MATH 87400:
ADDITIVE NUMBER THEORY
CUNY GRADUATE CENTER
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Class schedule. This class will meet on Tuesday, 2-4 p.m. and Thursday, 2-3 p.m.

1. Syllabus

1.1. **Overview.** Additive number theory includes not only classical problems, such as Waring's problem and the Goldbach conjecture, but also much recent work in additive combinatorics and combinatorial number theory. The focus of this course will be new problems and results, many of which are only available on arXiv or have not yet been published. Lecture notes will be printed and distributed at the first class meeting.

1.2. **Thin asymptotic bases of finite order.** An *asymptotic basis of order h* is a set A of nonnegative integers such that every sufficiently large integer can be represented as the sum of exactly h elements of A. The *counting function* of a set A of nonnegative integers counts the number of positive elements of A not exceeding x. A classical extremal problem for additive bases concerns the existence of "sparse" or "thin" asymptotic bases of finite order for the nonnegative integers. We shall consider various constructions of thin bases, and, in particular, a beautiful construction of J. W. S. Cassels of asymptotic bases $A = \{a_n\}$ of order h such that $a_n \sim \alpha n^h$ for some $\alpha > 0$. We shall also investigate the *additive spectrum* associated to Cassels bases.

1.3. **Minimal bases and maximal nonbases.** An asymptotic basis A of order h is *minimal* if no proper subset of A is an asymptotic basis of order h. Such extremal bases exist, but it is a nontrivial exercise to construct them. Dually, an *asymptotic nonbasis of order h* is a set A of nonnegative integers such that infinitely many positive integers cannot be represented as the sum of exactly h elements of A. An asymptotic nonbasis A of order h is *maximal* if no proper superset of A is an asymptotic nonbasis of order h. We can classify asymptotic nonbases constructed as unions of arithmetic progressions, but it is more difficult to construct nonperiodic maximal asymptotic nonbases. Indeed, we shall construct a set of nonnegative integers that oscillates from asymptotic basis to nonbasis to basis ... as integers are successively adjoined to and deleted from the set. We can even partition the nonnegative integers into a minimal asymptotic basis of order 2 and a maximal asymptotic nonbasis of order 2. Even more exotic additive phenomena will be described.

1.4. **Representation functions of bases.** Let A be a set of integers. The representation function $r_{A,h}(n)$ counts the number of representations of n as the sum of h elements of A. Many classical number theory problems are related to represenation functions. For example, how many ways is an integer the sum of s squares? The most famous unsolved problem in additive number theory is the *Erdős-Turán conjecture*: The representation function of an asymptotic basis of finite order is unbounded. The story is very different for sets of integers: If $f : \mathbf{Z} \to \{0, 1, 2, 3, \ldots\} \cup \{\infty\}$ is any function such that $f^{-1}(0)$ is a finite set, then there exist infinitely many sets A such that $r_{A,h}(n) = f(n)$ for all integers n, and it is an open problem to find the densest set with a given representation function.

1.5. **Complementing sets of integers.** The finite set A of integers is *complementing* if there exists a set B such that every integer n has a unique representation in the form $n = a + b$, where $a \in A$ and $b \in B$. The set B must be periodic, and it is an open problem to determine the least period of a set that is complementary to a given finite set A. The analogous problem for complementing pairs of sets of lattice points is unsolved. There is also no method to decide if a given finite set of integers has a complement, except in special cases where the cardinality of the finite set A is divisible by at most two distinct primes. These and other related results will be studied.

1.6. **Additive problems in metric geometry.** Questions about bi-Lipschitz equivalence and quasi-isometry in Cayley graphs and other metric spaces lead naturally to certain problems in additive number theory. In particular, they suggest the importance of studying growth of *infinitely* generated groups, which is largely an open problem in geometric group theory.

2. Text

M. B. Nathanson, *Additive Number Theory: Extremal Problems and the Combinatorics of Sumsets*, Graduate Texts in Mathematics, Springer, to appear in 2009. Prepublication copies will be distributed to students the first week of class.

Supplementary reading:

(1) M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Graduate Texts in Mathematics, Springer-Verlag, 1996.
(2) M. B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Graduate Texts in Mathematics, Springer-Verlag, 1996.
(3) T. Tao and V. Vu, *Additive Combinatorics*, Cambridge Univ. Press, 2006.