ENDOMORPHISM ALGEBRAS AND AUTOMORPHISM GROUPS OF CERTAIN COMPLEX TORI

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ABSTRACT. We study the endomorphism algebra and automorphism groups of complex tori, whose second rational cohomology group enjoys a certain Hodge property introduced by F. Campana.

1. INTRODUCTION

As usual, we write $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ for the fields of rational, real, complex numbers and $\mathbb{Z}$ for the ring of integers. We write $\bar{\mathbb{Q}}$ for the subfield of all algebraic nunbers in $\mathbb{C}$, which is an algebraic closure of $\mathbb{Q}$. If p is a prime then $\mathbb{F}_p, \mathbb{Z}_p, \mathbb{Q}_p$ stand for the p-element field, the ring of p-adic integers, the field of p-adic numbers respectively. If E is a number field of degree $n = [E : \mathbb{Q}]$ then r_E and s_E are nonnegative integers such that the $\mathbb{R}$-algebra $E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a product $\mathbb{R}^r \times \mathbb{C}^s$. (In other words, r_E is the number of "real" field embeddings $E \hookrightarrow \mathbb{R}$ and $2s_E$ is the number of "imaginary" field embeddings $E \hookrightarrow \mathbb{C}$, whose images do *not* lie in $\mathbb{R}$.) In particular,

$$[E : \mathbb{Q}] = r_E + 2s_E. \tag{1}$$

Let X be a connected compact complex Kähler manifold, $\mathrm{H}^2(X, \mathbb{Q})$ its second rational cohomology group equipped with the canonical rational Hodge structure, i.e., there is the Hodge decomposition

$$\mathrm{H}^2(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{H}^2(X, \mathbb{C}) = \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^{2,0}(X)$$

where $\mathrm{H}^{2,0}(X) = \Omega^2(X)$ is the space of holomorphic 2-forms on X, $\mathrm{H}^{0,2}(X)$ is the "complex-conjugate" of $\mathrm{H}^{2,0}(X)$ and $\mathrm{H}^{1,1}(X)$ coincides with its own "complex-conjugate" (see [13, Sections 2.1–2.2], [23, Ch. VI-VII])). The following property of X was introduced and studied by F. Campana [10, Definition 3.3]. (Recently, it was used in the study of coisotropic and lagrangian submanifolds of symplectic manifolds [1].)

Definition 1.1. A manifold X is *irreducible in weight 2* (irréductible en poids 2) if it enjoys the following property.

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Let H be a rational Hodge substructure of $\mathrm{H}^2(X,\mathbb{Q})$ such that

$$H_{\mathbb{C}} \cap \mathrm{H}^{2,0}(X) \neq \{0\}$$

where $H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C}$.

Then $H_{\mathbb{C}}$ contains the whole $\mathrm{H}^{2,0}(X)$.

Our aim is to study complex tori T that enjoy this property. Namely, we discuss their endomorphism algebras, automorphism groups and Hodge groups.

Let $T = V/\Lambda$ be a complex torus of positive dimension g where V is a g-dimensional complex vector space, and Λ is a discrete lattice of rank $2g$ in V. One may naturally identify Λ with the first integral homology group $\mathrm{H}_1(T,\mathbb{Z})$ of T and

$$\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q} = \{v \in V \mid \exists n \in \mathbb{Z} \setminus \{0\} \ \text{ such that } nv \in \Lambda\}$$

with the first rational homology group $\mathrm{H}_1(T,\mathbb{Q})$ of T. There are also natural isomorphisms of real vector spaces

$$\Lambda \otimes \mathbb{R} = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \to V, \ \lambda \otimes r \mapsto r\lambda$$

that may be viewed as isomorphisms related to the first real cohomology group $\mathrm{H}_1(T,\mathbb{R})$ of T:

$$\mathrm{H}_1(T,\mathbb{R}) = \mathrm{H}_1(T,\mathbb{Z}) \otimes \mathbb{R} = \mathrm{H}_1(T,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \to V.$$

In particular, there is a canonical isomorphism of real vector spaces

$$\mathrm{H}_1(T,\mathbb{R}) = V, \tag{2}$$

and a canonical isomorphism of complex vector spaces

$$\mathrm{H}_1(T,\mathbb{C}) = \mathrm{H}_1(T,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{H}_1(T,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C} =: V_{\mathbb{C}} \tag{3}$$

where $\mathrm{H}_1(T,\mathbb{C})$ is the first complex homology group of T.

There are natural isomorphisms of $\mathbb{R}$-algebras

$$\mathrm{End}_{\mathbb{Z}}(\Lambda) \otimes \mathbb{R} \cong \mathrm{End}_{\mathbb{R}}(V), \ u \otimes r \mapsto ru,$$

$$\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \otimes \mathbb{R} \cong \mathrm{End}_{\mathbb{R}}(V), \ u \otimes r \mapsto ru,$$

which give rise to the natural ring embeddings

$$\mathrm{End}_{\mathbb{Z}}(\Lambda) \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \subset \mathrm{End}_{\mathbb{R}}(V) \subset \mathrm{End}_{\mathbb{R}}(V) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}}). \tag{4}$$

Here the structure of an $2g$-dimensional *complex* vector space on $V_{\mathbb{C}}$ is defined by

$$z(v \otimes s) = v \otimes zs \quad \forall v \otimes s \in V \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}, \ z \in \mathbb{C}.$$

If $u \in \mathrm{End}_{\mathbb{R}}(V)$ then we write $u_{\mathbb{C}}$ for the corresponding $\mathbb{C}$-linear operator in $V_{\mathbb{C}}$, i.e.,

$$u_{\mathbb{C}}(v \otimes z) = u(v) \otimes z \quad \forall u \in V, z \in \mathbb{C}, v \otimes z \in V_{\mathbb{C}}. \tag{5}$$

Remark 1.2. Sometimes, we will identify $\mathrm{End}_{\mathbb{R}}(V)$ with its image $\mathrm{End}_{\mathbb{R}}(V) \otimes 1 \subset \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ and write u instead of $u_{\mathbb{C}}$, slightly abusing notation.

As usual, one may naturally extend the complex conjugation $z \mapsto \bar{z}$ on $\mathbb{C}$ to the $\mathbb{C}$-antilinear involution

$$V_{\mathbb{C}} \to V_{\mathbb{C}},\ w \mapsto \bar{w},\ v \otimes z \mapsto \overline{v \otimes z} = v \otimes \bar{z},$$

which is usually called the complex conjugation on $V_{\mathbb{C}}$. Clearly,

$$u_{\mathbb{C}}(\bar{w}) = \overline{u(w)} \quad \forall u \in \mathrm{End}_{\mathbb{R}}(V), w \in V_{\mathbb{C}}. \tag{6}$$

This implies easily that the set of fixed points of the involution is

$$V = V \otimes 1 \subset V_{\mathbb{C}}.$$

Let $\mathrm{End}(T)$ be the endomorphism ring of the complex commutative Lie group T and $\mathrm{End}^0(T) = \mathrm{End}(T) \otimes \mathbb{Q}$ the corresponding endomorphism algebra, which is a finite-dimensional algebra over$\mathbb{Q}$, see [20, 6, 3]. There are well known canonical isomorphisms

$$\mathrm{End}(T) = \mathrm{End}_{\mathbb{Z}}(\Lambda) \cap \mathrm{End}_{\mathbb{C}}(V), \quad \mathrm{End}^0(T) = \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \cap \mathrm{End}_{\mathbb{C}}(V).$$

Let $g \geq 2$ and

$$\mathrm{H}^2(T, \mathbb{Q}) = \wedge^2_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}, \mathbb{Q})$$

be the *second rational cohomology group* of T, which carries a natural rational Hodge structure of weight two:

$$\mathrm{H}^2(T, \mathbb{Q}) = \mathrm{H}^2(T, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{H}^{2,0}(T) \oplus \mathrm{H}^{1,1}(T) \oplus \mathrm{H}^{0,2}(T)$$

where $\mathrm{H}^{2,0}(T) = \Omega^2(T)$ is the $g(g-1)/2$-dimensional space of holomorphic 2-forms on T.

Definition 1.3. Let $g = \dim(T) \geq 2$. We say that T is 2-*simple* if it is *irreducible in weight 2*, i.e., enjoys the following property.

Let H be a rational Hodge substructure of $\mathrm{H}^2(T, \mathbb{Q})$ such that

$$H_{\mathbb{C}} \cap \mathrm{H}^{2,0}(T) \neq \{0\}$$

where $H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C}$.

Then $H_{\mathbb{C}}$ contains the whole $\mathrm{H}^{2,0}(T)$.

Remark 1.4. We call such complex tori 2-simple, because they are simple in the usual meaning of this word if $g > 2$, see Theorem 1.7(i) below.

Example 1.5. (See [10, Example 3.4(2)].) If $g = 2$ then $\dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T)) = 1$. This implies that (in the notation of Definition 1.3) if $H_{\mathbb{C}} \cap \mathrm{H}^{2,0}(T) \neq \{0\}$ then $H_{\mathbb{C}}$ contains the whole $\mathrm{H}^{2,0}(T)$. Hence, every 2-dimensional complex torus is 2-simple.

In what follows we write $\mathrm{Aut}(T) = \mathrm{End}(T)^*$ for the automorphism group of the complex Lie group T. We will need the following well known definition.

Definition 1.6. A number field is called *primitive* if its only proper subfield is $\mathbb{Q}$.

Our main result is the following assertion.

Theorem 1.7. *Let T be a complex torus of dimension $g \ge 3$. Suppose that T is 2-simple.*

Then T enjoys the following properties.

- (i) *T is simple.*
- (ii) *If E is any subfield of $\mathrm{End}^0(T)$ then it is a number field, whose degree over $\mathbb{Q}$ is either 1 or g or $2g$.*
- (iii) *$\mathrm{End}^0(T)$ is a number field E such that its degree $[E:\mathbb{Q}]$ is either 1 or g or $2g$.*
- (iv) *If $[E:\mathbb{Q}] = 1$ then*
$$\mathrm{End}^0(T) = \mathbb{Q}, \mathrm{End}(T) = \mathbb{Z}, \ \mathrm{Aut}(T) = \{\pm 1\}.$$
- (v) *If $E = \mathrm{End}^0(T)$ and $[E:\mathbb{Q}] = 2g$ then E is a purely imaginary number field and $\mathrm{Aut}(T) \cong \{\pm 1\} \times \mathbb{Z}^{g-1}$. In addition, if E is not primitive then it contains precisely one proper subfield except $\mathbb{Q}$ and this subfield has degree g.*
- (vi) *If $E = \mathrm{End}^0(T)$ and $[E:\mathbb{Q}] = g$ then E is a primitive number field and $\mathrm{Aut}(T) \cong \mathbb{Z}^d \times \{\pm 1\}$ where the positive integer d equals $r_E + s_E - 1$. In particular,*
$$\frac{1}{2} \le \frac{g}{2} - 1 \le d \le g - 1.$$
In addition, if T is a complex abelian variety then E is a primitive totally real number field and $d = g - 1$.

Remark 1.8.
- (i) It is well known (and can be easily checked) that T is *simple* if and only if the rational Hodge structure on $\Lambda_{\mathbb{Q}} = \mathrm{H}_1(T, \mathbb{Q})$ is *irreducible*.[1]
- (ii) We may view $\mathrm{H}^2(T, \mathbb{Q})$ as the $\mathbb{Q}$-vector subspace $\mathrm{H}^2(T, \mathbb{Q}) \otimes 1$ of $\mathrm{H}^2(T, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{H}^2(T, \mathbb{C})$. Let us consider the $\mathbb{Q}$-vector (sub)space
$$\mathrm{H}^{1,1}(T, \mathbb{Q}) := \mathrm{H}^2(T, \mathbb{Q}) \cap \mathrm{H}^{1,1}(T)$$
of 2-dimensional *Hodge cycles* on T. Notice that the *irreducibility* of the rational Hodge structure on $\Lambda_{\mathbb{Q}}$ implies the *complete reducibility* [2] of the rational Hodge structure on $\mathrm{H}^2(T, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}\left(\wedge^2_{\mathbb{Q}} \Lambda_{\mathbb{Q}}, \mathbb{Q}\right)$. (It follows from the reductiveness of the Mumford-Tate group of a simple torus [11, Sect. 2.2].) In light of (i) and Theorem 1.7(i), a complex torus T of dimension > 2 is 2-simple if and only if it is simple and $\mathrm{H}^2(T, \mathbb{Q})$ splits into a direct sum of $\mathrm{H}^{1,1}(T, \mathbb{Q})$ and an *irreducible* rational Hodge substructure.

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[1]A rational Hodge structure H is called *irreducible* or *simple* if its only rational Hodge substructures are H itself and $\{0\}$ [11, Sect. 2.2].

[2]A rational Hodge structure is called completely reducible it it splits into a direct sum of simple rational Hodge structures.

Theorem 1.9. *Let $g \geq 3$ be an integer. Let $\mathbf{r}, \mathbf{s}$ be nonnegative integers such that*

$$\mathbf{r} + 2\mathbf{s} = g.$$

Then there exists a 2-simple torus T of degree g that enjoys the following properties.

The endomorphism algebra $\mathrm{End}^0(T)$ is a number field E such that

$$[E : \mathbb{Q}] = g, \; r_E = \mathbf{r}, \; s_E = \mathbf{s}.$$

In particular, if d is an integer such that

$$\frac{g}{2} - 1 \leq d \leq g - 1$$

then there exists a g-dimensional 2-simple complex torus T such that

$$\mathrm{Aut}(T) \cong \mathbb{Z}^d \times \{\pm 1\}.$$

The paper is organized as follows. We prove Theorem 1.7 in Section 3, using explicit constructions related to the Hodge structure on $\Lambda_{\mathbb{Q}}$ that will be discussed in Section 2. Section 4 deals with (mostly well known) results about number fields that will be used in the computations of Hodge groups of complex tori. In Section 5 we discuss general properties of Hodge groups of 2-simple tori. In Section 6 we concentrate on the case when the endomorphism algebra is a number field of degree g.

This paper may be viewed as a follow up of [20] and [3].

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2. Hodge structures

2.1. It is well known that $\Lambda_{\mathbb{Q}} = \mathrm{H}_1(T, \mathbb{Q})$ carries the natural structure of a rational Hodge structure of weight -1. Let us recall the construction. Let $J : V \to V$ be the multiplication by $\mathbf{i} = \sqrt{-1}$, which is viewed as an element of $\mathrm{End}_{\mathbb{R}}(V)$ such that

$$J^2 = -1.$$

Hence, $J_{\mathbb{C}}^2 = -1$ in $\mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ and we define two mutually complex-conjugate $\mathbb{C}$-vector subspaces (of the same dimension) $\mathrm{H}_{-1,0}(T)$ and $\mathrm{H}_{0,-1}(T)$ of $V_{\mathbb{C}}$ as the eigenspaces $V_{\mathbb{C}}(\mathbf{i})$ and $V_{\mathbb{C}}(-\mathbf{i})$ of $J_{\mathbb{C}}$ attached to eigenvalues $\mathbf{i}$ and $-\mathbf{i}$ respectively. Clearly,

$$V_{\mathbb{C}} = V_{\mathbb{C}}(\mathbf{i}) \oplus V_{\mathbb{C}}(-\mathbf{i}) = \mathrm{H}_{-1,0}(T) \oplus \mathrm{H}_{0,-1}(T),$$

which defines the rational Hodge structure on $\Lambda_{\mathbb{Q}}$, in light of $V_{\mathbb{C}} = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. It also follows that both $\mathrm{H}_{-1,0}(T)$ and $\mathrm{H}_{0,-1}(T)$ have the same dimension $2g/2 = g$.

Recall that V is a complex vector space. I claim that the map

$$\Psi : V \to V_{\mathbb{C}}(\mathbf{i}) = \mathrm{H}_{-1,0}(T), \; v \mapsto Jv \otimes 1 + v \otimes \mathbf{i} \tag{7}$$

is an isomorphism of complex vector spaces. Indeed, first, Ψ defines a homomorphism of real vector spaces $V \to V_{\mathbb{C}}$. Second, if $v \in V$ then

$$J_{\mathbb{C}}(Jv \otimes 1 + v \otimes \mathbf{i}) = J^2 v \otimes 1 + Jv \otimes \mathbf{i} = -v \otimes 1 + Jv \otimes \mathbf{i} = \mathbf{i}(Jv \otimes 1 + v \otimes \mathbf{i}),$$

i.e., $Jv \otimes 1 + v \otimes \mathbf{i} \in V_{\mathbb{C}}(\mathbf{i}) = \mathrm{H}_{-,0}(T)$ and therefore the map (7) is defined correctly. Third, taking into account that J is an automorphism of V and $V_{\mathbb{C}} = V \otimes 1 \oplus V \otimes \mathbf{i}$, we conclude that Ψ is an injective homomorphism of real vector spaces and a dimension argument implies that it is actually an isomorphism. It remains to check that Ψ is $\mathbb{C}$-linear, i.e.,

$$\Psi(Jv) = \mathbf{i}\Psi(v).$$

Let us do it. We have

$$\Psi(Jv) = J(Jv) \otimes 1 + Jv \otimes \mathbf{i} = -v \otimes 1 + Jv \otimes \mathbf{i} = \mathbf{i}(Jv \otimes 1 + v \otimes \mathbf{i}) = \mathbf{i}\Psi(v).$$

Hence, Ψ is a $\mathbb{C}$-linear isomorphism and we are done.

Now suppose that $u \in \mathrm{End}_{\mathbb{R}}(V)$ commutes with J, i.e., $u \in \mathrm{End}_{\mathbb{C}}(V)$. Then

$$\Psi \circ u = u_{\mathbb{C}} \circ \Psi. \tag{8}$$

In particular, $\mathrm{H}_{-1,0}(T)$ is $u_{\mathbb{C}}$-invariant. Indeed, if $v \in V$ then

$$\Psi \circ u(v) = Ju(v) \otimes 1 + u(v) \otimes \mathbf{i} = uJ(v) \otimes 1 + u_{\mathbb{C}}(v \otimes \mathbf{i}) = u_{\mathbb{C}}(J(v) \otimes 1) + u_{\mathbb{C}}(v \otimes \mathbf{i}) = u_{\mathbb{C}} \circ \Psi(v),$$

which proves our claim.

Similarly, there is an anti-linear isomorphism of complex vector spaces

$$V \to V_{\mathbb{C}}(-\mathbf{i}) = \mathrm{H}_{0,-1}(T), \ v \mapsto Jv \otimes 1 - v \otimes \mathbf{i}.$$

It is also well known that there is a canonical isomorphism of rational Hodge structures of weight 2

$$\mathrm{H}^2(T, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(\wedge^2_{\mathbb{Q}} \mathrm{H}_1(T, \mathbb{Q}), \mathbb{Q})$$

where the Hodge components $\mathrm{H}^{p,q}(T)$ $(p, q \geq 0, p + q = 2)$ are as follows.

$$\mathrm{H}^{2,0}(T) = \mathrm{Hom}_{\mathbb{C}}(\wedge^2_{\mathbb{C}} \mathrm{H}_{-1,0}(T), \mathbb{C}), \quad \mathrm{H}^{0,2}(T) = \mathrm{Hom}_{\mathbb{C}}(\wedge^2_{\mathbb{C}} \mathrm{H}_{0,-1}(T), \mathbb{C}), \tag{9}$$

$$\mathrm{H}^{1,1}(T) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_{-1,0}(T), \mathbb{C}) \wedge \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_{0,-1}(T), \mathbb{C}) \cong$$

$$\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_{-1,0}(T), \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_{0,-1}(T), \mathbb{C}).$$

Clearly,

$$\dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T)) = \frac{g(g-1)}{2}.$$

3. Endomorphism Fields and Automorphism Groups

Proof of Theorem 1.7. Let T be a 2-simple complex torus and assume that

$$g = \dim(T) \geq 3.$$

(i) Suppose that T is *not* simple. This means that there is a proper complex subtorus $S = W/\Gamma$ where W is a complex vector subspace of V with

$$0 < d = \dim_{\mathbb{C}}(W) < \dim_{\mathbb{C}}(V) = g$$

such that

$$\Gamma = W \cap \Lambda$$

is a discrete lattice of rank $2d$ in W. Then the quotient T/S is a complex torus of positive dimension $g - d$.

Let $H \subset \mathrm{H}^2(T, \mathbb{Q})$ be the image of the canonical *injective* homomorphism of rational Hodge structures $\mathrm{H}^2(T/S, \mathbb{Q}) \hookrightarrow \mathrm{H}^2(T, \mathbb{Q})$ induced by the quotient map $T \to T/S$ of complex tori. Clearly, H is a rational Hodge substructure of $\mathrm{H}^2(T, \mathbb{Q})$ and its $(2,0)$-component

$$H^{2,0} \subset H_{\mathbb{C}}$$

has $\mathbb{C}$-dimension

$$\dim_{\mathbb{C}}(H^{2,0}) = \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T/S)) = \frac{(g-d)(g-d-1)}{2} < \frac{g(g-1)}{2} = \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T)).$$

In light of the 2-simplicity of T,

$$\dim_{\mathbb{C}}(\mathrm{H}^{2,0}) = 0,$$

which implies that

$$g - d = 1.$$

On the other hand, let $\tilde{H}$ be the kernel of the canonical *surjective* homomorphism of rational Hodge structures $\mathrm{H}^2(T, \mathbb{Q}) \twoheadrightarrow \mathrm{H}^2(S, \mathbb{Q})$ induced by the inclusion map $S \subset T$ of complex tori. Clearly, $\tilde{H}$ is a rational Hodge substructure of $\mathrm{H}^2(T, \mathbb{Q})$. Notice that the induced homomorphism of $(2,0)$-components $\mathrm{H}^{2,0}(T) \to \mathrm{H}^{2,0}(S)$ is also surjective, because every holomorphic 2-form on S obviously extends to a holomorphic 2-form on T. This implies that the $(2,0)$-component

$$\tilde{H}^{2,0} \subset \tilde{H}_{\mathbb{C}}$$

of $\tilde{H}$ has $\mathbb{C}$-dimension

$$\dim_{\mathbb{C}}(\tilde{H}^{2,0}) = \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T)) - \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(S)) = \frac{g(g-1)}{2} - \frac{d(d-1)}{2} > 0.$$

In light of the 2-simplicity of T,

$$\dim_{\mathbb{C}}(\tilde{H}^{2,0}) = \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T)) = \frac{g(g-1)}{2},$$

which implies that $\frac{d(d-1)}{2} = 0$, i.e., $d = 1$. Taking into account that $g - d = 1$, we get $g = 1 + 1 = 2$, which is not true. The obtained contradiction proves that T is simple and (i) is proven. In particular, $\mathrm{End}^0(T)$ is a division algebra over $\mathbb{Q}$.

In order to handle (ii), let us assume that E is a subfield of $\mathrm{End}^0(T)$. The simplicity of T implies that $1 \in E$ is the identity automorphism of T. Then $\Lambda_{\mathbb{Q}}$ becomes a faithful E-module. This implies that E is a number field and $\Lambda_{\mathbb{Q}}$ is an E-vector space of finite positive dimension

$$d_E = \frac{2g}{[E:\mathbb{Q}]}.$$

This implies that $V_{\mathbb{C}} = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ is a free $E \otimes_{\mathbb{Q}} \mathbb{C}$-module of rank d_E. Clearly, both $\mathrm{H}_{-1,0}(T)$ and $\mathrm{H}_{0,-1}(T)$ are $E \otimes_{\mathbb{Q}} \mathbb{C}$-submodules of its direct sum $V_{\mathbb{C}}$. Let

$$\mathrm{tr}_{E/\mathbb{Q}} : E \to \mathbb{Q}$$

be the trace map attached to the field extension $E/\mathbb{Q}$ of finite degree. Let

$$\mathrm{Hom}_E\big(\wedge_E^2 \Lambda_{\mathbb{Q}}, E\big)$$

be the $\frac{d_E(d_E-1)}{2}$-dimensional E-vector space of alternating E-bilinear forms on $\Lambda_{\mathbb{Q}}$; it carries a natural structure of a rational Hodge structure of $\mathbb{Q}$-dimension $[E:\mathbb{Q}] \cdot \frac{d_E(d_E-1)}{2}$. There is the natural embedding of rational Hodge structures

$$\mathrm{Hom}_E\left(\wedge_E^2 \Lambda_{\mathbb{Q}}, E\right) \hookrightarrow \mathrm{Hom}_{\mathbb{Q}}\left(\wedge_{\mathbb{Q}}^2 \Lambda_{\mathbb{Q}}, \mathbb{Q}\right) = \mathrm{H}^2(T,\mathbb{Q}),\ \phi_E \mapsto \phi := \mathrm{tr}_{E/\mathbb{Q}} \circ \phi_E, \tag{10}$$

i.e.,

$$\phi(\lambda_1, \lambda_2) = \mathrm{tr}_{E/\mathbb{Q}}\big(\phi_E(\lambda_1, \lambda_2)\big)\ \forall \lambda_1, \lambda_2 \in \Lambda_{\mathbb{Q}}. \tag{11}$$

The image of $\mathrm{Hom}_E\left(\wedge_E^2 \Lambda_{\mathbb{Q}}, E\right)$ in $\mathrm{Hom}_{\mathbb{Q}}\left(\wedge_{\mathbb{Q}}^2 \Lambda_{\mathbb{Q}}, \mathbb{Q}\right) = \mathrm{H}^2(T,\mathbb{Q})$ coincides with the $\mathbb{Q}$-vector subspace

$$H_E := \{\phi \in \mathrm{Hom}_{\mathbb{Q}}\left(\wedge_{\mathbb{Q}}^2 \Lambda_{\mathbb{Q}}, \mathbb{Q}\right) \mid \phi(u\lambda_1, \lambda_2) = \phi(\lambda_1, u\lambda_2)\ \forall u \in E,\ \lambda_1, \lambda_2 \in \Lambda_{\mathbb{Q}}\}. \tag{12}$$

Indeed, it is obvious that the image lies in H_E. In order to check that the image coincide with the whole subspace H_E, let us construct the inverse map

$$H_E \to \mathrm{Hom}_E\left(\wedge_E^2 \Lambda_{\mathbb{Q}}, E\right),\ \phi \mapsto \phi_E$$

to (10) as follows. If $\lambda_1, \lambda_2 \in \Lambda_{\mathbb{Q}}$ then there is a $\mathbb{Q}$-linear map

$$\Phi : E \mapsto \mathbb{Q},\ u \mapsto \phi(u\lambda_1, \lambda_2) = \phi(\lambda_1, u\lambda_2) = -\phi(u\lambda_2, \lambda_1) = -\phi(\lambda_2, u\lambda_1). \tag{13}$$

The properties of the trace map imply that there exists precisely one $\beta \in E$ such that

$$\Phi(u) = \mathrm{tr}_{E/\mathbb{Q}}(u\beta)\ \forall u \in E.$$

Let us put

$$\phi_E(\lambda_1, \lambda_2) := \beta.$$

It follows from (13) that $\phi_E \in \mathrm{Hom}_E\left(\wedge_E^2 \Lambda_{\mathbb{Q}}, E\right)$. In addition,

$$\mathrm{tr}_{E/\mathbb{Q}}(\phi_E(\lambda_1, \lambda_2)) = \mathrm{tr}_{E/\mathbb{Q}}(\beta) = \mathrm{tr}_{E/\mathbb{Q}}(1 \cdot \beta) = \Phi(1) = \phi(\lambda_1, \lambda_2),$$

which proves that $\phi \mapsto \phi_E$ is indeed the inverse map, in light of (11).

Clearly, H_E is a rational Hodge substructure of $\mathrm{H}^2(T,\mathbb{Q})$.

By 2-simplicity of T, the $\mathbb{C}$-dimension of the $(2,0)$-component $H_E^{(2,0)}$ of H_E is either 0 or $g(g-1)/2$. Let us express this dimension explicitly in terms of g and $[E:\mathbb{Q}]$.

In order to do that, let us consider the set Σ_E of all field embeddings $\sigma: E \hookrightarrow \mathbb{C}$, which consists of $[E:\mathbb{Q}]$-elements. We have

$$E_{\mathbb{C}} := E \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma_E} \mathbb{C}_\sigma \quad \text{where } \mathbb{C}_\sigma = E \otimes_{E,\sigma} \mathbb{C} = \mathbb{C}, \tag{14}$$

which gives us a splitting of $E_{\mathbb{C}}$-modules

$$V_{\mathbb{C}} = \oplus_{\sigma \in \Sigma_E} V_\sigma = \bigoplus_{\sigma \in \Sigma_E} (\mathrm{H}_{-1,0}(T)_\sigma \oplus \mathrm{H}_{0,-1}(T)_\sigma) \tag{15}$$

where for all $\sigma \in \Sigma_E$ we define

$$\mathrm{H}_{-1,0}(T)_\sigma := \mathbb{C}_\sigma \mathrm{H}_{-1,0}(T) = \{x \in \mathrm{H}_{-1,0}(T) \mid u_{\mathbb{C}} x = \sigma(u)x \ \forall u \in E\} \subset \mathrm{H}_{-1,0}(T);$$

$$n_\sigma := \dim_{\mathbb{C}}(\mathrm{H}_{-1,0}(T)_\sigma);$$

$$\mathrm{H}_{0,-1}(T)_\sigma := \mathbb{C}_\sigma \mathrm{H}_{0,-1}(T) = \{x \in \mathrm{H}_{0,-1}(T) \mid u_{\mathbb{C}} x = \sigma(u)x \ \forall u \in E\} \subset \mathrm{H}_{0,-1}(T);$$

$$m_\sigma := \dim_{\mathbb{C}}(\mathrm{H}_{0,-1}(T)_\sigma);$$

$$V_\sigma = \mathbb{C}_\sigma = \mathbb{C}_\sigma V_{\mathbb{C}} = \{x \in V_{\mathbb{C}} \mid u_{\mathbb{C}} x = \sigma(u)x \ \forall u \in E\} = \mathrm{H}_{-1,0}(T)_\sigma \oplus \mathrm{H}_{0,-1}(T)_\sigma$$

Since $\mathrm{H}_{-1,0}(T) \oplus \mathrm{H}_{0,-1}(T) = V_{\mathbb{C}}$ is a free $E_{\mathbb{C}}$-module of rank d_E, its direct summand V_σ is a vector space of dimension d_E over $\mathbb{C}_\sigma = \mathbb{C}$ and therefore

$$n_\sigma + m_\sigma = d_E \tag{16}$$

for all σ. Since $\mathrm{H}_{-1,0}(T)$ and $\mathrm{H}_{0,-1}(T)$ are mutually complex-conjugate subspaces of $V_{\mathbb{C}}$, it follows from (6) that

$$m_\sigma = n_{\bar{\sigma}} \quad \text{where } \bar{\sigma}: E \hookrightarrow \mathbb{C}, \ u \mapsto \overline{\sigma(u)}$$

is the *complex-conjugate* of σ. Therefore, in light of (16),

$$n_\sigma + n_{\bar{\sigma}} = d_E \ \forall \sigma. \tag{17}$$

We have

$$\sum_{\sigma \in \Sigma_E} n_\sigma = \sum_{\sigma \in \Sigma_E} \dim_{\mathbb{C}}(\mathrm{H}_{-1,0}(T)_\sigma) = \dim_{\mathbb{C}}(\mathrm{H}_{-1,0}(T)) = g. \tag{18}$$

Let us consider the complexification of H_E

$$H_{E,\mathbb{C}} := H_E \otimes_{\mathbb{Q}} \mathbb{C} \subset \mathrm{Hom}_{\mathbb{Q}}\left(\wedge^2 \Lambda_{\mathbb{Q}}, \mathbb{Q}\right) \otimes_{\mathbb{Q}} \mathbb{C} =$$

$$\mathrm{Hom}_{\mathbb{C}}\left(\wedge^2_{\mathbb{C}}(\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}), \mathbb{C}\right) = \mathrm{Hom}_{\mathbb{C}}\left(\wedge^2 V_{\mathbb{C}}, \mathbb{C}\right).$$

In light of (12),

$$H_{E,\mathbb{C}} = \{\phi \in \mathrm{Hom}_{\mathbb{C}}\left(\wedge^2_{\mathbb{C}} V_{\mathbb{C}}, \mathbb{C}\right) \mid \phi(u_{\mathbb{C}}x, y) = \phi(x, u_{\mathbb{C}}y)\ \forall u \in E, ; x, y \in V_{\mathbb{C}}\} \tag{19}$$
$$= \{\phi \in \mathrm{Hom}_{\mathbb{C}}\left(\wedge^2_{\mathbb{C}} V_{\mathbb{C}}, \mathbb{C}\right) \mid \phi(u_{\mathbb{C}}x, y) = \phi(x, u_{\mathbb{C}}y)\ \forall u \in E_{\mathbb{C}}; x, y \in V_{\mathbb{C}}\}.$$

In particular, if $\sigma, \tau \in \Sigma_E$ are *distinct* field embeddings then for all $\phi \in H_{E,\mathbb{C}}$

$$\phi(V_\sigma, V_\tau) = \phi(V_\tau, V_\sigma) = \{0\}.$$

This implies that

$$H_{E,\mathbb{C}} = \oplus_{\sigma\in\Sigma_E} \mathrm{Hom}_{\mathbb{C}}\left(\wedge^2_{\mathbb{C}} V_\sigma, \mathbb{C}\right) \tag{20}$$
$$= \bigoplus_{\sigma\in\Sigma_E} \mathrm{Hom}_{\mathbb{C}}\Big(\wedge^2_{\mathbb{C}} \left(\mathrm{H}_{-1,0}(T)_\sigma \bigoplus \mathrm{H}_{0,-1}(T)_\sigma\right), \mathbb{C}\Big).$$

In light of (9), the $(2,0)$-Hodge component of $H_{E,\mathbb{C}}$ is

$$H_E^{(2,0)} = \oplus_{\sigma\in\Sigma_E} \mathrm{Hom}_{\mathbb{C}}\left(\wedge^2_{\mathbb{C}} \mathrm{H}_{-1,0}(T)_\sigma, \mathbb{C}\right) \text{ and } \dim_{\mathbb{C}}(H_E^{(2,0)}) = \sum_{\sigma\in\Sigma_E} \frac{n_\sigma(n_\sigma - 1)}{2}. \tag{21}$$

This implies that $\dim_{\mathbb{C}}(H_E^{(2,0)}) = 0$ if and only if all n_σ are in $\{0, 1\}$. If this is the case then, in light of (17), $d_E \in \{1, 2\}$, i.e., $[E : \mathbb{Q}] = 2g$ or g.

On the other hand, it follows from (18) combined with the second formula in (21) that $\dim_{\mathbb{C}}(H_E^{(2,0)}) = g(g-1)/2$ if and only if there is precisely one σ with $n_\sigma = g$ (and all the other multiplicities n_τ are 0). This implies that either $d_E = 2g$ and $E = \mathbb{Q}$, or $d_E = g$ and E an imaginary quadratic field with the pair of the field embeddings

$$\sigma, \bar{\sigma} : E \hookrightarrow: \mathbb{C}$$

such that

$$n_\sigma = g,\ n_{\bar{\sigma}} = 0.$$

It is therefore enough to rule out the case $d_E = g$. By way of contradiction, assume that $d_E = g$. Then E is an imaginary quadratic field; in addition,

$$u \in E \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \subset \mathrm{End}_{\mathbb{R}}(V)$$

then $u_{\mathbb{C}}$ acts on $\mathrm{H}_{-1,0}(T)$ as multiplication by $\sigma(u) \in \mathbb{C}$. In light of (6), $u_{\mathbb{C}}$ acts on the complex-conjugate subspace $\mathrm{H}_{0,-1}(T)$ as multiplication by $\overline{\sigma(u)} = \bar{\sigma}(u) \in \mathbb{C}$. Since E is an imaginary quadratic field, there are a positive integer D and $\alpha \in E$ such that $\alpha^2 = -D$ and $E = \mathbb{Q}(\alpha)$. It follows that $\sigma(\alpha) = \pm\mathbf{i}\sqrt{D}$. Replacing if necessary α by $-\alpha$, we may and will assume that

$$\sigma(\alpha) = \mathbf{i}\sqrt{D}$$

and therefore $\alpha_{\mathbb{C}}$ acts on $\mathrm{H}_{-1,0}(T)$ as multiplication by $\mathbf{i}\sqrt{D}$. Hence, $\alpha_{\mathbb{C}}$ acts on $\mathrm{H}_{0,-1}(T)$ as multiplication by $\overline{\mathbf{i}\sqrt{D}} = -\mathbf{i}\sqrt{D}$. Since

$$V_{\mathbb{C}} = \mathrm{H}_{-1,0}(T) \oplus \mathrm{H}_{0,-1}(T),$$

we get $\alpha_{\mathbb{C}} = \sqrt{D} J_C$ and therefore

$$\alpha = \sqrt{D} J.$$

This implies that the centralizer $\mathrm{End}^0(T)$ of J in $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ coincides with the centralizer of α in $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$, which, in turn, coincides with the centralizer $\mathrm{End}_E(\Lambda_{\mathbb{Q}})$ of E in $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$, i.e.,

$$\mathrm{End}^0(T) = \mathrm{End}_E(\Lambda_{\mathbb{Q}}) \cong \mathrm{Mat}_{d_E}(E).$$

This is a matrix algebra, which is not a division algebra, because $d_E = g > 1$. This contradicts the simplicity of T. The obtained contradiction rules out the case $d_E = g$. This ends the proof of (ii).

In order to prove (iii), recall that $\mathrm{End}^0(T)$ is a division algebra over $\mathbb{Q}$, thanks to the simplicity of T [20]. Hence $\Lambda_{\mathbb{Q}}$ is a free $\mathrm{End}^0(T)$-module of finite positive rank and therefore

$$\dim_{\mathbb{Q}}(\mathrm{End}^0(T)) \mid 2g, \tag{22}$$

because $2g = \dim_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$. We will apply several times the already proven assertion (ii) to various subfields of $\mathrm{End}^0(T)$.

Suppose that $\mathrm{End}^0(T)$ is *not* a field and let $\mathcal{Z}$ be its center. Then $\mathcal{Z}$ is a number field and there is an integer $d > 1$ such that $\dim_{\mathcal{Z}}(\mathrm{End}^0(T)) = d^2$ and therefore

$$\dim_{\mathbb{Q}}(\mathrm{End}^0(T)) = d^2 \cdot [\mathcal{Z} : \mathbb{Q}]$$

divides $2g$, thanks to (22). Since $\mathcal{Z}$ is a subfield of $\mathrm{End}^0(T)$, the degree $[\mathcal{Z} : \mathbb{Q}]$ is either 1 or g or $2g$. If $[\mathcal{Z} : \mathbb{Q}] > 1$ then $2g$ is divisible by

$$d^2 \cdot [\mathcal{Z} : \mathbb{Q}] \geq 2^2 g = 4g,$$

which is nonsense. Hence, $[\mathcal{Z} : \mathbb{Q}] = 1$, i.e., $\mathcal{Z} = \mathbb{Q}$ and $\mathrm{End}^0(T)$ is a central division $\mathbb{Q}$-algebra of dimension d^2 with $d^2 | 2g$. Then every maximal subfield E of the central division $\mathbb{Q}$-algebra $\mathrm{End}^0(T)$ has degree d over $\mathbb{Q}$ [21, Sect. 13.1, Cor. b]. By the already proven assertion (ii), $d \in \{1, g, 2g\}$. Since $d > 1$, we obtain that either $d = g$ and $g^2 \mid 2g$ or $d = 2g$ and $(2g)^2 \mid 2g$. This implies that $d = g$ and $g = 1$ or 2. Since $g \geq 3$, we get a contradiction, which implies that $\mathrm{End}^0(T)$ is a field.

It follows from the already proven assertion (ii) that the degree $\dim_{\mathbb{Q}}(\mathrm{End}^0(T))$ of the number field $\mathrm{End}^0(T)$ is either 1 or g or $2g$.

Assertion (iv) is obvious and was included just for the sake of completeness.

In order to handle the structure of $\mathrm{Aut}(T)$, let us check first that the only roots of unity in $\mathrm{End}^0(T)$ are 1 and -1. If this is not the case then the field $\mathrm{End}^0(T)$ contains either $\sqrt{-1}$ or a primitive pth root of unity ζ where p is a certain odd prime. In the former case $\mathrm{End}^0(T)$ contains the quadratic field $\mathbb{Q}(\sqrt{-1})$, which contradicts (ii). In the latter case $\mathrm{End}^0(T)$ contains either the quadratic field $\mathbb{Q}(\sqrt{-p})$ or the quadratic field $\mathbb{Q}(\sqrt{p})$: each of these outcomes contradicts (ii) as well.

Now recall that $\mathrm{End}(T)$ is an order in the number field $E = \mathrm{End}^0(T)$ and $\mathrm{Aut}(T) = \mathrm{End}(T)^*$ is its group of units. By the Theorem of Dirichlet about units [5, Ch. II, Sect. 4, Th. 5], the group of units is

$$\mathrm{Aut}(T) \cong \mathbb{Z}^d \times \{\pm 1\} \quad \text{with} \quad d = r_E + s_E - 1 \tag{23}$$

where r_E is the number of real field embeddings $E \hookrightarrow \mathbb{R}$ and

$$r_E + 2s_E = [E : \mathbb{Q}], \quad \text{i.e.,} \quad s_E = \frac{[E : \mathbb{Q}] - r_E}{2}. \tag{24}$$

Let us prove (v). Assume that the number field $E = \mathrm{End}^0(T)$ has degree $2g$. A dimension argument implies that $\Lambda_{\mathbb{Q}}$ is a 1-dimensional E-vector space and $V = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is a free $E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R}$-module of rank 1. Hence $E_{\mathbb{R}}$ coincides with its own centralizer $\mathrm{End}_{E_{\mathbb{R}}}(V)$ in $\mathrm{End}_{\mathbb{R}}(V)$. Since J commutes with $\mathrm{End}^0(T) = E$, it also commutes with $E_{\mathbb{R}}$ and therefore

$$J \in \mathrm{End}_{E_{\mathbb{R}}}(V) = E_{\mathbb{R}}.$$

Recall that the $\mathbb{R}$-algebra $E_{\mathbb{R}}$ is isomorphic to a product of copies of $\mathbb{R}$ and $\mathbb{C}$. Since $J^2 = -1$, only copies of $\mathbb{C}$ appear in $E_{\mathbb{R}}$, i.e., E is purely imaginary, which means that $r_E = 0$ and therefore $2g = [E : \mathbb{Q}] = 2s_E$. This proves the first assertion of (v); the second one follows readily from (23) combined with (24).

In order to prove the last assertion, assume that E contains two distinct proper subfields E_1 and E_2, none of which coincides with $\mathbb{Q}$. Clearly,

$$[E_1 : \mathbb{Q}] = g = [E_2 : \mathbb{Q}],$$

which means that both field extensions E/E_1 and E/E_2 are quadratic. This implies that the (finite) automorphism group $G := \mathrm{Aut}(E/\mathbb{Q})$ of the field extension $E/\mathbb{Q}$ contains two distinct elements t_1 and t_2 of order 2 such that

$$E_1 = \{u \in E \mid t_1(u) = u_1\}, \quad E_2 = \{u \in E \mid t_2(u) = u_2\},$$

It follows that G is a group of order M where M is an even integer that is strictly greater than 2. Then the subfield $F := E^G$ of G-invariants is a proper subfield of E and its degree

$$[F : \mathbb{Q}] = \frac{[E : \mathbb{Q}]}{M} < \frac{2g}{2} = g.$$

It follows from (ii) that $F = \mathbb{Q}$ and therefore $M = [E : \mathbb{Q}] = 2g$.

If g is *not* a power of 2 then there is an odd prime p dividing g and therefore dividing M. It follows that G contains an element t of order p. Therefore the subfield E^t of t-invariants is a proper subfield of E and its degree $[E^t : \mathbb{Q}]$ is $2g/p < g$. By (ii), $E^t = \mathbb{Q}$ and therefore $2g = [E : \mathbb{Q}] = p$, which is wrong, since p is odd. Hence g is a power of 2 and therefore G is a finite 2-group. It follows that G has a normal subgroup H of index 2. Then the subfield F^{H_2} is a proper subfield of E and its degree $[F_2 : \mathbb{Q}]$ equals the index $[G : H] = 2$. This also contradicts (ii), which ends the proof of the last assertion of (v).

Let us prove (vi). Assume that $[E : \mathbb{Q}] = g$. Then the assertion about $\mathrm{Aut}(T)$ follows readily from (23) combined with (24). If $F \neq \mathbb{Q}$ is a proper subfield of E then

$$1 = [\mathbb{Q} : \mathbb{Q}] < [F : \mathbb{Q}] < [E : \mathbb{Q}] = g$$

and therefore $1 < [F : \mathbb{Q}] < g$, which contradicts (ii) applied to F instead of E. So, such an F does *not* exist, i.e., E is *primitive*.

Assume now that T is a complex abelian variety. By Albert's classification [18], $E = \mathrm{End}^0(T)$ is either a totally real number field or a CM field. If E is a CM field then it contains a subfield E_0 of degree $[E : \mathbb{Q}]/2 = g/2$. Since E_0 is a subfield of $\mathrm{End}^0(T)$ and $1 < g/2 < g$ (recall that $g \geq 3$), the existence of E_0 contradicts the already proven assertion (ii). This proves that E is a totally real number field, i.e., $s = 0, r = g$. Now the assertion about $\mathrm{Aut}(T)$ follows from (23). □

4. Number Fields and transitive permutation groups

All the results of this section are standard and pretty well known except, may be, the notion of almost double transitivity.

Definition 4.1. Let $\mathcal{T}$ be a set that consists of at least three elements. We write $\mathrm{Perm}(\mathcal{T})$ for the group of all permutations of S. Let G be a group that acts on S, i.e., we are given a group homomorphism

$$G \to \mathrm{Perm}(S),$$

whose image we denote by $\tilde{G}$, which is a subgroup of $\mathrm{Perm}(\mathcal{T})$. We say that a *transitive* action of G on $\mathcal{T}$ is *almost doubly transitive* if the action of G on the set of all two-element subsets of $\mathcal{T}$ is transitive.

Remarks 4.2.
1. Every doubly transitive action is almost doubly transitive.
2. Every almost doubly transitive action of G on $\mathcal{T}$ is primitive, i.e., the stabilizer of a point is a maximal subgroup. Indeed, suppose the action is *not* primitive, i.e., that $\mathcal{T}$ partitions into a disjoint union of r sets $\mathcal{T}_1, \dots, \mathcal{T}_r$ such that $r \geq 2$, each $\mathcal{T}_i$ consists of $m \geq 2$ elements, and G permutes $\mathcal{T}_i$s. Let A be a 2-element subset of T_1. Pick $b_1 \in \mathcal{T}_1$ and $b_2 \in \mathcal{T}_2$, and consider a 2-element subset $B = \{b_1, b_2\}$ of $\mathcal{T}$. Clearly, no $s \in G$ sends A to B, i.e., the action is not almost doubly transitive.
3. If S consists of three elements then every transitive action on S of any group G is almost doubly transitive.
4. Let $\tilde{G}$ be the image of G in the group $\mathrm{Perm}(S)$ of permutations of $\mathcal{T}$. If S is a finite set then the group $\tilde{G}$ is a finite group of permutations of $\mathcal{T}$ that is primitive (resp. almost doubly transitive, resp. doubly transitive) if and only if G is primitive (resp. almost doubly transitive, resp. doubly transitive)

(5) Suppose that $\mathcal{T}$ is a finite set that consists of $n \geq 3$ elements and G is a group that acts faithfully and almost doubly transitively on $\mathcal{T}$. Let N be the order of $\tilde{G}$.

Then N is divisible by $n(n-1)/2$. If N is even then $\tilde{G}$ contains an element $\tilde{\sigma}$ of order 2 and therefore there are two distinct elements $s_1, s_2 \in \mathcal{T}$ such that

$$\tilde{\sigma}(s_1) = s_2, \tilde{\sigma}(s_2) = s_1.$$

It follows that the action of $\tilde{G}$ on $\mathcal{T}$ is doubly transitive and therefore the action of G on $\mathcal{T}$ is also doubly transitive. This implies that if either $4|n$ or $n \equiv 1 \mod 4$ then the action of G on $\mathcal{T}$ is doubly transitive, because in these cases $n(n-1)/2$ is even.

(6) Let $n = q$ be a prime power that is congruent to 3 modulo 4. Let $\mathbb{F}_q$ be a q-element finite field and $\mathbb{F}_q^*$ the multiplicative group of nonzero elements of $\mathbb{F}_q$. Then $\mathbb{F}_q^*$ splits into a direct product $\mathbb{F}_q^* = H \times \{\pm 1\}$ where H is a cyclic group of odd order $(q-1)/2$. Let us put $\mathcal{S} = \mathbb{F}_q$ and let G be the group of affine transformations of $\mathbb{F}_q$

$$x \mapsto ax + b, \ a \in H \subset \mathbb{F}_q^*, b \in \mathbb{F}_q.$$

Then the action of G on $\mathbb{F}_q$ is almost doubly transitive but *not* doubly transitive.

Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and

$$\mathrm{Gal}(\mathbb{Q}) = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) = \mathrm{Aut}(\bar{\mathbb{Q}}/\mathbb{Q})$$

the absolute Galois group of $\mathbb{Q}$. Let us consider the humongous group $\mathrm{Aut}(\mathbb{C})$ of all automorphisms of the field $\mathbb{C}$. Obviously, the subfield $\bar{\mathbb{Q}}$ is $\mathrm{Aut}(\mathbb{C})$-invariant, which gives rise to the (restriction) homomorphism of groups

$$\mathrm{Aut}(\mathbb{C}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}), \quad s \mapsto \{\alpha \mapsto s(\alpha)\} \ \forall s \in \mathrm{Aut}(\mathbb{C}), \alpha \in \bar{\mathbb{Q}} \tag{25}$$

which is *surjective*.

Let E be a number field of degree $n = [E : \mathbb{Q}]$. We write Σ_E for the n-element set of all field embeddings $\sigma : E \hookrightarrow \mathbb{C}$. For each $\sigma \in \Sigma_E$ the image $\sigma(E)$ lies in $\bar{\mathbb{Q}}$. If t is an element of $\mathrm{Aut}(\mathbb{C})$ (or of $\mathrm{Gal}(\mathbb{Q})$) then the composition

$$t \circ \sigma : E \overset{\sigma}{\hookrightarrow} \bar{\mathbb{Q}} \overset{t}{\to} \bar{\mathbb{Q}} \subset \mathbb{C}$$

also lies in Σ_E. Then the map

$$\mathrm{Aut}(\mathbb{C}) \times \Sigma_E \to \Sigma_E, \quad (t, \sigma) \mapsto t \circ \sigma \tag{26}$$

is a *transitive group action* of $\mathrm{Aut}(\mathbb{C})$ on Σ_E, which factors through $\mathrm{Gal}(\mathbb{Q})$ via (25). This action is *primitive* (i.e., the stabilizer of a point is a maximal subgroup) if and only if E is a primitive number field. Similarly, the map

$$\mathrm{Gal}(\mathbb{Q}) \times \Sigma_E \to \Sigma_E, \quad (t, \sigma) \mapsto t \circ \sigma \tag{27}$$

is a *transitive group action* of $\mathrm{Gal}(\mathbb{Q})$ on Σ_E, which is primitive if and only if E is a primitive number field.

We say that E is a *doubly transitive* (respectfully *almost doubly transitive*) number field if the action (26) (or equivalently the action (27)) is doubly transitive (respectfully almost doubly transitive). The corresponding finite subgroup $\tilde{G} = \widetilde{\mathrm{Aut}(\mathbb{C})} = \widetilde{\mathrm{Gal}(\mathbb{Q})}$ of $\mathrm{Perm}(\Sigma_E)$ is isomorphic to the Galois group $\mathrm{Gal}(\tilde{E}/\mathbb{Q})$ where $\tilde{E}$ is a *normal closure* of E.

Remark 4.3. Clearly, if E and F are isomorphic number fields then E is primitive (resp. doubly transitive) (resp. almost doubly transitive) if and only if F is primitive (resp. doubly transitive) (resp. almost doubly transitive).

Example 4.4.
(i) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n \geq 2$ on $\mathbb{Q}$ and $E_f = \mathbb{Q}[x]/f(x)\mathbb{Q}[x]$ the corresponding number field of degree n. We write $\mathfrak{R}_f$ for the n-element set of roots of $f(x)$ in $\bar{\mathbb{Q}}$ and $\mathbb{Q}(\mathfrak{R}_f)$ for the subfield of $\bar{\mathbb{Q}}$ generated by $\mathfrak{R}_f$. By definition, $\mathbb{Q}(\mathfrak{R}_f)$ is a splitting field of $f(x)$ that is a finite Galois extension of $\mathbb{Q}$. We write $\mathrm{Gal}(f)$ for the Galois group $\mathrm{Gal}(\mathbb{Q}(\mathfrak{R}_f)/\mathbb{Q})$ of the field extension $\mathbb{Q}(\mathfrak{R}_f)/\mathbb{Q}$. It is well known that $\mathrm{Gal}(\mathbb{Q})$ acts transitively (through $\mathrm{Gal}(f)$) on $\mathfrak{R}_f$. There is a $\mathrm{Gal}(\mathbb{Q})$-equivariant bijection between Σ_{E_f} and $\mathfrak{R}_f$ that is defined as follows. To each $\alpha \in \mathfrak{R}_f$ corresponds the field embedding
$$\sigma_\alpha : E_f = \mathbb{Q}[x]/f(x)\mathbb{Q}[x] \hookrightarrow \bar{\mathbb{Q}} \subset \mathbb{C}, \; h(x) + f(x)\mathbb{Q}[x] \mapsto h(\alpha)$$
(in particular, the coset of x goes to α). This implies that the field E_f is doubly transitive (respectfully almost doubly transitive) if and only if the action of $\mathrm{Gal}(f)$ on $\mathfrak{R}_f$ is doubly transitive (respectfully almost doubly transitive). The similar characterization of primitive number fields is well known:
the field E_f is primitive if and only if the action of $\mathrm{Gal}(f)$ *on* $\mathfrak{R}_f$ *is primitive.*

(ii) Conversely, let F be a number field of degree n and $z \in$ is a primitive element of E, i.e., the small subfield $\mathbb{Q}(z)$ of E that contains z coincides with F (such an element always exists). Let $f(x) \in \mathbb{Q}[x]$ be the minimal polynomial of z, i.e., $f(x)$ is irreducible over $\mathbb{Q}$ and $f(z) = 0$; in addition, $\deg(f) = n$. Then there is a field isomorphism $E_f \cong F$ such that the coset $x + f(x)\mathbb{Q}[x] \in E_f$ goes to $z \in F$. Therefore the number field F is (almost) doubly transitive if and only if $\mathrm{Gal}(f)$ acts (almost) doubly transitively on $\mathfrak{R}_f$.

Theorem 4.5. *Let $n \geq 2$ be an integer. Let $\mathbf{r}, \mathbf{s}$ be nonnegative integers such that*
$$\mathbf{r} + 2\mathbf{s} = n. \tag{28}$$
Then there exists a number field E of degree n that enjoys the following properties.

(i) $r_E = \mathbf{r}, \quad s_E = \mathbf{s}$.
(ii) *E is doubly transitive.*

Proof. We will use a week approximation in E, approximating several polynomials with respect to several metrics in E.

First, fix a degree n monic polynomial $h_{\infty}(x) \in \mathbb{Z}[x]$ts that has precisely r distinct real roots and s distinct pairs of non-real complex-conjugate roots. (E.g., one may take

$$h_{\infty}(x) = \prod_{i=1}^{s}(x-i)\prod_{j=1}^{s}(x^2+j^2) \in \mathbb{Z}[x] \subset \mathbb{Q}[x].)$$

Second, take any prime p and choose a monic *p-adic Eisenstein polynomial* $h_p(x) \in \mathbb{Z}[x]$, all whose coefficients (except the leading one) are divisible by p while the constant term is *not* divisible by p^2. (E.g., one may take

$$h_2(x) = x^n - p \in \mathbb{Z}[x] \subset \mathbb{Q}[x].)$$

Third, take any prime $\ell =\neq p$ and choose a monic irreducible polynomial $\tilde{u}_{\ell}(x) \in \mathbb{F}_{\ell}[x]$ of degree $(n-1)$ over $\mathbb{F}_{\ell}$. (Such a polynomial always exists for any given ℓ and $n-1$.) Let $u_{\ell}(x) \in \mathbb{Z}[x]$ be any monic degree $(n-1)$ polynomial with integer coefficients, whose reduction modulo ℓ coincides with $\tilde{u}_{\ell}(x)$. Let us put

$$h_{\ell}(x) := x \cdot u_{\ell}(x) \in \mathbb{Z}[x] \subset \mathbb{Q}[x].$$

By a weak approximation theorem [2, Th. 1], there is a monic degree n polynomial $f(x) \in \mathbb{Q}[x]$ that enjoys the following properties.

(a) $f(x)$ is so close to $h_{\infty}(x)$ in the archimedean topology that it also has precisely r distinct real roots and s distinct pairs of non-real complex-conjugate roots.

(b) $f(x) - h_p(x) \in p^2 \cdot x \cdot \mathbb{Z}_p[x]$. This implies that $f(x)$ is irreducible over the field $\mathbb{Q}_p$ of p-adic numbers and therefore irreducible over $\mathbb{Q}$.

(c) $f(x) - h_{\ell}(x) \in \ell \cdot x \cdot \mathbb{Z}_{\ell}[x]$. This implies that

$$f(x) \in \mathbb{Z}_{\ell}[x], \ f(x) \bmod \ell = x \cdot \tilde{u}_{\ell}(x) \in \mathbb{F}_{\ell}[x].$$

By Hensel's Lemma, there are

$$\alpha \in \ell\mathbb{Z}_{\ell} \subset \mathbb{Z}_{\ell}$$

and a monic degree $(n-1)$ polynomial $v(x) \in \mathbb{Z}_{\ell}[x]$ such that

$$f(x) = (x-\alpha)v(x) \in \mathbb{Z}_{\ell}[x], \quad v(x) \bmod \ell = \tilde{u}_{\ell}(x) \in \mathbb{F}_{\ell}x]. \tag{29}$$

By [29, Sect. 66], the irreducibility of $\tilde{u}_{\ell}(x)$ combined with (29) imply that $\mathrm{Gal}(\mathfrak{R}_f)$, viewed as the certain permutation group of $\mathfrak{R}_f$, contains a permutation s that is a cycle of length $n-1$. In particular, if $\alpha \in \mathfrak{R}_f$ is the fixed point of s then the cyclic subgroup $< s >$ of $\mathrm{Gal}(\mathfrak{R}_f)$ generated by s acts transitively on $\mathfrak{R}_f \setminus \{\alpha\}$. Now the transitivity of $\mathrm{Gal}(\mathfrak{R}_f)$ implies its double transitivity, which ends the proof.

□

5. Hodge groups

Recall that $\Lambda_{\mathbb{R}} = V$ carries the natural structure of a complex vector space. This gives rise to the injective homomorphism of real Lie groups

$$h : \mathbb{C}^* \hookrightarrow \mathrm{Aut}_{\mathbb{R}}(\Lambda_{\mathbb{R}})$$

where $h(z)$ is multiplication by a nonzero complex number z in $\Lambda_{\mathbb{R}} = V$. Let $\mathbb{S}^1 \subset \mathbb{C}^*$ be the subgroup of all complex numbers z with $|z| = 1$. Clearly, $h(\mathbb{S}^1)$ is a one-dimensional closed connected real Lie subgroup of $\mathrm{Aut}_{\mathbb{R}}(\Lambda_{\mathbb{R}})$; in addition, the Lie algebra of $h(\mathbb{S}^1)$ is

$$\mathbb{R} \cdot J \subset \mathrm{End}_{\mathbb{R}}(\Lambda_{\mathbb{R}}).$$

Actually, $h(\mathbb{S}^1)$ lies in the special linear group $\mathrm{SL}(\Lambda_{\mathbb{R}})$ while $\mathbb{R} \cdot J$ lies in the Lie algebra $\mathrm{sl}(\Lambda_{\mathbb{R}})$ of traceless operators in $\Lambda_{\mathbb{R}}$.

By definition [17, 25] (see also [34]), the Hodge group $\mathrm{Hdg}(T)$ of the rational Hodge structure $\mathrm{H}_1(T, \mathbb{Q}) = \Lambda_{\mathbb{Q}}$ is the smallest algebraic $\mathbb{Q}$-subgroup G of $\mathrm{GL}(\Lambda_{\mathbb{Q}})$, whose group of real points

$$G(\mathbb{R}) \subset \mathrm{Aut}_{\mathbb{R}}(\Lambda_{\mathbb{R}})$$

contains $h(\mathbb{S}^1)$. One may easily check that $\mathrm{Hdg}(T)$ enjoys the following properties that we will freely use throughout the text.

(i) $\mathrm{Hdg}(T)$ is a *connected* algebraic $\mathbb{Q}$-group that is a subgroup of the special linear group $\mathrm{SL}(\Lambda_{\mathbb{Q}})$.

(ii) The centralizer of $\mathrm{Hdg}(T)$ in $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ coincides with $\mathrm{End}^0(T)$.

(iii) A $\mathbb{Q}$-vector subspace $H_{\mathbb{Q}}$ of $\Lambda_{\mathbb{Q}}$ is $\mathrm{Hdg}(T)$-invariant if and only if it is a rational Hodge substructure of $\Lambda_{\mathbb{Q}}$.

(iv) The subspace of $\mathrm{Hdg}(T)$-invariants

$$\mathrm{H}^2(T, \mathbb{Q})^{\mathrm{Hdg}(T)} \subset \mathrm{H}^2(T, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(\wedge^2_{\mathbb{Q}} \Lambda_{\mathbb{Q}}, \mathbb{Q})$$

coincides with the subspace $\mathrm{H}^2(T, \mathbb{Q}) \cap H^{1,1}(T)$ of two-dimensional Hodge classes on T.

(v) The group of $\mathbb{Q}$-points $\mathrm{Hdg}(T)(\mathbb{Q})$ is *Zariski dense* in $\mathrm{Hdg}(T)$, because Hdg is connected and the field $\mathbb{Q}$ in infinite (see [4, Cor. 18.3]).

Let us consider the $\mathbb{Q}$-Lie algebra hdg_T of the linear *algebraic* $\mathbb{Q}$-group $\mathrm{Hdg}(T)$. By definition, hdg_T is a linear *algebraic* Lie subalgebra of $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$.

Remark 5.1. Clearly, hdg_T is the *smallest algebraic* Lie $\mathbb{Q}$-subalgebra $\mathfrak{g}$ of $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ such that

$$J \in \mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{R}. \tag{30}$$

Properties (i) and (ii) above imply that

$$\mathrm{hdg}_T \subset \mathrm{sl}(\Lambda_{\mathbb{Q}}) \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \tag{31}$$

and the centralizer of hdg_T in $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ is described as follows.

$$\mathrm{End}_{\mathrm{hdg}_T}(\Lambda_{\mathbb{Q}}) = \mathrm{End}^0(T). \tag{32}$$

Clearly,

$$J \in \mathrm{hdg}_{T,\mathbb{R}} := \mathrm{hdg}_T \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \otimes \mathbb{R} = \mathrm{End}_{\mathbb{R}}(\Lambda_{\mathbb{R}})$$

contains J. Let us consider the *complexification*

$$\mathrm{hdg}_{T,\mathbb{C}} := \mathrm{hdg}_T \otimes_{\mathbb{Q}} \mathbb{C} \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \otimes \mathbb{C} = \mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}})$$

where

$$\Lambda_{\mathbb{C}} = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \Lambda_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

We have

$$J \in \mathrm{hdg}_{T,\mathbb{R}} = \mathrm{hdg}_{T,\mathbb{R}} \otimes 1 \subset \mathrm{hdg}_{T,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{hdg}_{T,\mathbb{C}} \subset \mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}}).$$

(See (5) and Remark 1.2.) In what follows, we will write J instead of $J_{\mathbb{C}}$, slightly abusing notation.

The group $\mathrm{Aut}(\mathbb{C})$ acts naturally, *semi-linearly* and compatibly on $\Lambda_{\mathbb{C}}$, $\mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}})$ and $\mathrm{hdg}_{T,\mathbb{C}}$.

The *minimality property* of $\mathrm{Hdg}(T)$ allows us to give the following "explicit" description of the complexification $\mathrm{hdg}_{T,\mathbb{C}}$ (compare with [34, Lemma 6.3.1]).

Theorem 5.2. *The complex Lie algebra* $\mathrm{hdg}_{T,\mathbb{C}}$ *coincides with the Lie subalgebra* $\mathfrak{u}$ *of* $\mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}})$ *generated by all* $s(J)$ *where* s *run over the group* $\mathrm{Aut}(\mathbb{C})$. *In particular,* hdg_T *coincides with the smallest* $\mathbb{Q}$*-Lie subalgebra* $\mathfrak{g} \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ *such that*

$$\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R} = \mathrm{End}_{\mathbb{R}}(\Lambda_{\mathbb{R}})$$

contains J.

Proof. Clearly, $\mathfrak{u} \subset \mathrm{hdg}_{T,\mathbb{C}}$. Let us prove that $\mathfrak{u}$ is an *algebraic* complex Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}})$.

Recall that

$$J \in \mathrm{End}_{\mathbb{R}}(\Lambda_{\mathbb{R}}) \subset \mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}});\ J^2 = -1. \tag{33}$$

Clearly, $J : \Lambda_{\mathbb{C}} \to \Lambda_{\mathbb{C}}$ is a semisimple $\mathbb{C}$-linear operator, whose spectrum consists of eigenvalues, $\mathbf{i}$ and $-\mathbf{i}$, because $J^2 = -1$ Similarly, for all $s \in \mathrm{Aut}(\mathbb{C})$ the $\mathbb{C}$-linear operator $s(J) : \Lambda_{\mathbb{C}} \to \Lambda_{\mathbb{C}}$ is also semisimple and its spectrum is also $\{\mathbf{i}, -\mathbf{i}\}$, because (in light of (33))

$$s(J)^2 = s(J^2) = s(-1) = -1. \tag{34}$$

It follows that the $\mathbb{Q}$-vector subspace $\mathbb{Q}(s(J))$ of $\mathbb{C}$ generated by the *spectrum* of $s(J)$ coincides with $\mathbb{Q} \cdot \mathbf{i}$; in particular, the $\mathbb{Q}$-vector (sub)space $\mathbb{Q}(s(J))$ is one-dimensional. This implies that each $\mathbb{C} \cdot s(J)$ is an algebraic $\mathbb{C}$-Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}})$, because each *replica* of $s(J)$ is a scalar multiple of $s(J)$. Thus, the linear $\mathbb{C}$-Lie algebra $\mathfrak{u}$ is generated by the algebraic Lie subalgebras $\mathbb{C} \cdot \sigma(f)$ and therefore is algebraic itself, thanks to [12, volume 2, Ch. 2, Sect. 14]. Clearly, $\mathfrak{u}$ is defined over $\mathbb{Q}$, i.e., there is an algebraic $\mathbb{Q}$-Lie subalgebra

$$\mathfrak{u}_0 \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$$

such that

$$\mathfrak{u} = \mathfrak{u}_0 \otimes_{\mathbb{Q}} \mathbb{C}.$$

Clearly,

$$\mathfrak{u} = \mathfrak{u}_0 \otimes_{\mathbb{Q}} \mathbb{R} \oplus \mathbf{i}\cdot = \mathfrak{u}_0 \otimes_{\mathbb{Q}} \mathbb{R}$$

as a real vector vector space. This implies that

$$\mathfrak{u}_0 \otimes_{\mathbb{Q}} \mathbb{R} = \mathfrak{u} \cap \mathrm{End}_{\mathbb{R}}(\Lambda_{\mathbb{R}}). \tag{35}$$

Let $\mathcal{U}$ be the connected algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(\Lambda_{\mathbb{Q}})$, whose Lie algebra coincides with $\mathfrak{u}_0$. We need to prove that

$$\mathfrak{u}_0 = \mathrm{hdg}_T.$$

Clearly, $\mathfrak{u}_0 \subset \mathrm{hdg}_T$, because the comlexification of $\mathfrak{u}_0$ lies in the complexification of hdg_T. We know that $J \in \mathfrak{u}_0 \otimes_{\mathbb{Q}} \mathbb{C}$. Since $J \in \mathrm{End}_{\mathbb{R}}(\Lambda_{\mathbb{R}})$,

$$J \in \mathfrak{u}_0 \otimes_{\mathbb{Q}} \mathbb{R}.$$

In light of Remark 5.1, $\mathfrak{u}_0 \supset \mathrm{hdg}_T$. This implies that $\mathfrak{u}_0 = \mathrm{hdg}_T$, which ends the proof. □

Corollary 5.3. *Let us put*

$$f_T := \frac{1}{\mathbf{i}} J \in \mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}}).$$

Then

$$s(f_T)^2 = 1 \ \forall s \in \mathrm{Aut}(\mathbb{C}) \tag{36}$$

and the complex Lie algebra $\mathrm{hdg}_{T,\mathbb{C}}$ *coincides with the Lie subalgebra* $\mathfrak{u}$ *of* $\mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}})$ *generated by all* $s(f_T)$ *where* s *run over the group* $\mathrm{Aut}(\mathbb{C})$. *In particular,* hdg_T *coincides with the smallest* $\mathbb{Q}$*-Lie subalgebra* $\mathfrak{g} \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ *such that*

$$\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{C} \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}})$$

contains f_T.

Proof. Since $\mathbf{i} = \sqrt{-1}$,

$$s(\mathbf{i}) = \pm \mathbf{i}, s(f_T) = \pm \mathbf{i} \cdot s(J) \ \forall s \in \mathrm{Aut}(\mathbb{C}). \tag{37}$$

Therefore

$$s(f_T)^2 = (\pm \mathbf{i} \cdot s(J))^2 = -s(J)^2.$$

It follows from (34) that

$$s(f_T)^2 = -(-1) = 1,$$

which proves our first assertion. It follows from (37) that

$$\mathbb{C} \cdot s(f_T) = \mathbb{C} \cdot s(J) \ \forall s \in \mathrm{Aut}(\mathbb{C}).$$

Now our second assertion follows from Theorem 5.2.

□

Remark 5.4. Let $\mathfrak{a}$ be the *smallest ideal* of hdg_T such that $\mathfrak{a}_{\mathbb{C}} := \mathfrak{a} \otimes_{\mathbb{Q}} \mathbb{C}$ contains f_T. Clearly, $\mathfrak{a}_{\mathbb{C}}$ contains $s(f_T)$ for all $s \in \mathrm{Aut}(\mathbb{C})$. It follows from Corollary 5.3 that $\mathfrak{a}_{\mathbb{C}} = \mathrm{hdg}_{T,\mathbb{C}}$. This implies that

$$\mathfrak{a} = \mathrm{hdg}_T.$$

The latter equality means that f_T is a *Hodge element* of the $\mathbb{Q}$-Lie algebra hdg_T in a sense of [36, Definition 1.1] where

$$k = \mathbb{Q}, \quad V = \Lambda_{\mathbb{Q}}, \quad C = \mathbb{C}.$$

For the sake of simplicity, from now on let us assume that T is *simple*. This means that the natural faithful representation of $\mathrm{Hdg}(T)$ in

$$\Lambda_{\mathbb{Q}} = \mathrm{H}_1(T, \mathbb{Q})$$

is *irreducible* and therefore $\mathrm{End}^0(T)$ is a division algebra over $\mathbb{Q}$. This implies that the $\mathbb{Q}$-algebraic (sub)group $\mathrm{Hdg}(T)$ is *reductive*. In addition, the $\mathbb{Q}$-Lie (sub)algebra

$$\mathrm{hdg}_T \subset \mathrm{sl}\left(\mathrm{H}_1(T, \mathbb{Q})\right) \subset \mathrm{End}_{\mathbb{Q}}\left(\mathrm{H}_1(T, \mathbb{Q})\right)$$

is *reductive algebraic*, the faithful hdg_T-module $\mathrm{H}_1(T, \mathbb{Q})$ is simple and the centralizer of hdg_T in $\mathrm{End}_{\mathbb{Q}}\left(\mathrm{H}_1(T, \mathbb{Q})\right)$ is the division $\mathbb{Q}$-algebra $\mathrm{End}^0(T)$. Then the *center* $\mathcal{Z}(T)$ of $\mathrm{End}^0(T)$ is a number field.

Let us split the *reductive* $\mathbb{Q}$-Lie algebra hdg_T into a direct sum

$$\mathrm{hdg}_T = \mathrm{hdg}_T^{\mathrm{ss}} \oplus \mathfrak{c}_T$$

of the semisimple $\mathbb{Q}$-Lie algebra

$$\mathrm{hdg}_T^{\mathrm{ss}} = [\mathrm{hdg}_T, \mathrm{hdg}_T]$$

and the *center* $\mathfrak{c}_T$ of hdg_T with

$$\mathfrak{c}_T \subset \mathcal{Z}(T) \subset \mathrm{End}^0(T).$$

The following useful assertion is well known in the case of abelian varieties.

Lemma 5.5. *Suppose that T is simple and $\mathcal{Z}(T) = \mathbb{Q}$ (e.g., $\mathrm{End}^0(T) = \mathbb{Q}$). Then $\mathfrak{c}_T = \{0\}$, i.e., the $\mathbb{Q}$-Lie algebra is semisimple and therefore $\mathrm{Hdg}(T)$ is a semisimple $\mathbb{Q}$-algebraic group.*

Proof. The result follows readily from the combination of inclusions

$$\mathfrak{c}_T \subset \mathcal{Z}(T) = \mathbb{Q}, \ \mathfrak{c}_T \subset \mathrm{hdg}_T \subset \mathrm{sl}\left(\mathrm{H}_1(T, \mathbb{Q})\right).$$

□

The next example deals with the opposite case when the endomorphism algebra of a simple torus T is a number field of (largest possible) degree $2\dim(T)$.

Example 5.6. Suppose that a complex torus $T = V/\Lambda$ is simple of dimension g and $\mathrm{End}^0(T)$ is a number field E of degree $2g$. Then $\Lambda_{\mathbb{Q}}$ becomes a one-dimensional vector space over E. Therefore

$$E = \mathrm{End}_E(\Lambda_{\mathbb{Q}}),\ E^* = \mathrm{Aut}_E(\Lambda_{\mathbb{Q}}).$$

This implies that

$$\mathrm{Hdg}(T)(\mathbb{Q}) \subset E^{(1)} = \{e \in E^* \mid \mathrm{Norm}_{E/\mathbb{Q}}(e) = 1\} \subset E^* = \mathrm{Aut}_E(\Lambda_{\mathbb{Q}}). \quad (38)$$

Here

$$\mathrm{Norm}_{E/\mathbb{Q}} : E^* \to \mathbb{Q}^*,\ e \mapsto \prod_{\sigma\in\Sigma_E} \sigma(e)$$

is the *norm homomorphism* of the multiplicative groups of fields attached to the field extension $E/\mathbb{Q}$

Let $\mathcal{S}_E = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ be the $2g$-dimensional *algebraic* torus over $\mathbb{Q}$ obtained from the multiplicative group $\mathbb{G}_m$ by the Weil's restriction of scalars from E to $\mathbb{Q}$. Then $\mathcal{S}_E(\mathbb{Q}) = E^*$ and for each $\sigma \in \Sigma_E$ there is a certain character

$$\delta_\sigma : \bar{\mathcal{S}}_E := \mathcal{S}_E \times_{\mathbb{Q}} \bar{\mathbb{Q}} \to \mathbb{G}_m \times_{\mathbb{Q}} \bar{\mathbb{Q}}$$

of the algebraic torus $\bar{\mathcal{S}_E}$ over $\bar{\mathbb{Q}}$ such that the restriction of δ_σ to

$$E^* = \mathcal{S}_E(\mathbb{Q}) \subset \mathcal{S}_E(\bar{\mathbb{Q}}) = \bar{\mathcal{S}}_E(\bar{\mathbb{Q}})$$

coincides with

$$\sigma : E^* \hookrightarrow \bar{\mathbb{Q}}^*$$

. In addition, the $2g$-element set $\{\delta_\sigma \mid \sigma \in \Sigma_K\}$ constitutes a *basis* of the free $\mathbb{Z}$-module $\mathrm{X}(\bar{\mathcal{S}}_E)$ of *characters of the algebraic torus* $\bar{\mathcal{S}}_E$ over $\bar{\mathbb{Q}}$. Since $\mathcal{S}_E$ is defined over $\mathbb{Q}$, the group $\mathrm{X}(\bar{\mathcal{S}}_E)$ is provided with the natural structure of a $\mathrm{Gal}(\mathbb{Q})$-module in such a way that

$$s(\delta_\sigma) = \delta_{s(\sigma)}\ \forall \sigma \in \Sigma_E, s \in \mathrm{Gal}(\mathbb{Q}) \quad (39)$$

([26, Ch. II, Sect. 1], [30, Ch. III, Sect. 5 and 6]). Clearly, the character

$$\chi = \prod_{\sigma\in\Sigma_E} \delta_\sigma \in \mathrm{X}(\bar{\mathcal{S}}_E)$$

is $\mathrm{Gal}(\mathbb{Q})$-invariant and may be viewed as the character of $\mathcal{S}_E$ such that

$$\chi(e) = \mathrm{Norm}_{E/\mathbb{Q}}(e) \in \mathbb{Q}^*\ \forall e \in E^* = \mathcal{S}_E(\mathbb{Q}).$$

Let us put

$$\mathcal{S}_E^1 = \ker(\chi).$$

Since χ is obviously non-divisible in the group of characters, $\mathcal{S}_E^1$ is an algebraic $\mathbb{Q}$-subtorus of dimension $2g-1$ in $\mathcal{S}_E$ such that

$$\mathcal{S}_E^1(\mathbb{Q}) = \ker(\mathrm{Norm}_{E/\mathbb{Q}}) = E^{(1)}. \quad (40)$$

Combining (38) and (40), and taking into account that $\mathrm{Hdg}(T)(\mathbb{Q})$ is Zariski dense in $\mathrm{Hdg}(T)$, we conclude that

$$\mathrm{Hdg}(T) \subset \mathcal{S}_E^1. \quad (41)$$

In particular, if $\mathcal{S}_E^1$ is a *simple* algebraic torus over $\mathbb{Q}$ then

$$\mathrm{Hdg}(T) = \mathcal{S}_E^1.$$

By definition of $\mathcal{S}_E^1$, the Galois module $\mathrm{X}(\bar{\mathcal{S}}_E^1)$ of characters of the algebraic $\bar{\mathbb{Q}}$-torus

$$\bar{\mathcal{S}}_E^1 = \mathcal{S}_E^1 \times_{\mathbb{Q}} \bar{\mathbb{Q}}$$

is the quotient $\mathrm{X}(\bar{\mathcal{S}}_E)/(\mathbb{Z}\cdot\chi)$. It follows from (39) that the $\mathrm{Gal}(\mathbb{Q})$-module $\mathrm{X}(\bar{\mathcal{S}}_E^1)$ is isomorphic to the quotient $\mathbb{Z}^{\Sigma_E}/\mathbb{Z}\cdot\mathbf{1}$ where $\mathbb{Z}^{\Sigma_E}$ is the free $\mathbb{Z}$-module of functions $\phi:\Sigma_E\to\mathbb{Z}$ and $\mathbf{1}$ is the constant function 1. It follows easily that the Galois module $\mathrm{X}(\bar{\mathcal{S}}_E^1)\otimes\mathbb{Q}$ is isomorphic to the $\mathbb{Q}$-vector space

$$\left(\mathbb{Q}^{\Sigma_E}\right)^0 := \{\phi:\Sigma_E\to\mathbb{Q} \mid \sum_{\sigma\in\Sigma_E}\phi(\sigma)=0\}$$

of $\mathbb{Q}$-valued functions on Σ_E with zero "integral". Recall that the action of $\mathrm{Gal}(E)$ on Σ_E is transitive and this action induces the structure of the Galois module on $\left(\mathbb{Q}^{\Sigma_E}\right)^0$. Notice that if the action of $\mathrm{Gal}(\mathbb{Q})$ on $\left(\mathbb{Q}^{\Sigma_E}\right)^0$ is *doubly transitive* (i.e., E is doubly transitive) then the representation of $\mathrm{Gal}(\mathbb{Q})$ in $\left(\mathbb{Q}^{\Sigma_E}\right)^0$ is *irreducible*, i.e., the Galois module $\mathrm{X}(\bar{\mathcal{S}}_E^1)\otimes\mathbb{Q}$ is simple, which means that the algebraic $\mathbb{Q}$-torus $\mathcal{S}_E^1$ is simple and therefore $\mathrm{Hdg}(T)=\mathcal{S}_E^1$. So we have proven that

$$\mathrm{Hdg}(T) = \mathcal{S}_E^1 \tag{42}$$

if E is *doubly transitive*. In particular, the algebraic $\mathbb{Q}$-torus is simple.

Theorem 5.7. *Let $T=V/\Lambda$ be a simple complex torus of dimension $g>2$ such that its endomorphism algebra is a number field E of degree $2g$ that is doubly transitive.*

Then:

(i) *The Hodge group $\mathrm{Hdg}(T)$ of T coincides with $\mathcal{S}_E^1$. In addition, $\mathrm{Hdg}(T)$ is a simple algebraic $\mathbb{Q}$-torus of dimension $2g-1$.*
(ii) *The $\mathrm{Hdg}(T)$-module $\mathrm{H}^2(T,\mathbb{Q})$ is simple. In particular, T is 2-simple.*

Proof. We keep the notation of Example 5.6 where the assertion (i) and the simplicity of $\mathrm{Hdg}(T)$ are already proven.

In order to prove (ii), notice that $\mathrm{H}^2(T,\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(\wedge^2_{\mathbb{Q}}\Lambda_{\mathbb{Q}},\mathbb{Q})$, so, it suffices to check that the $\mathrm{Hdg}(T)$-module $\wedge^2_{\mathbb{Q}}\Lambda_{\mathbb{Q}}$ is simple.

If $\sigma\in\Sigma_E$ then let us consider the character $\delta_\sigma^{(1)}$ of $\bar{\mathcal{S}}_E^1$ that is the restriction of the character δ_σ to $\bar{\mathcal{S}}_E^1$. Clearly,

$$\prod_{\sigma\in\Sigma_E}\delta_\sigma^{(1)} = 1 \in \mathrm{X}(\bar{\mathcal{S}}_E^1)$$

and this is the only "nontrivial" multiplicative relation between $\delta_\sigma^{(1)}$. In particular, if A and B are two *distinct* 2-element subsets of Σ_E then

$$\delta_A^1 := \prod_{\sigma\in A}\delta_\sigma^{(1)} \ne \prod_{\sigma\in B}\delta_\sigma^{(1)} =: \delta_B^1. \tag{43}$$

In other words, δ^1_A and δ^1_B are *distinct* characters of $\bar{\mathcal{S}}^1_E$.

Let us fix an order on the $2g$-element set Σ_E and consider the $\bar{\mathbb{Q}}$-vector space

$$\bar{\Lambda} = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}},$$

which is provided with the natural faithful action of $\bar{\mathcal{S}}^1_E$, and splits into a direct sum

$$\bar{\Lambda} = \oplus_{\sigma \in \Sigma_E} \bar{\Lambda}_\sigma$$

of *one-dimensional* weight subspaces $\bar{\Lambda}_\sigma$ defined by the condition that $\bar{\mathcal{S}}^1_E$ acts on $\bar{\Lambda}_\sigma$ by the character $\delta^{(1)}_\sigma$.

We have

$$\wedge^2_{\mathbb{Q}} \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} = \wedge^2_{\bar{\mathbb{Q}}} \bar{\Lambda} = \oplus_{A=\{\sigma,\tau \in \Sigma_E; \sigma<\tau\}} \bar{\Lambda}_A,$$

where

$$\bar{\Lambda}_A = \bar{\Lambda}_\sigma \wedge_{\bar{\mathbb{Q}}} \bar{\Lambda}_\tau \cong \bar{\Lambda}_\sigma \otimes_{\bar{\mathbb{Q}}} \bar{\Lambda}_\tau$$

are one-dimensional $\bar{\mathcal{S}}^1_E$-invariant subspaces; the action of $\bar{\mathcal{S}}^1_E$ on $\bar{\Lambda}_A$ is defined by the character δ^1_A

It follows from (43) that if W is a nonzero $\mathcal{S}^1_E$-invariant $\mathbb{Q}$-vector subspace of $\Lambda_{\mathbb{Q}}$ then $\bar{W} = W \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ is a direct sum of some of $\bar{\Lambda}_A$. The double transitivity condition implies that all $\bar{\Lambda}_A$'s are mutually Galois-conjugate over $\mathbb{Q}$. It follows that $\bar{W} = \wedge^2_{\bar{\mathbb{Q}}} \bar{\Lambda}$, i.e., $W = \wedge^2_{\mathbb{Q}} \Lambda_{\mathbb{Q}}$ and we are done.

□

Remark 5.8. See [3] for explicit examples of complex tori that satisfy the conditions of Theorem 5.7.

In the case of arbitrary simple complex tori (or even abelian varieties) the Hodge group may be neither semisimple nor commutative (see [31, 32, 33] for explicit examples). This is not the case for 2-simple tori in dimensions > 2, in light of the following assertion.

Proposition 5.9. *Let T be a 2-simple torus of dimension $g > 2$. (In particular, T is simple.) Then* $\mathrm{Hdg}(T)$ *is either semisimple or commutative. The latter case occurs if and only if* $\mathrm{End}^0(T)$ *is a number field of degree* $2g$.

Proof. We know (thanks to Theorem 1.7) that $E = \mathrm{End}^0(T)$ is a number field of degree

$$[E : \mathbb{Q}] \in \{1, g, 2g\}.$$

If $[E : \mathbb{Q}] = 1$ then $E = \mathbb{Q}$. In light of Lemma 5.5, $\mathrm{Hdg}(T)$ is semisimple.

If $[E : \mathbb{Q}] = 2g$ then $\Lambda_{\mathbb{Q}} = \mathrm{H}_1(T, \mathbb{Q})$ is a one-dimensional E-vector space, i.e.,

$$E = \mathrm{End}_E\left(\mathrm{H}_1(T, \mathbb{Q})\right).$$

This implies that

$$\mathrm{hdg}_T \subset \mathrm{End}_E\left(\mathrm{H}_1(T, \mathbb{Q})\right) \subset E$$

and therefore hdg_T is a commutative $\mathbb{Q}$-Lie algebra. It follows that $\mathrm{Hdg}(T)$ is commutative.

Assume that $[E:\mathbb{Q}]=g$, i.e., $\Lambda_{\mathbb{Q}}=\mathrm{H}_1(T,\mathbb{Q})$ is a two-dimensional E-vector space. Then

$$\mathrm{hdg}_T \subset \mathrm{End}_E\left(\mathrm{H}_1(T,\mathbb{Q})\right) \supset E \supset \mathfrak{c}_T.$$

Let

$$\mathrm{Tr}_E : \mathrm{End}_E\left(\mathrm{H}_1(T,\mathbb{Q})\right) \to E \tag{44}$$

be the (surjective) E-linear trace map, which is a homomorphism of $\mathbb{Q}$-Lie algebras (here we view E as a commutative $\mathbb{Q}$-Lie algebra); in addition, the restriction of Tr_E to E is multiplication by

$$\dim_E\left(\mathrm{H}_1(T,\mathbb{Q})\right)=2.$$

We write $\mathrm{sl}\left(\mathrm{H}_1(T,\mathbb{Q})/E\right)$ for $\ker(\mathrm{Tr}_E)$, which is an absolutely simple E-Lie algebra of traceless E-linear operators in $\mathrm{H}_1(T,\mathbb{Q})$. (Viewed as the $\mathbb{Q}$-Lie algebra, $\mathrm{sl}\left(\mathrm{H}_1(T,\mathbb{Q})/E\right)$ is a simple but not absolutely simple.)

On the other hand, let

$$\det_E : \mathrm{Aut}_E\left(\mathrm{H}_1(T,\mathbb{Q})\right) \to E^*$$

be the multiplicative *determinant* map. Clearly,

$$\mathrm{Hdg}(T)(\mathbb{Q}) \subset \mathrm{Aut}_E\left(\mathrm{H}_1(T,\mathbb{Q})\right)$$

and the group $\mathrm{Aut}_E\left(\mathrm{H}_1(T,\mathbb{Q})\right)$ acts naturally on the one-dimensional E-vector space $\mathrm{Hom}_E\left(\wedge_E^2\mathrm{H}_1(T,\mathbb{Q}),E\right)$ via the character $\det_E$. I claim that $\det_E$ kills $\mathrm{Hdg}(T)(\mathbb{Q})$. Indeed, if this is not the case, then the rational Hodge substructure $\mathrm{Hom}_E\left(\wedge_E^2\mathrm{H}_1(T,\mathbb{Q}),E\right)$ of $\mathrm{H}^2(T,\mathbb{Q})$ has *nonzero* $(2,0)$-component, whose $\mathbb{C}$-dimension

$$\leq \frac{[E:\mathbb{Q}]}{2}=\frac{g}{2}<g,$$

which contradicts the 2-simplicity of T. Hence, $\det_E$ kills $\mathrm{Hdg}(T)(\mathbb{Q})$. Taking into account that $\mathrm{Hdg}(T)(\mathbb{Q})$ is dense in $\mathrm{Hdg}(T)$ in Zariski topology and the minimality property in the definition of the Hodge group, we conclude that

$$\mathrm{Hdg}(T) \subset \mathrm{Res}_{E/\mathbb{Q}}\mathrm{SL}((\mathrm{H}_1(T,\mathbb{Q})/E)$$

where $\mathrm{SL}((\mathrm{H}_1(T,\mathbb{Q})/E)$ is the special linear group of the E-vector space $\mathrm{H}_1(T,\mathbb{Q})$, which is a *simple algebraic E-group*, and $\mathrm{Res}_{E/\mathbb{Q}}$ is the *Weil restrictions* of scalars. Taking into account that the $\mathbb{Q}$-Lie algebra $\mathrm{sl}\left(\mathrm{H}_1(T,\mathbb{Q})/E\right)$ is the Lie algebra of the $\mathbb{Q}$-algebraic group $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{SL}((\mathrm{H}_1(T,\mathbb{Q})/E)$, we conclude that

$$\mathrm{hdg}_T \subset \mathrm{sl}\left(\mathrm{H}_1(T,\mathbb{Q})/E\right) \cong \mathrm{sl}(2,E). \tag{45}$$

In particular, Tr_E kills $\mathfrak{c}_T$. Since $\mathfrak{c}_T \subset E$,

$$0=\mathrm{Tr}_E(c)=2c \ \forall c \in \mathfrak{c}_T \subset E.$$

This implies that $\mathfrak{c}_T=\{0\}$, i.e., hdg_T is semisimple, i.e., $\mathrm{Hdg}(T)$ is semisimple. This ends the proof. □

The following assertion may be viewed as a variant of a theorem of P. Deligne [14] about abelian varieties (see also [27]).

Theorem 5.10. *Let T be a simple complex torus. Let* $\mathrm{hdg}_T \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ *be the (reductive)* $\mathbb{Q}$*-Lie algebra of* $\mathrm{Hdg}(T)$*, whose natural representation in* $\Lambda_{\mathbb{Q}}$ *is irreducible.*

Let $\mathfrak{g}$ *be a simple (non-abelian) factor of the complex reductive Lie algebra*

$$\mathrm{hdg}_{T,\mathbb{C}} = \mathrm{hdg}_T \otimes_{\mathbb{Q}} \mathbb{C} \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}}) = \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}}).$$

Then

(i) *The simple complex Lie algebra* $\mathfrak{g}$ *is of classical type* $(\mathsf{A_l}, \mathsf{B_l}, \mathsf{C_l}, \mathsf{D_l})$ *of a certain positive rank l.*
(ii) *Let W be a nontrivial simple* $\mathfrak{g}$*-submodule of* $V_{\mathbb{C}}$*. Then its highest weight is minuscule one.*

Proof. The result follows readily from Corollary 5.3 combined with Proposition 2.4.1 of [34] applied to

$$k = \mathbb{C}, k_0 = \mathbb{Q}, W = \Lambda_{\mathbb{Q}}, g = \mathrm{hdg}_T, f = f_T, A = \{1, -1\}$$

and

$$n = 1, a_0 = 1, a_1 = -1.$$

□

The following assertion may be viewed as a variant of a theorem of M.V. Borovoi about abelian varieties [7], see also [36].

Theorem 5.11. *Suppose that T is a simple complex torus with* $\mathrm{End}^0(T) = \mathbb{Q}$.

Then its Hodge group $\mathrm{Hdg}(T)$ *is a* $\mathbb{Q}$*-simple linear algebraic group, i.e., its* $\mathbb{Q}$*-Lie algebra* hdg_T *is simple.*

Proof. Clearly, hdg_T is a semisimple $\mathbb{Q}$-Lie algebra, whose natural faithful representation in $\Lambda_{\mathbb{Q}}$ is *absolutely irreducible*. By Remark 5.4, $f_T \in \mathrm{hdg}_{T,\mathbb{C}}$ is a Hodge element of hdg_T. The spectrum of the linear semisimple operator f_T in $\Lambda_{\mathbb{Q}}$ consists of precisely two eigenvalues, 1 and -1. Now it follows from Theorem 1.5 of [34] that hdg_T is simple. This means that $\mathrm{Hdg}(T)$ is a $\mathbb{Q}$-simple algebraic group.

□

Corollary 5.12. *Suppose that T is a simple complex torus of dimension g with* $\mathrm{End}^0(T) = \mathbb{Q}$*. Assume also that $2g$ is not a power (e.g., g is odd).*

Then $\mathrm{Hdg}(T)$ *is an absolutely simple* $\mathbb{Q}$*-algebraic group that enjoys precisely one of the following two properties.*

- $\mathrm{Hdg}(T)$ *is of type* $\mathsf{A_{2g-1}}, \mathsf{C_g}, \mathsf{D_g}$.
- $\mathrm{Hdg}(T)$ *is of type or* $\mathsf{A_r}$ *where r is a positive integer that enjoys the following properties.*
  $1 < r < 2g-1$ *and there is an integer j such that* $1 < j < 2g-1$ *and* $2g = \binom{r+1}{j}$.

Proof. By Theorem 5.11, hdg_T is a simple $\mathbb{Q}$-Lie algebra. Suppose that hdg_T is *not* absolutely simple, i.e., the complex Lie algebra $\mathrm{hdg}_{T,\mathbb{C}}$ splits into a direct sum

$$\mathrm{hdg}_{T,\mathbb{C}} = \oplus_{\ell=1}^{d} \mathfrak{g}_\ell$$

of d simple complex Lie algebras $\mathfrak{g}_\ell$ where $d > 1$. We are going to prove that $2g$ is a dth power and get a contradiction. The simplicity of hdg_T means that $\mathrm{Aut}(\mathbb{C})$ permutes the set $\{\mathfrak{g}_\ell\}_{\ell=1}^{d}$ transitively. Namely, each $s \in \mathrm{Aut}(\mathbb{C})$ gives rise to the semi-linear automorphism of the $\mathbb{C}$-vector space

$$\Lambda_\mathbb{C} \to \Lambda_\mathbb{C}, x \otimes z \mapsto x \otimes s(z) \ \forall x \in \Lambda_\mathbb{Q}, z \in \mathbb{C}$$

and to the semi-linear automorphism of the $\mathbb{C}$-algebra

$$\mathrm{End}_\mathbb{C}(\Lambda_\mathbb{C}) \to \mathrm{End}_\mathbb{C}(\Lambda_\mathbb{C}), u \otimes z \mapsto u \otimes s(z) \ \forall x \in \mathrm{End}_\mathbb{Q}(\Lambda_\mathbb{Q}), z \in \mathbb{C}.$$

We continue to denote those automorphisms by s.

The simplicity of hdg_T implies that for each $\mathfrak{g}_\ell$ there is $s_\ell \in \mathrm{Aut}(\mathbb{C})$ such that

$$\mathfrak{g}_\ell = s_\ell\left(\mathfrak{g}_1\right).$$

We know that the $\mathrm{hdg}_{T,\mathbb{C}}$-module $\Lambda_\mathbb{C}$ is (absolutely) simple. Since each $\mathfrak{g}_\ell$ is a direct summand of $\mathfrak{g}_\ell$, the $\mathfrak{g}_\ell$-module $\Lambda_\mathbb{C}$ is isotypic, i.e., there is a simple $\mathfrak{g}_\ell$-submodule $W_\ell \subset \Lambda_\mathbb{C}$ such that all simple $\mathfrak{g}_\ell$-submodules of $\Lambda_\mathbb{C}$ are isomorphic to W_ℓ. In addition, the $\mathrm{hdg}_{T,\mathbb{C}} = \oplus_{\ell=1}^{d}\mathfrak{g}_\ell$-module $\Lambda_\mathbb{C}$ splits into a tensor product $\otimes_{\ell=1}^{d} W_\ell$. Let us prove that $\dim_\mathbb{C}(W_\ell)$ does not depend on ℓ.

Indeed, $s_\ell(W_1)$ is a simple $\mathfrak{g}_\ell$-submodule of $\Lambda_\mathbb{C}$ and therefore is isomorphic to W_1. This implies that $\dim(W_1) = \dim(W_\ell)$ and therefore

$$2g = \dim_\mathbb{C}(\Lambda_\mathbb{C}) = \dim_\mathbb{C}(W_1)^d,$$

which gives us a desired contradiction. So, $\mathrm{hdg}_{T,\mathbb{C}}$ is a *simple* complex Lie algebra and $\Gamma_\mathbb{C}$ is a faithful simple $\mathrm{hdg}_{T,\mathbb{C}}$-module. By Theorem 5.10, $\mathrm{hdg}_{T,\mathbb{C}}$ is a classical Lie algebra (of type $\mathsf{A}_\mathsf{r}, \mathsf{B}_\mathsf{r}, \mathsf{C}_\mathsf{r}$ or D_r), and the highest weight of $\Gamma_\mathbb{C}$ is *minuscule*. The remaining assertion follows readily from the inspection of dimensions of minuscule representations of classical Lie algebras [9, Tables].

□

Example 5.13. Suppose that T is a complex torus of dimension g such that one of the following conditions holds.

(i) $\mathrm{Hdg}(T) = \mathrm{SL}(\Lambda_\mathbb{Q})$.

(ii) There exists a nondegenerate quadratic form

$$\phi : \Lambda_\mathbb{Q} \to \mathbb{Q}$$

of even signature $(2p, 2q)$ with $p + q = g \geq 3$ such that $\mathrm{Hdg}(T)$ coincides with the corresponding special orthogonal group. $\mathrm{SO}(\Lambda_\mathbb{Q})$

(iii) There exists a nondegenerate alternating $\mathbb{Q}$-bilinear form

$$\Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}$$

such that $\mathrm{Hdg}(T)$ coincides with the corresponding symplectic group $\mathrm{Sp}(\Lambda_{\mathbb{Q}})$.

Then T is 2-simple.

Indeed, in the cases (i) and (ii), (in the obvious notation) the natural representation of $\mathrm{SL}(\Lambda_{\mathbb{C}})$ (resp. $\mathrm{SO}(\Lambda_{\mathbb{C}})$) in $\wedge^2_{\mathbb{C}}(\Lambda_{\mathbb{C}})$ is irreducible, see [9, Ch. 8, Sect. 13]. This implies that the natural representation of $\mathrm{SL}(\Lambda_{\mathbb{Q}})$ (resp. $\mathrm{SO}(\Lambda_{\mathbb{Q}}, \phi)$) in $\wedge^2_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ is absolutely irreducible. By duality, the same is true for $\mathrm{Hom}_{\mathbb{Q}}(\wedge^2_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}), \mathbb{Q}) = \mathrm{H}^2(T, \mathbb{Q})$, i.e., the $\mathrm{Hdg}(T)$-module $\mathrm{H}^2(T, \mathbb{Q})$ is simple. This implies that T is 2-simple. Notice that in these cases we deal with simple complex tori that are *not* abelian varieties (since they do not carry nonzero 2-dimensional Hodge classes), and whose endomorphism algebra is $\mathbb{Q}$.

In the case (iii), the natural representation of $\mathrm{Sp}(\Lambda_{\mathbb{C}})$ in $\wedge^2_{\mathbb{C}}(\Lambda_{\mathbb{C}})$ is a direct sum of an irreducible representation and a trivial one-dimension representation [28, Tables]. This implies that the natural representation of $\mathrm{Sp}(\Lambda_{\mathbb{Q}})$ in $\wedge^2_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ is a direct sum of an absolutely irreducible representation and a trivial one-dimension representation. By duality, the same is true for $\mathrm{Hom}_{\mathbb{Q}}(\wedge^2_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}), \mathbb{Q}) = \mathrm{H}^2(T, \mathbb{Q})$, i.e., the $\mathrm{Hdg}(T)$-module $\mathrm{H}^2(T, \mathbb{Q})$ is a direct sum of an absolutely simple simple module and a trivial module of $\mathbb{Q}$-dimension 1 The latter consists of all $\mathrm{Hdg}(T)$-invariants in $\mathrm{H}^2(T, \mathbb{Q})$, i.e., coincides with $\mathrm{H}^{1,1}(T, \mathbb{Q})$. The former is an irreducible rational Hodge structure. It follows from Remark 1.8 (ii) that T is 2-simple.[3] See [37] for explicit examples (in all dimensions) of complex abelian varieties T with $\mathrm{Hdg}(T) = \mathrm{Sp}(\Lambda_{\mathbb{Q}})$.

Theorem 5.14. *Let $\Pi_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space of positive even dimension $2g$, and $\mathcal{G}$ a $\mathbb{Q}$-simple algebraic subgroup of $\mathrm{GL}(\Pi_{\mathbb{Q}})$, whose $\mathbb{Q}$-Lie algebra $\mathfrak{g}$ may be viewed as a simple $\mathbb{Q}$-Lie subalgebra of $\mathrm{End}_{\mathbb{Q}}(\Pi_{\mathbb{Q}})$. Let us consider the real Lie subalgebra*

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathrm{End}_{\mathbb{Q}}(\Pi_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R} = \mathrm{End}_{\mathbb{R}}(\Pi_{\mathbb{R}})$$

where $\Pi_{\mathbb{R}} = \Pi_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is the corresponding $2g$-dimensional real vector space. Suppose that there exists an element

$$J_0 \in \mathfrak{g}_{\mathbb{R}} \subset \mathrm{End}_{\mathbb{R}}(\Pi_{\mathbb{R}})$$

such that $J_0^2 = -1$ in $\mathrm{End}_{\mathbb{R}}(\Pi_{\mathbb{R}})$. Then there exists $J \in \mathfrak{g}_{\mathbb{R}}$ that enjoys the following properties.

(i) $J^2 = -1$.

(ii) *Let us endow $\Pi_{\mathbb{R}}$ with the structure of a g-dimensional complex vector space by defining*

$$(a + b\mathbf{i})v = av + bJ(v) \quad \forall a + b\mathbf{i} \in \mathbb{C} \text{ with } a, b \in \mathbb{R}.$$

[3] For abelian varieties T the case (iii) was done in [1, Sect. 5.1].

Then for every discrete subgroup Λ of rank $2g$ in $\Pi_{\mathbb{Q}}$ the corresponding complex torus $T=\Pi_{\mathbb{R}}/\Lambda$ has Hodge group $\mathcal{G}$.

Proof. We will need the following auxiliary statement.

Lemma 5.15. *If $\mathfrak{v}$ is a proper nonzero $\mathbb{Q}$-vector subspace of $\mathfrak{g}$ then the real vector subspace*

$$\mathfrak{v}_{\mathbb{R}}=\mathfrak{v}\otimes_{\mathbb{Q}}\mathbb{R}\subset\mathfrak{g}\otimes_{\mathbb{Q}}\mathbb{R}=\mathfrak{g}_{\mathbb{R}}$$

does not contain a nonzero ideal of $\mathfrak{g}_{\mathbb{R}}$.

Notice that the the set of proper nonzero $\mathbb{Q}$-Lie subalgebras L of $\mathfrak{g}$ is *countable*. By Lemma 5.15, every $L_{\mathbb{R}}=L\otimes_{\mathbb{Q}}\mathbb{R}$ does not contain a nonzero ideal of $\mathfrak{g}_{\mathbb{R}}$. By [38, Lemma 2 on p. 494], the closed subset

$$\mathcal{G}(L_{\mathbb{R}},J_0)=\{u\in\mathcal{G}(\mathbb{R})\mid J_0\in u^{-1}L_{\mathbb{R}}u\}$$

is nowhere dense in $\mathcal{G}(\mathbb{R})$. It follows that there exists $u\in\mathcal{G}(\mathbb{R})$ such that J_0 does not lie in any of $u^{-1}L_{\mathbb{R}}u$. Let us put

$$J:=uJ_0J^{-1}\in\mathfrak{g}_{\mathbb{R}}\subset\mathrm{End}_{\mathbb{R}}(\Pi_{\mathbb{R}}).$$

Then J does not lie in any of $L_{\mathbb{R}}$ and $J^2=-1\in\mathrm{End}_{\mathbb{R}}(\Pi_{\mathbb{R}})$. It follows that $\mathfrak{g}$ coincides with the smallest $\mathbb{Q}$-Lie subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ such that $\mathfrak{u}_{\mathbb{R}}$ contains J. This implies that $\mathfrak{g}$ coincides with the smallest $\mathbb{Q}$-Lie subalgebra $\mathfrak{u}$ of $\mathrm{End}\mathbb{Q}(\Pi_{\mathbb{Q}})$ such that $\mathfrak{u}_{\mathbb{R}}$ contains J. It follows readily that $\mathfrak{g}$ coincides with the Lie algebra of the Hodge group of a complex torus $T=\Pi_{\mathbb{R}}/\Lambda$ where the complex structure on the real vector space $\Pi_{\mathbb{R}}$ is defined by J and Λ is any discrete subgroup Λ of rank $2g$ in $\Pi_{\mathbb{Q}}$. □

Proof of Lemma 5.15. Suppose that $L_{\mathbb{R}}$ contains a nonzero ideal $\mathfrak{a}$ of $\mathfrak{g}_{\mathbb{R}}$. Then the $\mathbb{C}$-vector subspace

$$L_{\mathbb{C}}=L_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}=L\otimes_{\mathbb{Q}}\mathbb{C}$$

contains a nonzero ideal $\mathfrak{a}_{\mathbb{C}}=\mathfrak{a}\otimes_{\mathbb{R}}\mathbb{C}$ of the complex Lie algebra

$$\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}=\mathfrak{g}\otimes_{\mathbb{Q}}\mathbb{C}.$$

Then

$$\tilde{\mathfrak{a}}=\sum_{s\in\mathrm{Aut}(\mathbb{C})}s(\mathfrak{a})$$

is a $\mathrm{Aut}(\mathbb{C})$-invariant ideal of $\mathfrak{g}_{\mathbb{C}}$ that lies in $L_{\mathbb{C}}$. Hence, there is a $\mathbb{Q}$-vector subspace $\mathfrak{a}_{\mathbb{Q}}$ such that

$$\tilde{\mathfrak{a}}=\mathfrak{a}_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C};$$

in addition, $\mathfrak{a}_{\mathbb{Q}}$ is an ideal of $\mathfrak{g}$, which contradicts the simplicity of the $\mathbb{Q}$-Lie algebra $\mathfrak{g}$. This ends the proof. □

Example 5.16. We keep the notation of Theorem 5.14. Let $g\geq 3$ be an integer, $\Pi_{\mathbb{Q}}$ a $2g$-dimensional vector space over $\mathbb{Q}$. Let $\mathcal{G}$ be a $\mathbb{Q}$-simple algebraic subgroup of $\mathrm{GL}(\Pi_{\mathbb{Q}})$ that enjoys one of the following properties.

(i) $\mathcal{G}=\mathrm{SL}(\Pi_{\mathbb{Q}})$.

(ii) There exists a nondegenerate quadratic form

$$\phi : \Pi_{\mathbb{Q}} \to \mathbb{Q}$$

of even signature $(2p, 2q)$ with $p + q = g$ such that $\mathcal{G}$ coincides with the corresponding special orthogonal group. $\mathrm{SO}(\Pi_{\mathbb{Q}})$.

In both cases there exists $J_0 \in \mathfrak{g}_{\mathbb{R}}$ with $J_0^2 = -1$. (Here $\mathfrak{g} \subset \mathrm{End}_{\mathbb{Q}}(\Pi_{\mathbb{Q}})$ is the Lie algebra of $\}$.) In light of Theorem 5.14, there exists a complex structure on the real vector space $\Pi_{\mathbb{R}}$ such that the Hodge group of corresponding complex tori $T = \Pi_{\mathbb{R}}/\Lambda$ coincides with $\mathcal{G}$. In light of Example 5.13, T is 2-simple.

6. The degree g case

In this section we discuss g-dimensional 2-simple tori, whose endomorphism algebra is a number field of degree g.

Theorem 6.1. *Let T a 2-simple torus of dimension $g > 2$. If $\mathrm{End}^0(T)$ is a number field E of degree g then*

$$\mathrm{Hdg}(T) = \mathrm{Res}_{E/\mathbb{Q}}\mathrm{SL}((\mathrm{H}_1(T, \mathbb{Q})/E)\,.$$

Proof. It suffices to check that

$$\mathrm{hdg}_T = \mathrm{sl}\,(\mathrm{H}_1(T, \mathbb{Q})/E)\,. \tag{46}$$

In light of (45), the desired equality (46) is an immediate corollary of the following observation applied to

$$k = \mathbb{Q},\ K = E, W = \mathrm{H}_1(T, \mathbb{Q}),\ \mathfrak{g} = \mathrm{hdg}_T.$$

Lemma 6.2. *Let g be a positive integer, W a $2g$-dimensional vector space over a field k of characteristic 0, $\mathfrak{g} \subset \mathrm{End}_k(W)$ a linear semisimple k-subalgebra such that the centralizer*

$$K := \mathrm{End}_{\mathfrak{g}} W \subset \mathrm{End}_k(W)$$

is an overfield of K such that $[K : k] = g$. Then $\mathfrak{g}$ coincides with the Lie algebra $\mathrm{sl}(W/K)$ of traceless K-linear operators in W.

Proof of Lemma 6.2. The semisimplicity of $\mathfrak{g}$ implies that

$$\mathfrak{g} \subset \mathrm{sl}(W/K). \tag{47}$$

In what follows we mimick the arguments of [24, pp. 790–791, Proof of Th. 4.4.10] where ℓ-adic Lie algebras are treated.

Let $\bar{k}$ be an algebraic closure of k, and Σ_K the g-element set of field embeddings $\sigma : K \hookrightarrow \bar{k}$ that coincide with the identity map on k. Let us consider the $2g$-dimensional $\bar{k}$-vector space $\bar{W} = W \otimes_k \bar{k}$ and the $\bar{k}$-Lie algebra

$$\bar{\mathfrak{g}} = \mathfrak{g} \otimes_k \bar{k} \subset \mathrm{End}_k(W) \otimes_k \bar{k} = \mathrm{End}_{\bar{k}}(\bar{W}). \tag{48}$$

The semisimplicity of the k-Lie algebra $\mathfrak{g}$ implies the *semisimplicity* of the $\bar{k}$-Lie algebra $\bar{\mathfrak{g}}$.

Clearly, the *centralizer* $\mathrm{End}_{\bar{\mathfrak{g}}}(\bar{W})$ of $\bar{\mathfrak{g}}$ in $\mathrm{End}_{\bar{k}}(\bar{W})$ equals

$$\mathrm{End}_{\mathfrak{g}}(W) \otimes_k \bar{k} = K \otimes_k \bar{k} \tag{49}$$

and $\bar{W}$ is a free $K \otimes_k \bar{k}$-module of rank 2, because W is a vector space over K of dimension 2. We have

$$K \otimes_k \bar{k} = \oplus_{\sigma \in \Sigma_K} K \otimes_{K,\sigma} \bar{k} = \oplus_{\sigma \in \Sigma_K} \bar{k}_\sigma \tag{50}$$

where

$$\bar{k}_\sigma = K \otimes_{K,\sigma} \bar{k} = \bar{k}.$$

We have

$$\bar{W} = \oplus_{\sigma \in \Sigma_K} \bar{W}_\sigma \quad \text{where } \bar{W}_\sigma = \bar{k}_\sigma \bar{W} \subset \bar{W}.$$

The freeness of the $K \otimes_k \bar{k}$-module $\bar{W}$ with rank 2 implies that each $\bar{W}_\sigma$ is a $\bar{k}_\sigma$-vector space of dimension 2. Since

$$\bar{k}_\sigma \subset K \otimes_k \bar{k} = \mathrm{End}_{\bar{\mathfrak{g}}}(\bar{W}),$$

each $\bar{W}_\sigma = \bar{k}_\sigma \bar{W}$ is a $\bar{\mathfrak{g}}$-invariant subspace of $\bar{W}$ and the centralizer of $\mathfrak{g}$

$$\mathrm{End}_{\bar{\mathfrak{g}}}(\bar{W}_\sigma) = \bar{k}_\sigma = \bar{k}. \tag{51}$$

Let

$$\bar{\mathfrak{g}}_\sigma \subset \mathrm{End}_{\bar{k}_\sigma}(\bar{W}_\sigma) = \mathrm{End}_{\bar{k}}(\bar{W}_\sigma)$$

be the image of the natural $\bar{k}$-Lie algebra homomorphism

$$\bar{\mathfrak{g}} \to \mathrm{End}_{\bar{k}}(\bar{W}_\sigma).$$

The semisimplicity of $\bar{\mathfrak{g}}$ implies the semisimplicity of the Lie algebra $\bar{\mathfrak{g}}_\sigma$, because the latter is isomorphic to a quotient of the former. This implies that

$$\bar{\mathfrak{g}}_\sigma \subset \mathrm{sl}(\bar{W}_\sigma) \cong \mathrm{sl}(2, \bar{k}_\sigma) = \mathrm{sl}(2, \bar{k}).$$

Taking into account (51) and the semisimplicity of $\bar{\mathfrak{g}}_\sigma$, we conclude that

$$\bar{\mathfrak{g}}_\sigma = \mathrm{sl}(\bar{W}_\sigma) \cong \mathrm{sl}(2, \bar{k}). \tag{52}$$

This implies that

$$\bar{\mathfrak{g}} \subset \oplus_{\sigma \in \Sigma_K} \bar{\mathfrak{g}}_\sigma = \oplus_{\sigma \in \Sigma_K} \mathrm{sl}(\bar{W}_\sigma) \subset \oplus_{\sigma \in \Sigma_K} \mathrm{End}_{\bar{k}}(\bar{W}_\sigma). \tag{53}$$

Let σ and τ be *distinct* elements of Σ_K. Clearly, $\bar{W}_\sigma \oplus \bar{W}_\tau$ is a $\bar{\mathfrak{g}}$-invariant subspace of $\bar{W}$. Let $\bar{\mathfrak{g}}_{\sigma,\tau}$ be the image of $\bar{\mathfrak{g}}$ in $\mathrm{End}_{\bar{k}}(\bar{W}_\sigma \oplus \bar{W}_\tau)$. Since $\bar{\mathfrak{g}}_{\sigma,\tau}$ is isomorphic to a quotient of $\bar{\mathfrak{g}}$, it is a semisimple $\bar{k}$-Lie algebra such that

$$\bar{\mathfrak{g}}_{\sigma,\tau} \subset \mathrm{sl}(\bar{W}_\sigma) \oplus \mathrm{sl}(\bar{W}_\tau) \subset \mathrm{End}_{\bar{k}}(\bar{W}_\sigma) \oplus \mathrm{End}_{\bar{k}}(\bar{W}_\tau) \subset \mathrm{End}_{\bar{k}}(\bar{W}_\sigma \oplus \bar{W}_\tau).$$

Notice that $\bar{\mathfrak{g}}_{\sigma,\tau}$ projects *surjectively* on both

$$\bar{\mathfrak{g}}_\sigma = \mathrm{sl}(\bar{W}_\sigma) \quad \text{and } \bar{\mathfrak{g}}_\tau = \mathrm{sl}(\bar{W}_\tau),$$

because $\bar{\mathfrak{g}}$ does. The simplicity of both mutually isomorphic Lie algebras $\mathrm{sl}(\bar{W}_\sigma)$ and $\mathrm{sl}(\bar{W}_\tau)$ and the semisimplicity of $\bar{\mathfrak{g}}_{\sigma,\tau}$ implies that either

$$\bar{\mathfrak{g}}_{\sigma,\tau} = \mathrm{sl}(\bar{W}_\sigma) \oplus \mathrm{sl}(\bar{W}_\tau) \tag{54}$$

or

$$\bar{\mathfrak{g}}_{\sigma,\tau} \cong \mathrm{sl}(\bar{W}_\sigma) \cong \mathrm{sl}(\bar{W}_\tau) \cong \mathrm{sl}(2, \bar{k}).$$

In the latter case the $\bar{\mathfrak{g}}_{\sigma,\tau}$-modules $\bar{W}_\sigma$ and $\bar{W}_\tau$ are isomorphic, because the Lie algebra $\mathrm{sl}(2,\bar{k})$ has precisely one nontrivial 2-dimensional representation over $\bar{k}$, up to an isomorphism. This implies that the $\bar{\mathfrak{g}}$-modules $\bar{W}_\sigma$ and $\bar{W}_\tau$ are isomorphic as well and therefore the centralizer $\mathrm{End}_{\bar{\mathfrak{g}}}(\bar{W})$ is noncommutative, which is not the case. The obtained contradiction proves that the equality (54) holds for any σ, τ. Now, it follows from Lemma on p. 790-791 of [24] that

$$\bar{\mathfrak{g}} = \oplus_{\sigma\in\Sigma_K}\bar{\mathfrak{g}}_\sigma = \oplus_{\sigma\in\Sigma_K}\mathrm{sl}(\bar{W}_\sigma).$$

This implies that

$$\dim_{\bar{k}}(\bar{\mathfrak{g}}) = 3g = \dim_k \mathrm{sl}(W/K).$$

By (47), $\mathfrak{g} \subset \mathrm{sl}(W/K)$. Taking into account that $\dim_k(\mathfrak{g}) = \dim_{\bar{k}}(\bar{\mathfrak{g}})$, we conclude that $\dim_k(\mathfrak{g}) = \dim_k \mathrm{sl}(W/K)$. This implies that $\mathfrak{g} = \mathrm{sl}(W/K)$, which ends the proof.

□

□

Theorem 6.3. *Let T a simple complex torus of dimension $g > 2$. Suppose that* $\mathrm{End}^0(T)$ *is a number field E of degree g and*

$$\mathrm{Hdg}(T) = \mathrm{Res}_{E/\mathbb{Q}}\mathrm{SL}((\mathrm{H}_1(T,\mathbb{Q})/E)\,.$$

Then the following conditions are equivalent.

(i) *T is 2-simple.*
(ii) *E is almost doubly transitive.*

Theorem 6.3 is an immediate corollary of the following observation applied to

$$k = \mathbb{Q},\ K = E,\ W = \mathrm{H}_1(T,\mathbb{Q}).$$

Lemma 6.4. *Let g be a positive integer ≥ 2, W a $2g$-dimensional vector space over a field k of characteristic 0, $\mathfrak{g} \subset \mathrm{End}_k(W)$ a linear semisimple k-subalgebra such that the centralizer of $\mathfrak{g}$*

$$K := \mathrm{End}_{\mathfrak{g}} W \subset \mathrm{End}_k(W)$$

is an overfield of k such that $[K : k] = g$, and $\mathfrak{g} = \mathrm{sl}(W/K)$ is the Lie algebra of traceless K-linear operators in W.

Then the following conditions are equivalent.

(i) *The $\mathfrak{g}$-module $\wedge_k^2 W$ is a direct sum of its submodule $\left(\wedge_k^2 W\right)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants and a simple $\mathfrak{g}$-module.*
(ii) *The $\mathfrak{g}$-module $\mathrm{Hom}(\wedge_k^2 W, k)$ is a direct sum of its submodule $\mathrm{Hom}(\wedge_k^2 W, k)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants and a simple $\mathfrak{g}$-module.*
(iii) *Let $\mathrm{Gal}(k) = \mathrm{Aut}(\bar{k}/k)$ be the absolute Galois group of k. Let Σ_K be the set of k-linear field embeddings $K \hookrightarrow \bar{k}$. Then the natural action of $\mathrm{Gal}(k)$ on Σ_K is almost doubly transitive.*

Remark 6.5. The equivalence of (i) and (ii) follows readily from the semisimplicity of $\mathfrak{g}$.

Corollary 6.6. *Let T a simple complex torus of dimension 3. Suppose that* $\mathrm{End}^0(T)$ *is a cubic number field E and*

$$\mathrm{Hdg}(T) = \mathrm{Res}_{E/\mathbb{Q}}\mathrm{SL}((\mathrm{H}_1(T,\mathbb{Q})/E)\,.$$

Then T is 2-simple.

Proof of Corollary 6.6. The result follows readily from Theorem 6.3, since every transitive action on the 3-element set Σ_E is almost doubly transitive. □

Proof of Lemma 6.4. We use the notation of the Proof of Lemma 6.2. In particular,

$$\bar{\mathfrak{g}}_\sigma = \mathrm{sl}(\bar{W}_\sigma),\ \bar{\mathfrak{g}}_{\sigma,\tau} = \mathrm{sl}(\bar{W}_\sigma)\oplus \mathrm{sl}(\bar{W}_\tau)\ \forall \sigma,\tau\in\Sigma_K, \sigma\ne\tau;$$
$$\bar{W} = \oplus_{\sigma\in\Sigma_K}\bar{W}_\sigma,\ \bar{\mathfrak{g}} = \oplus_{\sigma\in\Sigma_K}\bar{\mathfrak{g}}_\sigma,\ .$$

Let us start with the $\mathfrak{g}$-module

$$W^{\otimes 2} := W\otimes_k W \to W.$$

There is an involution

$$\delta : W^{\otimes 2}\to W^{\otimes 2},\ u\otimes v\mapsto v\otimes u,$$

whose subspace of invariants is the symmetric square $\mathbf{S}^2_k W$ of W and the subspace of anti-invariants is the exterior square $\wedge^2_k W$. Clearly, δ commutes with the action of $\mathfrak{g}$; in particular, both $\mathbf{S}^2_k W$ and $\wedge^2_k W$ are $\mathfrak{g}$-invariant subspaces of the tensor square of W.

Let us consider the $\bar{\mathfrak{g}}$-module

$$\bar{W}^{\otimes 2} := \bar{W}\otimes_{\bar{k}}\bar{W}.$$

Extending by $\bar{k}$-linearity the involution δ, we get the involution

$$\bar{\delta} : \bar{W}^{\otimes 2}\to \bar{W}^{\otimes 2},\ u\otimes v\mapsto v\otimes u,$$

whose subspace of invariants is the symmetric square $\mathbf{S}^2_{\bar{k}}\bar{W}$ of $\bar{W}$ and the subspace of anti-invariants is the exterior square $\wedge^2_{\bar{k}}\bar{W}$. Clearly, $\bar{\delta}$ commutes with the action of $\bar{\mathfrak{g}}$; in particular, both $\mathbf{S}^2_{\bar{k}}\bar{W}$ and $\wedge^2_{\bar{k}}\bar{W}$ are $\bar{\mathfrak{g}}$-invariant subspaces of $\bar{W}^{\otimes 2}$.

Let us choose an *order* on Σ_K. Let $\Sigma_{K,2}$ be the set of all two-element subsets B of Σ_K with

$$B = \{\sigma,\tau\};\ \sigma,\tau\in\Sigma_K;\ \sigma<\tau. \tag{55}$$

Let us consider the following decomposition of the $\bar{\mathfrak{g}}$-module $\bar{W}^{\otimes 2}$ into a direct sum of $\bar{\delta}$-invariant $\bar{\mathfrak{g}}$-submodules

$$\bar{W}^{\otimes 2} = \Big(\oplus_{\sigma\in\Sigma_K}\bar{W}_\sigma\otimes_{\bar{k}}\bar{W}_\sigma\Big)\oplus\Big(\oplus_{B=\{\sigma,\tau\}\in\Sigma_{K,2}}(\bar{W}_\sigma\otimes_{\bar{k}}\bar{W}_\tau)\oplus(\bar{W}_\tau\otimes_{\bar{k}}\bar{W}_\sigma)\Big). \tag{56}$$

Clearly, the action of the Lie algebra $\bar{\mathfrak{g}}$ on the tensor product $\bar{W}_\sigma\otimes_{\bar{k}}\bar{W}_\sigma$ factors through

$$\bar{\mathfrak{g}}_\sigma = \mathrm{sl}(\bar{W}_\sigma)$$

while the action of $\bar{\mathfrak{g}}$ on

$$\bar{W}^{(B)} := \left(\bar{W}_\sigma \otimes_{\bar{k}} \bar{W}_\tau\right) \oplus \left(\bar{W}_\tau \otimes_{\bar{k}} \bar{W}_\sigma\right)$$

factors through

$$\bar{\mathfrak{g}}_{(B)} := \bar{\mathfrak{g}}_{\sigma,\tau} = \mathrm{sl}(\bar{W}_\sigma) \oplus \mathrm{sl}(\bar{W}_\tau) \text{ with } B = \{\sigma, \tau\}.$$

We have

$$\bar{W}_\sigma \otimes_{\bar{k}} \bar{W}_\sigma = \mathbf{S}^2_{\bar{k}} \bar{W}_\sigma \oplus \wedge^2_{\bar{k}} \bar{W}_\sigma$$

where first summand is a simple $\bar{\mathfrak{g}}_\sigma$-module that lies in $\mathbf{S}^2_{\bar{k}}\bar{W}$ while the action of $\bar{\mathfrak{g}}_\sigma$ (and therefore of $\bar{\mathfrak{g}}$) on the second one is *trivial*.

Both $\bar{\mathfrak{g}}_{(B)} = \bar{\mathfrak{g}}_{\sigma,\tau}$-modules $\bar{W}_\sigma \otimes_{\bar{k}} \bar{W}_\tau$ and $\bar{W}_\tau \otimes_{\bar{k}} \bar{W}_\sigma$ are faithful simple; in addition, they are mutually isomorphic. Let us split $\bar{W}^{(B)}$ into a direct sum

$$\bar{W}^{(B)} = \bar{W}^{(B)}_+ \oplus \bar{W}^{(B)}_-$$

of the subspaces $\bar{W}^{(B)}_+$ of $\bar{\delta}$-invariants and $\bar{W}^{(B)}_-$ of $\bar{\delta}$-anti-invariants. Clearly, both subspaces are nonzero $\bar{\mathfrak{g}}_{\sigma,\tau}$-invariant subspaces and therefore are *non-trivial simple* $\bar{\mathfrak{g}}_{\sigma,\tau}$-modules that are isomorphic to

$$\bar{W}_\sigma \otimes_{\bar{k}} \bar{W}_\tau \cong \bar{W}_\tau \otimes_{\bar{k}} \bar{W}_\sigma.$$

The last sentence remains true if we replace "$\bar{\mathfrak{g}}_{\sigma,\tau}$-modules" by "$\bar{\mathfrak{g}}$-modules". Obviously,

$$\bar{W}^{(B)}_+ \subset \mathbf{S}^2_{\bar{k}}\bar{W}, \ \bar{W}^{(B)}_- \subset \wedge^2_{\bar{k}}\bar{W}.$$

It follows from (56) that the $\bar{\mathfrak{g}}$-module $\wedge^2_{\bar{k}}\bar{W}$ splits into a direct sum of the trivial $\bar{\mathfrak{g}}$-module $\oplus_{\sigma\in\Sigma_K} \wedge^2_{\bar{k}} \bar{W}_\sigma$ and a direct sum of nontrivial mutually non-isomorphic simple $\bar{\mathfrak{g}}$-modules $\oplus_{B\in\Sigma_{K,2}}\bar{W}^B_-$. So,

$$\mathbf{S}^2_{\bar{k}}\bar{W} = \left(\oplus_{\sigma\in\Sigma_K}\mathbf{S}^2_{\bar{k}}\bar{W}_\sigma\right) \oplus \left(\oplus_{B\in\Sigma_{K,2}}\bar{W}^{(B)}_+\right); \tag{57}$$

$$\wedge^2_{\bar{k}}\bar{W} = \left(\oplus_{\sigma\in\Sigma_K} \wedge^2_{\bar{k}} \bar{W}_\sigma\right) \oplus \left(\oplus_{B\in\Sigma_{K,2}}\bar{W}^{(B)}_-\right). \tag{58}$$

Clearly,

$$\bar{W}^{(0)}_- := \oplus_{\sigma\in\Sigma_K} \wedge^2_{\bar{k}} \bar{W}_\sigma$$

coincides with the subspace of all $\bar{\mathfrak{g}}$-invariants in $\wedge^2_{\bar{k}}\bar{W}$.

Notice that $\mathrm{Gal}(k)$ acts naturally on both Σ_K and $\Sigma_{K,2}$ in such a way that for all $s \in \mathrm{Gal}(K)$

$$s\left(\wedge^2_{\bar{k}}\bar{W}_\sigma\right) = \wedge^2_{\bar{k}}\bar{W}_{s\sigma}, \ s\left(\bar{W}^{(B)}_-\right) = \bar{W}^{(sB)}_- \tag{59}$$

for all $s \in \mathrm{Gal}(k)$; it follows that

$$s\bar{W}^{(0)}_- = \bar{W}^{(0)}_-.$$

It is also clear that if we put

$$\bar{U} := \oplus_{B\in\Sigma_{K,2}}\bar{W}^{(B)}_- \subset \wedge^2_{\bar{k}}W. \tag{60}$$

then $s\bar{U} = \bar{U}$ for all $s \in \mathrm{Gal}(k)$. This implies that both $\bar{W}_{-}^{(0)}$ and $\bar{U}$ are defined over k, i.e., there are vector k-subspaces $W_{-1}^{(0)}$ and U of $\wedge_k^2 W$ such that

$$\bar{W}_{-}^{(0)} = W_{-}^{(0)} \otimes_k \bar{k},\ \ \bar{U} = U \otimes_k \bar{k}.$$

It follows from (58) and (60) that

$$\wedge_k^2 W = W_{-}^{(0)} \oplus U. \tag{61}$$

The $\bar{\mathfrak{g}}$-invariance of both $\bar{k}$-vector subspaces $\bar{W}_{-}^{(0)}$ and $\bar{U}$ implies that both k-vector subspaces $W_{-}^{(0)}$ and U are $\mathfrak{g}$-submodules of $\wedge_k^2 W$. It is also clear that $W_{-}^{(0)}$ coincides with the subspace $\left(\wedge_k^2 W\right)^{\mathfrak{g}}$ of all $\mathfrak{g}$-invariants in $\wedge_k^2 W$. Combining this with (61), we obtain that the property (i) of our Lemma is equivalent to the simplicity of the $\mathfrak{g}$-module U.

Let O be a $\mathrm{Gal}(k)$-*orbit* in $\Sigma_{K,2}$. Let us consider the corresponding $\bar{\mathfrak{g}}$-*submodule* of $\bar{U}$ defined by

$$\bar{U}^O = \sum_{B \in O} \bar{W}_{-}^{(B)}. \tag{62}$$

Clearly, $s\bar{U}^O = \bar{U}^O$ for all $s \in \mathrm{Gal}(K)$. This means that $\bar{U}^O$ is defined over k, i.e., there is a $\mathfrak{g}$-submodule U^O of U such that

$$\bar{U}^O = U^O \otimes_k \bar{k}.$$

Since all the summands in the RHS of (62) are mutually non-isomorphic simple $\bar{\mathfrak{g}}$-modules that (in light of (59)) are permuted transitively by $\mathrm{Gal}(k)$, we conclude that U^O is a *simple* $\mathfrak{g}$-submodule of U. Clearly, $U^O = U$ if and only if $O = \Sigma_{K,2}$, i.e., if and only if the action of $\mathrm{Gal}(k)$ on $\Sigma_{K,2}$ is *transitive*. This implies that the $\mathfrak{g}$-module U is simple if and only if the action of $\mathrm{Gal}(k)$ on $\Sigma_{K,2}$ is *transitive*. It follows that conditions (i) and (iii) of our Lemma are equivalent. We have already seen that conditions (ii) and (iii) are equivalent. This ends the proof.

□

Theorem 6.7. *Let E be a number field of degree $g > 2$. Then there exists a simple g-dimensional complex torus $T = V/\Lambda$ such that*

$$\mathrm{End}^0(T) = E,\ \ \mathrm{Hdg}(T) = \mathrm{Res}_{E/\mathbb{Q}}\mathrm{SL}((\mathrm{H}_1(T,\mathbb{Q})/E)\,.$$

In particular, T is 2-simple if and only if E is almost doubly transitive.

Proof. Let us consider the matrix

$$J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{Q}) \subset \mathrm{Mat}_2(E) \subset \mathrm{Mat}_2(E_{\mathbb{R}})$$

where $E_{\mathbb{R}} := E \otimes_{\mathbb{Q}} \mathbb{R}$ is the realification of E. By [20, Prop. 2.8 on p. 19], there is $u \in \mathrm{Mat}_2(E_{\mathbb{R}})$ such that

$$J = \exp(u) J_0 \exp(-u) = \exp(u) J_0 \exp(u)^{-1} \in \mathrm{Mat}_2(E_{\mathbb{R}})$$

enjoys the following property. If D is a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_2(E)$ such that $D_{\mathbb{R}} = D \otimes_{\mathbb{Q}}$ contains J then $D = \mathrm{Mat}_2(E)$. Notice that

$$J_0^2 = -1, \quad J_0 \in \mathrm{sl}_2(E_{\mathbb{R}}) \subset \mathrm{Mat}_2(E_{\mathbb{R}})$$

It follows that

$$J^2 = -1, \quad J \in \mathrm{sl}_2(E_{\mathbb{R}})\mathrm{Mat}_2(E_{\mathbb{R}}).$$

Let $\mathfrak{g}$ be the smallest $\mathbb{Q}$-Lie subalgebra of $\mathrm{Mat}_2(E)$ such that its realification

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathrm{Mat}_2(E) \otimes_{\mathbb{Q}} \mathbb{R} = \mathrm{Mat}_2(E_{\mathbb{R}})$$

contains J. Clearly,

$$\mathfrak{g} \subset \mathrm{sl}_2(E) \tag{63}$$

and the $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_2(E)$ generated by $\mathfrak{g}$ coincides with $\mathrm{Mat}_2(E)$. It makes the $2g$-dimensional $\mathbb{Q}$-vector space

$$E^2 = E \oplus E$$

a faithful simple $\mathfrak{g}$-module such that the centralizer of $\mathfrak{g}$ in $\mathrm{End}_{\mathbb{Q}}(E^2)$ coincides with E. This implies that $\mathfrak{g}$ is a reductive $\mathbb{Q}$-Lie algebra and its center lies in E. In light of (63), this center is $\{0\}$, i.e., $\mathfrak{g}$ is a semisimple $\mathbb{Q}$-Lie algebra. Applying Lemma 6.2 to

$$k = \mathbb{Q}, \ K = E, \ W = E^2,$$

we conclude that

$$\mathfrak{g} = \mathrm{sl}_2(E). \tag{64}$$

Now we are ready to construct the desired complex torus T. The operator J provides the structure of a complex vector space on

$$V := E^2 \otimes_{\mathbb{Q}} \mathbb{R} = E_{\mathbb{R}}^2 = E_{\mathbb{R}} \oplus E_{\mathbb{R}}$$

such that $J \in \mathrm{End}_{\mathbb{R}}(V)$ defines multiplication by $\mathbf{i}$. Pick any $\mathbb{Z}$-lattice of rank $2g$ in E^2 and put $T := V/\Lambda$. One may naturally identify $\Lambda \otimes \mathbb{Q}$ with E^2. In light of Theorem 5.2, the $\mathbb{Q}$-Lie algebra hdg_T coincides with $\mathfrak{g}$, i.e., $\mathrm{hdg}_T =_2 (E)$. It follows that $\mathrm{Hdg}(T) = \mathrm{Res}_{E/\mathbb{Q}}\mathrm{SL}((\mathrm{H}_1(T, \mathbb{Q})/E)$, which ends the proof of the forst assertion of our Theorem. Now the second one follows from Theorem 6.3.

□

Proof of Theorem1.9. The first assertion follows readily from Theorem 6.7 combined with Theorem 4.5 applied to $n = g$.

In order to prove the second assertion, one should take

$$\mathbf{s} = g - d - 1 \geq 0, \quad \mathbf{r} = g - 2\mathbf{s} = g - 2(g - d - 1) = 2(d + 1) - g \geq 0.$$

□

7. Semi-linear algebra

This section contains auxiliary results that will be used for the study of Hodge groups of 2-simple tori without nontrivial endomorphisms. In what follows k stands for a field of characteristic 0 and K for an overfield of k such that the automorphism group $\mathrm{Aut}(K/k)$ of k-linear automorphisms of K enjoys the following property.

The subfield $K^{\mathrm{Aut}(K/k)}$ of $\mathrm{Aut}(K/k)$-invariants coincides with k. (This property holds if K is an algebraically closed field.)

Definition 7.1. Let $\mathcal{V}$ be a finite-dimensional vector space over K and $\sigma \in \mathrm{Aut}(K/k)$. Then the finite-dimensional vector space ${}^{\sigma}V$ over K is defined as follows. Viewed as an additive group, ${}^{\sigma}V$ coincides with V but multiplication by elements $a \in K$ is defined in ${}^{\sigma}V$ by the formula

$$a, v \mapsto \sigma^{-1}(a)v.$$

Clearly,

$$\dim_K(\mathcal{V}) = \dim_K({}^{\sigma}\mathcal{V}).$$

Remark 7.2. (1) If $x \in \mathrm{End}_K(\mathcal{V})$ is a K-linear operator in $\mathcal{V}$ then

$$x(\sigma^{-1}(a)v) = \sigma^{-1}(a)x(v) \quad \forall a \in K, v \in^{\sigma} \mathcal{V}.$$

In other words, one may view x as a K-linear operator in $\mathcal{V}$ that we denote by ${}^{\sigma}\mathrm{id}(x) \in \mathrm{End}_K({}^{\sigma}\mathcal{V})$.

(2) Let $m := \dim_K(\mathcal{V}) > 0$, and $\{e_1, \dots, e_m\}$ be a basis of $\mathcal{V}$. Then one may view $\{e_1, \dots, e_m\}$ as a basis of ${}^{\sigma}\mathcal{V}$.

If $A = (a_{ij})_{i,j=1}^{m}$ is the matrix of $x \in \mathrm{End}_K(\mathcal{V})$ with respect to $\{e_1, \dots, e_m\}$ then obviously $\sigma(A) = (\sigma(a_{ij}))_{i,j=1}^{m}$ is the matrix of ${}^{\sigma}x \in \mathrm{End}_K(\mathcal{V})$ with respect to $\{e_1, \dots, e_m\}$.

Lemma 7.3. *The formula*

$${}^{\sigma}\mathrm{id} : \mathrm{End}_K(\mathcal{V}) \to \mathrm{End}_K({}^{\sigma}\mathcal{V}), \quad x \mapsto \{v \mapsto x(v)\} \, \forall x \in \mathrm{End}_K(\mathcal{V}), \; v \in {}^{\sigma}\mathcal{V} = \mathcal{V}$$

defines a ring isomorphism that enjoys the following properties.

(i) ${}^{\sigma}\mathrm{id}(ax) = \sigma(a) \cdot {}^{\sigma}\mathrm{id}(x) \quad \forall a \in K, x \in \mathrm{End}_K(\mathcal{V})$.

(ii) *Let*

$$\mathcal{P}_{x,\min}(t), \; \mathcal{P}_{x,\mathrm{char}}(t) \in K[t]$$

be the minimal and characteristic polynomials of x respectively.

Then the minimal and characteristic polynomials of ${}^{\sigma}\mathrm{id}(x)$ coincide with $\sigma(\mathcal{P}_{x,\min}(t))$ and $\sigma(\mathcal{P}_{x,\mathrm{char}}(t))$ respectively.

(iii) *If $a \in K$ is the trace of $x \in \mathrm{End}_K(\mathcal{V})$ then $\sigma(a)$ is the trace of ${}^{\sigma}\mathrm{id}(x) \in \mathrm{End}_K({}^{\sigma}\mathcal{V})$.*

Proof. (i) is obvious. Both assertions (ii) and (iii) follow from Remark 7.2. □

Let $\mathcal{V}_0$ be a finite-dimensional k-vector space and

$$\mathcal{V} := \mathbf{T}_{k,K}(\mathcal{V}) = \mathcal{V}_0 \otimes_k K$$

the corresponding K-vector space endowed by the following semi-linear action of $\mathrm{Aut}(K/k)$.

$$\sigma(v_0 \otimes a) = v_0 \otimes \sigma(a) \quad \forall a \in K, v_0 \in \mathcal{V}_0.$$

We will identify $\mathcal{V}_0$ with the k-subspace

$$\mathcal{V}_0 \otimes 1 = \{v_0 \otimes 1 \mid v_0 \in \mathcal{V}_0\} \subset \mathcal{V}_0 \otimes_k K = \mathcal{V}.$$

Clearly, the k-vector subspace $\mathcal{V}_0 = \mathcal{V}_0 \otimes 1$ coincides with the k-vector subspace $\mathcal{V}^{\mathrm{Aut}(K/k)}$ of $\mathrm{Aut}(K/k)$-invariants.

The next asertion is probably known but I was unable to find a reference.

Lemma 7.4. *Let $\mathcal{W}$ be a K-vector subspace of $\mathcal{V}$. Then the following conditions are equivalent.*

(i) *$\mathcal{V}$ is $\mathrm{Aut}(K/k)$-invariant.*

(ii) *There exists a k-vector subspace $\mathcal{W}_0$ of $\mathcal{V}_0$ such that*

$$\mathcal{W} = \mathbf{T}_{k,K}(\mathcal{W}_0) = \mathcal{W}_0 \otimes_k K = \{w_0 \otimes a \mid w_0 \in \mathcal{W}_0,\ a \in K\} \subset \mathcal{V}_0 \otimes_k K = \mathbf{T}_{k,K}(\mathcal{V}_0).$$

If this is the case then $\mathcal{W}_0 = \mathcal{W} \cap \mathcal{V}_0$.

Proof. Let us put

$$m := \dim_k(\mathcal{V}_0) = \dim_K(\mathcal{V}); \quad n := \dim_K(\mathcal{W}) \le m.$$

If either $n = 0$ or $n = m$ then the desired result is obvious. So, we may and will assume that $0 < n < m$, i.e.,

$$1 \le n \le m - 1; \quad m \ge 2.$$

Let us fix a k-basis $\{e_1, \dots, e_m\}$ of $\mathcal{V}_0$, which we will view as a K-basis of $\mathcal{V}$.

Step 1. Assume that $n = 1$. Take a nonzero vector $w \in \mathcal{W}$. Then at least one of its coefficients with respect to our basis is not 0, i.e.,

$$w = \sum_{i=1}^{n} a_i e_i,\ a_i \in K$$

and $\exists j \in \{1, \dots, n\}$ such that $a_j \ne 0$. Replacing w by $a_j^{-1} w \in \mathcal{W}$, we may and will assume that $a_j = 1$. Then

$$K \cdot w = \mathcal{W} \ni \sigma(w) = \sum_{i=1}^{n} \sigma(a_i) e_i \quad \forall \sigma \in \mathrm{Aut}(K/k).$$

We have

$$\sigma(a_j) = \sigma(1) = 1 = a_j$$

and

$$\sigma(w) \in \mathcal{W} = K \cdot w.$$

Since both w and $\sigma(w)$ have the same (non-zero) jth coordinate, we conclude that $\sigma(w) = w$ for all σ, i.e., all the coefficients $a_i \in k$ and therefore $w \in \mathcal{V}_0$ and $\mathcal{W} = \mathcal{W}_0 \otimes_k K$ with

$$\mathcal{W}_0 = k \cdot w \in \mathcal{V}_0.$$

So, we have proven our assertion in the case of $n = 1$.

Step 2. Let us prove that the k-vector subspace of $\mathrm{Aut}(K)$-invariants

$$\tilde{\mathcal{W}}_0 := \mathcal{W}^{\mathrm{Aut}(K/k)} = \mathcal{W} \cap V_0 \tag{65}$$

is *not* $\{0\}$. Let us use induction by n and m. By Step 1, our assertion is true for $n = 1$. This implies its validity for for $m = 2$. So, we may assume that

$$1 < n < m > 2.$$

Let us consider the hyperplanes

$$\mathcal{H}_0 = \sum_{i=1}^{m-1} k \cdot e_i \subset V_0, \quad \mathcal{H} = \mathbf{T}_{k,K}(\mathcal{H}_0) = \mathcal{H}_0 \otimes_k K = \sum_{i=1}^{m-1} K \cdot e_i \subset \mathcal{V}_0.$$

Clearly, the intersection

$$\mathcal{W}_H = \mathcal{W} \cap \mathcal{H} \subset \mathcal{H}$$

is an $\mathrm{Aut}(K/k)$-invariant subspace of both $\mathcal{W}$ and $\mathcal{H}$. Clearly, either $\mathcal{W}_H = \mathcal{W}$ or $\dim_K(\mathcal{W}_H) = n-1 > 0$. In the former case $\mathcal{W}_H$ is $\mathrm{Aut}(K/k)$-invariant subspace of the $(m-1)$-dimensional K-vector space $\mathcal{H} = \mathcal{H}_0 \otimes_k K$. Now the induction assumption for m (applied to $\mathcal{H}$ instead of $\mathcal{V}$) implies that

$$\tilde{\mathcal{W}}_0 = \mathcal{W}^{\mathrm{Aut}(K/k)} = (\mathcal{W}_H)^{\mathrm{Aut}(K/k)} \neq 0.$$

In the latter case, the induction assumption for n applied to $\mathcal{W}_H$ implies that $(\mathcal{W}_H)^{\mathrm{Aut}(K/k)} \neq 0$. Since $\mathcal{W} \supset \mathcal{W}_H$, we get $\mathcal{W}^{\mathrm{Aut}(K/k)} \neq 0$, which ends the proof.

Step 3 We have

$$\mathbf{T}_{k,K}(\tilde{\mathcal{W}}_0) = \tilde{\mathcal{W}}_0 \otimes_k K \subset \mathcal{W}.$$

This implies that

$$n_0 = \dim_k(\tilde{\mathcal{W}}_0) \leq \dim_K(W) = n.$$

The assertion of our Lemma actually means that the equality holds. By Step 2, $n_0 > 0$. Suppose that $n_0 < n$ and choose in $\mathcal{V}_0$ a $(n - n_0)$-dimensional k-vector subspace $\mathcal{U}_0$ such that $\mathcal{U}_0 \cap \tilde{\mathcal{W}}_0 = \{0\}$ (i.e., $\mathcal{V}_0 = \mathcal{W}_0 \oplus \mathcal{U}_0$). Let us consider the $(m - n_0)$-dimensional K-vector subspace

$$\mathcal{U} = \mathbf{T}_{k,K}(\mathcal{U}_0) = \mathcal{U}_0 \otimes_k K \subset \mathcal{V}.$$

Clearly,

$$\mathcal{V} = \mathbf{T}_{k,K}(\tilde{\mathcal{W}}_0) \oplus \mathbf{T}_{k,K}(\tilde{\mathcal{U}}_0) = \mathbf{T}_{k,K}(\tilde{\mathcal{W}}_0) \oplus \mathcal{U},$$

and therefore

$$\mathcal{U} \cap \mathbf{T}_{k,K}(\tilde{\mathcal{W}}_0) = \{0\}.$$

Dimension arguments imply that $\mathcal{U}_1 := \mathcal{U} \cap \mathcal{W}$ is a nonzero $\mathrm{Aut}(K/k)$-invariant K-vector subspace of $\mathcal{V}$. By Step 2, the subspace $\tilde{\mathcal{U}}_1 := \mathcal{U}_1^{\mathrm{Aut}(K/k)} \neq \{0\}$; on the other hand, $\tilde{\mathcal{U}}_1$ obviously lies in $\mathcal{W}^{\mathrm{Aut}(K/k)}$ but meets the latter only at $\{0\}$. The obtained contradiction proves that $n_0 = n_1$, which ends the proof. □

Remark 7.5. Let us consider the dual vector spaces

$$V_0^* = \mathrm{Hom}_k(V_0, k), \quad V^* = \mathrm{Hom}_K(V, K).$$

Obviously, the restriction map

$$\mathrm{res}_{K,k} : V^* = \mathrm{Hom}_K(V, K) = \mathrm{Hom}_K(V_0 \otimes_k K, K) \to \mathrm{Hom}_k(V_0, K), \phi \mapsto \{v_0 \mapsto \phi(v_0 \otimes 1)\}$$

is a $\mathrm{Aut}(K/k)$-equivariant isomorphism of K-vector spaces where the actions of $\mathrm{Aut}(K/k)$ are defined as follows.

$$\sigma : \phi \mapsto \sigma \circ \phi \sigma^{-1} \quad \forall \phi \in \mathrm{Hom}_K(V, K),$$

$$\sigma : \phi_0 \mapsto \{v_0 \mapsto \sigma(\phi_0(v_0)) \quad \forall \phi_0 \in \mathrm{Hom}_k(V_0, K)$$

for all $\sigma \in \mathrm{Aut}(K/k)$. As usual, we have

$$\sigma(\phi)(\sigma(v)) = \sigma(\phi(v)) \quad \forall v \in V, \phi \in V^*, \sigma \in \mathrm{Aut}(K/k).$$

7.6. What is discussed in this section (and in Theorem 7.13 below) is pretty well known in the case of $k = \mathbb{R}$ and $K = \mathbb{C}$, see [19].

Let $\mathfrak{u}$ be a Lie k-algebra of finite dimension and

$$\bar{\mathfrak{u}} := \mathfrak{u} \otimes_k K$$

the corresponding finite-dimensional Lie K-algebra. Let

$$\rho : \mathfrak{u} \to \mathrm{End}_K(\mathcal{V})$$

be a homomorphism of Lie k-algebras. Extending ρ by K-linearity, we get the homomorphism of Lie K-algebras

$$\bar{\rho} : \bar{\mathfrak{u}} \to \mathrm{End}_K(\mathcal{V}),$$

which coincides with ρ on

$$\mathfrak{u} = \mathfrak{u} \otimes 1 \subset \mathfrak{u} \otimes_k K = \bar{\mathfrak{u}}.$$

Thus $\bar{\rho}$ endows $\mathcal{V}$ with the structure of a $\bar{\mathfrak{u}}$-module.

If $\sigma \in \mathrm{Aut}(K/k)$ then we may define the composition

$${}^{\sigma}\rho : \mathfrak{u} \xrightarrow{\rho} \mathrm{End}_K(\mathcal{V}) \xrightarrow{{}^{\sigma}\mathrm{id}} \mathrm{End}_K({}^{\sigma}\mathcal{V}),$$

which is a homomorphism of k-Lie algebras. Then the corresponding homomorphism of Lie K-algebras

$$\overline{{}^{\sigma}\rho} : \bar{\mathfrak{u}} \to \mathrm{End}_K({}^{\sigma}\mathcal{V}),$$

provides ${}^{\sigma}\mathcal{V}$ with the structure of a $\bar{\mathfrak{u}}$-module.

Remark 7.7. Let $\mathcal{W}$ be a K-vector subspace of $\mathcal{V}$. Clearly, $\mathcal{W}$ is $\mathfrak{u}$-invariant if and only if it is $\bar{\mathfrak{u}}$-invariant. It follows easily that $\mathcal{W}$ is a $\bar{\mathfrak{u}}$-submodule of $\mathcal{V}$ if and only if it is a $\bar{\mathfrak{u}}$-submodule of ${}^{\sigma}\mathcal{V}$. This implies that the $\bar{\mathfrak{u}}$-module $\mathcal{V}$ is *simple* if and only if the $\bar{\mathfrak{u}}$-module ${}^{\sigma}\mathcal{V}$ is *simple*.

7.8. Let $\mathcal{V}_0$ be a finite dimensional k-vector space endowed with a homomorphism of k-Lie algebras

$$\rho_0 : \mathfrak{u} \to \mathrm{End}_k(\mathcal{V}_0)$$

that endowed $\mathcal{V}_0$ with the structure of a $\mathfrak{u}$-module. Let us consider the K-vector space $V := V_0 \otimes_k K$ and the obvious homomorphism of k-Lie algebras

$$\rho_0 \otimes 1 : \mathfrak{u} = \mathfrak{u} \otimes 1 \to \mathrm{End}_k(\mathcal{V}_0) \otimes_k K = \mathrm{End}_K(\mathcal{V}_0 \otimes K) = \mathrm{End}_K(\mathcal{V}).$$

obtained from ρ_0 by extension of scalars.

Let $\mathcal{W}$ be a $\mathfrak{u}$-invariant K-vector subspace of $\mathcal{V}$. If $\sigma \in \mathrm{Aut}(K/k)$ then obviously $\sigma(\mathcal{W})$ is also a $\mathfrak{u}$-invariant K-vector subspace of $\mathcal{V}$. Clearly, both $\mathcal{W}$ and $\sigma(\mathcal{W})$ carry the natural structure of modules over the Lie K-algebra

$$\bar{\mathfrak{u}} = \mathfrak{u} \otimes_k K.$$

We will need the following assertion.

Proposition 7.9. *The* $\bar{\mathfrak{u}}$*-modules* $\sigma^{-1}(\mathcal{W})$ *and* ${}^{\sigma}\mathcal{W}$ *are isomorphic.*

Proof. It suffices to check that the $\mathfrak{u}$-modules $\sigma(\mathcal{W})$ and ${}^{\sigma}\mathcal{V}$ are isomorphic. Let us consider the k-linear isomorphism

$$\Pi : \sigma(\mathcal{W}) \to {}^{\sigma}\mathcal{W}, \quad \sigma(w) \mapsto \sigma(w) \ \forall w \in \mathcal{W}.$$

Actually, Π is K-linear, because for all $a \in K, w \in \mathcal{W}$ the vector

$$a\sigma(w) = \sigma^{-1}(a)w) \in \mathcal{W}$$

(recall that in ${}^{\sigma}\mathcal{W}$ multiplication by a is defined as multiplication by $\sigma^{-1}(a)$). Clearly, the actions of $\mathfrak{u}$ and $\mathrm{Aut}(K/k)$ on $\mathcal{V}$ do commute. This implies that

$$\Pi \circ \sigma = \sigma \circ \Pi \quad \forall \sigma \in \mathrm{Aut}(K/k).$$

It follows that Π is an isomorphism of $\mathfrak{u}$-modules, which ends the proof. □

Till the end of this section we assume that K is *algebraically closed* (e.g., K is an algebraic closure of k). Let $\mathfrak{g}$ be a nonzero semisimple finite-dimensional Lie algebra over k of rank l and consider the corresponding semisimple finite-dimensional Lie algebra

$$\bar{\mathfrak{g}} := \mathfrak{g} \otimes_k K$$

over K. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ then

$$\dim_k(\mathfrak{h}) = l.$$

We write

$$\bar{\mathfrak{h}} := \mathfrak{h} \otimes_k K \subset \mathfrak{g} \otimes_k K = \bar{\mathfrak{g}}$$

for the corresponding Cartan subalgebra of $\bar{\mathfrak{g}}$; we have

$$\dim_K(\bar{\mathfrak{h}}) = l.$$

As usual, let us consider the dual K-vector space

$$\bar{\mathfrak{h}}^* := \mathrm{Hom}_K(\bar{\mathfrak{h}}, K)$$

of K-dimension l endowed by the action of $\mathrm{Aut}(K/k)$ defined by the formula

$$\sigma \mapsto \{\phi :\mapsto \sigma \circ \phi \circ \sigma^{-1}\} \quad \forall \phi : \bar{\mathfrak{h}} \to K$$

and $\sigma \in \mathrm{Aut}(K/k)$. As above, the restriction map

$$\mathrm{res}_{K,k} : \bar{\mathfrak{h}}^* := \mathrm{Hom}_K(\bar{\mathfrak{h}}, K) \to \mathrm{Hom}_k(\mathfrak{h}, K)$$

is an isomorphism of K-vector spaces. (Here as above we identify $\mathfrak{h}$ with

$$\mathfrak{h} \otimes 1 \subset \mathfrak{h} \otimes_k K = \bar{\mathfrak{h}}.)$$

The inverse map

$$\mathrm{res}_{K,k}^{-1} : \mathrm{Hom}_k(\mathfrak{h}, K) \to \mathrm{Hom}_K(\bar{\mathfrak{h}}, K) = \mathrm{Hom}_K(\mathfrak{h} \otimes_k K, K),$$

is described explicitly by the formula

$$\mu \mapsto \{h \otimes a \mapsto a \cdot \mu(h)\} \quad \forall h \in \mathfrak{h}, a \in K.$$

Let $R \subset \bar{\mathfrak{h}}^*$ be the *root system* of $(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$ [9]. By definition, R consists of all *nonzero* $\alpha \in \bar{\mathfrak{h}}^*$ such that

$$\bar{\mathfrak{g}}_\alpha := \{x \in \bar{\mathfrak{g}} \mid [H, x] = \alpha(H)x \ \forall H \in \bar{\mathfrak{h}}\} \neq \{0\}.$$

Clearly,

$$\bar{\mathfrak{g}}_\alpha = \{x \in \bar{\mathfrak{g}} \mid [H, x] = \alpha(H)x \ \forall H \in \mathfrak{h}\}$$

and therefore

$$\sigma(\bar{\mathfrak{g}}_\alpha) = \bar{\mathfrak{g}}_{\sigma(\alpha)} \quad \forall \sigma \in \mathrm{Aut}(K/k).$$

It follows that the subset R of $\bar{\mathfrak{h}}^*$ is $\mathrm{Aut}(K/k)$-invariant. We write

$$\mathrm{W}(R) \subset \mathrm{Aut}_K(\bar{\mathfrak{h}}^*)$$

for the *Weyl group* of the root system R. Notice that $\mathrm{W}(R)$ permutes elements of R.

Let us choose a *basis* (a *simple root system*) B of R. The l-element set B is a basis of the K-vector space $\bar{\mathfrak{h}}^*$. Every root $\alpha \in R$ is a linear combination of elements of B with integer coefficients; in addition, the nonzero coefficients are either all positive or all negative. (Actually, these properties characterize a *basis* of R.) This implies the equality of abelian subgroups

$$\mathbb{Z} \cdot R := \sum_{\alpha \in R} \mathbb{Z} \cdot \alpha = \sum_{\beta \in B} \mathbb{Z} \cdot \beta; \tag{66}$$

$\mathbb{Z} \cdot R$ is a free abelian group of rank l that is a $\mathrm{W}(R)$-invariant subgroup of $\bar{\mathfrak{h}}^*$.

The set B does *not* have to be $\mathrm{Aut}(K/k)$-invariant. However, if $\sigma \in \mathrm{Aut}(K/k)$ then $\sigma(B)$ is a basis of R as well. Since the Weyl group $\mathrm{W}(R)$ acts transitively on the set of all simple root systems of R, there is $w_\sigma \in \mathrm{W}(R)$ such that

$$w_\sigma(\sigma(B)) = B$$

(compare with [22, p. 203]). In particular,

$$s_\sigma := w_\sigma \circ \sigma \in \mathrm{Aut}_k(\bar{\mathfrak{h}}^*)$$

permutes elements of B. It is also clear that s_σ permutes elements of R. Hence, $\mathbb{Z} \cdot R$ is s_σ-invariant.

7.10. Throughout this subsection we use the notation and constructions of Subsection 7.6 applied to

$$\mathfrak{u} = \mathfrak{g}, \quad \bar{\mathfrak{u}} = \bar{\mathfrak{g}}.$$

Let $\mathcal{V}$ be a nonzero finite-dimensional vector space over K endowed by the homomorphism of K-algebras

$$\bar{\mathfrak{g}} \to \mathrm{End}_K(\mathcal{V}), \tag{67}$$

which may be viewed (in the notation of Subsection 7.6) as $\bar{\rho}$ where

$$\rho : \mathfrak{g} \to \mathrm{End}_K(\mathcal{V})$$

is the restriction of the homomorphism (67) to $\mathfrak{g} = \mathfrak{g} \otimes 1$. The homomorphism $\bar{\rho}$ that appeared in (67) provides $\mathcal{V}$ with the structure of a $\bar{\mathfrak{g}}$-module. Let us assume that this module is *simple*.

Let us consider the set

$$\mathrm{Supp}(\mathcal{V}) \subset \bar{\mathfrak{h}}^*$$

of weights of the $\bar{\mathfrak{g}}$-module $\mathcal{V}$, i.e., $\mu \in \bar{\mathfrak{h}}^*$ lies in $\mathrm{Supp}(\mathcal{V})$ if and only if the *weight subspace*

$$\mathcal{V}_\mu := \{v \in \mathcal{V} \mid \rho(H)(x) = \mu(H)v \quad \forall H \in \bar{\mathfrak{h}}\} \neq \{0\}.$$

Then

$$\mathrm{Supp}(\mathcal{V}) \subset \mathbb{Q} \cdot R := \sum_{\alpha \in R} \mathbb{Q} \cdot \alpha =: \sum_{\beta \in B} \mathbb{Q} \cdot \beta \subset \bar{\mathfrak{h}}^*, \tag{68}$$

and there exists the *highest weigh*t λ of the $\mathfrak{g}$-module $\mathcal{V}$ that enjoys the following properties.

(i) $\lambda \in \mathrm{Supp}(\mathcal{V})$.
(ii) If $\mu \in \mathrm{Supp}(\mathcal{V})$ then $\lambda - \mu$ is a linear combination of elements of B with nonnegative integer coefficients.

Remark 7.11. It is well known that:

(i)
$$\mathrm{Supp}(\mathcal{V}) \subset \sum_{\beta \in B} \mathbb{Q} \cdot \beta = \mathbb{Q} \cdot R.$$

(ii) The subset $\mathrm{Supp}(\mathcal{V})$ is $\mathrm{W}(R)$-invariant.

Remark 7.12. It follows from the $\mathrm{W}(R)$-invariance of $\mathbb{Z} \cdot R$ (defined in (66)) that the l-dimensional $\mathbb{Q}$-vector (sub)space $\mathbb{Q} \cdot R$ is $\mathrm{W}(R)$-invariant.

Recall (Subsection 7.6) that one may attach to each $\sigma \in \mathrm{Aut}(K/k)$ the homomorphism of Lie K-algebras

$$\overline{^{\sigma}\rho} : \bar{\mathfrak{g}} \to \mathrm{End}_K(^{\sigma}\mathcal{V}),$$

and the corresponding $\bar{\mathfrak{g}}$-module $^{\sigma}\mathcal{V}$ is simple.

Theorem 7.13. *Suppose that λ is the dominant weight of a simple $\mathfrak{g}$-module $\mathcal{V}$ of finite dimension. If $\sigma \in \mathrm{Aut}(K/k)$ then $s_{\sigma^{-1}}(\lambda)$ is the dominant weight of the simple $\mathfrak{g}$-module $^{\sigma}\mathcal{V}$.*

Proof. First, notice that

$$\mathrm{Supp}(^{\sigma}\mathcal{V}) = \sigma(\mathrm{Supp}(\mathcal{V})).$$

Indeed, for any

$$H \in \mathfrak{h} \subset \bar{\mathfrak{h}},$$

the spectrum of the diagonalizable operator $\rho(H)$ in $\mathcal{V}$ is the collection $\{\mu(H) \mid \mu \in \mathrm{Supp}(\mathcal{V})\}$ (with multiplicities). In light of Lemma 7.3, the spectrum of the diagonalizable operator $^{\sigma}\mathrm{id} \circ \rho(H)$ in $^{\sigma}\mathcal{V}$ is the collection

$$\{\sigma(\mu(H)) \mid \mu \in \mathrm{Supp}(\mathcal{V})\}$$

(with multiplicities). More precisely, let $n = \dim(\mathcal{V}$ and $\{e_1, \dots, e_n\}$ be a common (weight) eigenbasis of all elements of $\bar{\mathfrak{h}}$ in $\mathcal{V}$, i.e., for each index $i \in \{1, \dots, n\}$ there is a weight

$$\mu_i \in \mathrm{Supp}(\mathcal{V}) \in \bar{\mathfrak{h}}^*$$

such that

$$\rho(H)(e_i) = \mu_i(H)e_i \ \forall H \in \bar{\mathfrak{h}}.$$

(Clearly, the collection $\{\mu_1, \dots, \mu_n\}$ coincides with $\mathrm{Supp}(\mathcal{V})$.) In light of Lemma 7.3, $\{e_1, \dots, e_n\}$ is a basis of $^{\sigma}\mathcal{V}$, and if $H \in \mathfrak{h}$, then $H = \sigma^{-1}H$ and

$$^{\sigma}\mathrm{id} \circ \rho(H)(e_i) = (\sigma^{-1})^{-1}(\mu_i(H))e_i = \sigma(\mu_i(\sigma^{-1}H))e_i = (\sigma(\mu_i))(H)(e_i). \tag{69}$$

Since $\bar{\mathfrak{h}} = \mathfrak{h} \otimes_k K$, we conclude that

$$\overline{^{\sigma}\rho}(H)(e_i) = (\sigma(\mu_i))(H)e_i \ \forall H \in \bar{\mathfrak{h}}. \tag{70}$$

In other words,

$$\mathrm{Supp}(^{\sigma}\mathcal{V}) = \{\sigma \circ \mu_i \mid i = 1, \dots, n\} = \{\sigma \circ \mu \mid \mu \in \mathrm{Supp}(\mathcal{V})\}.$$

Second, the $\mathrm{W}(R)$-invariance of $\mathrm{Supp}(^{\sigma}\mathcal{V})$ implies that

$$\mathrm{Supp}(^{\sigma}\mathcal{V}) = w_{\sigma} \circ \sigma(\mathrm{Supp}(\mathcal{V})) = (w_{\sigma} \circ \sigma)(\mathrm{Supp}(\mathcal{V})) = s_{\sigma}(\mathrm{Supp}(\mathcal{V})).$$

It follows that $\mathrm{Supp}(^{\sigma}\mathcal{V})$ contains $s_{\sigma}(\lambda)$, and all the other weights in $\mathrm{Supp}(^{\sigma}\mathcal{V})$ are of the form $s_{\sigma}(\mu)$ where $\lambda - \mu$ is is a linear combination of elements of B with nonnegative integer coefficients. Since s_{σ} permutes elements of B, the difference $s_{\sigma}(\lambda) - s_{\sigma}(\mu)$ is also a linear combination of elements of B with nonnegative integer coefficients. It follows that $s_{\sigma}(\lambda)$ is the dominant weight of the simple $\mathfrak{g}$-module $^{\sigma}\mathcal{V}$.

□

8. 2-Simple complex tori without nontrivial endomorphisms

Theorem 8.1. *Suppose that T is a 2-simple complex torus of dimension $g \ge 3$ with $\mathrm{End}^0(T) = \mathbb{Q}$. Assume also that $g \ne 10$ and $2g$ is not a power (e.g., g is odd) . Then $\mathrm{Hdg}(T)$ enjoys one of the following properties.*

(i) $\mathrm{Hdg}(T) = \mathrm{SL}(\Lambda_{\mathbb{Q}})$;

(ii) *There exists a nondegenerate symmetric $\mathbb{Q}$-bilinear form*

$$\Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}$$

such that $\mathrm{Hdg}(T)$ coincides with the corresponding special orthoginal group $\mathrm{SO}(\Lambda_{\mathbb{Q}})$.

(iii) *There exists a nondegenerate alternating $\mathbb{Q}$-bilinear form*

$$\Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}$$

such that $\mathrm{Hdg}(T)$ coincides with the corresponding symplectic group $\mathrm{Sp}(\Lambda_{\mathbb{Q}})$.

Proof. It follows from Corollary 5.12 that $\mathrm{hdg}_{T,\mathbb{C}}$ is a complex *simple* classical Lie algebra, whose natural faithful representation in $\Lambda_{\mathbb{C}}$ has a minuscule weight as the highest weight, thanks to Theorem 5.10. Since $2g = \dim_{\mathbb{C}}(\Lambda_{\mathbb{C}})$ is not a power of 2, one should exclude the cases when either $\mathrm{hdg}_{T,\mathbb{C}}$ is of type $\mathsf{B_l}$, or $\mathrm{hdg}_{T,\mathbb{C}}$ is of type $\mathsf{D_l}$ and $\Gamma_{\mathbb{C}}$ is one of its two semi-spinorial representations. Let us list the remaining cases.

(i) $\mathrm{hdg}_{T,\mathbb{C}}$ is of type $\mathsf{C_g}$ or $\mathsf{D_g}$, and there is a nondegenerate alternating or symmetric bilinear form on $\Lambda_{\mathbb{C}}$ such that $\mathrm{hdg}_{T,\mathbb{C}}$ coincides with the corresponding symplectic Lie algebra $\mathrm{sp}(\Lambda_{\mathbb{C}})$ or the corresponding orthogonal Lie algebra $\mathrm{so}(\Lambda_{\mathbb{C}})$.

(ii) $\mathrm{hdg}_{T,\mathbb{C}}$ is of type $\mathsf{A_l}$, i.e., $\mathrm{hdg}_{T,\mathbb{C}}$ may be identified with the Lie algebra $\mathrm{sl}(W)$ of a $(l+1)$-dimensional complex vector space W in such a way that the $\mathrm{sl}(W)$-module $\Lambda_{\mathbb{C}}$ is isomorphic to the jth exterior power $\wedge^j_{\mathbb{C}}(W)$ of W for some integer j with $1 \le j \le l$. We may assume that $1 < j < l$.

Let us handle the case (i). In this situation the $\mathrm{hdg}_{T,\mathbb{C}}$-module $\Lambda_{\mathbb{C}}$ is self-dual, which implies that there is a non-zero homomorphism between the $\mathrm{hdg}_{T,\mathbb{C}}$-module $\Lambda_{\mathbb{C}}$ and its dual. This, in turn, implies that there is a non-zero homomorphism between the $\mathrm{hdg}_{T,\mathbb{Q}}$-module $\Lambda_{\mathbb{Q}}$ and its dual. Now the simplicity of the hdg_T-module $\Lambda_{\mathbb{Q}}$ implies that $\Lambda_{\mathbb{Q}}$ and its dual are isomorphic, i.e., there is a nondegenerate hdg_T-invariant bilinear form

$$\Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}.$$

The absolutely simplicity of the $\mathrm{hdg}_{T,\mathbb{Q}}$-module $\Lambda_{\mathbb{Q}}$ implies that this form is unique (up to multiplication by a non-zero rational number) and therefore is alternating if hdg_T is of type $\mathsf{C_g}$ or symmetric if hdg_T is of type $\mathsf{D_g}$. Now the dimension arguments imply that $\mathrm{hdg}_T = \mathrm{sp}(\Lambda_{\mathbb{C}})$ in the former case and $\mathrm{hdg}_T = \mathrm{so}(\Lambda_{\mathbb{C}})$ in the latter case.

Let us handle the case (ii). We know that $T = V/\Lambda$ where the $\mathrm{hdg}_{T,\mathbb{C}} = \mathfrak{sl}(W)$-module $V_{\mathbb{C}}$ is isomorphic to $\wedge^j_{\mathbb{C}}(W)$.

If $l = 1$ then the inequality $1 < j < l = 1$ implies that this case does not occur.

If $l = 2$ then the inequality $1 < j < l = 2$ implies that $j = 1$ and $V_{\mathbb{C}}$ is isomorphic to W, which is a 3-dimensional complel vector space. Since 3 is an odd integer and the $\mathbb{C}$-dimension of $V_{\mathbb{C}}$ is even, this case also does not occur.

If $l = 3$ then the inequality $1 < j < l = 3$ implies that $j = 2$ and $V_{\mathbb{C}}$ is isomorphic to $\wedge^2_{\mathbb{C}} W$ where W is a 4-dimensional complex vector space and $\wedge^2_{\mathbb{C}} W$ is an irreducible 6-dimensional orthogonal representation of the Lie algebra $\mathfrak{sl}(W)$. This implies that the representation of $\mathrm{hdg}_{T,\mathbb{C}}$ in the 6-dimensional complex vector space $V_{\mathbb{C}}$ is orthogonal irreducible. It follows that

$$\dim_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) = \dim_{\mathbb{C}}(W) = 6$$

and hdg_T is a $\mathbb{Q}$-Lie subalgebra $\mathfrak{so}(\Lambda_{\mathbb{Q}})$ of the corresponding special orthogonal group $\mathrm{SO}(\Lambda_{\mathbb{Q}})$. It follows that the representation of hdg_T in the 6-dimensional $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}}$ is orthogonal irreducible and therefore

$$\dim_{\mathbb{Q}}(\mathrm{hdg}_T) \le \dim_{\mathbb{Q}}(\mathfrak{so}(\Lambda_{\mathbb{Q}}) = 15.$$

However,

$$\dim_{\mathbb{Q}}(\mathrm{hdg}_T) = \dim_{\mathbb{C}}(\mathrm{hdg}_{T,\mathbb{C}}) = \dim_{\mathbb{C}}(\mathfrak{sl}(W)) = 15.$$

This implies that hdg_T coincides with $\mathfrak{so}(\Lambda_{\mathbb{Q}})$, i.e., Hdg_T coincides with $\mathrm{SO}(\Lambda_{\mathbb{Q}})$.

So, we may and will assume that $l > 3$. Then there is an element $u \in \mathfrak{sl}(W)$ that acts on $V_{\mathbb{C}}$ as J. Since J is a nonzero semisimple linear operator in $V_{\mathbb{C}}$, the element u is also a semisimple (i.e., diagonalizable) nonzero linear operator in W. Let $\{e_1, \dots, e_{l+1}\}$ be an eigenbasis of W and $\{z_1, \dots, z_{l+1}\} \subset \mathbb{C}$ be the corresponding eigenvalues of u, i.e,

$$u(e_i) = z_i e_i \ i = 1, \dots, l+1$$

and the trace

$$\sum_{i=1}^{l+1} z_i = 0.$$

This implies that u has at least two distinct eigenvalues.

Then the collections of eigenvalues of J in $V_{\mathbb{C}} \cong \wedge^j_{\mathbb{C}}(W)$ listed with multiplicities coincides with

$$\{t_A := \sum_{i \in A} z_i\}_A$$

where A runs through all j-element subsets A of $\{1, \dots, l+1\}$. On the other hand, we know that the spectrum of J in $V_{\mathbb{C}}$ consists of two eigenvalues $\mathbf{i}$ and $-\mathbf{i}$, whose multiplicities coincide. It follows almost immediately that u has precisely two (distinct) eigenvalues, say, $\mathbf{a}$ and $\mathbf{b}$, and none of them is 0. Indeed, suppose that that the spectrum of u contains (at least) three eigenvalues, say, $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Reordering the eigenbasis if necessary, we may assume that

$$z_1 = \mathbf{a},\ z_2 = \mathbf{b},\ z_3 = \mathbf{c}.$$

Let B be any $(j-2)$-element subset of $\{4, \dots, l+1)$ Let us consider three distinct j-element subsets

$$A_1 = \{2,3\} \cup B,\ A_2 = \{1,3\} \cup B,\ A_3 = \{1,2\} \cup B$$

of $\{1, \dots, l+1\}$. If we put

$$C := \{1,2,3\} \cup B \subset \{1, \dots, l+1\}, \quad c := \sum_{i \in C} z_i \in \mathbb{C}$$

then we get three distinct eigenvalues

$$t_{A_1} = c - \mathbf{a},\ t_{A_2} = c - \mathbf{b},\ t_{A_3} = c - \mathbf{c}$$

of J, which do not exist. This proves the spectrum of u consists of precisely two eigenvalues, say, $\mathbf{a}, \mathbf{b} \in \mathbb{C}$. Since the trace of *nonzero* u is 0, both $\mathbf{a}$ and $\mathbf{b}$ are *not* zero.

Let p be the multiplicity of the eigenvalue $\mathbf{a}$ and q the multiplicity of the eigenvalue $\mathbf{b}$. Both p and q are positive integers, whose sum

$$p + q = l + 1 > 3 + 1 = 4.$$

Since u is traceless,

$$p\mathbf{a} + q\mathbf{b} = 0.$$

I claim that either $p = 1$ or $q = 1$. Indeed, suppose that

$$p \ge 2,\ q \ge 2.$$

Since $p + q > 4$, we may assume that $p \ge 3$. Notice also that all three complex numbers

$$2\mathbf{a},\ 2\mathbf{b},\ \mathbf{a} + \mathbf{b}$$

are distinct. Reordering the eigenbasis if necessary, we may assume that

$$z_1 = \mathbf{a},\ z_2 = \mathbf{a}, z_3 = \mathbf{a},\ z_l = \mathbf{b},\ z_{l+1} = \mathbf{b}$$

(recall that $l + 1 > 4$).

Let $\mathbf{B}$ be a $(j-2)$-element subset of the $(l-3)$-element subset of $\{3, 4, \dots, l-1\}$ and $b := \sum_{i \in \mathbf{B}} z_i$. Let us consider three distinct j-element subsets

$$A_1 = \{1,2\} \cup B,\ A_2 = \{l, l+1\} \cup B,\ A_3 = \{1, l\} \cup B$$

of $\{1, \dots, l+1\}$. Then we get three *distinct* eigenvalues

$$t_{A_1} = b + 2\mathbf{a},\ t_{A_2} = b + 2\mathbf{b},\ t_{A_3} = b + (\mathbf{a} + \mathbf{b})$$

of J, which could not be the case. The obtained contradiction proves that either $p = 1$ or $q = 1$.

Without loss of generality we may assume that $p = 1$. Reordering the eigenbasis if necessary, we may assume that $z_1 = \mathbf{a}$ and all other $z_i = \mathbf{b}$ (for all $i > 1$). It follows easily that the spectrum of J consists of two eigenvalues, namely, $j\mathbf{b}$ of multiplicity $\binom{l}{j}$ and $\mathbf{a}+(j-1)\mathbf{b}$ of multiplicity $\binom{l}{j-1}$. It follows that

$$\binom{l}{j}=\binom{l}{j-1},$$

i.e.,

$$\frac{l-j+1}{j}=1, \quad l-j+1=j, \quad l=2j-1.$$

It remains to put $m=j$ and we get that

$$l=2m-1,\ j=m,\ 2g=\binom{2m}{m}.$$

Since $2m-1=l>3$, we get

$$m\ge 3. \tag{71}$$

Now it is natural to look at the structure of the $\mathfrak{sl}(W)$-module $\wedge^2_{\mathbb{C}}(\wedge^m_{\mathbb{C}}(W))$. We are going to apply results of Section 7 with

$$k=\mathbb{Q},\ K=\mathbb{C},\ \mathrm{Aut}(K/k)=\mathrm{Aut}(\mathbb{C}),$$

$$\mathfrak{g}=\mathrm{hdg}_T,\ \bar{\mathfrak{g}}=\mathrm{hdg}_{T,\mathbb{C}},\ \mathcal{V}=\wedge^2_{\mathbb{Q}}(\wedge^m_{\mathbb{Q}}\Lambda_{\mathbb{Q}}),\ \bar{\mathcal{V}}=\wedge^2_{\mathbb{C}}(\wedge^m_{\mathbb{C}}\Lambda_{\mathbb{C}}).$$

Let us fix a Cartan subalgebra $\mathfrak{h}$ of the simple Lie $\mathbb{Q}$-algebra hdg_T, which is a l-dimensional $\mathbb{Q}$-vector space. Then

$$\bar{\mathfrak{h}}=\mathfrak{h}\otimes_{\mathbb{Q}}\otimes\mathbb{C}$$

is a Cartan subalgebra of the complex simple Lie algebra $\mathrm{hdg}_{T,\mathbb{C}}$ that is a l-dimensional complex vector space endowed with the natural semi-linear $\mathrm{Aut}(\mathbb{C})$-action; its subalgebra of invariants coincides with $\mathfrak{h}\otimes 1=\mathfrak{h}$.

As in Section 7, let us consider the dual complex vector space

$$\bar{\mathfrak{h}}^*=\mathrm{Hom}_{\mathbb{C}}(\bar{\mathfrak{h}},\mathbb{C}).$$

Let

$$R\subset\bar{\mathfrak{h}}^*$$

be the root system of $(\mathrm{hdg}_{T,\mathbb{C}},\bar{\mathfrak{h}})$.

Let us choose a simple root system B of simple roots (basis) of R and let $P_{++}(R)\subset\bar{\mathfrak{h}}^*$ be the corresponding semigroup of dominant weights [8].

If $\mu\in P_{++}(R)$ then we write $\mathbf{V}(\mu)$ for the simple $\mathrm{hdg}_{T,\mathbb{C}}$-module with highest weight μ [9]. In particular, $\mathbf{V}(0)$ stands for the one-dimensional $\bar{\mathbb{Q}}$-vector space $\bar{\mathbb{Q}}$ with trivial (zero) action of $\mathrm{hdg}_{T,\mathbb{C}}$. Then the $\binom{2m}{m}$-dimensional $\bar{\mathbb{Q}}$-vector space

$$\bar{\Lambda}=\Lambda_{\mathbb{Q}}\otimes_{\mathbb{Q}}\bar{\mathbb{Q}}$$

becomes a simple $\mathrm{hdg}_{T,\mathbb{C}}$-module that is isomorphic to $\mathbf{V}(\bar{\omega}_m)$. Hereafter we use the notation of Bourbaki ([8, Tables], [9, Tables]). In particular,

$$B=\{\alpha_1,\dots,\alpha_l\}=\{\alpha_1,\dots,\alpha_{2m-1}\}$$

(see Root systems of type $\mathsf{A_l}$ in [8, Tables]), and $\bar{\omega}_i$ is the dominant weight of a fundamental representation of dimension $\binom{2m}{i}$ (when $1 \le i \le l = 2m-1$), see [9, Table 2]. In addition, we put

$$\bar{\omega}_0 := 0 =: \bar{\omega}_{2m}.$$

Notice that the only nontrivial automorphism of (R, B) is the involution

$$\alpha_i \to \alpha_{2m-i} \quad \forall i = 1, \dots, 2m-1 = l.$$

Hence, each dominant weight $\bar{\omega}_{m+i} + \bar{\omega}_{m-i}$ is $\mathrm{Aut}(R, \mathbf{B})$-invariant for all $i = 0, \dots, 2m$.

It follows from results of [16, p. 140, Example 9a, last displayed formula] (see also [15, Exercises 6.16 on p. 81 and 15.32 on p. 226]) that the $\bar{\mathfrak{g}} = \mathrm{hdg}_{T,\mathbb{C}}$-module

$$\bar{\mathcal{V}} = \wedge^2_{\mathbb{C}}(\mathbf{V}(\bar{\omega}_m))$$

is isomorphic to a direct sum

$$\oplus_{i \text{ odd}, \ 1 \le i \le m} \mathbf{V}(\bar{\omega}_{m+i} + \bar{\omega}_{m-i}). \tag{72}$$

This implies that the $\bar{\mathfrak{g}}$-module $\bar{\mathcal{V}}$ splits into a direct sum of mutually non-isomorphic simple $\bar{\mathfrak{g}}$-modules; one of them is trivial if and only if m is odd (one should take $i = m$ in order to get the summand $\mathbf{V}(0)$.)

Let $\mathcal{W}$ be a simple $\bar{\mathfrak{g}}$-submodule of $\bar{\mathcal{V}}$. Let $\lambda_{\mathcal{W}}$ be the highest weight of $\mathcal{W}$. We know that $\lambda_{\mathcal{W}}$ is $\mathrm{Aut}(R, B)$-invariant. It follows from Theorem 7.13 combined with Proposition 7.9 that the simple $\bar{\mathfrak{g}}$-submodules $\mathcal{W}$ and $\sigma(\mathcal{W})$ have the same highest weight and therefore are isomorphic. This implies that

$$\sigma\mathcal{W} = \mathcal{W} \ \forall \sigma \in \mathrm{Aut}(\mathbb{C}).$$

By Lemma 7.4, $\mathcal{W}$ is defined over $\mathbb{Q}$, i.e., there is a $\mathbb{Q}$-vector subspace $\mathbf{W}$ of $\mathcal{V}$ such that

$$\mathcal{W} = \mathbf{W} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Clearly, such $\mathbf{W}$ is a simple hdg_T-submodule of $\mathcal{V}$. It follows from (72) that the hdg_T-module $\mathcal{V}$ splits into a direct sum

$$\oplus_{i \text{ odd}, \ 1 \le i \le m} \mathbf{W}_i. \tag{73}$$

of hdg_T-modules such that

$$\mathbf{V}(\bar{\omega}_{m+i} + \bar{\omega}_{m-i}) = \mathbf{T}_{\mathbb{Q},\mathbb{C}}(\mathbf{W}_i) = \mathbf{W}_i \otimes_{\mathbb{Q}} \mathbb{C}.$$

This implies that all $\mathbf{W}_i$ are mutually non-isomorphic simple hdg_T-modules. In adddition, one of them is trivial if and only if m is odd. (Namely, if m is odd then $\mathbf{W}_m$ is a trivial hdg_T-module of $\mathbb{Q}$-dimension 1.)

Thus, if m is *even*, then the hdg_T-module $\mathcal{V}$ splits into a direct sum of $(m/2)$ simple modules, none of which is trivial. If m is *odd*, then the hdg_T-module $\mathcal{V}$ splits into a direct sum of $(m+1)/2)$ simple modules, and precisely one of them is trivial. It follows that hdg_T-module $\mathcal{V}$ is simple if and only if $m = 2$. Since $m \ge 3$ (71), we conclude that $\mathcal{V}$ is never simple. On the other hand, it's clear that $\mathcal{V}$ is a direct sum of a simple hdg_T-module and a trivial one if and only if $m = 3$.

Recall that we are actually interested in the dual hdg_T-module

$$\mathrm{H}^2(T, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(\mathcal{V}, \mathbb{Q}).$$

By duality, the hdg_T-module is never simple; it is a direct sum of a simple hdg_T-module and a trivial one if and only if $m = 3$. Now the 2-simplicity of T implies that $m = 3$ and therefore

$$2g = \binom{2 \cdot 3}{3} = 20,$$

i.e., $g = 10$. □

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