F_{\overline{k}} = F \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is rational over $\overline{k}$. Equivalent cohomological characterization of those smooth complete surfaces due to Castelnuovo: F is rational if and only if $q(F) = P_2(F) = 0$, where $q(F) = \dim_k H^1(F, \mathcal{O}_F)$ is the irregularity, $P_2(F) = \dim_k H^0(F, \omega_F^{\otimes 2})$ is the 2-nd genus and ω_F is the canonical invertible sheaf of F [AS][Isk, §2]. Main problems discussed in both papers are about birational classification of rational surfaces F and about their birational invariants. Recall that two surfaces F, F' over k are *birationally equivalent* if there exists a birational isomorphism $F \dashrightarrow F'$ over k, equivalently, the fields of rational functions $k(F)$ and $k(F')$ are isomorphic over k. Thus every surface F' in the birational equivalence class is a *model of* F or *of* its field of rational functions $k(F)$; and the birational invariants are invariants of the field $k(F)$ too. For the surface F, the field $k(F)$ is a finitely generated field of the transcendence degree 2 over k. Actually, Manin considers also the quasi-birational equivalence which adds purely inseparable dominant morphisms, *radical dominant* in terminology of [Manin66, 1.1], e.g., Frobenius morphisms, to usual birational isomorphisms. The quasi-birational equivalence preserves rationality of varieties, in particuar, of surfaces. For the field k of characteristic 0 both equivalences are coincide. Notice that in algebraic geometry in positive characteristic Frobenius morphisms play very important role. However, in our review we consider only the usual birational equivalence.

A good model F' of F should be *smooth* and *complete*, so projective. Manin assumes this and to construct such a model, especially, in the positive characteristic, he uses the Abhyankar result [Ab].

Another important property of a good model is to be *minimal*. In modern terminology a surface F is a *minimal model* if it is smooth complete and its canonical divisor K_F is nef (numerically eventually free). Since F is smooth, every canonical divisor K_F is Cartier and its intersection number $(K_F.D)$ with any other divisor D is well-defined. The nef property means that $(K_F.D) \geq 0$ for any effective divisor D or, equivalently, $(K_F.C) \geq 0$ for any curve C on F. Notice that the canonical divisor K_F is defined up to a linear equivalence on F and $(K_F.D)$ is invariant under the equivalence. Manin uses the canonical invertible sheaf $\omega_F = \mathcal{O}_F(K_F)$ instead of K_F. Surfaces F under consideration in [Manin66] [Manin67] and in other papers are not such minimal models because they are rational over $\overline{k}$ and their Kodaira dimension is negative. However it is well-known as a slightly different concept: a surface F over k is *minimal* or k-minimal, or, we can say, a *minimal model in the Italian sense* if it is smooth, complete and if every birational contractions $F \to F'$ over k to another smooth surface F' over k is an isomorphism. The minimal property over algebraically closed field k means the absence of *exceptional curves of the first kind* C on F, that is, nonsingular curves with $C^2 = (K_F.C) = -1$. The first condition $C^2 = -1$ is more traditional but as Mori explained the second condition $(K_F.C) = -1$ is really crucial. After him the negativity of intersection property with K_F is one of the coner stones of modern birational geometry [Mori]. For a general field the minimal property has a similar interpretation [Manin66, Lemma in 0.4]: there are no exceptional curves of the first kind X on F over $\overline{k}$ such that for every $s \in G$ the conjugated curve $s(X)$ does not intersect X, where G denotes the Galois group $G(\overline{k}/k)$.

The minimal property ditto make sense for every proper relative surface F/Z, that is, with respect to a proper morphism $F \to Z$.

Notice that the minimal model surfaces are also minimal in the Italian sense. The converse does not hold and the difference between this two classes are exactly *Mori fibrations for surfaces*. They have two types:

(1) a contraction $f\colon F \to C$, where C is a complete geometrically irreducible and reduced smooth curve C over k, $-K_F$ is ample over C and F over C is a relative minimal surface; or

(2) F is a del Pezzo surface with the trivial contraction $F \to \mathrm{pt.} = \operatorname{Spec} k$ and the Picard number 1 over k.

The minimal surface property and its interpretation [Manin66, Lemma in 0.4] do not have direct generalization in higher dimensions. Except for the Hirzebruch surface $\mathbb{F}_1$ with its natural contraction $\mathbb{F}_1 \to \mathbb{P}^1$ in (1), this property for rational surfaces F is equivalent to the *extremal* property of contractions $F \to C$ in (1) and of $F \to \mathrm{pt.}$ in (2): the relative Picard number of F over C in (1) and of F/k in (2) is 1. This is the Mori property. It is easy to generalize it in any dimension and it is another corner-stone of modern birational geometry. Italian algebraic geometers and Manin already well understood its importance.

Notice that the generic fiber F_η in type (1) is a complete, geometrically irreducible and reduced curve. For every closed point $t \in F$, the fiber F_t is irreducible over k and geometrically reduced with $H^1(F_t, \mathcal{O}_t) = 0$. Every smooth geometric fiber F_t is isomorphic to $\mathbb{P}^1_{\overline{k}}$ and nonsmooth geometric fiber is isomorphic to the pair of lines intersecting in one point $\mathbb{P}^1_{\overline{k}} \vee \mathbb{P}^1_{\overline{k}}$. In other words, in type (1) the surface F over C is geometrically a conic bundle. (In modern terminology $F, F_{\overline{k}}$ over $C, C_{\overline{k}}$ are *central models* over $k, \overline{k}$ of ranks $1, \rho(F_{\overline{k}}/C_{\overline{k}})$ respectively, where $\rho(F_{\overline{k}}/C_{\overline{k}})$ denotes the relative Picard number of $F_{\overline{k}}$ over $C_{\overline{k}}$.)

Recall that by definition $-K_F$ is ample on a del Pezzo surface F. Notice that the extremal property is very restrictive. For example, if k is algebraically closed then $\mathbb{P}^2$ up to isomorphism is the only del Pezzo surface with Picard number 1 over k.

In [Manin66] Manin shows that every rational surface F has a model F' and a contraction $f\colon F' \to C$ such that C is a proper, geometrically irreducible and reduced, smooth curve over k of the genus 0, F' is a minimal surface over C and the reduction of generic fiber of f has the arithmetic genus 0 or 1. In the case genus 0 fibers, f is a Mori fibration of type (1) with geometrically reduced fibers. In the case genus 1 fibers, the generic fiber of f is not necessarily reduced that is why Manin uses quasi-birational equivalence. Under this equivalence he can assume that the generic fiber of f is also reduced in the case of genus 1 fibers. The rational surfaces with such a fibration or pencil in more traditional terminology have the Picard numbe ≤ 10 and form a bounded family over k. The boundedness of this class of surfaces with a genus 1 fibration and the rational curve base has interesting generalization in higher dimensions which plays an important role in modern mathematical physics. But still it is not a final product as a Mori fibration. And really on every such surface there exists another fibration which is a Mori fibration of type (1) or (2) possibly after a contraction of exceptional curves as in [Manin66, Lemma in 0.4]. But this fact was established later by Manin's former student Iskovskihk [Isk, Theorem 1] and even for any field k. That was done two year before the Mori theory [Mori]. Conversely, every del Pezzo surface has a pencil of genus 1 curves which can be converted into a genus 1 fibration up to quasi-birational equivalence after a blowup of the fixed points of the pencil.

However, as we undestand now it is better to replace fibrations of genus 1 curves by del Pezzo surfaces with the Picard number 1 as in (2). Actually Manin himself uses a reduction to del Pezzo surfaces in the proof of [Manin66, Theorem 1.7] about fibrations of genus 1 curves. The class of Mori fibrations for surfaces gives the complete description of minimal rational surfaces.

To prove the existence of genus 0 and 1 fibrations over a genus 0 curve Manin develops an updated version of Enriques results based on modern cohomological technique. Today this approch is known as Enriques-Manin method. It considers adjoint linear systems $|nK_F + H|$, where H is a very ample divisor and n is a non-negative integral threshold such that $\dim |nK_F + H| \geq 1$ while $\dim |(n+1)K_F + H| \leq 0$. A linear subsystem of dimension 1, a *pencil*, in the complete linear system $|nK_F + H|$ gives a required contraction to a curve C. This method is very classical and in general also known as an "adjunction method"and respectively linear systems $|nK_F + H|$ is called *adjoint*; we adjoin here a canonical divisor K_F to a divisor H. Using this method Manin constructed a contraction $f\colon F' \to C$ where F' is a projective resolution of F, C is a rational curve [Manin66, Theorem 1.2]. After that he makes the fibration minimal over C. If F' is minimal over C and the generic fiber f has genus 0, then f' gives a Mori fibration of type (1). Its properties described in [Manin66, Theorem 1.6]: the fibers over $\overline{k}$ are isomorphic to $\mathbb{P}^1_{\overline{k}}$ or to the pair of lines $\mathbb{P}^1_{\overline{k}} \vee \mathbb{P}^1_{\overline{k}}$ intersecting in one point. In other word f is a conic bundle over $\overline{k}$.

Twenty five years later Mori proposed a numerical version of Enriques-Manin method [Mori] where n is not necessarily integral.

In the case of minimal f with the generic fiber of genus 1 after reduction in positive characteristics, we can suppose that the generic fiber is geometrically smooth, in particular, reduced up to quasi-birational equivalence. Under the last assumption Manin proved that F' has Picard number 10 over $\overline{k}$ [Manin66, Theorem 1.7]. Notice that in this case f over $\overline{k}$ is a usual minimal fibration with the generic genus 1 curve. Moreover, for a positive integer $a = (X.F'_x)$, a linear equivalence

$$aK_{F'} \sim -F'_x \tag{1}$$

holds, where X is an exceptional curve of the first kind and F'_x is a geometric fiber of $F'_{\overline{k}}$ over a closed point $x \in C_{\overline{k}}$. Actually the relation (1) shows that a is independent of X and of $x \in C_{\overline{k}}$ because the Picard group of F' and of $F_{\overline{k}}$ do not have torsions.

The simplest examples of non-trivial rational surfaces are Severi-Brauer surfaces, forms of $\mathbb{P}^2$ over k, that is, such surfaces F that $F_{\overline{k}}$ is isomorphic to $\mathbb{P}_{\overline{k}}$ over $\overline{k}$. Similarly we can consider forms over k of other rational surfaces over $\overline{k}$. For instance, the forms of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ over k are quadrics of dimension 2. Manin classified the forms of Hirzebruch surfaces $\mathbb{F}_n$ for $n \geq 1$ in [Manin66, Theorem 1.10] (cf. with [Manin63, Theorem 2]):

a) if $n \equiv 1 \mod 2$, then any form of $\mathbb{F}_n$ over k is trivial, that is, isomorphic to $\mathbb{F}_n$ over k;

b) if $n \equiv 0 \mod 2$, then the forms F of $\mathbb{F}_n$ over k are in 1-to-1 correspondence with the forms of $\mathbb{P}^1$ over k; the correspondence is given by mapping F to a curve $C \subset F$ with $C^2 = -n$.

Recall, that the forms of $\mathbb{P}^1$ over k are in 1-to-1 correspondence with the quaternion algebras over k.

Manin also classifies del Pezzo surfaces of degree $n = K_F^2 \geq 3$. In this case $-K_F$ is *very* ample and F has the anti-canonical embedding $F \hookrightarrow \mathbb{P}^3$. This gives the 1-to-1 correspondence between the lines on F and the exceptional curves of the first kind [Manin66, Corollary 3.3]. To bound the del Pezzo surfaces Manin uses the Noether formula:

$$K_F^2 + n(F_{\overline{k}}) = 10,$$

where $n(F_{\overline{k}})$ is the rank of Neron-Severi group $N(F_{\overline{k}})$ for $F_{\overline{k}}$ or the Picard number of $F_{\overline{k}}$ because $N(F_{\overline{k}}) = \mathrm{Pic}(F_{\overline{k}})$ for rational surfaces F. Thus $9 \geq n \geq 3$ [Manin66, Theorem 3.4] and:

a) for $n = 9$, $F_{\overline{k}}$ is isomorphic to $\mathbb{P}^2$;

b) for $n = 8$, $F_{\overline{k}}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or to a blowup of $\mathbb{P}^2$ in a closed point;

c) for $3 \leq n \leq 7$, $F_{\overline{k}}$ is isomorphic to a blowup of $\mathbb{P}^2$ in $9 - n$ closed points, for which none of three points lie on a line and none of six points lie on a conic; and

d) every exceptional curve of the first kind on $F_{\overline{k}}$ is a blowup of a closed point in $\mathbb{P}^2$, or a proper birational transformation of a line through two of such points, or of a conic through five of such points.

This classification implies that, for $n \leq 7$, the exceptional curves of the first kind of a del Pezzo surface $F_{\overline{k}}$ generate the Neron-Severy group $N(F_{\overline{k}})$ of $F_{\overline{k}}$ (cf. [Manin66, следствие стр. 73]). This also implies that every del Pezzo surface over k is rational (over $\overline{k}$) and is a form of del Pezzo surfaces in the above classification for degrees $n \geq 6$. The large part of [Manin66] is devoted to the description of forms of simplest del Pezzo surfaces of degree $9 \geq n \geq 4$ and of line configurations on them.

For $9 \geq n \geq 7$, Manin obtained the following [Manin66, Theorem 3.7]: let F be a del Pezzo surface over k of degree n then

a) for $n = 9$, F is isomorphic to a Severi-Brauer surface;

b) for $n = 8$, F is a form of $\mathbb{P}^1 \times \mathbb{P}^1$, or is a blowup of $\mathbb{P}^2$ in a closed k-point, i.e., isomorphic to $\mathbb{F}_1$; and

c) for $n = 7$, F is a blowup of $\mathbb{P}^2$ of two closed k-points, or of one closed point $x \in \mathbb{P}^2$ of degree 2, that is, $[k(x) : k] = 2$.

This implies that every del Pezzo surface of degree 7 has a closed k-point [Manin66, следствие 1, стр. 75]; if k is a number field of finite degree or a functional field of transcendence degree 1 over a finite field, then the Hasse principle holds for del Pezzo surfaces over k of degree 8 and 9 [Manin66, следствие 2, стр. 75]; a del Pezzo surface of degree $7, 8$ or 9 is rational over k if and only if it has a k-point [Manin66, следствие 3, стр. 75]. Actually, every del Pezzo surface of degree 7 has a closed k-point and is rational over k.

The del Pezzo surfaces of degree 6 are related to toric geometry. Indeed, any such surface F is a form of a blowup of $\mathbb{P}^2$ in three closed points in general position which is a toric variety with the open torus $(\overline{k}^\times)^2$ of dimension 2, the complement to three lines in general position in $\mathbb{P}^2_{\overline{k}}$. Respectively, for surface F, the open subset $U = F \setminus X$ has a natural structure of a principal homogeneous space for $(\overline{k}^\times)^2$, where X denotes the divisor $\sum_{i=1}^6 X_i$ on F which is a sum of the exceptional divisors X_i on $F_{\overline{k}}$, a hexagon [Manin66, Theorem 3.10].

This implies that a del Pezzo surface of degree 6 is rational over k if and only if it has a closed k-point [Manin66, следствие 1, стр. 77].

For del Pezzo surfaces of degree 5, Manin generalizes the Enriques theorem: every del Pezzo surface of degree 5 over an infinite field k is rational over k [Enr] and partially revises its proof. For any field k, Manin shows that if a del Pezzo surface F of degree 5 has a closed k point then it is rational over k [Manin66, теорема 3.15]. He uses for this double projection from a closed k-point of F. On the other hand, for any finite field k, it was already known that every rational surface has a closed k-point due to A. Weil formula for the number of such points [W]. In the case of an infinite field k, Manin gave a sufficient general condition for the existence of such a k-point on a del Pezzo surface F in terms of the group $N(F_{\overline{k}})$ with the action of Galois group G of $\overline{k}$ over k [Manin66, теорема 3.12, a)]. However, it was insufficient for the existence of a closed k-point on a del Pezzo surface of degree 5 [Manin66, замечание на стр. 83]. A few year later the existence of such a point was established by Swinnerton-Dyer [SD]. A short proof of the Enriques theorem see also in [ShB].

To investigate birational geometry of del Pezzo surfaces of degree 4 and of other algebraic surfaces Manin developes certain birational invariants of algebraic surfaces (and actually it works in higher dimensions too). Let F be a smooth complete algebraic surface over a perfect field k. Recall that $N(F_{\overline{k}})$ denotes the Neron-Severi group of $F_{\overline{k}}$, equivalently, $N(F_{\overline{k}})$ is the group of classes of divisors on $F_{\overline{k}}$ modulo numerical equivalence. The group has the following additional structures:

(a) A continuous G-module structure, where G is the Galois group of $\overline{k}$ over k. As a $\mathbb{Z}$-module $N(F_{\overline{k}})$ is free of a finite rank.

(b) A symmetric G-invariant pairing $N(F_{\overline{k}}) \times N(F_{\overline{k}}) \to \mathbb{Z}$ given by intersection of divisors on $F_{\overline{k}}$.

(c) The numerical equivalence class ω_F of a canonical divisor of $F_{\overline{k}}$. For the rational surfaces the notation is correct because in this case the numerical equivalence is the linear one, and the classes can be identified with invertible sheaves up to isomorphism.

Denote by $\mathfrak{C}(k)$ the category of $\mathbb{Z}$-free continuous G-modules with a G-invariant pairing. A module in $\mathfrak{C}(k)$ is called *trivial* if it is a finite direct sum of modules $\mathbb{Z}[G]_{\mathbb{Z}[H]} \otimes \mathbb{Z}$, where H are open subgroups in G.

Manin established the following cretion [Manin66, теорема 2.2]: smooth complete surfaces F, F' over k are birationally equivalent over k if there exist trivial G-modules M, M' such that

$$N(F_{\overline{k}}) + M' \simeq N(F_{\overline{k}'}) + M.$$

This is an easy corollary of factorization for a birational isomorphism between to smooth complete surfaces into elementary modification: blowups of closed points and inverse modifications to them.

Changes of the canonical class ω_F do not have such a simple description in general. However Italina geometers and Manin already understood that the difference for an elementary modification $f\colon F' \to F$, blowup in $x \in F$, is easy comutable and related to the discrepancy in the exceptional divisor in modern terminology:

$$\omega_{F'} = f^*\omega_F + f^{-1}x.$$

Ono described some invariants of G-modules in $\mathfrak{C}(k)$ which independent of adding trivial G-modules [O]. For instance the cohomology group $H^1(G/H, N)$ independent of an open subgroup H and $H^1(G/H, N + M) = H^1(G/H, N)$, if M is trivial [O, §3]. This group is denoted by $H^1(K, N)$. As it was notice to Manin by Shafarevich the above criterion implies the canonical isomorphism

$$H^1(K, N(F_{\overline{k}})) = H^1(K, N(F'_{\overline{k}}))$$

for birationally equivalent smooth complete surfaces F, F'.

The group $H^1(k, N(F_{\overline{k}}))$ can be explicitly computed in many interesting cases that gives a powerful tool to distinguish birational equivalence classes.

Manin [Manin66] also discuss some other invariants of Ono's paper [O]. See also his survey with Tsfasman [MT].

The necessary cohomological condition does not implies non-trivial results for del Pezzo surfaces of degree ≥ 5. For del Pezzo surfaces of degree $n \leq 4$ the existence of a closed k-point is not equivalent anymore to its rationality over k. In particular, B. Segre discovered [Seg] that minimal del Pezzo surfaces of degree 3, *cubic* surfaces, are not rational over k while it is easy to construct examples of such surfaces with a closed k-point. We discuss this result later in connection with Noether-Fano-Iskovskihk-Manin method [Manin67] [IM]. Actually this methods works also for del Pezzo surfaces of degree 4.

To compute $H^1(k, N(F_{\overline{k}})$ for a minimal del Pezzo surface F of degree 4, Manin uses decomposition of lines on $F_{\overline{k}}$ into orbits under the Galois action (cf. [Manin66, лемма 2.9]). A result similar to Segre's ones, i.e., rigidity does not hold for those del Pezzo surfaces. As we know today all such del Pezzo surfaces are not rational and this is an easy corollary (a graduate student level exercise) of the classification of Sarkisov links for algebraic surfaces [Isk96, теорема 2.6]. The paper of Iskovskikh was dededicated to 60th birthday of Manin and its better title would be "Factorization of birational morphisms of rational algebraic surfaces from the Sarkisov point of view."For example, every del Pezzo surface F of degree 4 with the Picard number 1 over k and a closed k-point $x \in F$ has a link $f\colon F \dashrightarrow F'$ of type I into a relative minimal conic bundle $F' \to C$ on a cubic surface F' with a unique line on F', where f is the blowup of x. The surface F can be treated as a Mori fibration of type (2) and respectively the conic bundle $F' \to C$ can be treated as a Mori fibration of type (1) over a rational curve C. That is, the link transforms one Mori fibration into another one. Conversely, every cubic surface F' with a unique line has a relative minimal conic bundle structure and a link f^{-1} of type III, the blowdown of the line into the k-point x [Isk96, ibid]. This implies the positive answer on a Segre question [Seg]: whether a cubic surface with a unique line over k is non-rational. Contracting the line we get a del Pezzo surface of degree 4 with the Picard number 1 over k. Thus the non-rationality question for cubics is equivalent to the same question for such a del Pezzo surface.

Manin classifies the configurations of lines and their orbits on minimal del Pezzo surfaces F of degree 4 and computes respectively cohomology $H^1(k, N(F_{\overline{k}})$ [Manin66, таблица 3.28]. He obtaines 19 types and for 12 of them the cohomology are non-trivial. (Corrections to the classification see in [MT, 7.3].) Thus corresponding surfaces are non-rational over k. This gives a partial answer on the Segre question too. This shows also that, for this type of problems, the cohomological (formal) method is good but is not so effective as (geometrical) Noether-Fano-Iskovskihk-Manin method that we start to discuss.

A *central model* is a non-birational contraction $Y \to T$ of projective algebraic varieties over a field k such that Y has only terminal singularities and an anti-canonical divisor $-K_Y$ of Y is ample over T. The rank of the model Y/T is the relative Picard number $\rho(Y/T)$. The central models of rank 1 are exactly Mori fibrations. Notice that they are $\mathbb{Q}$-factorial. Central models of rank $r \geq 2$ are not necessarily $\mathbb{Q}$-factorial in dimension ≥ 3. The central models of rank 2 are not Sarkisov links but every Sarkisov can be constructed from a central model Y/T. Indeed, let $Y' \to Y$ be a projective $\mathbb{Q}$-factoralization of Y. Then the Picard number of Y'/T is also 2 and we can apply the two-ray game to Y'/T. This gives us two Mori fibrations $Y_1/T_1, Y_2/T_2$ over T in the following commutative diagram:

$$\begin{array}{ccccc}
Y_1 & \dashleftarrow & Y' & \dashrightarrow & Y_2 \\
\downarrow & & \downarrow & & \downarrow \\
T_1 & & Y & & T_2 \\
& \searrow & \downarrow & \swarrow & \\
& & T & &
\end{array}$$

with birational transformations $Y' \dashrightarrow Y_1, Y' \dashrightarrow Y_2$. Their composition $Y_1 \dashrightarrow Y_2$ (actually with inverse $Y_1 \dashrightarrow Y'$) is a birational transformation of the Mori fibration Y_1/T_1 into the Mori fibration Y_2/T_2 *over* T which is called a *Sarkisov link over* T (in modern terminology). Notice that the link is always not an isomorphism, i.e., not regular. Actually each central model of rank 2 gives two Sarkisov links: $Y_1 \dashrightarrow Y_2$ and its inverse $Y_1 \dashleftarrow Y_2$. Conversely, any Sarkisov link over T has a *unique* central model over T of rank 2 which gives the link [ShCh, Central model p. 525]. A type of Sarkisov link is determined by the dichotomy: contraction $T_i \to T, i = 1, 2$, is an isomorphism or not. Thus we have four types: I–IV. E.g., a type I link has a canonical isomorphism $T_1 = T$ and a non-isomorphism $T_2 \to T$.

Central models of rank 3 give *elementary relations* of Sarkisov links and *elementary syzygies* for higher ranks [He].

It is easy to classify central models for algebraic surfaces over an algebraically closed field k (cf. [Manin67, теорема 3.2]):

(a) conic bundles $X \to C$ over a curve C with the rank equal to 1 plus the number of degenerate fibers $\mathbb{P}^1_k \vee \mathbb{P}^1_k$;

(b) del Pezzo surfaces $X \to$ pt. with the rank equal to the Picard number of X.

In this situation, the only Mori fibrations are $\mathbb{P}^1$-bundles over a curve and $\mathbb{P}^2$ over a point pt. Respectively, the only rank 2 central models are conic bundles $X \to C$ over a curve C with a singe degenerate fiber and two del Pezzo surfaces $\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_1$ over pt. It easy to find corresponding links. For example, $\mathbb{F}_1$ gives two well-known links over pt.:

$$\begin{array}{ccc} \mathbb{P}^1 & \leftarrow & \mathbb{F}_1 \\ \downarrow & & \downarrow \\ \text{pt.} & \leftarrow & \mathbb{P}^1 \end{array} \quad \text{and inverse} \quad \begin{array}{ccc} \mathbb{F}^1 & \rightarrow & \mathbb{P}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \rightarrow & \text{pt.} \end{array}$$

of types I and III respectively. In [Manin67, теорема 3.2] Manin stumble at $\mathbb{F}_1$ because it is minimal over $\mathbb{P}^1$ but not over pt. Today we clearly undestand that $\mathbb{F}_1$ over pt. or itself is a central model over pt. related to the above link which actually appears in [Manin67, теорема 3.2].

If k is not algebraically closed the complete classification of Sarkisov links is given in [Isk96, теорема 2.6]: about forty classes, and about twenty rank 2 central model classes. Unfortunately, in dimension 3 we have hundreds already known classes Mori fibrations and of Sarkisov links and it is expected tens of thousands of them. Of course, the most interesting in this case is the classes of Mori fibrations and of Sarkisov links of rational threefolds over an algebraically closed fields k, say, $\mathbb{C}$. This is a similar but much more challenging task as Manin's one to classify all rational minimal models for surfaces over a perfect or general field k. This was essentially done in [Manin66] [Manin67].

Let X be an algebraic variety and Y/T be a Mori fibration or, more generally, a central model with a birational isomorphism $Y \dashrightarrow X$. Then we say that Y/T is a *Mori fibration* model or, respectively, a *central model of* X. The central models of X of rank 2 give the Sarkisov links of X. If $Y_1/T_1, Y_2/T_2$ are two models of X with birational isomorphisms $Y_1 \dashrightarrow X, Y_2 \dashrightarrow X$ then a *canonical* birational isomorphism $Y_1 \dashrightarrow Y_2$ is defined. Models $Y_1/T_1, Y_2/T_2$ are *equal* and we write $Y_1/T_1 = Y_2/T_2$ if the canonical birational isomorphism is an isomorphism, that is, regular and gives the commutative diagram

$$\begin{array}{ccc} Y_1 & = & Y_2 \\ \downarrow & & \downarrow \\ T_1 & = & T_2 \end{array},$$

where $Y_1 = Y_2, T_1 = T_2$ denote canonical isomorphisms. Notice that the isomorphim $T_1 = T_2$ is unique if exists. For any Sarkisov link $Y_1/T_1 \dashrightarrow Y_2/T_2$, the birational isomorphism $Y_1 \dashrightarrow Y_2$ is canonical and models $Y_1/T_1, Y_2/T_2$ are not equal. Even if $Y_1 = Y_2$ then $Y_1/T_1 \neq Y_2/T_2$. For example, $\pi_1 \dashrightarrow \pi_2$ is the link of type IV over pt. from the projection π_1 on the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$ into the projection π_2 on the second one:

$$\begin{array}{ccc} \mathbb{P}^1 & \times & \mathbb{P}^1 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array}$$

with $\pi_1 \neq \pi_2$.

The same applies to central models. Notice that $Y_1/T_1 = Y_2/T_2$ and Y_1/T_1 is a Mori fibration then Y_2/T_2 is also a Mori fibration.

The main resalt of a weak version of the Sarkisov theory states that for any algebraic variety X the canonical birational isomorphism $Y/T \dashrightarrow Y'/T'$ of two Mori fibrations $Y/T, Y'/T'$ of X can be factorise into a sequence of Sarkisov links of X: there exist Sarkisov links $Y_i/T_i \dashrightarrow Y_{i+1}/T_{i+1}, i = 1, \dots, n-1$, of X such that $Y \dashrightarrow Y'$ is the following composition

$$Y = Y_1 \dashrightarrow Y_2 \dashrightarrow \cdots \dashrightarrow Y_{n-1} \dashrightarrow Y_n = Y'.$$

It was established over any field k of characteristic 0; and even in the relative case [ShCh, Theorem 7.2] [HM]. According to the geography of log models [ShCh] and the LMMP, it holds in dimension ≤ 3 in positive characteristic too. In particular, this holds for algebraic surfaces at least over any perfect field k.

The Sarkisov theory or Sarkisov Program states a little bit stricter result that the above sequence also decreases the Sarkisov degree which is given by an appropriate very ample linear system on Y'. This theory is actually a direct generalization of the Noether-Fano-Iskovskikh-Manin method to untwist the birational isomorphism $Y \dashrightarrow Y'$. We will discuss the method below. The Sarkisov theory is still established in dimensions ≤ 4 in characteristic 0 and in dimensions ≤ 3 in positive characteristic. Thus the weak Sarkisov theory shows that any two Mori fibrations of a variety X can be connected by elementary birational transformations, called Sarkisov links while in the Sarkisov theory the choice of links should be stricter. For most of applications and, in particular, for Manin and Manin-Iskovskihk, and even for Sarkisov results on rigidity [Sar] it is enough the weak Sarkisov theory. However the

Neother-Fano-Iskovskikh-Manin inequality from the method and its generalizations play very important role to ditect rigid and superrigid varieties.

An algebraic variety X is called *birationally rigid* if it has only one Mori fibration up to isomorphism possibly non-canonical. Moreover, it is *superrigid* if it has only one Mori fibration, i.e., its any two Mori fibrations are canonically isomorphic. It is easy to see that a birationally rigid algebraic variety X does not have Mori fibrations Y/T with $\dim T \geq 2$. There are a lot of rigid varieties with a Mori fibrations over a point: $T = \mathrm{pt}$. I do not know whether exist any such examples over a curve? At least they do not exist for surfaces. Thus in the following for a birationally rigid variety X we usually suppose that $X = Y/\mathrm{pt}.$ is itself Mori-Fano variety: terminal $\mathbb{Q}$-factorial Fano variety with the Picard number 1 over k.

If X is a rigid variety then its group $\mathrm{Bir}(X)$ of birational automorphisms is generated by birational automorphisms given by isomorphisms of Mori models. Moreover, it is enough isomorphisms of Mori models $\gamma\colon Y_1 \simeq Y_2$ for Sarkisov links $\beta\colon Y_1/T_1 \dashrightarrow Y_2/T_2$. Such an isomorphism γ exists by the rigidity. The corresponding birational automorphism of X is

$$X \overset{\alpha^{-1}}{\dashrightarrow} Y_1 \overset{\beta}{\dashrightarrow} Y_2 \overset{\gamma^{-1}}{\simeq} Y_1 \overset{\alpha}{\dashrightarrow} X,$$

where $\alpha\colon Y_1 \dashrightarrow X$ is the canonical (structure) birational isomorphism for the model Y_1/T_1. The only non-canonical isomorphism in the above chain is γ. The constructed isomorphism $X \dashrightarrow X$ depends on it choice. The isomorphism γ is unique up to automorphism of Y_1 and of Y_2; it is enough to take automorphisms on one side. If $\delta\colon Y/T \dashrightarrow X$ is a Mori fibration model of X and $\varepsilon\colon X \dashrightarrow X$ a birational isomorphism of X then

$$\begin{array}{ccccc} Y' = Y & \overset{\delta}{\dashrightarrow} & X & \overset{\varepsilon}{\dashrightarrow} & X \\ \downarrow & & & & \\ T & & & & \end{array}$$

is also a Mori fibration model $\varepsilon \circ \delta\colon Y' = Y/T \dashrightarrow X$ of X. This allows to convert ε into a canonical birational isomorphism

$$Y' = Y \overset{\delta}{\dashrightarrow} X \overset{\varepsilon}{\dashrightarrow} X \overset{\delta^{-1}}{\dashrightarrow} Y$$

between models $Y/T, Y' = Y/T$ and conversely. Under this correspondence the Sarkisov links go into above generators in $\mathrm{Bir}(X)$ and composition of links into product in $\mathrm{Bir}(X)$. Thus the factorization theorem into Sarkisov links implies the generation of $\mathrm{Bir}(X)$. Notice that if $X = Y$ for a Mori fibration Y/T of X then for each Sarkisov link we can chose one generator but to these generators we need to add the automorphisms of $X = Y$.

For a superrigid variety X every birational automorphism of X is regular when $X = Y/\mathrm{pt.}$ is a Mori-Fano variety (model of itself): $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. Indeed, in this case the set of generators coming from Sarkisov links is empty. Thus we have only generaters from $\mathrm{Aut}(X)$. Moreover, if $-K_X$ is very ample then $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(\mathbb{P}^n)$ under the monomorphism corresponding to the anti-canonical imbedding $X \hookrightarrow \mathbb{P}^n$.

The same applies to G-varieties and, in particular, to G-surfaces where G is a finite group of automorphisms of X. In positive characteristics G can be a group scheme of finite type over k of dimension 0. In [Manin67, §§1,2,4] Manin depolops birational geometry of G-surfaces. A G-variety according to Manin is an algebraic variety V over a field k and group G acting on $V_{\overline{k}} = V \times_k \mathrm{Spec}(\overline{k})$ such that one of the following holds

а) *Algebraic case.* The field k is perfect, G is the Galois group of $\overline{k}$ over k and acts on $V_{\overline{k}}$.

б) *Geometric case.* The field k is algebraically closed, G is a finite group and acts on V.

A *G-morphism* $V_1 \to V_2$ of G-varieties V_1, V_2 in the algebraic case is a morphism $V_1 \to V_2$ of algebraic varieties V_1, V_2 over k. Respectively, a *G-morphism* $V_1 \to V_2$ of G-varieties V_1, V_2 in the geometric case is a G-equivariant morphism $V_1 \to V_2$ of algebraic varieties V_1, V_2 over k. Similarly, we can define *rational* G-morphisms, *birational G-isomorphisms*, *G-contractions*, *G-invariant curves of the first kind* (not necessarily irreducible) on G-surfaces, *rational* G-varieties, *minimal* G-surfaces, *Mori G-fibrations*, *central G-models*, etc. The algebraic surfaces over a perfect field k are now included into more general notion of a G-surface.

Manin shows that the adjunction Enriques-Manin method works for rational G-surfaces and the classification of minimal rational G-surfaces is similar to the geometric case with $G = \{1\}$ and the case of rational surfaces over a field k. For this Manin introduces *standard* G-surfaces [Manin67, 4.1]:

а) *G-surfaces with a rational pencil.* G-surface F is smooth, has a G-contraction $F \to C$ on a rational G-curve with ample $-K_F$ over C and F is relative minimal G-surface over C. The generic fiber of contraction is rational, equivalently, is geometrically reduced irreducible of genus 0.

б) *Non-degenerate del Pezzo G-surfaces.* The G-surface F is smooth, its anticanonical divisor $-K_F$ is ample and the G-invariant Picard number of F is 1.

They are exactly Mori G-fibrations for rational G-surfaces and prototypes of Mori G-fibrations in higher dimensions.

Manin also add to standard G-surfaces degenerate del Pezzo G-surfaces [Manin67, стр. 175]. They are weak del Pezzo G-surfaces in modern terminology and not considered as good (minimal) G-models. Manin actually struggle with what is right class of del Pezzo surfaces (see [Manin67, замечание на стр. 175]).

Manin proves that every rational G-surface is birationally G-isomorphic to a standard G-surface up to unessential drawbacks [Manin67, теорема 4.2]. Namely, Manin supposes here that G is Abelian in the geometric case and to standard G-surfaces he adds also degenerate del Pezzo G-surfaces. However, he obtains a more precise and geometrical result (cf. [Manin67, основная лемма 2.1]). The result on an isomorphism with a standard G-surface as it was stated above was established later by Iskovskikh [Isk, теорема 1G]. Notice also that the groups G acting effectively on geometric G-surfaces satisfies the Jordan property and so are close to (finite) Abelian groups. Thus the groups should have an explicit description. For this in [Manin67, §4] Manin investigates the representation of G in the group $N(F_{\overline{k}})$ for standard G-surfaces. He gave the complete description of such representations in terms of root systems [Manin67, теорема 4.5]. For del Pezzo G-surfaces F of degree ≤ 5 the representation is faithful if G acts effectively on F. On the other hand degenerate del Pezzo G-surfaces can be G-contracted to non-degenerate ones. The standard G-surfaces with a rational pencil have the G-invariant Picard number 2 and are minimal G-surfaces if $K_F^2 < 0$ (actually ≤ 2) [Manin67, теорема 4.3] [Isk96, теорема 2.6].

After this lengthy intermission we are ready to continue our discussion of Manin's contributions to birational rigidity. Manin constracted first examples of rigid variery – minimal del Pezzo surfaces of degree ≤ 3, and first examples of superrigid variety – minimal del Pezzo surfaces of degree 1 [Manin67], the part II of [Manin66], and (with Iskovskikh) quartic threeforlds [IM]. Of course, such examples of minimal del Pezzo surfaces exist only for algebraically non-closed fields. In dimension 3 there are a lot rigid and superrigid varieties.

The main results of [Manin67] are about cubic surfaces – del Pezzo surfaces of degree 3. Under the anti-canonical imbedding such a surface is a cubic hypersurface in $\mathbb{P}^3$. Manin proves that minimal cubic surfaces over k are not rational over k [Manin67, теорема A, a)]. Moreover, two minimal cubic surfaces are birationally isomorphic if and only if they are isomorphic under a projective transformation of $\mathbb{P}^3$ [Manin67, теорема A, б)]. The first statement belongs to Segre [Seg] and is an immediate corollary of the second one in a slightly more general form (cf. [Manin67, теоремы 5.7, 5.8]). This more general form means that every minimal Mori fibration of a minimal cubic surface F is a minimal cubic surface F' too, in particular, there are no Mori fibration models over curves. Thus degenerate del Pezzo surfaces in [Manin67, следствие стр. 185] are impossible. This follows form the table of Sarkisov links for algebraic surfaces [Isk96, теорема 2.6] or the Nother-Fano-Iskovskihk-Manin method. But the second statement is much stronger: every cubic model F' is projectively isomorphic to F, equivalently, F' is isomorphic to F. This means that F is rigid. Actually, F is superrigid if there no closed points of degree 1 and 2 on F. Indeed, if F is not superrigid then there exists a Sarkisov link into another model. Thus there a central model F'' over F which is a blowup of a closed point $x \in F$, and $K_{F''}^2 = 3-d$. Since F'' is also a del Pezzo surface, we get $3-d \geq 1$ and $d \leq 2$. Notice now that if x has degree 1, i.e., is a k-point then there exists a birational automorphism $t_x\colon F \dashrightarrow F$ – Geiser involution: x does not lie on a line on F and for any line l through x in $\mathbb{P}^3$, the divisor $F_{|l} - x$ is invariant under the action of t_x. Actually, this involution corresponds to the link given by the central model F''. Similarly, every closed point $x \in F$ of degree 2 does not lie on a line on F and determines a Bertini involution $s_x\colon F \dashrightarrow F \in \mathrm{Bir}(F)$ which corresponds to F''. Thus we have only links of these two types. In particular, this shows the stated rigidity if we know the weak form of Sarkisov theory: every canonical birational isomorphisms between Mori fibraions models can be factorised into Sarkisov links. This explains also the next Manin result [Manin67, теорема Б]: $\mathrm{Bir}(F)$ is generated by involutions t_x, s_x and by the finite group of projective automorphisms of F. Actually, every automorphisms of F is projective and the group $\mathrm{Aut}(F)$ is finite, e.g., because the set of exceptional curves of the first kind on $F_{\overline{k}}$ is finite. The generation implies that F has a closed k-point if and only if $\mathrm{Bir}(F)$ is infinite. Indeed, even when $\mathrm{Bir}(F)$ is finitely generated (e.g., over a finite field [Manin67, 5.16]) subgroup $< t_x, s_x >$ generated by involutions t_x, s_x is infinite. The subgroup $< s_x, t_x >$ is normal in $\mathrm{Bir}(F)$ and the group $\mathrm{Bir}(F)$ is a semidirect product of $< t_x, s_x >$ with the finite group $\mathrm{Aut}(F)$ permuting separately Geiser and Bertini involutions. In introduction [Manin67, стр. 162] Manin states also that $< s_x, t_x >$ is close to a free product of cyclic groups $\mathbb{Z}/2\mathbb{Z}$ of order 2. Every Bertini involution s_x does not have relations with the other involutions and relations are only possible for Geiser involutions. Indeed, the only central models of rank 3 for minimal cubics are blowups of two closed k-points $x, x' \in F$. A more conclusive statement will be given below for del Pezzo surfaces of degree 2.

The rigidity holds for the minimal G-cubics and, moreover, for the minimal Del Pezzo G-surfaces F of degree ≤ 3 [Manin67, теорема 5.9]. The group $\mathrm{Bir}(F)$ is generated by $\mathrm{Aut}(F)$ and by Geiser and Bertini involutions t_x and s_x for F of degree 3 where x is a G-invariant closed point of $F_{\overline{k}}$ for t_x and x is a G-invariant pair of closed point of $F_{\overline{k}}$; the point x should be *non-special*, i.e., not on a line on $F_{\overline{k}}$ and the pair x should be not on a line and not on a conic on $F_{\overline{k}}$. Respectively, for degree 2, the group $\mathrm{Bir}(F)$ is generated by $\mathrm{Aut}(F)$ and by Bertini involutions s_x where x are the non-special G-invariant closed points of $F_{\overline{k}}$. Additionally, for degree 2, the subgroup $< s_x >$ generated by Bertini involutions is normal in $\mathrm{Bir}(F)$, a free product of order 2 subroups corresponding to each involution and $\mathrm{Bir}(F)$ is a semidirect product of $< s_x >$ with the finite group $\mathrm{Aut}(F)$ permuting Bertini involutions [Manin67, замечание 5.15]. Finally, the minimal del Pezzo G-surfaces of degree 1 are superrigid and $\mathrm{Bir}(F) = \mathrm{Aut}(F)$ is finite. Thus for every minimal del Pezzo surfaces of degree ≤ 3 over a finite field k, the group $\mathrm{Bir}(F)$ is finitely generated.

In [Manin67, теорема 6.1] Manin shows unirationality of del Pezzo surfaces F over a field k of degree $n = 2, 3$ and 4. More precisely, if F has a non-special closed k-point, then there esxists a rational dominant morphism $\mathbb{P}^2 \dashrightarrow F$ of degree 2 for $n = 4$ and of degree 6 for $n = 3$. Thus if k is infinite and such an F has a non-special k-point then the set of closed k-points in dense in the Zariski topology. For $n = 3$ the same result was established by Segre without the non-special assumption [Seg]. Manin also proved that established degrees of unirationality are optimal [Manin67, теорема 6.3].

Quartic threefolds. The year 1971 was a turning point for higher dimensional birational algebraic geometry. Independently, it was constructed three types of counterexamples to the Lüroth problem: whether a unirational algebraic variery is rational. Lüroth proved himself that this is true in dimension 1: for any field k, any finitely generated subfield of the field $k(x)$ of rational functions in one variable x is isomorphic to $k(x)$ over k, equivalently, for any rational dominant morphism $\mathbb{P}^1 \dashrightarrow C$ over k, C is rational over k or $C = \mathrm{pt.}$. In dimension 2, the Lüroth problem has the positive answer if k is algebraically closed. This follows from the Castelnuovo cohomological characterization of smooth rational projective surfaces F over an algebraically closed field k. But this is not true for surfaces over any algebraically non-closed field k, e.g., for the minimal del Pezzo surfaces of degree 3 and 2. Each of counterexamples [AM] [CG] [IM] constitutes a new direction in birational algebraic geometry. The direction due to Iskovskikh-Manin related to the Noether-Fano-Iskovskikh-Manin method and to the Minimal Model Program.

The main theorem of [IM]: if $\chi\colon V \dashrightarrow V'$ is a birational isomorphism of two smooth hypersurfaces of degree 4 (quartics) in $\mathbb{P}^4$ over any field then χ is an projective isomorphism, i.e., is regular. Thus the group of birational automorphisms of any smooth quartic V is finite and V is not rational [IM, следствие стр. 141]. On the other hand, Segre constructed examples of smooth unirational quartics at least for fields k with characteristic $\neq 2, 3$ [Seg]. This gives examples of unirational but not rational varieties in dimension 3.

It was noticed in 1988 that every smooth quartic threefold V is superrigid that follows from the proof in [IM]. Since the Picard number of V is 1, V is a Mori fibration over pt.. E.g., it is not birationally isomorphic to a Mori fibration of conic bundle type because the latter has infinitely many birational isomorphisms. After this example appeared many other examples of rigid and superrigid varieties and this terminology. Actually, Iskovskihk and Manin noticed that quartics do not have non-trivial birational automorphisms and this means extremal "rigidity" of these unirational and possibly non-unirational varieties. Still we do not know whether any smooth quartic threefold is unirational.

The statement of [IM, основная теорема] and many fundamental ideas in its proof belongs to Fano [F07] [F15]. The Nother-Fano-Iskovskikh method uses the singularities of linear systems of good models, e.g., of Mori fibrations. It is based on the uniqueness of canonical model [ISh, теорема 2.3]. We explain this method for a Mori-Fano variety X. Suppose that $\varphi'\colon X' \to T'$ is its Mori fibration model with a canonical birational isomorphism $\chi\colon X \dashrightarrow X'$. To prove a superrigidity of X we need to verify that χ is an isomorphism, that is, is regular. Let

$$\mathcal{H}' = |-\mu' K_{X'} + \varphi'^* A'|$$

be a very ample complete linear system on X', where μ' is a positive integer ≥ 2 and A' is a very ample divisor on T'; such a liner system exists for every Mori fibration. Let $\mathcal{H} = \chi_*^{-1}\mathcal{H}'$ be its proper birational transform on X. Then $\mathcal{H}$ is a mobil linear subsystem in $|-\mu K^s|$ with a positive rational number μ and K^s is a semi-canonical divisor, that is, a canonical divisor up to $\mathbb{Q}$-linear or numerical equivalence; μ is integral if X is smooth and $K^s = K_X$ generates the Picard group of X up to linear equivalence. For sufficiently general effective divisor $H' \in \mathcal{H}'$, the pair $(X', \frac{1}{\mu'}H')$ is a minimal model of $(X, \frac{1}{\mu'}H)$ where H is the proper birational transform of H', i.e., the pair $(X', \frac{1}{\mu'}H')$ has only log canonical, terminal in codimension ≥ 2 singularities and $K_{X'} + \frac{1}{\mu'}H'$ is nef. If the $(X, \frac{1}{\mu})$ is weak log canonical, i.e., has only log canonical, canonical in codimension ≥ 2 singularities and $K_X + \frac{1}{\mu}H$ is nef, then χ is an isomorphism by [ISh, теорема 2.3]. Thus to prove the superrigidity it is sufficent to show the weak log canonical property of $(X, \frac{1}{\mu}H)$. Otherwise X is not superrigid and we can construct a Sarkisov link of $X/\text{pt.}$ to another Mori fibration with a smaller invariant μ, the first component in the Sarkisov degree. After finitely many such steps conjecturally we get X'/T'. This hold in dimensions ≤ 3 but unknown in higher dimensions – the Sarkisov theory.

For a rigid variety X we should check that each Sarkisov link gives and isomorphic variety. This type of constructions Manin develops in [Manin67] for minimal del Pezzo surfaces of degree ≤ 3.

In the superrigid case, since $K^s + \frac{1}{\mu}H \equiv K_X + \frac{1}{\mu}H \equiv 0$ and nef, we need to verify that $(X, \frac{1}{\mu}H)$ is log canonical and canonical in codimension ≥ 2, where $\equiv$ denotes the numerical equivalence. This is to investigate singularities of H. The tool used by Iskovskikh and Manin goes back to Noether and Fano and after its sharpening becomes the Noether-Fano-Iskovskikh-Manin inequality. It depends on a situatation. Thus we consider now a smooth threefold quartic $X = V$. In this case $K^s = K_V$ and generates the Picard group of V up to the linear equivalence. Thus μ is a positive integer and $H \in \mathcal{H} \subseteq |-\mu K_V|$. In particular, every component of effective divisor H has multiplicity $\leq \mu$ and $(V, \frac{1}{\mu}H)$ has only log canonical singularities in codimension 1. The remaining possible non-canonical singularities are along curves or in points of V. On the other hand, V is smooth and has only terminal singularities. This implies the existence of the *canonical thereshold*

$$c(V,H) = \max\{t \in \mathbb{Q} \mid K_X + tH \text{ has only canonical singularities in codimesion } \geq 2\}.$$

This is the second component of the Sarkisov degree. Instead, of $c((V,H)$ Iskovskikh and Manin as Noether and Fano use the *maximal multiplicity* $\lambda = 1/c(V,H)$ *of* H or *of the linear system* $\mathcal{H}$ [IM, §5]. The threshold attains on an exceptional divisor E and it or its center on V is called the *maximal singularity*. By our construction $c(V,H) < 1/\mu$ and we need to verify absence of maximal singularities.

If the maximal singularity is a curve C on V then in the case $c(V,H) < 1/\mu$ we get the Noether inquality $\operatorname{ord}_C H > \mu$. The same holds for any smooth Fano variety X, an effective divisor $H \in |-\mu K_X|$ and a maximal point P of codimension ≥ 2 (closed or not): if $c(X,H;P) < 1/\mu$ then

$$\operatorname{ord}_P H > \mu \quad \text{(Neother)}$$

where $c(X,D;P)$ is the canonical threshold at P. The today proof is very simple: it follows from the fact that the discrepancy in blown up divisor in a point P of codimension ≥ 2 is ≥ 1, and for any effective divisor D on X

$$c = c(X,D;P) \geq \frac{1}{\operatorname{ord}_P D} \quad \text{or} \quad \lambda \leq \operatorname{ord}_P D$$

where $\lambda = 1/c$ is the *maximal multiplicity at* P. For P of codimension 2, the equality holds

$$c = c(X,D;P) = \frac{1}{\operatorname{ord}_P D} \quad \text{or} \quad \lambda = \operatorname{ord}_P D$$

This explains the terminology. Now for a quartic threefold $X = V$ and a line l on V but not on H passing through a point P on a maximal singularity curve C, $(l, -K_V) = 1$ and $\mu = (l,H) \geq \operatorname{ord}_C H > \mu$ hold, a contradiction by the Noether inequality where $(-,-)$ denotes the intersection form for curves with divisors.

Thus every maximal singularity for H on V is a closed point P. In this case we have stronger inequality: $\operatorname{ord}_P H > 2\mu$. As above, more generally, for a maximal points P of codimension ≥ 3 if $c(X,H) < 1/\mu$ then

$$\operatorname{ord}_P H > 2\mu \quad \text{(Fano)}.$$

In this situation the discrepancy in blown up divisor in P is ≥ 2 and for any effective divisor D on X

$$c = c(X, D; P) \geq \frac{2}{\operatorname{ord}_P D} \quad \text{or} \quad \lambda \leq \frac{\operatorname{ord}_P D}{2}.$$

Unfortunately, the Fano inequality is not optimal in codimensions ≥ 3 vs codimension 2 and leads to absence of maximal singualarities only in general cases, e.g., when the maximal singularity on the first blowup of a point.

Iskovskikh and Manin discovered a quadratic version of the Fano inequality that works for all maximal singularities. Let $\mathcal{H}$ be a mobile linear system on a smooth threefold X and a closed point $P \in X$ be a maximal singularity of $(X, \frac{1}{\mu}H)$ for sufficiently general $H \in \mathcal{H}$. Let H_1, H_2 be two members of $\mathcal{H}$ without common irreducible components and S be an effective surface through P not containing any irreducible component of $H_1 \cap H_2$. Then the local intersection number $(H_1, H_2, S)_P$ at P is well defined and

$$(H_1, H_2, S)_P > 4m^2 \qquad \text{(Iskovskihk-Manin)}.$$

For the quartic V take sufficiently general divisors $H_1, H_2 \in \mathcal{H}$ and a sufficiently general hyperplane section S through P. Then

$$4m^2 = (H_1, H_2, S) \geq (H_1, H_2, S)_P > 4m^2,$$

a contradition, where $(-, -, -)$ is the intersection form for divisors on V.

In conclusion the impact of Manin to birational geometry is enormous and produce a lot further generalizations and developments. The flow of mathematical research is like a river with many contributions as influxes which make flow full-flowing. In the case of birational rigidity: Neother, Fano, Manin, Iskovslikh, Sarkisov, Pukhlikov, etc. And this flow goes on and on to the see of perfection.

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