ON THE LAU GROUP SCHEME

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Abstract. In a 2013 article, Eike Lau constructed a canonical morphism from the stack of n-truncated Barsotti-Tate groups over $\mathbb{F}_p$ to the stack of n-truncated displays. He also proved that this morphism is a gerbe banded by a commutative group scheme. In this paper we describe the group scheme explicitly.

The stack of n-truncated Barsotti-Tate groups over $\mathbb{F}_p$ has a generalization related to any pair (G, μ), where G is a smooth group scheme over $\mathbb{Z}/p^n\mathbb{Z}$ and μ is a 1-bounded cocharacter of G. The same is true for the stack of n-truncated displays. We conjecture that in this more general situation the first stack is a gerbe over the second one banded by a commutative group scheme, and we give a conjectural description of this group scheme.

We also give a conjectural description of the stack of n-truncated Barsotti-Tate groups over $\operatorname{Spf} \mathbb{Z}_p$ and of its (G, μ)-generalization.

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1. Introduction

Throughout this article, we fix a prime p.

Let d, d' be integers such that $0 \le d' \le d$. Let $n \in \mathbb{N}$.

1.1. **A theorem of E. Lau.** The notion of n-truncated Barsotti-Tate group was introduced by Grothendieck [Gr]. We recall it in §3.

n-truncated Barsotti-Tate groups of height d and dimension $d' \le d$ form an algebraic stack over $\mathbb{Z}$, denoted by $\mathscr{BT}_n^{d,d'}$ (see §3.1.4 for more details). This stack is rather mysterious.

Let $\overline{\mathscr{BT}}_n^{d,d'} := \mathscr{BT}_n^{d,d'} \otimes \mathbb{F}_p$. Using (a covariant version of) crystalline Dieudonné theory, E. Lau defined in [L13] a canonical morphism $\phi_n : \overline{\mathscr{BT}}_n^{d,d'} \to \mathrm{Disp}_n^{d,d'}$, where $\mathrm{Disp}_n^{d,d'}$ is a certain explicit algebraic stack[1] over $\mathbb{F}_p$ (which is called the stack of n-truncated displays of height d and dimension d'). Moreover, he proved the following

Theorem 1.1.1. *(i) The morphism $\phi_n : \overline{\mathscr{BT}}_n^{d,d'} \to \mathrm{Disp}_n^{d,d'}$ is a gerbe banded by a commutative locally free finite group scheme over $\mathrm{Disp}_n^{d,d'}$, which we denote by $\mathrm{Lau}_n^{d,d'}$.*

(ii) The group scheme $\mathrm{Lau}_n^{d,d'}$ has order $p^{nd'(d-d')}$ and is killed by F^n, where F is the geometric Frobenius.

This theorem of E. Lau is a combination of [L13, Thm. B] and [L13, Rem. 4.8].

1.2. **Main results.** In this article we describe $\mathrm{Lau}_n^{d,d'}$ explictly. We also prove a property of $\mathrm{Lau}_n^{d,d'}$, which we call *n-smoothness*[2]. A group scheme G over an $\mathbb{F}_p$-scheme S is said to

[1] $\mathrm{Disp}_n^{d,d'}$ was defined in [L13] using Th. Zink's ideas.

[2] In [Gr, §VI.2] Grothendieck used a different name for this property.

be n-*smooth* if locally on S, there exists an isomorphism of pointed S-schemes

$$G \xrightarrow{\sim} S \times \operatorname{Spec} A_{n,r}, \quad A_{n,r} := \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^n}, \ldots, x_r^{p^n})$$

for some $r \in \mathbb{Z}_+$ (here $\operatorname{Spec} A_{n,r}$ is viewed as a pointed $\mathbb{F}_p$-scheme). If $n > 1$ then n-smoothness is stronger than being a finite locally free group scheme killed by F^n.

1.3. **The description of** $\mathrm{Lau}_n^{d,d'}$**.** The stack $\mathrm{Disp}_n^{d,d'}$ is very explicit: it is the quotient of the $\mathbb{F}_p$-scheme $GL(d, W_n)$ by an explicit action of a certain group scheme, see §B.0.6 of Appendix B (which goes back to [LZ]). In this article we give an explicit description of the group scheme $\mathrm{Lau}_n^{d,d'}$, see Theorem 9.1.5. Namely, $\mathrm{Lau}_n^{d,d'}$ is obtained by applying the *Zink functor* (see §7) to a certain n-truncated *semidisplay* on $\mathrm{Disp}_n^{d,d'}$ (in the sense of §6).

Moreover, in §C.7 of Appendix C we describe the Cartier dual $(\mathrm{Lau}_n^{d,d'})^*$ as an explicit closed subgroup of a very simple smooth group scheme over $\mathrm{Disp}_n^{d,d'}$. In fact, we describe there the Cartier duals of more general group schemes $\mathrm{Lau}_n^{G,\mu}$ discussed in the next subsection.

1.4. **A "Shimurian" generalization of** $\mathrm{Lau}_n^{d,d'}$ **and two conjectures.** For any smooth affine group scheme G over $\mathbb{Z}/p^n\mathbb{Z}$ and any 1-bounded[3] homomorphism $\mu : \mathbb{G}_m \to G$, one has certain stacks[4] $\mathrm{Disp}_n^{G,\mu}$ and $\mathrm{BT}_n^{G,\mu}$ (hopefully, they are related to Shimura varieties); in the case $G = GL(d)$ these are the stacks $\mathrm{Disp}_n^{d,d'}$ and $\mathscr{BT}_n^{d,d'}$, where d' depends on μ. In Appendix C we formulate Conjecture C.5.3, which says that $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{F}_p$ is a gerbe over $\mathrm{Disp}_n^{G,\mu}$ banded by a certain commutative group scheme $\mathrm{Lau}_n^{G,\mu}$. The definition of $\mathrm{Lau}_n^{G,\mu}$ given in Appendix C is inspired by our description of $\mathrm{Lau}_n^{d,d'}$; in fact, $\mathrm{Lau}_n^{d,d'}$ is a particular case of $\mathrm{Lau}_n^{G,\mu}$. In some sense, the main result of this article is that Conjecture C.5.3 is true for $G = GL(d)$.

We also give a conjectural description of the stack $\mathrm{BT}_n^{G,\mu}$ for any 1-bounded μ and any n, see Conjecture D.8.4.

1.5. **Organization.** In §2 we discuss the notion of n-smoothness and the Cartier-dual notion of n-cosmoothness. In §2.4.4-2.4.9 we study the natural tensor structure on the category of 1-cosmooth group schemes; Corollary 2.4.8 is used in the proof of Theorems 4.2.2 and 4.4.2.

In §3.1 we recall the notion of n-truncated Barsotti-Tate group. In §3.2 we discuss automorphisms of such groups.

In §4 we formulate Theorems 4.2.2 and 4.4.2, which provide some information about $\mathrm{Lau}_n^{d,d'}$ (including n-smoothness). To transform Theorem 4.4.2 into an explicit description of $\mathrm{Lau}_n^{d,d'}$, we need further steps, which are briefly discussed in §4.5.

In §5 we prove Theorems 4.2.2 and 4.4.2.

In §6 we introduce the stack of n-truncated semidisplays and two related stacks (weak and strong n-truncated semidisplays).

In §7 we discuss the Zink functor $\mathfrak{Z}$.

In §8 we prove Proposition 8.1.1, which plays a key role in the proof of Theorem 9.1.5.

In §9 we formulate and prove Theorem 9.1.5, which gives an explicit description of $\mathrm{Lau}_n^{d,d'}$. The proof uses [LZ, Lemma 3.12].

In §10 we discuss the category of n-truncated higher displays from [L21]. We use it to reformulate a certain construction from §9.1.3 in a way convenient for Appendix C.

[3] 1-boundedness means that all weights of the action of $\mathbb{G}_m$ on $\mathrm{Lie}(G)$ are ≤ 1.

[4] $\mathrm{Disp}_n^{G,\mu}$ was defined in [BP, L21]; we recall the definition in Appendix C. For $\mathrm{BT}_n^{G,\mu}$ see [GM].

In Appendix A we recall the description of the Cartier dual of the group of Witt vectors (p-typical or "big" ones).

In Appendix B we describe the stacks from §6 in very explicit terms.

In Appendix C we define the group scheme $\mathrm{Lau}_n^{G,\mu}$ mentioned in §1.4 and formulate Conjecture C.5.3. Using §10, we show that $\mathrm{Lau}_n^{d,d'}$ is a particular case of $\mathrm{Lau}_n^{G,\mu}$. We also describe $(\mathrm{Lau}_n^{G,\mu})^*$ very explicitly.

In Appendix D we formulate Conjecture D.8.4 describing the stack $\mathrm{BT}_n^{G,\mu}$ for any 1-bounded μ and any n.

1.6. **Simplifications if $n = 1$.** A complete description of $\mathrm{Lau}_1^{d,d'}$ is given already by Theorem 4.4.2(ii) combined with §9.2.2. Moreover, in the $n = 1$ case §8 and §10 are unnecessary and §7 is almost unnecessary (because if $n = 1$ then the Zink functor $\mathfrak{Z}$ from §7 is just the classical functor from restricted Lie algebras to group schemes of height 1).

1.7. **Acknowledgements.** This paper is based on the theory of displays developed by Th. Zink and E. Lau. I also benefited from discussions with them.

On the other hand, Conjecture D.8.4 is strongly influenced by the theory of sheared prismatization [BMVZ, BKMVZ] and by my discussions with D. Arinkin and N. Rozenblyum.

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2. n-smoothness and n-cosmoothness

2.1. Recollections on exact sequences of group schemes.

2.1.1. *Definition of exactness.* Let S be a scheme. A sequence

$$G' \xrightarrow{f} G \xrightarrow{h} G'' \tag{2.1}$$

of affine group S-schemes is said to be exact if the corresponding sequence of fpqc-sheaves on the category of S-schemes is exact. If S is the spectrum of a field, exactness just means that $\operatorname{Ker} h = \operatorname{Im} f$.

2.1.2. *How to check exactness.* Exactness of (2.1) clearly implies fiberwise exactness. Now suppose we have a diagram (2.1) such that $h \circ f = 0$ and G', G, G'' are flat schemes of finite presentation over S. In this situation it is well known (e.g., see [dJ, Prop. 1.1] and its proof) that if the sequence (2.1) is fiberwise exact then it is exact, $\operatorname{Ker} h$ is flat (and finitely presented) over S and the morphism $G' \to \operatorname{Ker} h$ is faithfully flat. The latter implies that $\operatorname{Ker} f$ is flat (and finitely presented) over S.

From now on, let us assume that the group schemes G', G, G'' from the exact sequence (2.1) are finite and locally free. Then $\operatorname{Ker} h$ and $\operatorname{Ker} f$ are also finite and locally free. Moreover, the morphism $G/\operatorname{Ker} h \to G''$ is a closed immersion, so if G'' is commutative we also have the group scheme $\operatorname{Coker} h$, which is finite and locally free.

Note that exactness of the fiber of (2.1) over $s \in S$ is equivalent to the condition

$$|\operatorname{Ker} h_s| = |\operatorname{Im} f_s|;$$

this condition is open because $|\operatorname{Ker} h_s|$ is upper-semicontinuous and $|\operatorname{Im} f_s|$ is lower-semicontinuous (since $|\operatorname{Im} f_s| \cdot |\operatorname{Ker} f_s| = |G'_s|$).

If the groups G, G', G'' in a complex (2.1) are commutative then exactness of (2.1) is equivalent to exactness of the Cartier-dual complex.

2.2. **n-smooth group schemes.** From now on, *we assume that S is an $\mathbb{F}_p$-scheme.*

2.2.1. *Pointed S-schemes.* By a pointed S-scheme we mean an S-scheme equipped with a section. We have a forgetful functor from the category of group S-schemes to that of pointed S-schemes (forget multiplication but remember the unit).

2.2.2. *Definition.* Let $n \in \mathbb{N}$. A group S-scheme G is said to be *n-smooth*[5] if Zariski-locally on S, there exists an isomorphism of pointed S-schemes

$$G \xrightarrow{\sim} S \times \operatorname{Spec} A_{n,r}, \quad A_{n,r} := \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^n}, \ldots, x_r^{p^n}) \tag{2.2}$$

for some $r \in \mathbb{Z}_+$ (here $\operatorname{Spec} A_{n,r}$ is viewed as a pointed $\mathbb{F}_p$-scheme).

If G is an n-smooth group S-scheme then the $\mathcal{O}_S$-module $\operatorname{coLie}(G)$ is a vector bundle. Its rank is a locally constant function on S, which is called the *rank* of G. (This is the number r from (2.2)). It is clear that an n-smooth group S-scheme is finite and locally free; moreover, its order equals p^{nr}, where r is the rank.

The next lemma implies that the property of n-smoothness is fpqc-local with respect to S.

Lemma 2.2.3. *Let $r \in \mathbb{Z}_+$ and G a group scheme over an $\mathbb{F}_p$-scheme S. The following are equivalent:*

(a) G is n-smooth of rank r;

(b) G is a finite locally free group scheme of order p^{nr} killed by F^n, and $\dim \operatorname{Lie}(G_s) = r$ *for all $s \in S$.* □

The following variant of Lemma 2.2.3 will be used in §5.1.2.

Lemma 2.2.4. *Let $r \in \mathbb{Z}_+$. Let H be a finite group scheme over an $\mathbb{F}_p$-scheme S. Let $H' \subset H$ be a closed subgroup. Suppose that*

(i) H is killed by F^n;

(ii) H' is a finite locally free group S-scheme of order p^{nr};

(iii) $\dim \operatorname{Lie}(H_s) \leq r$ *for all $s \in S$.*

Then $H' = H$ and H is n-smooth of rank r. □

Proposition 2.2.5. *A group S-scheme G is n-smooth if and only if it is finite, locally free, killed by $F^n : G \to (\operatorname{Fr}_S^n)^*G$, and satisfies the following condition: the complex*

$$G \xrightarrow{F^m} (\operatorname{Fr}_S^m)^*G \xrightarrow{F^{n-m}} (\operatorname{Fr}_S^n)^*G \tag{2.3}$$

is an exact sequence for every $m \in \{1, \ldots, n-1\}$.

This is due to W. Messing [Me72, Prop. II.2.1.2] and Grothendieck (see Proposition 2.1 of [Gr, Ch. VI]). Here is a slight improvement of the above proposition.

Proposition 2.2.6. *Let G be a finite locally free S-scheme killed by $F^n : G \to (\operatorname{Fr}_S^n)^*G$. If the complex* (2.3) *is exact for at least one $m \in \{1, \ldots, n-1\}$ then G is n-smooth.*

Proof. This was proved by Grothendieck (see Proposition 2.1 of [Gr, Ch. VI]). On the other hand, the argument from [Me72] works with the following modification: in line 10 of p.29 of [Me72] replace $1 \otimes T_1$ by $1 \otimes T_1^{p^i}$, where $i = \max(0, n_1 - m)$. □

[5]In [Gr, §VI.2] Grothendieck used a different name for the class of n-smooth group schemes.

2.2.7. *The case* $n = 1$. A group scheme G over S is 1-smooth if and only if it is finite, locally free, and killed by Frobenius. This is a part of Theorem 7.4 of [SGA3, Exp.VIIA] (more precisely, the equivalence (ii)⇔(iv) of the theorem). On the other hand, this follows from Proposition 2.2.5.

2.2.8. *The kernel of* F^n. For a group scheme (or ind-scheme) G over S, we set

$$G^{(F^n)} := \operatorname{Ker}(G \xrightarrow{F^n} (\mathrm{Fr}_S^n)^* G).$$

It is easy to see that $G^{(F^n)}$ is n-smooth if G is either smooth or N-smooth for some $N \geq n$.

2.2.9. *Notation.* Let $\mathrm{Sm}_n^{\mathrm{all}}(S)$ be the category of n-smooth group schemes over S. Let $\mathrm{Sm}_n(S) \subset \mathrm{Sm}_n^{\mathrm{all}}(S)$ be the full subcategory of commutative group schemes. The categories $\mathrm{Sm}_n^{\mathrm{all}}(S)$ form a projective system: for $N \geq n$ the functor $\mathrm{Sm}_N^{\mathrm{all}}(S) \to \mathrm{Sm}_n^{\mathrm{all}}(S)$ is $G \mapsto G^{(F^n)}$. The same is true for the categories $\mathrm{Sm}_n(S)$.

2.2.10. *Relation to formal Lie groups.* By a *formal Lie group* over S we mean a group ind-scheme G over S such that Zarsiski-locally on S, there exists an isomorphism of pointed S-ind-schemes $G \xrightarrow{\sim} (\hat{\mathbb{A}}_S^r, 0)$ (here $\hat{\mathbb{A}}_S^r$ is the formal completion of $\mathbb{A}_S^r$ along the zero section).

Let $\mathrm{Sm}_\infty^{\mathrm{all}}(S)$ (resp. $\mathrm{Sm}_\infty^{\mathrm{all}}(S)$) be the category of all (resp. commutative) formal Lie groups over S. If $G \in \mathrm{Sm}_\infty^{\mathrm{all}}(S)$ then $G^{(F^n)} \in \mathrm{Sm}_n^{\mathrm{all}}(S)$. As noted in [Me72, Ch. II], this construction defines equivalences

$$\mathrm{Sm}_\infty^{\mathrm{all}}(S) \xrightarrow{\sim} \varprojlim_n \mathrm{Sm}_n^{\mathrm{all}}(S), \quad \mathrm{Sm}_\infty(S) \xrightarrow{\sim} \varprojlim_n \mathrm{Sm}_n(S). \tag{2.4}$$

2.3. n**-cosmooth group schemes.** Recall that every n-smooth group scheme over S is finite and locally free.

2.3.1. *Definitions.* Let $n \in \mathbb{N}$. A group S-scheme G is said to be n*-cosmooth* if it is Cartier dual to a commutative n-smooth group S-scheme. The *rank* of G is defined to be the rank of G^*.

2.3.2. *Notation.* The category of n-cosmooth group schemes over S is denoted by $\mathrm{Sm}_n^*(S)$. By definition, it is anti-equivalent to $\mathrm{Sm}_n(S)$.

If $S = \operatorname{Spec} R$ we write $\mathrm{Sm}_n(R), \mathrm{Sm}_n^*(R)$ instead of $\mathrm{Sm}_n(S), \mathrm{Sm}_n^*(S)$.

2.3.3. *Remarks.* By definition, any n-cosmooth group S-scheme G is commutative, finite, and locally free; moreover, its order equals p^{nr}, where r is the rank of G (this follows from a similar statement about n-smooth group schemes).

2.3.4. *Example.* Let $m, n \in \mathbb{N}$. The group S-scheme $W_{n,S}^{(F^m)} := \operatorname{Ker}(W_{n,S} \xrightarrow{F^m} W_{n,S})$ is clearly m-smooth. It is also n-cosmooth because its Cartier dual is isomorphic to $W_{m,S}^{(F^n)}$.

2.4. **The category** $\mathrm{Sm}_1^*(R)$**.** Let R be an $\mathbb{F}_p$-algebra.

2.4.1. Let $\mathcal{B}(R)$ be the category of pairs (P, φ), where P is a finitely generated projective R-module and $\varphi : P \to P$ is a p-linear map. If $(P, \varphi) \in \mathcal{B}(R)$ and $\tilde{R}$ is an R-algebra, let $A_{P,\varphi}(\tilde{R})$ be the group of R-linear maps $\xi : P \to \tilde{R}$ such that $\xi(\varphi(x)) = \xi(x)^p$ for all $x \in P$. For any $(P, \varphi) \in \mathcal{B}(R)$ the functor $A_{P,\varphi}$ is an affine group R-scheme. The following theorem is well known (see §2 of [dJ], which refers to [SGA3, Exp.VIIA]).

Theorem 2.4.2. *(i) For any $(P, \varphi) \in \mathcal{B}(R)$, the group scheme $A_{P,\varphi}$ is 1-cosmooth. Its rank equals the rank of P.*

(ii) The functor

$$\mathcal{B}(R)^{\mathrm{op}} \to \mathrm{Sm}_1^*(R), \quad (P, \varphi) \mapsto A_{P,\varphi}$$

is an equivalence.

(iii) The inverse functor takes $A \in \mathrm{Sm}_1^*(R)$ to (P, φ), where $P = \mathrm{Hom}(A, (\mathbb{G}_a)_R)$ and $\varphi : P \to P$ is given by composition with $F : (\mathbb{G}_a)_R \to (\mathbb{G}_a)_R$. Equivalently, (P, φ) is the restricted Lie algebra of $A^* \in \mathrm{Sm}_1(R)$. □

2.4.3. $\mathrm{Sm}_1^*(R)$ *as a tensor category.* $\mathcal{B}(R)$ is clearly a tensor category (i.e., a symmetric monoidal additive category). So Theorem 2.4.2(ii) provides a structure of tensor category on $\mathrm{Sm}_1^*(R)$. We are going to describe this structure directly (without using $\mathcal{B}(R)$), see Propositions 2.4.5(ii) and 2.4.7. A similar tensor structure on $\mathrm{Sm}_n^*(R)$ is briefly mentioned in §8.1.2(ii-iii) below.

2.4.4. Let $(P_i, \varphi_i) \in \mathcal{B}(R)$, where $1 \le i \le m$. Let $(P, \varphi) := \bigotimes_i (P_i, \varphi_i)$. Then we have a poly-additive morphism

$$A_{P_1,\varphi_1} \times \ldots \times A_{P_m,\varphi_m} \to A_{P,\varphi}\,, \tag{2.5}$$

where the Cartesian product is over R: namely, $(\xi_1, \ldots, \xi_m) \in A_{P_1,\varphi_1}(\tilde{R}) \times \ldots \times A_{P_m,\varphi_m}(\tilde{R})$ goes to the R-linear map

$$P_1 \otimes \ldots \otimes P_m \to \tilde{R}, \quad x_1 \otimes \ldots \otimes x_m \mapsto \prod_{i=1}^{m} \xi_i(x_i).$$

Proposition 2.4.5. *(i) The map*

$$\mathrm{Hom}(A_{P,\varphi}, (\mathbb{G}_a)_R) \to \text{Poly-add}(A_{P_1,\varphi_1} \times \ldots \times A_{P_m,\varphi_m}, (\mathbb{G}_a)_R)) \tag{2.6}$$

induced by (2.5) *is an isomorphism (here* Poly-add *stands for the group of poly-additive maps).*

(ii) For any $A \in \mathrm{Sm}_1^*(R)$, the map

$$\mathrm{Hom}(A_{P,\varphi}, A) \to \text{Poly-add}(A_{P_1,\varphi_1} \times \ldots \times A_{P_m,\varphi_m}, A)$$

induced by (2.5) *is an isomorphism*

Note that (i) would become false if $\mathbb{G}_a$ is replaced by $\mathbb{G}_m$, see §2.4.6 below.

Proof. To simplify the notation, we assume that $m = 2$.

(i) We can assume that the R-modules P_1 and P_2 are free. Let $x_1, \ldots, x_r$ (resp. $y_1, \ldots, y_s$) be a basis in P_1 (resp. in P_2). Then the monomials $x_1^{a_1} \ldots x_l^{a_r}$ with $a_1, \ldots a_r \le p - 1$ form a basis in the R-module of regular functions on A_{P_1,φ_1}. Using this fact and a similar fact for (P_2, φ_2), we see that a bi-additive morphism $A_{P_1,\varphi_1} \times A_{P_2,\varphi_2} \to (\mathbb{G}_a)_R$ is the same as a polynomial $f \in R[x_1, \dots, x_r, y_1, \dots, y_s]$ which has degree $< p$ with respect to each variable and is bi-additive in the usual sense. Such f is bilinear.

(ii) If $A = A_{P',\varphi'}$ then

$$\text{Poly-add}(A_{P_1,\varphi_1} \times A_{P_2,\varphi_2}, A_{P',\varphi'}) = \text{Hom}_{R[F]}(P', \text{Poly-add}(A_{P_1,\varphi_1} \times A_{P_2,\varphi_2}, (\mathbb{G}_a)_R)),$$

where $R[F] := \text{End}(\mathbb{G}_a)_R$ acts on P' via φ'. Similarly,

$$\text{Hom}(A_{P,\varphi}, A_{P',\varphi'}) = \text{Hom}_{R[F]}(P', \text{Hom}(A_{P,\varphi}, (\mathbb{G}_a)_R)).$$

So statement (ii) follows from (i). $\square$

2.4.6. In the situation of §2.4.4, the map (2.5) induces a homomorphism

$$A^*_{P,\varphi} = \underline{\text{Hom}}(A_{P,\varphi}, (\mathbb{G}_m)_R) \to \underline{\text{Poly-add}}(A_{P_1,\varphi_1} \times \ldots \times A_{P_m,\varphi_m}, (\mathbb{G}_m)_R)), \tag{2.7}$$

where $\underline{\text{Poly-add}}$ stands for the group scheme of poly-additive maps. Note that (2.7) is *not an isomorphism*, in general. E.g., let $(P_1, \varphi_1) = (P_2, \varphi_2) = (P, \varphi) = (R, 0)$. Then

$$A_{P_1,\varphi_1} = A_{P_2,\varphi_2} = A_{P,\varphi} = (\alpha_p)_R\,, \quad \underline{\text{Hom}}((\alpha_p)_R, (\mathbb{G}_m)_R) = (\alpha_p)_R,$$

$$\underline{\text{Poly-add}}((\alpha_p)_R \times (\alpha_p)_R, (\mathbb{G}_m)_R)) = \underline{\text{Hom}}((\alpha_p)_R, ((\alpha_p)_R)^*) = \underline{\text{Hom}}((\alpha_p)_R, (\alpha_p)_R) = (\mathbb{G}_a)_R\,.$$

Proposition 2.4.7. *The homomorphism* (2.7) *induces an isomorphism*

$$A^*_{P,\varphi} \xrightarrow{\sim} \underline{\text{Poly-add}}(A_{P_1,\varphi_1} \times \ldots \times A_{P_m,\varphi_m}, (\mathbb{G}_m)_R)^{(F)}. \tag{2.8}$$

As usual, $\underline{\text{Poly-add}}(A_{P_1,\varphi_1} \times \ldots \times A_{P_m,\varphi_m}, (\mathbb{G}_m)_R)^{(F)}$ *stands for the kernel of Frobenius in the group scheme* $\underline{\text{Poly-add}}(A_{P_1,\varphi_1} \times \ldots \times A_{P_m,\varphi_m}, (\mathbb{G}_m)_R)$.

Proof. Let $H := \underline{\text{Poly-add}}(A_{P_1,\varphi_1} \times \ldots \times A_{P_m,\varphi_m}, (\mathbb{G}_m)_R)^{(F)}$. Since $A^*_{P,\varphi}$ is killed by F, the map (2.7) induces a homomorphism $f : A^*_{P,\varphi} \to H$. The problem is to show that f is an isomorphism.

(i) By definition, H is killed by F. Moreover, H has finite type over R because the scheme parametrizing *all* invertible functions on $A_{P_1,\varphi_1} \times \ldots \times A_{P_m,\varphi_m}$ has finite type over R. So H is finite over R.

(ii) Assume that R is a field. Then the theory of height 1 group schemes tells us that to prove that $f : A^*_{P,\varphi} \to H$ is an isomorphism it suffices to show that $\text{Lie}(f)$ is an isomorphism. But $\text{Lie}(f)$ is the map (2.6), which is an isomorphism by Proposition 2.4.5(i).

(iii) Let R be any ring. Let C and C' be the coordinate rings of the finite schemes $A^*_{P,\varphi}$ and H. Then $f^* : C' \to C$ is a homomorphism of finitely generated R-modules. By (i), f^* is a fiberwise isomorphism. But C is projective, so f^* is an isomorphism. $\square$

Corollary 2.4.8. *For any* $A_1, \dots, A_m \in \text{Sm}^*_1(R)$ *the group scheme*

$$\underline{\text{Poly-add}}(A_1 \times \ldots \times A_m, (\mathbb{G}_m)_R)^{(F)}$$

is 1*-smooth. Its rank equals the product of the ranks of* $A_1, \dots, A_m$*. One has a canonical isomorphism of restricted Lie* R*-algebras*[6] $\text{Lie}\,\underline{\text{Poly-add}}(A_1 \times \ldots \times A_m, (\mathbb{G}_m)_R)^{(F)} \xrightarrow{\sim} \bigotimes_i \text{Lie}(A^*_i)$.

[6]The commutator in these restricted Lie algebras is, of course, zero.

2.4.9. *Remarks.* (i) In the case $m = 2$ one has

$$\underline{\text{Poly-add}}(A_1 \times A_2, (\mathbb{G}_m)_R)^{(F)} = \underline{\text{Hom}}(A_1, A_2^*)^{(F)}.$$

(ii) If $A_1, A_2 \in \text{Sm}_1^*(R)$ then the group scheme $\underline{\text{Hom}}(A_1, A_2^*)$ does not have to be either finite or flat, see §3.2.2-3.2.3 below.

2.5. **A general principle.** Objects of $\text{Sm}_n^*(R)$ (i.e., n-cosmooth group schemes) are easier to handle than objects of the dual category $\text{Sm}_n(R)$.

E.g., the 1-cosmooth group scheme $A_{P,\varphi}$ from §2.4.1 is a subgroup of the very simple group scheme $\mathbb{V}(P) := \operatorname{Spec} \operatorname{Sym}(P)$, and this subgroup is defined by very simple equations. On the other hand, $A^*_{P,\varphi}$ is a *quotient* of the group *ind*-scheme $\mathbb{V}(P)^*$; this is less elementary. Note that $\mathbb{V}(P)^*$ is the PD-nilpotent PD neighborhood of zero in $\mathbb{V}(P^*)$.

3. n-truncated Barsotti-Tate groups

3.1. Recollections.

3.1.1. *Definition of n-truncated Barsotti-Tate group.* The notion of n-truncated Barsotti-Tate group was introduced by Grothendieck [Gr]. It is reviewed in [Me72, Il85, dJ].

Let $n \in \mathbb{N}$. By definition, an n-truncated Barsotti-Tate group (in short, a BT_n group) over a scheme S is a finite locally free commutative group scheme G over S which is killed by p^n, is $(\mathbb{Z}/p^n\mathbb{Z})$-flat (as an fpqc sheaf) and satisfies an extra condition in the case $n = 1$. If $n = 1$ and S is an $\mathbb{F}_p$-scheme we have complexes

$$G \xrightarrow{F} \operatorname{Fr}_S^* G \xrightarrow{V} G, \quad \operatorname{Fr}_S^* G \xrightarrow{V} G \xrightarrow{F} \operatorname{Fr}_S^* G, \tag{3.1}$$

and these complexes are required to be exact sequences; as noted by Grothendieck, exactness of one of them implies exactness of the other.[7] If $n = 1$ and S is any scheme then the above condition is required for the restriction of G to $S \otimes \mathbb{F}_p$.

The above-mentioned $(\mathbb{Z}/p^n\mathbb{Z})$-flatness condition is equivalent to exactness of the sequences

$$G \xrightarrow{p^{n-m}} G \xrightarrow{p^m} G$$

for all $m < n$. By §2.1.2, the group scheme $\operatorname{Ker}(G \xrightarrow{p^m} G)$ is finite and locally free over S. Moreover, it is a BT_m group (even if $m = 1$), see [Gr, III.3.3] or [Me72, II.3.3.11].

If G is a BT_n group then so is its Cartier dual G^*.

The groupoid of BT_n groups over S will be denoted by $\text{BT}_n(S)$. If $S = \operatorname{Spec} R$ we write $\text{BT}_n(R)$ instead of $\text{BT}_n(S)$.

3.1.2. *Height and dimension.* From now on, *we assume that S is an $\mathbb{F}_p$-scheme.* If G is a BT_n group over S then the group schemes $\operatorname{Ker}(G \xrightarrow{p} G)$ and $\operatorname{Ker}(G \xrightarrow{F} \operatorname{Fr}_S^* G)$ are finite and locally free by §2.1.2, and they are killed by p. So there exist locally constant functions $d : S \to \mathbb{Z}_+$ and $d' : S \to \mathbb{Z}_+$ such that the orders of these groups equal p^d and $p^{d'}$, respectively; clearly $d' \leq d$. Names: d is the *height* of G, and d' is the *dimension* of G.

The group scheme $G^{(F)} := \operatorname{Ker}(G \xrightarrow{F} \operatorname{Fr}_S^* G)$ is 1-smooth (see §2.2.7), so its Lie algebra is a vector bundle of rank d'. The Lie algebra of G is the same.

[7]See [Il85, §1.3 (b)]. The idea is to use §2.1.2 and look at the orders of the kernels and images.

3.1.3. *Some results of Messing.* Let $G \in \mathrm{BT}_n(S)$. Similarly to (3.1), we have a complex

$$G \xrightarrow{F^n} (\mathrm{Fr}_S^n)^* G \xrightarrow{V^n} G \xrightarrow{F^n} (\mathrm{Fr}_S^n)^* G. \tag{3.2}$$

By [Me72, II.3.3.11 (b)], this complex is exact. So it gives rise to exact sequences of finite locally free group S-schemes

$$0 \to G^{(F^n)} \to G \to G/G^{(F^n)} \to 0, \tag{3.3}$$

$$0 \to G/G^{(F^n)} \to (\mathrm{Fr}_S^n)^* G \to G^{(F^n)} \to 0. \tag{3.4}$$

Moreover, $G^{(F^n)}$ is n-smooth (see [Me72, II.3.3.11 (a)] and [Me72, II.2.1.2]). On the other hand, $(G/G^{(F^n)})^* = (G^*)^{(F^n)}$, so $G/G^{(F^n)}$ is n-cosmooth.

Thus (3.3) exhibits G as an extension of an n-cosmooth group scheme by an n-smooth one, and (3.4) exhibits $(\mathrm{Fr}_S^n)^* G$ as an extension of an n-smooth group scheme by an n-cosmooth one.

3.1.4. *The stacks $\mathscr{BT}_n^{d,d'}$ and $\overline{\mathscr{BT}}_n^{d,d'}$.* For any scheme S, let $\mathscr{BT}_n^{d,d'}(S)$ be the groupoid of BT_n groups over S whose restriction to $S \otimes \mathbb{F}_p$ has height d and dimension d'. The assignment $S \mapsto \mathscr{BT}_n^{d,d'}(S)$ is an algebraic stack of finite type over $\mathbb{Z}$ with affine diagonal, see [W, Prop. 1.8]. Moreover, the stack $\mathscr{BT}_n^{d,d'}$ is *smooth* over $\mathbb{Z}$ by a deep theorem of Grothendieck, whose proof is given in Illusie's article [Il85].

3.2. A subgroup of $\operatorname{Aut} G$, where $G \in \mathrm{BT}_n(S)$ and S is over $\mathbb{F}_p$.

3.2.1. As before, let S be an $\mathbb{F}_p$-scheme and $G \in \mathrm{BT}_n(S)$. Then one has a monomorphism $\mathrm{Hom}(G/G^{(F^n)}, G^{(F^n)}) \hookrightarrow \operatorname{Aut} G$ defined by $f \mapsto 1 + \tilde{f}$, where $\tilde{f}$ is the composite morphism

$$G \twoheadrightarrow G/G^{(F^n)} \xrightarrow{f} G^{(F^n)} \hookrightarrow G.$$

Thus we get a homomorphism

$$\underline{\mathrm{Hom}}(G/G^{(F^n)}, G^{(F^n)}) \hookrightarrow \underline{\mathrm{Aut}}\, G,$$

which is a closed immersion.

3.2.2. *Example.* Suppose that $S = \operatorname{Spec} \bar{\mathbb{F}}_p$ and $G = \mathrm{Ker}(E \xrightarrow{p} E)$, where E is an elliptic curve over S. If E is not supersingular then

$$\underline{\mathrm{Hom}}(G/G^{(F)}, G^{(F)}) \simeq \mu_p, \quad \underline{\mathrm{Aut}}\, G = (\mathbb{F}_p^\times \times \mathbb{F}_p^\times) \ltimes \underline{\mathrm{Hom}}(G/G^{(F)}, G^{(F)}),$$

where the action of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ on $\underline{\mathrm{Hom}}(G/G^{(F)}, G^{(F)})$ is nontrivial for $p > 2$. On the other hand, if E is supersingular then

$$\underline{\mathrm{Hom}}(G/G^{(F)}, G^{(F)}) \simeq \underline{\mathrm{Hom}}(\alpha_p, \alpha_p) = \mathbb{G}_a, \quad \underline{\mathrm{Aut}}\, G = \mathbb{F}_{p^2}^\times \ltimes \underline{\mathrm{Hom}}(G/G^{(F)}, G^{(F)}),$$

where the action of $\mathbb{F}_{p^2}^\times$ on $\underline{\mathrm{Hom}}(G/G^{(F)}, G^{(F)})$ is nontrivial.

3.2.3. *Remark.* The above example implies that $\underline{\mathrm{Hom}}(G/G^{(F^n)}, G^{(F^n)})$ is neither finite nor flat, in general.

4. Formulating Theorems 4.2.2 and 4.4.2

4.1. **The group scheme $\mathcal{I}_n^{d,d'}$.** Let $\mathcal{I}_n^{d,d'}$ be the inertia stack of $\overline{\mathscr{BT}}_n^{d,d'}$. Thus an S-point of $\mathcal{I}_n^{d,d'}$ is a pair consisting of an S-point of $\overline{\mathscr{BT}}_n^{d,d'}$ and an automorphism of this point.

It is clear that $\mathcal{I}_n^{d,d'}$ is an affine group scheme of finite type over the stack $\overline{\mathscr{BT}}_n^{d,d'}$.

4.2. The first main theorem.

4.2.1. *The group scheme* $\mathrm{Lau}_n^{d,d'}$. E. Lau defined in [L13, §4] a canonical morphism

$$\phi_n : \overline{\mathscr{BT}}_n^{d,d'} \to \mathrm{Disp}_n^{d,d'}.$$

He proved that ϕ_n is a gerbe banded by a commutative locally free finite group scheme over $\mathrm{Disp}_n^{d,d'}$, which we denote by $\mathrm{Lau}_n^{d,d'}$. He also proved some properties of $\mathrm{Lau}_n^{d,d'}$; we formulated them in Theorem 1.1.1.

Note that $\phi_n^* \mathrm{Lau}_n$ is a subgroup of $\mathcal{I}_n^{d,d'}$. This subgroup is closed because Lau_n is finite over $\mathrm{Disp}_n^{d,d'}$ and the morphism $\mathcal{I}_n^{d,d'} \to \overline{\mathscr{BT}}_n^{d,d'}$ is separated.

Theorem 4.2.2. *(i)* $\phi_n^* \mathrm{Lau}_n^{d,d'} = (\mathcal{I}_n^{d,d'})^{(F^n)}$, *where*

$$(\mathcal{I}_n^{d,d'})^{(F^n)} := \mathrm{Ker}(\mathcal{I}_n^{d,d'} \xrightarrow{F^n} (\mathrm{Fr}^n)^* \mathcal{I}_n^{d,d'}).$$

(ii) The group scheme $\mathrm{Lau}_n^{d,d'}$ *is n-smooth of rank $d'(d-d')$.*

The proof will be given in §5.

The above theorem and Theorem 4.4.2 below are steps towards an explicit description of $\mathrm{Lau}_n^{d,d'}$.

4.2.3. *Remarks.* (i) Theorem 4.2.2(i) improves Proposition 4.2 of [LZ].

(ii) Theorem 4.2.2 implies that the group scheme $(\mathcal{I}_n^{d,d'})^{(F^n)}$ is finite, locally free, and commutative. On the other hand, by §3.2.2, the group scheme $\mathcal{I}_1^{2,1}$ is neither finite nor flat, nor commutative.

4.3. Reconstructing $\mathrm{Lau}_n^{d,d'}$ from $\phi_n^* \mathrm{Lau}_n^{d,d'}$.

Lemma 4.3.1. *Let $\phi : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks. If ϕ is an fppf gerbe then the functor*

$$\phi^* : \{\textit{Group schemes over } \mathscr{Y}\} \to \{\textit{Group schemes over } \mathscr{X}\} \tag{4.1}$$

is fully faithful.

(In fact, the lemma and its proof given below remain valid if the words "group scheme" are replaced by "scheme", "vector bundle", etc.)

Proof. We can assume that the gerbe is trivial, so $\mathscr{X}$ is the classifying stack of a flat group scheme H over $\mathscr{Y}$. Then a group scheme over $\mathscr{X}$ is the same as a group scheme over $\mathscr{Y}$ equipped with an action of H, and the functor ϕ^* takes a group scheme over $\mathscr{Y}$ to the same group scheme equipped with the trivial action of H. □

By Lemma 4.3.1, $\mathrm{Lau}_n^{d,d'}$ can be reconstructed from $\phi_n^* \mathrm{Lau}_n^{d,d'}$, in principle. In practice, this requires some work.

4.3.2. *The essential image of the functor* (4.1). In the situation of Lemma 4.3.1, let $\mathcal{I}_{\mathscr{X}/\mathscr{Y}}$ be the relative inertia stack; this is the group scheme over $\mathscr{X}$ defined by

$$\mathcal{I}_{\mathscr{X}/\mathscr{Y}} := \operatorname{Ker}(\mathcal{I}_{\mathscr{X}} \to \phi^*\mathcal{I}_{\mathscr{Y}}),$$

where $\mathcal{I}_{\mathscr{X}}$ and $\mathcal{I}_{\mathscr{Y}}$ are the inertia stacks of $\mathscr{X}$ and $\mathscr{Y}$. The proof of Lemma 4.3.1 implies the following description of the essential image of the functor (4.1) (assuming that $\phi : \mathscr{X} \to \mathscr{Y}$ is a gerbe): a group scheme H over $\mathscr{X}$ belongs to the essential image of (4.1) if and only if the canonical action of $\mathcal{I}_{\mathscr{X}/\mathscr{Y}}$ on H is trivial.

4.4. **A complement to Theorem 4.2.2.**

4.4.1. *The group schemes* $\tilde{\mathcal{A}}_n^{d,d'}, \tilde{\mathcal{B}}_n^{d,d'}$. Let S be an $\mathbb{F}_p$-scheme and G a BT_n group scheme over S of height d and dimension d'. Then $G^{(F^n)}$ and $(G/G^{(F^n)})^* = (G^*)^{(F^n)}$ are n-smooth group schemes over S (see §3.1.3); their ranks equal d' and $d-d'$, respectively. As S and G vary, $G^{(F^n)}$ and $(G^*)^{(F^n)}$ determine commutative n-smooth group schemes $\tilde{\mathcal{A}}_n^{d,d'}$ and $\tilde{\mathcal{B}}_n^{d,d'}$ over $\overline{\mathscr{BT}}_n^{d,d'}$, whose ranks equal d' and $d-d'$, respectively.

Theorem 4.4.2. *(i)* $\tilde{\mathcal{A}}_n^{d,d'}$ *and* $\tilde{\mathcal{B}}_n^{d,d'}$ *descend*[8] *to commutative n-smooth group schemes* $\mathcal{A}_n^{d,d'}$ *and* $\mathcal{B}_n^{d,d'}$ *over* $\mathrm{Disp}_n^{d,d'}$.

(ii) One has a canonical isomorphism of (commutative) restricted Lie algebras

$$\operatorname{Lie}(\mathrm{Lau}_n^{d,d'}) \xrightarrow{\sim} \operatorname{Lie}(\mathcal{A}_n^{d,d'}) \otimes \operatorname{Lie}(\mathcal{B}_n^{d,d'}).$$

(iii) One has a canonical isomorphism

$$\mathrm{Lau}_n^{d,d'} \xrightarrow{\sim} \underline{\mathrm{Hom}}((\mathcal{B}_n^{d,d'})^*, \mathcal{A}_n^{d,d'})^{(F^n)}, \tag{4.2}$$

where as usual,

$$\underline{\mathrm{Hom}}(((\mathcal{B}_n^{d,d'})^*, \mathcal{A}_n^{d,d'})^{(F^n)} := \operatorname{Ker}(F^n : \underline{\mathrm{Hom}}(((\mathcal{B}_n^{d,d'})^*, \mathcal{A}_n^{d,d'}) \longrightarrow (\mathrm{Fr}^n)^* \underline{\mathrm{Hom}}(\mathcal{B}_n^*, \mathcal{A}_n^{d,d'})).$$

The proof will be given in §5.6. Note that Theorem 4.4.2(ii) is really informative if $n=1$ (indeed, the group scheme $\mathrm{Lau}_n^{d,d'}$ is 1-smooth, so it is uniquely determined by its restricted Lie algebra).

4.5. **Improving formula** (4.2)**.** Eventually, we will transform formula (4.2) into an *explicit* description of $\mathrm{Lau}_n^{d,d'}$ (see Theorem 9.1.5) using two ingredients. The first one is an explicit description (due to E. Lau and T. Zink) of the group schemes $\mathcal{A}_n^{d,d'}$ and $\mathcal{B}_n^{d,d'}$ defined in Theorem 4.4.2(i). The other ingredient is Proposition 2.4.7 for $n=1$ and Proposition 8.1.1 for an arbitrary n. In the case $n=1$ Proposition 2.4.7 allows one to rewrite the r.h.s of (4.2) as a tensor product of $\mathcal{A}_1^{d,d'}$ and $\mathcal{B}_1^{d,d'}$ in the sense of the tensor structure on $\mathrm{Sm}_1^*(R)$ defined in §2.4.3. For $n>1$ we will do something similar.

5. Proofs of Theorems 4.2.2 and 4.4.2

5.1. **Deducing Theorem 4.2.2 from the key lemma.** Here is the key lemma, which will be proved in §5.5.

Lemma 5.1.1. *The Lie algebra of the fiber of* $(\mathcal{I}_n^{d,d'})^{(F^n)}$ *over any geometric point of* $\overline{\mathscr{BT}}_n^{d,d'}$ *has dimension* $d'(d-d')$.

[8]By Lemma 4.3.1, the descent is unique up to unique isomorphism.

5.1.2. *Deducing Theorem 4.2.2 from the key lemma.* $\phi_n^* \operatorname{Lau}_n^{d,d'}$ is a closed subgroup of the group scheme $\mathscr{I}_n^{d,d'}$. Moreover, $\phi_n^* \operatorname{Lau}_n^{d,d'} \subset (\mathscr{I}_n^{d,d'})^{(F^n)}$ by Theorem 1.1.1(ii). $(\mathscr{I}_n^{d,d'})^{(F^n)}$ is finite over $\overline{\mathscr{BT}}_n^{d,d'}$ because $\mathscr{I}_n^{d,d'}$ has finite type. By Theorem 1.1.1(ii), $\phi_n^* \operatorname{Lau}_n^{d,d'}$ is finite and locally free over $\overline{\mathscr{BT}}_n^{d,d'}$ of order $p^{nd'(d-d')}$. So Theorem 4.2.2 follows from Lemma 5.1.1 combined with Lemma 2.2.4 (the latter has to be applied to the pullbacks of $\phi_n^* \operatorname{Lau}_n^{d,d'}$ and $(\mathscr{I}_n^{d,d'})^{(F^n)}$ to a scheme S equipped with a faithfully flat morphism to $\overline{\mathscr{BT}}_n^{d,d'}$).

5.2. **A description of** $(\mathscr{I}_n^{d,d'})^{(F^n)}$**.** The goal of this subsection is to prove Corollary 5.2.2, which can be regarded as a description of $(\mathscr{I}_n^{d,d'})^{(F^n)}$.

Let S be an $\mathbb{F}_p$-scheme and $G \in \operatorname{BT}_n(S)$.

Lemma 5.2.1. *If* $f \in \operatorname{End} G$ *and* $(\operatorname{Fr}^n)^*(f) = 0$ *then* $f : G \to G$ *factors as*

$$G \twoheadrightarrow G/G^{(F^n)} \to G^{(F^n)} \hookrightarrow G.$$

In particular, $f^2 = 0$.

Proof. One has $F^n \circ f = (\operatorname{Fr}^n)^*(f) \circ F^n = 0$ and $f \circ V^n = V^n \circ (\operatorname{Fr}^n)^*(f) = 0$. So $\operatorname{Im} f \subset G^{(F^n)}$ and $\operatorname{Ker}(f) \supset \operatorname{Im}((\operatorname{Fr}^n)^* G \xrightarrow{V^n} G) = G^{(F^n)}$. □

Corollary 5.2.2. *One has group isomorphisms*

$$(\underline{\operatorname{Aut}}\, G)^{(F^n)} \xrightarrow{\sim} (\underline{\operatorname{End}}\, G)^{(F^n)} \xrightarrow{\sim} \underline{\operatorname{Hom}}(G/G^{(F^n)}, G^{(F^n)})^{(F^n)},$$

where the first map is $h \mapsto h - 1$*. As usual,* $\underline{\operatorname{Hom}}(G/G^{(F^n)}, G^{(F^n)})^{(F^n)}$ *denotes the kernel of* $F^n : \underline{\operatorname{Hom}}(G/G^{(F^n)}, G^{(F^n)}) \longrightarrow (\operatorname{Fr}^n)^* \underline{\operatorname{Hom}}(G/G^{(F^n)}, G^{(F^n)})$. □

Note that by §3.2.2, the group scheme $\underline{\operatorname{Hom}}(G/G^{(F^n)}, G^{(F^n)})$ is neither finite nor flat, in general.

5.3. **The case** $n = 1$**.** In this case, Lemma 5.1.1 follows from the next one.

Lemma 5.3.1. *Let* $G \in \operatorname{BT}_1(S)$*. Then the group scheme* $(\underline{\operatorname{Aut}}\, G)^{(F)}$ *is* 1*-smooth (in particular, finite and locally free), and one has a canonical isomorphism of (commutative) restricted Lie* $\mathcal{O}_S$*-algebras*

$$\operatorname{Lie}((\underline{\operatorname{Aut}}\, G)^{(F)}) \xrightarrow{\sim} \operatorname{Lie}(G^{(F)}) \otimes \operatorname{Lie}((G/G^{(F)})^*) = \operatorname{Lie}(G^{(F)}) \otimes \operatorname{Lie}((G^*)^{(F)}).$$

Proof. Use Corollary 5.2.2, Corollary 2.4.8, and §2.4.9(i). □

5.4. **On** $\operatorname{End} G_m$**, where** $m \le n$ **and** $G_m := \operatorname{Ker}(p^m : G \to G)$**.** As before, let $G \in \operatorname{BT}_n(S)$, where S is an $\mathbb{F}_p$-scheme. Let $m \le n$ and

$$G_m := \operatorname{Ker}(p^m : G \to G).$$

The subgroup $G_m \subset G$ can also be regarded as a quotient of G via $p^{n-m} : G \twoheadrightarrow G_m$. We have an *additive* homomorphism

$$\operatorname{End} G_m \to \operatorname{End} G, \quad f \mapsto \tilde{f}, \tag{5.1}$$

where $\tilde{f} : G \to G$ is the composition $G \overset{p^{n-m}}{\twoheadrightarrow} G_m \xrightarrow{f} G_m \hookrightarrow G$. The map (5.1) induces additive isomorphisms

$$\operatorname{End} G_m \xrightarrow{\sim} \operatorname{Ker}(\operatorname{End} G \xrightarrow{p^m} \operatorname{End} G), \tag{5.2}$$

$$\underline{\mathrm{End}}\, G_m \xrightarrow{\sim} \mathrm{Ker}(\underline{\mathrm{End}}\, G \xrightarrow{p^m} \underline{\mathrm{End}}\, G). \tag{5.3}$$

Moreover, (5.3) induces an additive isomorphism

$$(\underline{\mathrm{End}}\, G_m)^{(F^m)} \xrightarrow{\sim} (\underline{\mathrm{End}}\, G)^{(F^m)} \tag{5.4}$$

because $(\underline{\mathrm{End}}\, G)^{(F^m)} \subset \mathrm{Ker}(\underline{\mathrm{End}}\, G \xrightarrow{p^m} \underline{\mathrm{End}}\, G)$.

Using the group isomorphism

$$(\underline{\mathrm{End}}\, G)^{(F^n)} \xrightarrow{\sim} (\underline{\mathrm{Aut}}\, G)^{(F^n)}, \quad f \mapsto 1+f$$

and a similar group isomorphism $(\underline{\mathrm{End}}\, G_m)^{(F^m)} \xrightarrow{\sim} (\underline{\mathrm{Aut}}\, G_m)^{(F^m)}$, we get from (5.4) a group isomorphism

$$(\underline{\mathrm{Aut}}\, G_m)^{(F^m)} \xrightarrow{\sim} (\underline{\mathrm{Aut}}\, G)^{(F^m)}. \tag{5.5}$$

5.5. **Proof of Lemma 5.1.1.** The problem is to show that for any field k of characteristic p and any $G \in \mathrm{BT}_n(k)$, one has

$$\dim \mathrm{Lie}((\underline{\mathrm{Aut}}\, G)^{(F^n)}) = d'(d-d'), \tag{5.6}$$

where d, d' are the height and the dimension of G. Note that $(\underline{\mathrm{Aut}}\, G)^{(F^n)}$ and $(\underline{\mathrm{Aut}}\, G)^{(F)}$ have the same Lie algebra. So applying (5.5) for $m=1$, we reduce the proof of (5.6) to the case $n=1$, which was treated in §5.3.

5.6. **Proof of Theorem 4.4.2.**

5.6.1. *Proof of Theorem 4.4.2(i).* By §4.3.2 and Theorem 4.2.2(i), it suffices to show that the action of $(\mathscr{I}_n^{d,d'})^{(F^n)}$ on $\tilde{\mathscr{A}}_n d, d'$ and $\tilde{\mathscr{B}}_n^{d,d'}$ is trivial. This follows from Corollary 5.2.2. ☐

5.6.2. *Proof of Theorem 4.4.2(ii).* By Theorem 4.2.2(i), $\phi_n^* \mathrm{Lau}_n^{d,d'} = (\mathscr{I}_n^{d,d'})^{(F^n)}$. So Lemma 4.3.1 implies that proving Theorem 4.4.2(ii) amounts to constructing an isomorphism

$$\mathrm{Lie}((\mathscr{I}_n^{d,d'})^{(F^n)}) \xrightarrow{\sim} \mathrm{Lie}(\tilde{\mathscr{A}}_n^{d,d'}) \otimes \mathrm{Lie}(\tilde{\mathscr{B}}_n^{d,d'}).$$

For $n=1$, such an isomorphism is provided by Lemma 5.3.1. It remains to note that the pullbacks of $(\mathscr{I}_1^{d,d'})^{(F)}$, $\tilde{\mathscr{A}}_1^{d,d'}$, $\tilde{\mathscr{B}}_1^{d,d'}$ via the morphism $\overline{\mathscr{BT}}_n^{d,d'} \to \overline{\mathscr{BT}}_1^{d,d'}$ are canonically isomorphic to $(\mathscr{I}_n^{d,d'})^{(F)}$, $(\tilde{\mathscr{A}}_n^{d,d'})^{(F)}$, $(\tilde{\mathscr{B}}_n^{d,d'})^{(F)}$ (in the case of $(\mathscr{I}_1^{d,d'})^{(F)}$ this is the isomorphism (5.5)). ☐

5.6.3. *Proof of Theorem 4.4.2(iii).* Combining Theorem 4.2.2(i) and Corollary 5.2.2, we get a canonical isomorphism $\phi_n^* \mathrm{Lau}_n^{d,d'} \xrightarrow{\sim} \underline{\mathrm{Hom}}(\tilde{\mathscr{B}}_n^*, \tilde{\mathscr{A}}_n^{d,d'})^{(F^n)}$. It remains to use Lemma 4.3.1. ☐

6. n-TRUNCATED SEMIDISPLAYS

Let R be an $\mathbb{F}_p$-algebra and $n \in \mathbb{N}$. Let $\mathrm{Disp}_n(R)$ be the additive category of n-truncated displays in the sense of [L13, Def. 3.4]. In §6.1-6.2 we will construct a diagram of additive categories

$$\mathrm{Disp}_n(R) \hookrightarrow \mathrm{sDisp}_n^{\mathrm{strong}}(R) \to \mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n^{\mathrm{weak}}(R)$$

in which the first functor is fully faithful and the other functors are essentially surjective; $\mathrm{sDisp}_n(R)$ and $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$ are defined in §6.1, $\mathrm{Disp}_n(R)$ and $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$ are discussed in §6.2. Objects of $\mathrm{sDisp}_n(R)$ are called n-truncated *semidisplays*.

Unlike $\mathrm{Disp}_n(R)$, the categories $\mathrm{sDisp}_n(R)$, $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$, and $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$ are tensor categories, see §6.3 and §6.2.3.

The category $\mathrm{Disp}_n(R)$ is equipped with a duality functor[9] $\mathscr{P} \mapsto \mathscr{P}^t$ (see [L13, Rem. 4.4]). There is no such functor on $\mathrm{sDisp}_n(R)$, $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$, or $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$.

In §7.1-7.2 we discuss a functor

$$\mathrm{sDisp}_n(R) \to \mathrm{Sm}_n(R),$$

where $\mathrm{Sm}_n(R)$ is the category of commutative n-smooth group schemes over R. The corresponding functor $\mathrm{Disp}_n(R) \to \mathrm{Sm}_n(R)$ was defined in [LZ].

6.1. n-truncated semidisplays.

6.1.1. *Notation.* Let R be an $\mathbb{F}_p$-algebra. Fix $n \in \mathbb{N}$. We have natural epimorphisms

$$W_n(R) \twoheadrightarrow W_{n-1}(R) \twoheadrightarrow \ldots \twoheadrightarrow W_1(R) = R \twoheadrightarrow W_0(R) = 0.$$

Let $I_{n,R} := \mathrm{Ker}(W_n(R) \twoheadrightarrow W_1(R))$, $J_{n,R} := \mathrm{Ker}(W_n(R) \twoheadrightarrow W_{n-1}(R))$.

6.1.2. *Definition.* An *n-truncated semidisplay* over R is a quadruple (P, Q, F, F_1), where P is a finitely generated projective $W_n(R)$-module, $Q \subset P$ a submodule, $F : P \to P$ and $F_1 : Q \to P/J_{n,R} \cdot P$ are semilinear with respect to $F : W(R) \to W(R)$, and the following conditions hold:

(i) P/Q is a projective module over $W_1(R) = R$ (in particular, $Q \supset I_{n,R} \cdot P$);

(ii) for $a \in W(R)$, $x \in P$, one has $F_1(V(a) \cdot x) = a \cdot \bar{F}(x)$, where $\bar{F}(x)$ is the image of $F(x)$ in $P/J_{n,R} \cdot P$;

(iii) $F(x) = pF_1(x)$ for $x \in Q$.

Let $\mathrm{sDisp}_n(R)$ be the additive category of n-truncated semidisplays over R.

6.1.3. *Remarks.* (a) By §6.1.2(ii), one has

$$\bar{F}(x) = F_1(V(1) \cdot x) = F_1(px). \tag{6.1}$$

(b) If $x \in Q$ then (6.1) implies that $F(x) - pF_1(x) \in J_{n,R} \cdot P$, which is weaker than §6.1.2(iii).

(c) We have

$$F_1(J_{n,R} \cdot Q) = 0, \quad F(J_{n,R} \cdot Q) = 0. \tag{6.2}$$

Indeed, the first equality is clear because $P/J_{n,R} \cdot P$ is killed by $J_{n,R}$, and the second equality follows from the first one by §6.1.2(iii).

(d) By §6.1.2(iii), $F : P \to P$ induces a p-linear map $P.Q \to P/Q$ (so P/Q is a commutative restricted Lie R-algebra).

6.1.4. *Normal decompositions.* By §6.1.2(i), there exists a decomposition $P = T \oplus L$ such that $Q = I_{n,R} \cdot T \oplus L$. Following Th. Zink and E. Lau [Zi02, L13, LZ], we call this a *normal decomposition.*

Let us note that the notation T, L for the terms of a normal decomposition is standard. Mnemonic rule: T stands for "tangent" (in fact, the R-module $T/I_{n,R} \cdot T = P/Q$ is the Lie algebra of the n-smooth group scheme discussed in §7.1 below, see Lemma 7.1.4).

[9]As explained to me by E. Lau, this functor can be defined by formula (10.17) below; in this formula $\mathrm{Disp}_n(R)$ is identified with $\mathrm{DISP}_n^{[0,1]}(R)$ as explained in §10.7.2.

6.1.5. *Weak n-truncated semidisplays.* Given $(P, Q, F, F_1) \in \mathrm{sDisp}_n(R)$, set

$$M := P/J_{n,R} \cdot Q, \quad \mathcal{Q} := Q/J_{n,R} \cdot Q.$$

By (6.2), the maps $F : P \to P$ and $F_1 : Q \to P/J_{n,R} \cdot P$ induce maps

$$F : M \to M, \quad F_1 : \mathcal{Q} \to M/J_{n,R} \cdot M.$$

The quadruple $(M, \mathcal{Q}, F : M \to M, F_1 : \mathcal{Q} \to M/J_{n,R} \cdot M)$ has the following properties:

(a) F and F_1 are semilinear with respect to $F : W_n(R) \to W_n(R)$ and satisfy the relations from §6.1.2(ii)-(iii).

(b) The pair $(M, \mathcal{Q})$ is isomorphic to $(T, I_{n,R} \cdot T) \oplus (\bar{L}, \bar{L})$ for some finitely generated projective $W_n(R)$-module T and some finitely generated projective $W_{n-1}(R)$-module $\bar{L}$.

Quadruples $(M, \mathcal{Q}, F, F_1)$ satisfying (a)-(b) are called *weak n-truncated semidisplays.* They form an additive category, denoted by $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$. We have defined a functor

$$\mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n^{\mathrm{weak}}(R).$$

Using normal decompositions, one checks that it is essentially surjective.

6.1.6. *Example: $n = 1$.* $\mathrm{sDisp}_1(R)$ is the category of triples (P, Q, F), where P is a finitely generated projective R-module, $Q \subset P$ is a direct summand of P, and $F : P/Q \to P$ is a p-linear map. On the other hand, $\mathrm{sDisp}_1^{\mathrm{weak}}(R)$ is the category of finitely generated projective R-modules equipped with a p-linear endomorphism. The functor $\mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n^{\mathrm{weak}}(R)$ takes (P, Q, F) to P/Q equipped with the composite map $P/Q \xrightarrow{F} P \twoheadrightarrow P/Q$.

6.2. **n-truncated displays.** As before, let R be an $\mathbb{F}_p$-algebra and $n \in \mathbb{N}$. Let $\mathrm{Disp}_n(R)$ be the category of *n-truncated displays* in the sense of [L13, Def. 3.4]. This category is studied in [L13, §3] and [LZ, §1]. (Note that the setting of [LZ] is more general than we need: the ring R is required there to be a $(\mathbb{Z}/p^m\mathbb{Z})$-algebra for some m.)

6.2.1. *The functor $\mathrm{Disp}_n(R) \to \mathrm{sDisp}_n(R)$.* According to [L13, Def. 3.4], an object of $\mathrm{Disp}_n(R)$ is a collection

$$(P, Q, \iota : Q \to P, \varepsilon : I_{n+1,R} \otimes_{W_n(R)} P \to Q,\ F : P \to P,\ F_1 : Q \to P) \tag{6.3}$$

with certain properties. We consider the functor

$$\mathrm{Disp}_n(R) \to \mathrm{sDisp}_n(R) \tag{6.4}$$

that takes a collection (6.3) to (P, Q', F, F_1'), where $Q' := \iota(Q)$ and $F_1' : Q' \to P/J_{n,R} \cdot P$ is induced by $F_1 : Q \to P$.

6.2.2. *Strong n-truncated semidisplays.* One of the properties of a collection (6.3) required in [L13, Def. 3.4] is as follows: P has to be generated by $F_1(Q)$ as a module. Skipping this property, one gets a generalization of the notion of n-truncated display, which we call *strong n-truncated semidisplay.* The category of strong n-truncated semidisplays over R is denoted by $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$. The functor (6.4) extends from $\mathrm{Disp}_n(R)$ to the bigger category $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$.

In the last paragraph on p.141 of [L13] it is explained that the pair (F, F_1) from (6.3) is described by a semilinear map $\Psi : L \oplus T \to P$, where (L, T) is a normal decomposition[10] of $(P, Q, \iota, \varepsilon)$. In the case of an n-truncated display, the linear map corresponding to Ψ has to

[10]In this context, the notion of normal decomposition is defined in [L13] before Lemma 3.3.

be invertible; in the case of a strong n-truncated semidisplay, this is not required. Roughly speaking, the difference between n-truncated displays and strong n-truncated semidisplays amounts to the difference between the group of invertible matrices and the semigroup of all matrices.

Using normal decompositions, one checks that the functor $\mathrm{sDisp}_n^{\mathrm{strong}}(R) \to \mathrm{sDisp}_n(R)$ is essentially surjective.

6.2.3. *Remark.* According to E. Lau, a good way of dealing with $\mathrm{Disp}_n(R)$ is to replace it by the equivalent category $\mathrm{DISP}_n^{[0,1]}(R)$ (see §10.5.1 and §10.7.2 below). Similarly, one can work with $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$ using the equivalence $\mathrm{preDISP}_n^{[0,1]}(R) \xrightarrow{\sim} \mathrm{sDisp}_n^{\mathrm{strong}}(R)$ from §10.7.2; e.g., one can use it to define the structure of tensor category on $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$ (see §10.7.3 below).

6.3. **Tensor product in $\mathrm{sDisp}_n(R)$ and $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$.**

Lemma 6.3.1. *Let (P, Q, F, F_1) and (P', Q', F', F_1') be objects of $\mathrm{sDisp}_n(R)$. Let*

$$P'' := P \otimes P', \quad Q'' := P \otimes Q' + Q \otimes P' = \mathrm{Ker}(P \otimes P' \twoheadrightarrow (P/Q) \otimes (P'/Q')). \tag{6.5}$$

Define $F'' : P'' \to P''$ by

$$F'' := F \otimes F'. \tag{6.6}$$

Then there exists a (unique) additive map $F_1'' : Q'' \to P''/J_{n,R} \cdot P''$ such that

$$F_1''|_{P\otimes Q'} = F \otimes F_1', \quad F_1''|_{Q\otimes P'} = F_1 \otimes F'. \tag{6.7}$$

Moreover, $(P'', Q'', F'', F_1'') \in \mathrm{sDisp}_n(R)$.

Proof. The composite maps

$$Q \otimes Q' \to Q \otimes P' \xrightarrow{F_1 \otimes F'} P \otimes P' \text{ and } Q \otimes Q' \to P \otimes Q' \xrightarrow{F \otimes F_1'} P \otimes P'$$

are both equal to $pF_1 \otimes F_1'$ by §6.1.2(iii). If $x \in P$, $x' \in P'$, $a \in W_n(R)$ then

$$(F \otimes F_1')(V(a)(x \otimes x')) = a\bar{F}(x \otimes x') = (F_1 \otimes F')(V(a)(x \otimes x')). \tag{6.8}$$

These facts imply the existence of F_1'' because

$$(P \otimes Q') \cap (Q \otimes P') = \mathrm{Im}(Q \otimes Q' \to P \otimes P') + I_R \cdot (P \otimes P').$$

By (6.8), F'' and F_1'' satisfy the relation from §6.1.2(ii). Finally, it is easy to check that $F''|_{Q''} = pF_1''$. □

6.3.2. $\mathrm{sDisp}_n(R)$ *as a tensor category.* In the situation of Lemma 6.3.1, (P'', Q'', F'', F_1'') is called the *tensor product* of (P, Q, F, F_1) and (P', Q', F', F_1'); it is denoted by

$$(P, Q, F, F_1) \otimes (P', Q', F', F_1').$$

This tensor product makes $\mathrm{sDisp}_n(R)$ into a tensor category (by which we mean a symmetric monoidal additive category). The object

$$\mathbf{1}_{n,R} := (W_n(R), I_{n,R}, F, V^{-1} : I_{n,R} \to W_{n-1}(R)) \tag{6.9}$$

is the unit in $\mathrm{sDisp}_n(R)$.

Let us note that in the case $n = \infty$ the object (6.9) appears in [Zi02, Example 16] under the name of "multiplicative display" (because it corresponds to the multiplicative formal group).

6.3.3. $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$ *as a tensor category.* If (M, Q, F, F_1) and (M', Q', F', F_1') are objects of the category $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$ defined in §6.1.5, we set

$$(M, Q, F, F_1) \otimes (M', Q', F', F_1') := (M'', Q'', F'', F_1''),$$

where

$$M'' := M \otimes M', \quad Q'' := \mathrm{Ker}(M \otimes M' \twoheadrightarrow (M/Q') \otimes (M'/Q')), \quad F'' := F \otimes F',$$

and $F_1'' : Q'' \to M''/J_{n,R} \cdot M''$ is the unique additive map such that

$$F_1''(x \otimes y) = F(x) \otimes F_1'(y) \text{ if } x \in M, y \in Q',$$

$$F_1''(x \otimes y) = F_1(x) \otimes F'(y) \text{ if } x \in Q, y \in M'.$$

The existence of F_1'' follows from Lemma 6.3.1 and essential surjectivity of the functor $\mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n^{\mathrm{weak}}(R)$.

Thus $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$ becomes a tensor category equipped with a tensor functor

$$\mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n^{\mathrm{weak}}(R).$$

The quadruple $\mathbf{1}_{n,R}$ given by (6.9) is the unit object of $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$.

6.4. sDisp_n **and** $\mathrm{sDisp}_n^{\mathrm{weak}}$ **as stacks of tensor categories.** In [L13] it is proved that the assignment $R \mapsto \mathrm{Disp}_n(R)$ is an fpqc stack of categories.[11] The same is true for $\mathrm{sDisp}_n^{\mathrm{strong}}$, sDisp_n, $\mathrm{sDisp}_n^{\mathrm{weak}}$ and proved in the same way.

For a morphism of $\mathbb{F}_p$-algebras $f : R \to \tilde{R}$, the corresponding base change functor $\mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n(\tilde{R})$ takes (P, Q, F, F_1) to $(\tilde{P}, \tilde{Q}, \tilde{F}, \tilde{F}_1)$, where

$$\tilde{P} = W_n(\tilde{R}) \otimes_{W_n(R)} P, \quad \tilde{P}/\tilde{Q} = W_n(\tilde{R}) \otimes_{W_n(R)} (P/Q),$$

$\tilde{F} : \tilde{P} \to \tilde{P}$ is the base change of F, and $\tilde{F}_1 : \tilde{Q} \to \tilde{P}/J_{n,R} \cdot \tilde{P}$ is the unqiue F-linear map such that the diagram

$$\begin{array}{ccc} Q & \xrightarrow{F_1} & P/J_{n,R} \cdot P \\ \downarrow & & \downarrow \\ \tilde{Q} & \xrightarrow{\tilde{F}_1} & \tilde{P}/J_{n,\tilde{R}} \cdot \tilde{P} \end{array}$$

commutes and $\tilde{F}_1(V(a) \otimes x) = a \otimes \bar{F}(x)$ for all $a \in W(\tilde{R})$ and $x \in P$. The existence of $\tilde{F}_1$ can be proved using a normal decomposition of (P, Q, F, F_1).

The above description of the base change functor $\mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n(\tilde{R})$ and the quite similar description of the functor $\mathrm{sDisp}_n^{\mathrm{weak}}(R) \to \mathrm{sDisp}_n^{\mathrm{weak}}(\tilde{R})$ shows that these are *tensor* functors. Thus sDisp_n and $\mathrm{sDisp}_n^{\mathrm{weak}}$ as fpqc stacks of tensor categories

[11]Given a morphism of $\mathbb{F}_p$-algebras $f : R \to \tilde{R}$, there is an obvious notion of f-morphism from an object of $\mathrm{Disp}_n(R)$ to an object of $\mathrm{Disp}_n(\tilde{R})$. Existence of a (unique) base change functor is proved in [L13, Lemma 3.6]. Descent for n-truncated displays is proved in [L13, §3.3].

6.5. **Algebraicity of the stacks.** It is easy to show that the stacks of categories Disp_n, sDisp_n, $\mathrm{sDisp}_n^{\mathrm{weak}}$, $\mathrm{sDisp}_n^{\mathrm{strong}}$ are algebraic c-stacks in the sense of [D20, §2.3]. For the purposes of this paper, it is enough to know that the corresponding stacks of groupoids are algebraic (in the usual sense). We will formulate a more precise statement, see Proposition 6.5.1 below.

Given integers d and d' such that $0 \le d' \le d$, let $\mathrm{sDisp}_n^{d,d'}(R)$ be the full subgroupoid of the underlying groupoid of $\mathrm{sDisp}_n(R)$ whose objects are quadruples $(P, Q, F, F_1) \in \mathrm{sDisp}_n(R)$ such that $\operatorname{rank} P = d$ and $\operatorname{rank}(P/Q) = d'$.

Define $\mathrm{sDisp}_n^{d,d',\mathrm{weak}}(R)$, $\mathrm{sDisp}_n^{d,d',\mathrm{strong}}(R)$, and $\mathrm{Disp}_n^{d,d'}(R)$ similarly but with the following changes:

(i) in the case of $\mathrm{Disp}_n^{d,d'}(R)$ and $\mathrm{sDisp}_n^{d,d',\mathrm{strong}}(R)$ replace P/Q by $\operatorname{Coker}(Q \to P)$;

(ii) in the case of $\mathrm{sDisp}_n^{d,d',\mathrm{weak}}(R)$ the condition for P is that $\operatorname{rank}(P/I_{n,R}P) = d$.

Note that $\mathrm{sDisp}_1^{d,d'} \ne \emptyset$ only if $d' = d$; this follows from condition (b) from §6.1.5.

Proposition 6.5.1. *Each of the stacks* $\mathrm{sDisp}_n^{d,d'}$, $\mathrm{sDisp}_n^{d,d',\mathrm{weak}}$, $\mathrm{sDisp}_n^{d,d',\mathrm{strong}}$, *and* $\mathrm{Disp}_n^{d,d'}$ *is a smooth algebraic stack of finite type over* $\mathbb{F}_p$. *Moreover,* $\mathrm{sDisp}_n^{d,d'}$ *has pure dimension* $-(d-d')^2$, *and the other three stacks have pure dimension* 0.

In the case of $\mathrm{Disp}_n^{d,d'}$, this is [L13, Prop. 3.15].

Proof. Follows from the explicit presentation of the four stacks given in Appendix B. □

7. Zink's functor

7.1. **Zink's functor** $\mathrm{sDisp}_n(R) \to \mathrm{Sm}_n(R)$**.** As before, $\mathrm{Sm}_n(R)$ stands for the category of commutative n-smooth group schemes over R, see §2.2. In this subsection we recall the functor $\mathrm{sDisp}_n(R) \to \mathrm{Sm}_n(R)$, which was essentially[12] constructed in [LZ, §3.4]. In §7.3.3 we will decompose this functor as $\mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n^{\mathrm{weak}}(R) \to \mathrm{Sm}_n(R)$.

7.1.1. *Format of the construction.* To an object $\mathscr{P} = (P, Q, F.F_1) \in \mathrm{sDisp}_n(R)$ one functorially associates a diagram of commutative group ind-schemes

$$C_{\mathscr{P}}^{-1} \xrightarrow{\Phi} C_{\mathscr{P}}^{0}, \tag{7.1}$$

in which $C_{\mathscr{P}}^{-1}$ is a closed subgroup of $C_{\mathscr{P}}^{0}$. It is proved in [LZ, Prop. 3.11] that $\operatorname{Ker}(1-\Phi) = 0$ and the functor $\tilde{R} \mapsto \operatorname{Coker}(C_{\mathscr{P}}^{-1}(\tilde{R}) \xrightarrow{1-\Phi} C_{\mathscr{P}}^{0}(\tilde{R}))$ (where $\tilde{R}$ is an R-algebra) is representable by an n-smooth group scheme. We denote this group scheme by $\mathfrak{Z}_{\mathscr{P}}$. Thus

$$\mathfrak{Z}_{\mathscr{P}} := \operatorname{Coker}(C_{\mathscr{P}}^{-1} \xrightarrow{1-\Phi} C_{\mathscr{P}}^{0}). \tag{7.2}$$

The functor $\mathscr{P} \mapsto \mathfrak{Z}_{\mathscr{P}}$ will be called the *Zink functor*; in the case $n = \infty$ it was defined by Th. Zink in [Zi02, §3] (under the name of $BT_{\mathscr{P}}$). The complex of group ind-schemes

$$0 \to C_{\mathscr{P}}^{-1} \xrightarrow{1-\Phi} C_{\mathscr{P}}^{0} \to 0 \tag{7.3}$$

will be called the *Zink complex* of $\mathscr{P}$.

[12] The caveat is due to the fact that the authors of [LZ] worked with displays rather than semidisplays.

7.1.2. *Defining* (7.1). Let $\hat{W}^{(F^n)} := \operatorname{Ker}(F^n : \hat{W} \to \hat{W})$, where $\hat{W}$ is the formal Witt group (see Appendix A). For any $\mathbb{F}_p$-algebra $\tilde{R}$, the subgroup $\hat{W}^{(F^n)}(\tilde{R}) \subset W(\tilde{R})$ consists of Witt vectors $x \in W(\tilde{R})$ such that $F^n(x) = 0$ and almost all components of x are zero. Note that $\hat{W}^{(F^n)}(\tilde{R})$ is a $W(\tilde{R})$-submodule of $W(\tilde{R})$ and moreover, a $W_n(\tilde{R})$-submodule.

The group ind-schemes $C^0_{\mathscr{P}}, C^{-1}_{\mathscr{P}}$ from (7.1) are as follows: for any R-algebra $\tilde{R}$,

$$C^0_{\mathscr{P}}(\tilde{R}) := \hat{W}^{(F^n)}(\tilde{R}) \otimes_{W_n(R)} P,$$

$$C^{-1}_{\mathscr{P}}(\tilde{R}) := \operatorname{Ker}(C^0_{\mathscr{P}}(\tilde{R}) \stackrel{\pi}{\twoheadrightarrow} \mathbb{G}_a^{(F^n)}(\tilde{R}) \otimes_R (P/Q)),$$

where π comes from the map $\hat{W}^{(F^n)} \to \mathbb{G}_a^{(F^n)}$ that takes a Witt vector to its 0-th component.

The additive homomorphism $\Phi : C^{-1}_{\mathscr{P}}(\tilde{R}) \to C^0_{\mathscr{P}}(\tilde{R})$ is uniquely determined by the following properties:

$$\Phi(V(a) \otimes x) = a \otimes F(x) \quad \text{for } a \in \hat{W}^{(F^n)}(\tilde{R}),\ x \in P, \tag{7.4}$$

$$\Phi(a \otimes y) = F(a) \otimes F_1(y) \quad \text{for } a \in \hat{W}^{(F^n)}(\tilde{R}),\ y \in Q. \tag{7.5}$$

The r.h.s. of (7.5) makes sense (despite the fact that $F_1(y)$ is defined only modulo $J_{n,\tilde{R}} \cdot P$) because for any $a \in \hat{W}^{(F^n)}(\tilde{R})$ and $b \in W(\tilde{R})$ one has $V^{n-1}(b) \cdot F(a) = V^{n-1}(b \cdot F^n(a)) = 0$.

7.1.3. *On the proof of Proposition 3.11 of* [LZ]. In §7.1.1 we formulated a result from [LZ]. In [LZ] it is deduced from Theorem 81 of [Zi02], whose proof (given on p.80-81 of [Zi02]) does not use the surjectivity assumption[13] from Zink's definition of display. So Theorem 81 of [Zi02] and Proposition 3.11 of [LZ] are valid for *semi*displays[14].

The idea behind the proof of the result mentioned in §7.1.1 is roughly as follows: the group ind-schemes $C^0_{\mathscr{P}}, C^{-1}_{\mathscr{P}}$ are n-smooth in a certain sense[15], and one checks that the map

$$f : \operatorname{Lie}(C^{-1}_{\mathscr{P}}) \to \operatorname{Lie}(C^0_{\mathscr{P}}) \tag{7.6}$$

induced by $1 - \Phi$ is a monomorphism whose cokernel is a finitely generated projective R-module. For completeness, let us prove these properties of f; we will also give an explicit description of $\operatorname{Coker} f = \operatorname{Lie}(\mathfrak{Z}_{\mathscr{P}})$.

Lemma 7.1.4. *(i) The map* (7.6) *is injective.*

(ii) One has a canonical isomorphism of restricted Lie R-algebras

$$\operatorname{Lie}(\mathfrak{Z}_{\mathscr{P}}) \xrightarrow{\sim} P/Q, \tag{7.7}$$

where the structure of restricted Lie algebra on P/Q is as in §6.1.3(d). In particular, $\operatorname{rank}(\mathfrak{Z}_{\mathscr{P}}) = \operatorname{rank}(P/Q)$.

Proof. A commutative restricted Lie R-algebra $\mathfrak{g}$ is the same as a left $R[F]$-module, where $Fa = a^pF$ for all $a \in R$. If $\mathfrak{g} = \operatorname{Lie}(G)$, where G is a commutative group (ind)-scheme over R, then $F : \mathfrak{g} \to \mathfrak{g}$ comes from $V : \operatorname{Fr}^* G \to G$.

Let us describe (7.6) as a homomorphism of $R[F]$-modules. One has a canonical isomorphism of $R[F]$-modules

$$R[F] \xrightarrow{\sim} \operatorname{Lie}(\hat{W}_R) = \operatorname{Lie}(\hat{W}_R^{(F^n)});$$

[13]Part (ii) of [Zi02, §1, Def. 1].

[14]More details about this can be found in the proof of [L25, Prop. 11.13].

[15]We defined n-smoothness only for group *schemes.*

namely, $1 \in R[F]$ corresponds to the the derivative of the Teichmüller map $\hat{\mathbb{A}}^1_R \to \hat{W}_R$ at $0 \in \hat{\mathbb{A}}^1_R$. Let $\bar{P} = P/I_{n,R}P$, $\bar{Q} = Q/I_{n,R}P$. Then $\mathrm{Lie}(C^0_{\mathscr{P}}) = R[F] \otimes_R \bar{P}$, and $\mathrm{Lie}(C^1_{\mathscr{P}})$ is the $R[F]$-submodule of $R[F] \otimes_R \bar{P}$ generated by $1 \otimes \bar{Q}$ and the elements $F \otimes x$, where $x \in \bar{P}$. Our map $f : \mathrm{Lie}(C^{-1}_{\mathscr{P}}) \to \mathrm{Lie}(C^0_{\mathscr{P}})$ equals $\mathrm{Lie}(1 - \Phi)$, where Φ is given by (7.4)-(7.5). So

$$f(1 \otimes x) = 1 \otimes x \;\text{ for } x \in \bar{Q}, \quad f(F \otimes x) = F \otimes x - 1 \otimes F(x) \;\text{ for } x \in \bar{P}.$$

This description of f implies the lemma. □

7.2. **The Cartier dual of $\mathfrak{Z}_{\mathscr{P}}$.**

7.2.1. *Goal of this subsection.* Let $\mathscr{P} = (P, Q, F, F_1) \in \mathrm{sDisp}_n(R)$. Let $\mathfrak{Z}^*_{\mathscr{P}} := \underline{\mathrm{Hom}}(\mathfrak{Z}_{\mathscr{P}}, \mathbb{G}_m)$, where $\mathfrak{Z}_{\mathscr{P}} \in \mathrm{Sm}_n(R)$ is as in §7.1.1-7.1.2. According to the general principle from §2.5, the group scheme $\mathfrak{Z}^*_{\mathscr{P}} \in \mathrm{Sm}^*_n(R)$ is easier to understand than $\mathfrak{Z}_{\mathscr{P}}$ itself. Namely, we are going to prove Proposition 7.2.3, which establishes a canonical isomorphism

$$\mathfrak{Z}^*_{\mathscr{P}} \xrightarrow{\sim} \underline{\mathrm{Hom}}(\mathscr{P}, \mathbf{1}_{n,R}) \tag{7.8}$$

where $\mathbf{1}_{n,R} \in \mathrm{sDisp}_n(R)$ is given by (6.9). Here $\underline{\mathrm{Hom}}(\mathscr{P}, \mathbf{1}_{n,R})$ is the group R-scheme whose group of points over any R-algebra R' is $\mathrm{Hom}(\mathscr{P}', \mathbf{1}_{n,R'})$, where $\mathscr{P}'$ is the base change of $\mathscr{P}$ to R'. Explicitly, $\mathrm{Hom}(\mathscr{P}', \mathbf{1}_{n,R'})$ is the group of $W_n(R)$-linear maps $\eta : P \to W_n(R')$ such that

$$\eta(F(x)) = F(\eta(x)) \quad \text{for all } x \in P, \tag{7.9}$$

$$\eta(y) = V(\eta(F_1(y))) \quad \text{for all } y \in Q. \tag{7.10}$$

Note that

$$\underline{\mathrm{Hom}}(\mathscr{P}, \mathbf{1}_{n,R}) \subset \mathrm{Hom}_{W_n(R)}(P, W_{n,R}). \tag{7.11}$$

7.2.2. $\mathfrak{Z}^*_{\mathscr{P}}$ *as a subgroup of* $\mathrm{Hom}_{W_n(R)}(P, W_{n,R})$. By §7.1.1-7.1.2, $\mathfrak{Z}_{\mathscr{P}}$ is a quotient of the group ind-scheme $C^0_{\mathscr{P}} := \hat{W}^{(F^n)}_R \otimes_{W_n(R)} P$, so $\mathfrak{Z}^*_{\mathscr{P}}$ is a subgroup of $(C^0_{\mathscr{P}})^*$.

We will be using the canonical nondegenerate pairing

$$\hat{W}^{(F^n)}_R \times W_{n,R} \to (\mathbb{G}_m)_R\,, \tag{7.12}$$

which comes from the usual Cartier duality between W_R and $\hat{W}_R$, see Appendix A. This pairing induces an isomorphism

$$(C^0_{\mathscr{P}})^* \xrightarrow{\sim} \mathrm{Hom}_{W_n(R)}(P, W_{n,R}).$$

Thus

$$\mathfrak{Z}^*_{\mathscr{P}} \subset \mathrm{Hom}_{W_n(R)}(P, W_{n,R}). \tag{7.13}$$

Proposition 7.2.3. *The subgroups of* $\mathrm{Hom}_{W_n(R)}(P, W_{n,R})$ *given by* (7.11) *and* (7.13) *are equal to each other.*

Proof. Let R' be an R-algebra and η an R'-point of $\mathrm{Hom}_{W_n(R)}(P, W_{n,R})$, i.e., $\eta : P \to W_n(R')$ is a $W_n(R)$-linear map. The problem is to show that $\eta \in \mathfrak{Z}^*_{\mathscr{P}}(R')$ if and only if (7.9) and (7.10) hold.

Looking at formulas (7.1)-(7.5), we see that $\eta \in \mathfrak{Z}^*_{\mathscr{P}}(R')$ if and only if for every R'-algebra R'' and every $a \in \hat{W}^{(F^n)}(R'')$ the following conditions hold:

$$\langle V(a), \eta(x)\rangle = \langle a, \eta(F(x))\rangle \quad \text{for all } x \in P, \tag{7.14}$$

$$\langle a, \eta(y)\rangle = \langle F(a), \eta(F_1(y))\rangle \quad \text{for all } y \in Q. \tag{7.15}$$

Here $\langle\, , \rangle$ stands for the pairing (7.12).

Finally, conditions (7.14)-(7.15) are equivalent to (7.9)-(7.10). This follows from the equalities

$$\langle V(a), \eta(x)\rangle = \langle a, F(\eta(x))\rangle, \quad \langle F(a), \eta(F_1(y))\rangle = \langle a, V(\eta(F_1(y)))\rangle$$

(see formula (A.8) from Appendix A) and the nondegeneracy of the pairing (7.12). $\square$

7.2.4. *Remark.* $\mathfrak{Z}_{\mathscr{P}} = \underline{\mathrm{Hom}}(\mathfrak{Z}^*_{\mathscr{P}}, \mathbb{G}_{m,R})$, so $\mathrm{Lie}(\mathfrak{Z}_{\mathscr{P}}) = \mathrm{Hom}(\mathfrak{Z}^*_{\mathscr{P}}, \mathbb{G}_{a,R})$. Therefore (7.7) is a canonical isomorphism $\mathrm{Hom}(\mathfrak{Z}^*_{\mathscr{P}}, \mathbb{G}_{a,R}) \xrightarrow{\sim} P/Q$. Looking at the proof of Lemma 7.1.4(ii), we see that the the following diagram commutes:

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Hom}(\mathfrak{Z}^*_{\mathscr{P}}, W_{n,R}) \\ \downarrow & & \downarrow \\ P/Q & \xrightarrow{\sim} & \mathrm{Hom}(\mathfrak{Z}^*_{\mathscr{P}}, \mathbb{G}_{a,R}) \end{array} \tag{7.16}$$

(the upper row of the diagram comes from the embedding $\mathfrak{Z}^*_{\mathscr{P}} \hookrightarrow \mathrm{Hom}_{W_n(R)}(P, W_{n,R})$, see formula (7.13)).

7.3. **The functor** $\mathrm{sDisp}_n^{\mathrm{weak}}(R) \to \mathrm{Sm}_n(R)$**.** Recall that $\mathbf{1}_{n,R}$ is the unit object in both $\mathrm{sDisp}_n(R)$ and $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$ (see §6.3.2-6.3.3). For $\mathscr{P} \in \mathrm{sDisp}_n^{\mathrm{weak}}(R)$ we define the group R-scheme $\underline{\mathrm{Hom}}(\mathscr{P}, \mathbf{1}_{n,R})$ just as in §7.2.1.

Lemma 7.3.1. *Let $\mathscr{P}' \in \mathrm{sDisp}_n(R)$. Let $\mathscr{P} \in \mathrm{sDisp}_n^{\mathrm{weak}}(R)$ be the image of $\mathscr{P}'$ (see §6.1.5). Then the natural map $\underline{\mathrm{Hom}}(\mathscr{P}, \mathbf{1}_{n,R}) \to \underline{\mathrm{Hom}}(\mathscr{P}', \mathbf{1}_{n,R})$ is an isomorphism.*

Proof. Write $\mathscr{P}' = (P, Q, F, F_1)$. The problem is to show that for every $f \in \mathrm{Hom}(\mathscr{P}', \mathbf{1}_{n,R})$ one has $f(J_{n,R} \cdot Q) = 0$. By the definition of $\mathbf{1}_{n,R}$ (see formula (6.9)), we have $f(Q) \subset I_{n,R}$, so $f(J_{n,R} \cdot Q) \subset J_{n,R} \cdot I_{n,R} = 0$. $\square$

Lemma 7.3.2. *For every $\mathscr{P} \in \mathrm{sDisp}_n^{\mathrm{weak}}(R)$, the group scheme $\underline{\mathrm{Hom}}(\mathscr{P}, \mathbf{1}_{n,R})$ is n-cosmooth.*

Proof. As noted in §6.1.5, the functor $\mathrm{sDisp}_n^{\mathrm{weak}}(R) \to \mathrm{sDisp}_n(R)$ is essentially surjective. So by Lemma 7.3.1, it suffices to prove n-cosmoothness of $\underline{\mathrm{Hom}}(\mathscr{P}', \mathbf{1}_{n,R})$ for $\mathscr{P}' \in \mathrm{sDisp}_n(R)$. By (7.8), $\underline{\mathrm{Hom}}(\mathscr{P}', \mathbf{1}_{n,R}) = \mathfrak{Z}^*_{\mathscr{P}'}$. Finally, $\mathfrak{Z}_{\mathscr{P}'}$ is n-smooth by the result of [LZ] mentioned in §7.1.1. $\square$

7.3.3. *The functor* $\mathrm{sDisp}_n^{\mathrm{weak}}(R) \to \mathrm{Sm}_n(R)$. Let $\mathscr{P} \in \mathrm{sDisp}_n^{\mathrm{weak}}(R)$. By Lemma 7.3.2, $\underline{\mathrm{Hom}}(\mathscr{P}, \mathbf{1}_{n,R}) = \mathfrak{Z}^*_{\mathscr{P}}$ for some $\mathfrak{Z}_{\mathscr{P}} \in \mathrm{Sm}_n(R)$. The assignment $\mathscr{P} \mapsto \mathfrak{Z}_P$ is a functor $\mathrm{sDisp}_n^{\mathrm{weak}}(R) \to \mathrm{Sm}_n(R)$. By Lemma 7.3.1, precomposing this functor with the functor $\mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n^{\mathrm{weak}}(R)$, we get the functor $\mathfrak{Z}$ from §7.1.1.

7.3.4. *Example:* $n = 1$. An object of $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$ is a quadruple $(M, \mathcal{Q}, F, F_1)$ see §6.1.5. Now suppose that $n = 1$. Then $\mathcal{Q} = 0$, $F_1 = 0$, and M is a projective R-module. So an object $\mathscr{P} \in \mathrm{sDisp}_1^{\mathrm{weak}}(R)$ is just a pair (M, F), where M is a projective R-module and $F : M \to M$ is a p-linear map. To such a pair we associated in §2.4 a 1-cosmooth group scheme $A_{M,F}$. It is easy to see that

$$\mathfrak{Z}^*_{\mathscr{P}} = A_{M,F}. \tag{7.17}$$

7.4. $\mathfrak{Z}^*_{\mathscr{P}}$ **via Dieudonné modules.** Let R be an $\mathbb{F}_p$-algebra.

7.4.1. *The ring* $\mathfrak{D}_{n,R}$. Let $\mathfrak{D}_{n,R}$ be the n-truncated Dieudonné-Cartier ring of R. It is generated by the ring $W_n(R)$ and elements F, V; the defining relations in $\mathfrak{D}_{n,R}$ are

$$V^n = 0, \quad FV = p,$$

$$F \cdot a = F(a) \cdot F, \quad a \cdot V = V \cdot F(a), \quad V \cdot a \cdot F = V(a) \quad \text{for all } a \in W_n(R).$$

Note that $VF = V \cdot 1 \cdot F = V(1) = p$. For every $a \in W_n(R)$ and every $i \in \{1, \ldots, n-1\}$ we have $V^i \cdot (V^{n-i}(a)) = V^n \cdot a \cdot F^{n-i} = 0$; in particular,

$$V \cdot J_{n,R} = 0. \tag{7.18}$$

7.4.2. *The goal.* Let $\mathfrak{D}_{n,R}$-mod be the category of left $\mathfrak{D}_{n,R}$-modules. We will define a functor[16] $D : \mathrm{sDisp}_n^{\mathrm{weak}}(R) \to \mathfrak{D}_{n,R}$-mod such that for every $\mathscr{P} \in \mathrm{sDisp}_n^{\mathrm{weak}}(R)$ one has

$$\mathfrak{Z}^*_{\mathscr{P}} = \mathrm{Hom}_{\mathfrak{D}_{n,R}}(D(\mathscr{P}), W_{n,R}). \tag{7.19}$$

The r.h.s. of (7.19) makes sense because $\mathfrak{D}_{n,R}$ acts on the group scheme $W_{n,R}$.

7.4.3. *Definition of* $D(\mathscr{P})$. By §6.1.5, an object $\mathscr{P} \in \mathrm{sDisp}_n^{\mathrm{weak}}(R)$ is a quadruple

$$(M, \mathcal{Q}, F : M \to M, F_1 : \mathcal{Q} \to M/J_{n,R} \cdot M).$$

$D(\mathscr{P}) \in \mathfrak{D}_{n,R}$-mod is defined as follows: if $N \in \mathfrak{D}_{n,R}$-mod then $\mathrm{Hom}_{\mathfrak{D}_{n,R}}(D(\mathscr{P}), N)$ is the group of $W_n(R)[F]$-homomorphisms $f : M \to N$ such that

$$V(f(F_1(y)) = f(y) \quad \text{for all } y \in \mathcal{Q}. \tag{7.20}$$

Although $F_1(y)$ lives in $M/J_{n,R} \cdot M$ rather than in M, the l.h.s. of (7.20) makes sense by virtue of (7.18).

Formula (7.19) immediately follows from the definition of $\mathfrak{Z}^*_{\mathscr{P}}$ (see §7.3.3) and the definition of $\mathbf{1}_{n,R}$ (see formula (6.9)).

Proposition 7.4.4. *The* $\mathfrak{D}_{n,R}$*-module* $D(\mathscr{P})$ *is n-cosmooth in the sense of* [KM, Def. 1.0.2].

The proposition will be proved in §7.4.8.

Corollary 7.4.5. $D(\mathscr{P})$ *is the n-cosmooth* $\mathfrak{D}_{n,R}$*-module corresponding to* $\mathfrak{Z}_{\mathscr{P}}$ *via the "n-truncated Cartier theory" developed in* [KM].

Proof. Combine Proposition 7.4.4 with formula (7.19). ☐

7.4.6. *An economic presentation of* $D(\mathscr{P})$. Let $\mathscr{P}, M, \mathcal{Q}, F, F_1$ be as in §7.4.3. Choose a normal decomposition

$$M = T \oplus \bar{L}, \quad \mathcal{Q} = I_{n,R} \cdot T \oplus \bar{L},$$

where T is a finitely generated projective $W_n(R)$-module and $\bar{L}$ is a finitely generated projective $W_{n-1}(R)$-module. Then $F : T \to M$ is a pair $(\varphi_{TT} : T \to T, \varphi_{\bar{L}T} : T \to \bar{L})$, and $F_1 : \bar{L} \to M$ is a pair $(\varphi_{T\bar{L}} : \bar{L} \to T/J_{n,R} \cdot T, \varphi_{\bar{L}\bar{L}} : \bar{L} \to \bar{L})$. The following proposition represents $D(\mathscr{P})$ as a quotient of $\mathfrak{D}_{n,R} \otimes_{W_n(R)} T$ by an explicit submodule.

[16]Without n-truncation, the functor D is well known: see [Zi02, Prop. 90] and formula (4) on p.141 of [Zi01] (this formula goes back to the article [N], which inspired Zink's notion of display).

Proposition 7.4.7. *If $N \in \mathfrak{D}_{n,R}$-mod then $\operatorname{Hom}_{\mathfrak{D}_{n,R}}(D(\mathscr{P}), N)$ is the group of $W_n(R)$-linear maps $g : T \to N$ such that*

$$F(g(x)) = g(\varphi_{TT}(x)) + \sum_{i=1}^{n-1} V^i \varphi_{T\bar{L}} \varphi_{\bar{L}\bar{L}}^{i-1} \varphi_{\bar{L}T}(x) \quad \textit{for all } x \in T. \tag{7.21}$$

Note that by (7.18), the r.h.s. of (7.21) is well-defined even though $\varphi_{T\bar{L}} \varphi_{\bar{L}\bar{L}}^{i-1} \varphi_{\bar{L}T}$ is a map $T \to T/J_{n,R} \cdot T$.

Proof. Write the map $f : M \to N$ from §7.4.3 as a pair (g, h), where $g \in \operatorname{Hom}_{W_n(R)}(T, N)$, $h \in \operatorname{Hom}_{W_n(R)}(\bar{L}, N)$. The conditions for g, h are as follows:

$$F \circ g = g \circ \varphi_{TT} + h \circ \varphi_{\bar{L}T}, \tag{7.22}$$

$$h - V \circ h \circ \varphi_{\bar{L}\bar{L}} = V \circ g \circ \varphi_{T\bar{L}}. \tag{7.23}$$

Since $V^n = 0$, the map $h \mapsto V \circ h \circ \varphi_{\bar{L}\bar{L}}$ is nilpotent. So using (7.23), one can express h in terms of g. Then (7.22) becomes condition (7.21). □

7.4.8. *Proof of Proposition 7.4.4.* Proposition 7.4.4 follows from Proposition 7.4.7 combined with [KM, Prop. 4.3.1]. (On the other hand, C. Kothari noticed that one can deduce Proposition 7.4.4 from [Zi02, Prop. 90].) □

8. A formula for $\mathfrak{Z}_{\mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_l}$, where $\mathscr{P}_i \in \mathrm{sDisp}_n^{\mathrm{weak}}(R)$

Let R be an $\mathbb{F}_p$-algebra.

8.1. **Formulation of the result.** Recall that $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$ is a tensor category, see §6.3. By §7.3, we have a functor

$$\mathrm{sDisp}_n^{\mathrm{weak}}(R)^{\mathrm{op}} \to \mathrm{Sm}_n^*(R), \quad \mathscr{P} \mapsto \mathfrak{Z}_{\mathscr{P}}^* = \underline{\operatorname{Hom}}(\mathscr{P}, \mathbf{1}_{n,R}). \tag{8.1}$$

For each $\mathscr{P}_1, \ldots, \mathscr{P}_l \in \mathrm{sDisp}_n^{\mathrm{weak}}(R)$, we have the tensor product map

$$\underline{\operatorname{Hom}}(\mathscr{P}_1, \mathbf{1}_{n,R}) \times \ldots \times \underline{\operatorname{Hom}}(\mathscr{P}_l, \mathbf{1}_{n,R}) \to \underline{\operatorname{Hom}}(\mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_l, \mathbf{1}_{n,R}), \tag{8.2}$$

which is a poly-additive map

$$\mathfrak{Z}_{\mathscr{P}_1}^* \times \ldots \mathfrak{Z}_{\mathscr{P}_l}^* \to \mathfrak{Z}_{\mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_l}^*. \tag{8.3}$$

The map (8.3) induces a group homomorphism

$$\mathfrak{Z}_{\mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_l} \to \underline{\text{Poly-add}}(\mathfrak{Z}_{\mathscr{P}_1}^* \times \ldots \mathfrak{Z}_{\mathscr{P}_l}^*, \mathbb{G}_m)^{(F^n)}; \tag{8.4}$$

as before, the superscript (F^n) means passing to the kernel of F^n.

Proposition 8.1.1. *The map* (8.4) *is an isomorphism.*

The proposition will be proved in §8.3.

8.1.2. *Remarks.* (i) In the $n = 1$ case Proposition 8.1.1 is equivalent to Proposition 2.4.7; this follows from §7.3.4.

(ii) Proposition 8.1.1 is a part of a bigger and "cleaner" picture, which is more or less described in §8 of an older version of this paper[17] (but without detailed proofs). The main point is that $\mathrm{Sm}_n^*(R)$ has a natural structure of a tensor category and the functor (8.1) has a natural structure of a tensor functor. In the case $n = 1$ this was proved in §2.4.

(iii) Let $\mathfrak{D}_{n,R}$ be the n-truncated Dieudonné-Cartier ring of R, see §7.4. The "n-truncated Cartier theory" developed in [KM] provides a fully faithful functor from $\mathrm{Sm}_n(R)$ to the category of $\mathfrak{D}_{n,R}$-modules; the functor is $G \mapsto \underline{\mathrm{Hom}}(G^*, W_n)$. It is natural to expect that this is a tensor functor if $\mathrm{Sm}_n(R)$ is equipped with the tensor product mentioned in the previous remark and the category of $\mathfrak{D}_{n,R}$-modules is equipped with the Antieau-Nikolaus tensor product [AN, §4.2-4.3]. By §2.4, this is true for $n = 1$ (in this case a $\mathfrak{D}_{n,R}$-module is just an R-module equipped with a p-linear endomorphism, and the Antieau-Nikolaus tensor product is just the tensor product over R).

8.2. **Some lemmas.**

Lemma 8.2.1. *Let $G, \ldots, G_l \in \mathrm{Sm}_n^*(R)$. Then the natural map*

$$\mathrm{Hom}(G_1, \mathbb{G}_a) \otimes_R \ldots \otimes_R \mathrm{Hom}(G_l, \mathbb{G}_a) \to \text{Poly-add}(G_1 \times \ldots G_l, \mathbb{G}_a)$$

is an isomorphism.

Proof. To simplify the notation, assume that $l = 2$. Let $G_i' := \mathrm{Coker}(V : \mathrm{Fr}^* G_i \to G_i)$, then $G_i' \in \mathrm{Sm}_1^*(R)$. Since $\mathbb{G}_a$ is killed by V, we have $\mathrm{Hom}(G_i, \mathbb{G}_a) = \mathrm{Hom}(G_i', \mathbb{G}_a)$. Since

$$\text{Poly-add}(G_1 \times G_2, \mathbb{G}_a) = \mathrm{Hom}(G_1, \underline{\mathrm{Hom}}(G_2, \mathbb{G}_a)) = \mathrm{Hom}(G_2, \underline{\mathrm{Hom}}(G_1, \mathbb{G}_a))$$

and $\mathbb{G}_a$ is killed by V, we see that $\text{Poly-add}(G_1 \times G_2, \mathbb{G}_a) = \text{Poly-add}(G_1' \times G_2', \mathbb{G}_a)$. So the lemma reduces to the particular case $n = 1$, which was treated in Proposition 2.4.5(i). ☐

Lemma 8.2.2. *The map*

$$\mathrm{Hom}(\mathfrak{Z}^*_{\mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_l}, \mathbb{G}_a) \to \text{Poly-add}(\mathfrak{Z}^*_{\mathscr{P}_1} \times \ldots \mathfrak{Z}^*_{\mathscr{P}_l}, \mathbb{G}_a) \tag{8.5}$$

induced by (8.3) *is an isomorphism.*

Proof. To simplify the notation, assume that $l = 2$. By essential surjectivity of the functor $\mathrm{sDisp}_n(R) \to \mathrm{sDisp}_n^{\mathrm{weak}}(R)$, we can assume that $\mathscr{P}_1, \mathscr{P}_2$ are objects of $\mathrm{sDisp}_n(R)$ (rather than $\mathrm{sDisp}_n^{\mathrm{weak}}(R)$). Then so is the tensor product $\mathscr{P} := \mathscr{P}_1 \otimes \mathscr{P}_2$.

Recall that $\mathscr{P}_i$ is a quadruple (P_i, Q_i, F, F_1), see §6.1.2. Then $\mathscr{P}$ is a quadruple $(P, Q, \ldots)$ with $P = P_1 \otimes P_2$, $Q = (P_1 \otimes Q_2) + (Q_1 \otimes P_2)$, so we have a canonical isomorphism

$$(P_1/Q_1) \otimes (P_2/Q_2) \xrightarrow{\sim} P/Q. \tag{8.6}$$

By §7.2.4, $\mathrm{Hom}(\mathfrak{Z}^*_{\mathscr{P}}, \mathbb{G}_a) = P/Q$, $\mathrm{Hom}(\mathfrak{Z}^*_{\mathscr{P}_i}, \mathbb{G}_a) = P_i/Q_i$. So by Lemma 8.2.1, (8.5) can be viewed as a map

$$P/Q \to (P_1/Q_1) \otimes_R (P_2/Q_2). \tag{8.7}$$

Commutativity of diagram (7.16) implies that (8.7) is the inverse of (8.6). ☐

[17]See version 5 of the e-print arXiv:2307.06194.

8.3. **Proof of Proposition 8.1.1.** $\underline{\text{Poly-add}}(\mathfrak{Z}^*_{\mathscr{P}_1} \times \dots \mathfrak{Z}^*_{\mathscr{P}_l}, \mathbb{G}_m)$ is a scheme of finite type over R. So the scheme $H := \underline{\text{Poly-add}}(\mathfrak{Z}^*_{\mathscr{P}_1} \times \dots \mathfrak{Z}^*_{\mathscr{P}_l}, \mathbb{G}_m)^{(F^n)}$ is a finite group R-scheme killed by F^n. We have $\text{Lie}(H) = \text{Poly-add}(\mathfrak{Z}^*_{\mathscr{P}_1} \times \dots \mathfrak{Z}^*_{\mathscr{P}_l}, \mathbb{G}_a)$.

Let $H' := \mathfrak{Z}_{\mathscr{P}_1 \otimes \dots \otimes \mathscr{P}_l}$, then $\text{Lie}(H') = \text{Lie}(\underline{\text{Hom}}(\mathfrak{Z}^*_{\mathscr{P}_1 \otimes \dots \otimes \mathscr{P}_l}, \mathbb{G}_m)) = \text{Hom}(\mathfrak{Z}^*_{\mathscr{P}_1 \otimes \dots \otimes \mathscr{P}_l}, \mathbb{G}_a)$.

The homomorphism $\text{Lie}(H') \to \text{Lie}(H)$ corresponding to our homomorphism $f : H' \to H$ is the map (8.5), which is an isomorphism by Lemma 8.2.2. Moreover, for any R-algebra $\tilde{R}$, the map $\text{Lie}(H' \otimes_R \tilde{R}) \to \text{Lie}(H \otimes_R \tilde{R})$ corresponding to f is an isomorphism.

Thus $f : H' \to H$ is a closed immersion satitsfying the conditions of Lemma 2.2.4. So f is an isomorphism. $\square$

9. Explicit description of $\text{Lau}_n^{d,d'}$

9.1. Formulation of the result.

9.1.1. *The goal.* Let Sm_n^r denote the stack of groupoids formed by commutative n-smooth group schemes of rank r. Our goal is to describe the commutative group scheme $\text{Lau}_n^{d,d'}$ on the stack $\text{Disp}_n^{d,d'}$. By Theorem 4.2.2(ii), this group scheme is n-smooth of rank $d'(d-d')$. Thus $\text{Lau}_n^{d,d'}$ corresponds to a morphism $\text{Disp}_n^{d,d'} \to \text{Sm}_n^{d'(d-d')}$. Our goal is to describe this morphism explicitly.

9.1.2. *Remark.* By the second part of Lemma 7.1.4(ii), Zink's functor defines a morphism

$$\mathfrak{Z} : \text{sDisp}_n^{d,d'} \to \text{Sm}_n^{d'},$$

where $\text{sDisp}_n^{d,d'}$ is the stack of groupoids defined in §6.5.

9.1.3. *A morphism* $\text{Disp}_n^{d,d'} \to \text{sDisp}_n^{d^2,d'(d-d')}$. According to [L13, Rem. 4.4], there is a duality functor[18]

$$\text{Disp}_n(R)^{\text{op}} \xrightarrow{\sim} \text{Disp}_n(R), \quad \mathscr{P} \mapsto \mathscr{P}^t$$

parallel to the duality functor for non-truncated displays defined in [Zi02, Def. 19]. It induces an isomorphism of stacks of groupoids

$$\text{Disp}_n^{d,d'} \xrightarrow{\sim} \text{Disp}_n^{d,d-d'}\,. \tag{9.1}$$

Combining it with the morphisms $\text{Disp}_n^{d,d'} \to \text{sDisp}_n^{d,d'}$ and $\text{Disp}_n^{d,d-d'} \to \text{sDisp}_n^{d,d-d'}$, we get a morphism

$$\text{Disp}_n^{d,d'} \to \text{sDisp}_n^{d,d'} \times \text{sDisp}_n^{d,d-d'}\,. \tag{9.2}$$

The tensor product from §6.3.2 gives a morphism

$$\text{sDisp}_n^{d,d'} \times \text{sDisp}_n^{d,d-d'} \xrightarrow{\otimes} \text{sDisp}_n^{d^2,d'(d-d')}\,. \tag{9.3}$$

Composing (9.2) and (9.3), we get a morphism

$$\text{Disp}_n^{d,d'} \to \text{sDisp}_n^{d^2,d'(d-d')}\,. \tag{9.4}$$

[18]As explained to me by E. Lau, this functor can be defined by formula (10.17) below; in this formula $\text{Disp}_n(R)$ is identified with $\text{DISP}_n^{[0,1]}(R)$ as explained in §10.7.2.

9.1.4. *Remark.* In §10.8 we will give another description of the above morphism (9.4).

Theorem 9.1.5. *The n-smooth group scheme* $\mathrm{Lau}_n^{d,d'}$ *on* $\mathrm{Disp}_n^{d,d'}$ *corresponds to the composite morphism*

$$\mathrm{Disp}_n^{d,d'} \to \mathrm{sDisp}_n^{d^2,d'(d-d')} \xrightarrow{\mathfrak{Z}} \mathrm{Sm}_n^{d'(d-d')},$$

where the first arrow is (9.4) *and the second one is given by Zink's functor (see §9.1.2).*

The proof will be given in §9.3. It is based on the result of E. Lau and T. Zink described in the next subsection.

9.2. **The group schemes $\mathcal{A}_n^{d,d'}$ and $\mathcal{B}_n^{d,d'}$.** In §4.4 we defined commutative n-smooth group schemes $\mathcal{A}_n^{d,d'}$ and $\mathcal{B}_n^{d,d'}$ on $\mathrm{Disp}_n^{d,d'}$. The rank of $\mathcal{A}_n^{d,d'}$ (resp. $\mathcal{B}_n^{d,d'}$) equals d' (resp. $d-d'$). Thus $\mathcal{A}_n^{d,d'}$ corresponds to a morphism

$$\mathrm{Disp}_n^{d,d'} \to \mathrm{Sm}_n^{d'}, \tag{9.5}$$

and $\mathcal{B}_n^{d,d'}$ corresponds to a morphism

$$\mathrm{Disp}_n^{d,d'} \to \mathrm{Sm}_n^{d-d'}. \tag{9.6}$$

Proposition 9.2.1. *(i) The morphism* (9.5) *equals the composition*

$$\mathrm{Disp}_n^{d,d'} \to \mathrm{sDisp}_n^{d,d'} \xrightarrow{\mathfrak{Z}} \mathrm{Sm}_n^{d'}.$$

(ii) The morphism (9.6) *equals the composition*

$$\mathrm{Disp}_n^{d,d'} \xrightarrow{\sim} \mathrm{Disp}_n^{d,d-d'} \to \mathrm{sDisp}_n^{d,d-d'} \xrightarrow{\mathfrak{Z}} \mathrm{Sm}_n^{d-d'},$$

where the first arrow comes from the duality functor for n-truncated displays.

Proof. Statement (i) is [LZ, Lemma 3.12]. Statement (ii) follows from (i) and commutativity of the diagram

$$\begin{array}{ccc} \overline{\mathscr{BT}}_n^{d,d'} & \xrightarrow{\sim} & \overline{\mathscr{BT}}_n^{d,d-d'} \\ \downarrow & & \downarrow \\ \mathrm{Disp}_n^{d,d'} & \xrightarrow{\sim} & \mathrm{Disp}_n^{d,d-d'} \end{array} \tag{9.7}$$

whose horizontal arrows are given by the duality functors. A proof of commutativity of (9.7) was communicated to me by E. Lau; for the non-truncated version of this statement, see [L13, Rem. 2.3] and references therein. □

9.2.2. *Remark.* Combining Proposition 9.2.1 with Lemma 7.1.4(ii), one gets an explicit description of the restricted Lie algebras $\mathrm{Lie}(\mathcal{A}_n^{d,d'})$ and $\mathrm{Lie}(\mathcal{B}_n^{d,d'})$ from Theorem 4.4.2(ii).

9.3. **Proof of Theorem 9.1.5.** Theorem 4.4.2(iii) provides an isomorphism

$$\mathrm{Lau}_n^{d,d'} \xrightarrow{\sim} \underline{\text{Poly-add}}((\mathcal{A}_n^{d,d'})^* \times (\mathcal{B}_n^{d,d'})^*, \mathbb{G}_m)^{(F^n)}.$$

Proposition 9.2.1 provides isomorphisms $\mathcal{A}_n^{d,d'} \xrightarrow{\sim} \mathfrak{Z}_{\mathscr{P}_1}$, $\mathcal{B}_n^{d,d'} \xrightarrow{\sim} \mathfrak{Z}_{\mathscr{P}_2}$, where $\mathscr{P}_1, \mathscr{P}_2$ are certain n-truncated semidisplays over the stack $\mathrm{Disp}_n^{d,d'}$. Finally, by Proposition 8.1.1, $\underline{\text{Poly-add}}((\mathcal{A}_n^{d,d'})^* \times (\mathcal{B}_n^{d,d'})^*, \mathbb{G}_m)^{(F^n)} = \mathfrak{Z}_{\mathscr{P}_1 \otimes \mathscr{P}_2}$. □

10. The morphism $\mathrm{Disp}_n^{d,d'} \to \mathrm{sDisp}_n^{d^2,d'(d-d')}$ via higher displays

10.1. **The goal and plan of this section.**

10.1.1. *The goal and the idea.* In §9.1.3 we defined a canonical morphism

$$\mathrm{Disp}_n^{d,d'} \to \mathrm{sDisp}_n^{d^2,d'(d-d')}\,. \tag{10.1}$$

Our goal is to describe this morphism in "Shimurizable" terms, i.e., in a way which makes it clear how to generalize[19] (10.1) to a morphism $\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n$, where G is a smooth affine group scheme over $\mathbb{Z}/p^n\mathbb{Z}$ and $\mu : \mathbb{G}_m \to G$ is a 1-bounded homomorphism.

The idea is roughly as follows. The definition of the morphism (10.1) given in §9.1.3 involves something like $\rho \otimes \rho^*$, where ρ is the standard d-dimensional representation of $GL(d)$. Since $\rho \otimes \rho^*$ is the adjoint representation, it has an analog for any G (while the decomposition of the adjoint representation as a tensor product is specific for $GL(d)$).

To implement this idea, we will use the notion of higher display, which was developed by Langer-Zink and then by Lau [L21].

10.1.2. *The plan.* In §10.2-10.5 we recall some material from [L21]: the n-truncated Witt frame, the category of finitely generated projective graded modules over it, and the category of n-truncated higher (pre)displays. In §10.6 we construct a tensor functor from finitely generated projective higher predisplays to semidisplays. In §10.7 we discuss the interpretation of Disp_n and $\mathrm{sDisp}_n^{\mathrm{strong}}$ via higher predisplays. In §10.8 we implement the idea from §10.1.1.

§10.9-10.11 can be skipped by the reader. The goal of §10.9-10.10 is to introduce the tensor structure on $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$ promised in §6, and §10.11 will be used in a single sentence in §C.5.

10.2. **The Lau equivalence.** In this subsection we retell a part of E. Lau's paper [L21] (but not quite literally).

10.2.1. *The category $\mathcal{C}$.* Let $\mathcal{C}$ be the category of triples (A,t,u), where $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded ring and $t \in A_{-1}$, $u \in A_1$ are such that

(i) multiplication by u induces an isomorphism $A_i \xrightarrow{\sim} A_{i+1}$ for $i \geq 1$;

(ii) multiplication by t induces an isomorphism $A_i \xrightarrow{\sim} A_{i-1}$ for $i \leq 0$.

Because of (i) and (ii), $\mathcal{C}$ has an "economic" description. To formulate it, we will define a category $\mathcal{C}^{\mathrm{ec}}$ (where "ec" stands for "economic") and construct an equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{C}^{\mathrm{ec}}$.

10.2.2. *The category $\mathcal{C}^{\mathrm{ec}}$.* Let $\mathcal{C}^{\mathrm{ec}}$ be the category of diagrams

$$A_0 \underset{V}{\overset{F}{\rightleftarrows}} A_1, \tag{10.2}$$

where A_0 and A_1 are rings, F is a ring homomorphism, and V is an additive map such that

$$a \cdot V(a') = V(F(a)a') \quad \text{for } a \in A_0,\ a' \in A_1 \tag{10.3}$$

and for $a' \in A_1$ we have

$$F(V(a')) = \mathbf{p}a', \quad \text{where } \mathbf{p} := F(V(1)) \in A_1. \tag{10.4}$$

Note that by (10.3) we have $VF = V(1)$, which implies (10.4) if $a' \in F(A_0)$ (but not in general).

[19] For the actual generalization, see §C.4 of Appendix C.

10.2.3. *The functor $\mathcal{C} \to \mathcal{C}^{\mathrm{ec}}$.* Given a triple $(A, t, u) \in \mathcal{C}$, we construct a diagram (10.2) as follows:

(i) A_0 is the 0-th graded component of A;

(ii) A_1 is the first graded component of A, and the product of $x, y \in A_1$ is as follows: first multiply x by y in A, then apply the isomorphism $A_2 \xrightarrow{\sim} A_1$ inverse to $u : A_1 \xrightarrow{\sim} A_2$; equivalently, the product in A_1 comes from the product in $A/(u-1)A$ and the natural map $A_1 \to A/(u-1)A$, which is an isomorphism by virtue of §10.2.1(i);

(iii) $F : A_0 \to A_1$ is multiplication by u, and $V : A_1 \to A_0$ is multiplication by t.

Proposition 10.2.4. *The above functor $\mathcal{C} \to \mathcal{C}^{\mathrm{ec}}$ is an equivalence. The inverse functor $\mathfrak{L} : \mathcal{C}^{\mathrm{ec}} \to \mathcal{C}$ takes a diagram $A_0 \underset{V}{\overset{F}{\rightleftarrows}} A_1$ to a certain graded subring of the graded ring*

$$A_0[t, t^{-1}] \times A_1[u, u^{-1}], \quad \deg t := -1, \ \deg u = 1;$$

namely, the i-th graded component of the subring is the set of pairs $(at^{-i}, a'u^i)$, where $a \in A_0$ and $a' \in A_1$ satisfy the relation

$$a' = \mathbf{p}^{-i} F(a) \ \text{ if } i \leq 0, \quad a = V(\mathbf{p}^{i-1} a') \ \text{ if } i > 0. \tag{10.5}$$

(As before, $\mathbf{p} := F(V(1)) \in A_1$.) □

The functor $\mathfrak{L} : \mathcal{C}^{\mathrm{ec}} \to \mathcal{C}$ will be called the *Lau equivalence.*

The proof of the proposition is left to the reader. However, let us make some remarks.

10.2.5. *Remarks.* (i) The description of $\mathfrak{L}$ from Proposition 10.2.4 is motivated by the following observation: if $(A, t, u) \in \mathcal{C}$ then the natural map $A \to A[1/t] \times A[1/u]$ is injective, $A[1/t] = A_0[t, t^{-1}]$, and $A[1/u] = A_1[u, u^{-1}]$, where the ring structure on A_1 is as in §10.2.3(ii).

(ii) If $(A, t, u) \in \mathcal{C}$ then the nonpositively graded part of A identifies with $A_0[t]$ and the positively graded one identifies with $uA_1[u]$, where the ring structure on A_1 is as in §10.2.3(ii). So instead of describing A as a subring of $A_0[t, t^{-1}] \times A_1[u, u^{-1}]$, one could describe A as the group $A[t] \oplus uA_1[u]$ equipped with a "tricky" multiplication operation.

(iii) There exists a natural situation in which $\mathbf{p} \neq p$ and moreover, $\mathbf{p} \notin p \cdot A_1^\times$. Namely, in the case $p = 2$ this happens for the triple $({}^sW, F, \tilde{V})$ mentioned in §D.7.1.

10.2.6. *The Witt frame.* For any ring R the maps $F, V : W(R) \to W(R)$ satisfy the properties from §10.2.2 (with $\mathbf{p} = p$). Applying the Lau equivalence to the diagram $W(R) \underset{V}{\overset{F}{\rightleftarrows}} W(R)$, one gets an object of $\mathcal{C}$. Following [L21], we call it the *Witt frame.* Following [Da], we denote it by $W(R)^\oplus$ (in [L21, Example 2.1.3] it is denoted by $\underline{W}(R)$).

10.2.7. *The n-truncated Witt frame.* Let $n \in \mathbb{N}$ and let R be an $\mathbb{F}_p$-algebra. Then we have a map $F : W_n(R) \to W_n(R)$ (in addition to $V : W_n(R) \to W_n(R)$). Applying the Lau equivalence to the diagram $W_n(R) \underset{V}{\overset{F}{\rightleftarrows}} W_n(R)$, one gets an object of $\mathcal{C}$. Following [L21], we call it the *n-truncated Witt frame.* Following [Da], we denote it by $W_n(R)^\oplus$ (in Example 2.1.6 of [L21] it is denoted by $\underline{W_n}(R)$).

Let us note that $W(R)^\oplus$ and $W_n(R)^\oplus$ are particular examples of "higher frames" in the sense of [L21, §2]. We will not use more general higher frames.

10.3. **Recollections on the n-truncated Witt frame.** Let $n \in \mathbb{N}$ and let R be an $\mathbb{F}_p$-algebra. Let us recall the material from [L21] about the n-truncated Witt frame $W_n(R)^\oplus$.

10.3.1. The definition of $W_n(R)^\oplus$ given in §10.2.7 amounts to the following. We equip the rings $W_n(R)[t, t^{-1}]$ and $W_n(R)[u, u^{-1}]$ with the $\mathbb{Z}$-grading such that $\deg t = -1$, $\deg u = 1$, and $\deg a = 0$ for all $a \in W_n(R)$. Then

$$W_n(R)^\oplus \subset W_n(R)[u, u^{-1}] \times W_n(R)[t, t^{-1}]$$

is the graded subring whose i-th graded component consists of all pairs $(au^i, a't^{-i})$, where $a, a' \in W_n(R)$ are related as follows:

$$a = p^{-i}F(a') \text{ if } i \leq 0, \tag{10.6}$$

$$a' = p^{i-1}V(a) \text{ if } i > 0. \tag{10.7}$$

It is easy to check that $W_n(R)^\oplus$ is indeed a subring of $W_n(R))[u, u^{-1}] \times W_n(R))[t, t^{-1}]$; this is believable because (10.7) is an "avatar" of (10.6) (since $FV = VF = p$).

10.3.2. (i) The projection $W_n(R)^\oplus \to W_n(R)[t, t^{-1}]$ identifies the nonpositively graded part of $W_n(R)^\oplus$ with $W_n(R)[t]$. In particular, the 0-th graded component of $W_n(R)^\oplus$ identifies with $W_n(R)$.

(ii) The projection $W_n(R)^\oplus \to W_n(R)[u, u^{-1}]$ identifies the positively graded part of $W_n(R)^\oplus$ with $u \cdot W_n(R)[u]$.

10.3.3. By §10.3.2, we can view t, u as elements of $W_n(R)^\oplus$. As such, they satisfy the relation $tu = p$. The 0-th graded component of $W_n(R)^\oplus$ identifies with $W_n(R)$. So we get a graded homomorphism $W_n(R)[t, u]/(tu - p) \to W_n(R)^\oplus$. If R is perfect, this is an isomorphism.

10.3.4. As before, consider t, u as elements of $W_n(R)^\oplus$. Then we have canonical isomorphisms

$$W_n(R)^\oplus[u^{-1}] \xrightarrow{\sim} W_n(R)[u, u^{-1}], \quad W_n(R)^\oplus[t^{-1}] \xrightarrow{\sim} W_n(R)[t, t^{-1}]$$

induced by the projections $W_n(R)^\oplus \to W_n(R)[u, u^{-1}]$ and $W_n(R)^\oplus \to W_n(R)[t, t^{-1}]$.

10.3.5. The quotient of $W_n(R)^\oplus$ by the ideal generated by its graded components of nonzero degrees identifies with $W_n(R)/V(W_n(R)) = R$. Thus we get a canonical homomorphism $W_n(R)^\oplus \to R$.

10.3.6. *The homomorphisms $\sigma, \tau : W_n(R)^\oplus \to W_n(R)$.* Let $\tau : W_n(R)^\oplus \to W_n(R)$ be the composite map

$$W_n(R)^\oplus \to W_n(R)[t, t^{-1}] \xrightarrow{t=1} W_n(R).$$

Let $\sigma : W_n(R)^\oplus \to W_n(R)$ be the composite map

$$W_n(R)^\oplus \to W_n(R)[u, u^{-1}] \xrightarrow{u=1} W_n(R).$$

The homomorphisms $\sigma, \tau : W_n(R)^\oplus \to W_n(R)$ are surjective, and one has

$$\operatorname{Ker} \tau = (t - 1)W_n(R)^\oplus, \quad \operatorname{Ker} \sigma = (u - 1)W_n(R)^\oplus.$$

The restriction of τ to the 0-th graded component $W_n(R) \subset W_n(R)^\oplus$ equals the identity map. But the restriction of σ to $W_n(R) \subset W_n(R)^\oplus$ equals $F : W_n(R) \to W_n(R)$. Presumably, this was the motivation for introducing the notation σ in the paper [L21] (where the Witt vector Frobenius is also denoted by σ).

10.3.7. *Relation with the notation of Bhatt's lectures* [Bh]. In [Bh, §3.3] the letters t, u have essentially the same meaning as above. However, his grading is opposite (i.e., $\deg t = 1$, $\deg u = -1$).

10.4. **The category** $\mathrm{Vec}_n(R)$**.**

10.4.1. Let R be an $\mathbb{F}_p$-algebra. Let $S := W_n(R)^{\oplus}$. Let S_i be the i-th graded component of S, so $S_0 = W_n(R)$. If $M = \bigoplus\limits_j M_j$ is a graded S-module, we write[20] $M\{i\}$ for the graded S-module whose i-th graded component is M_{i+j}.

Let $\mathrm{Vec}_n(R)$ be the S_0-linear category of finitely generated projective graded S-modules. By [L21, Lemma 3.1.4], every $M \in \mathrm{Vec}_n(R)$ can be represented as

$$M = \bigoplus_i L_i \otimes_{S_0} S\{-i\}, \tag{10.8}$$

where L_i's are finitely generated projective S_0-modules. As noted in [L21, §3], the ranks of L_i's do not depend on the choice of a decomposition (10.8) (to see this, use the homomorphism $S \to R$ from §10.3.5).

10.4.2. For every $a \geq 0$, let $\mathrm{Vec}_n^{[0,a]}(R)$ be the full subcategory of all $M \in \mathrm{Vec}_n(R)$ such that $L_i \neq 0$ only if $0 \leq i \leq a$. Let $\mathrm{Vec}_n^{+}(R)$ be the union of $\mathrm{Vec}_n^{[0,a]}(R)$ for all $a \geq 0$. It is easy to see that an object $M \in \mathrm{Vec}_n(R)$ belongs to $\mathrm{Vec}_n^{+}(R)$ if and only if the map $t : M_i \to M_{i-1}$ is an isomorphism for all $i \leq 0$; this condition is often called *effectivity.*

10.4.3. *Tensor structure.* $\mathrm{Vec}_n(R)$ is a rigid tensor category. $\mathrm{Vec}_n^{+}(R)$ is a tensor subcategory of $\mathrm{Vec}_n(R)$, which is not rigid if $R \neq 0$. The subcategory $\mathrm{Vec}_n^{[0,a]}(R) \subset \mathrm{Vec}_n^{+}(R)$ is not closed under tensor product if $a > 0$ and $R \neq 0$. (Nevertheless, $\mathrm{Vec}_n^{[0,a]}(R)$ has a natural structure of tensor category, see §10.9.4 below.)

10.5. n**-truncated higher (pre)displays.** Let us recall the notion of higher (pre)display from [L21, Def. 3.2.1].

10.5.1. *Definitions.* If M is a $W_n(R)^{\oplus}$-module then we write M^{σ}, M^{τ} for the $W_n(R)$-modules obtained from M by base change via the homomorphisms $\sigma, \tau : W_n(R)^{\oplus} \to W_n(R)$ from §10.3.6. Let $\mathbf{preDISP}_n(R)$ be the category of pairs (M, f), where M is a graded $W_n(R)^{\oplus}$-module and $f \in \mathrm{Hom}(M^{\sigma}, M^{\tau})$. Let $\mathrm{preDISP}_n(R) \subset \mathbf{preDISP}_n(R)$ be the full subcategory defined by the condition $M \in \mathrm{Vec}_n(R)$. Let $\mathrm{DISP}_n(R)$ be the full subcategory of objects $(M, f) \in \mathrm{preDISP}_n(R)$ such that f is an isomorphism; thus $\mathrm{DISP}_n(R)$ is the category whose objects are pairs

$$(M, f), \quad M \in \mathrm{Vec}_n(R),\ f \in \mathrm{Isom}(M^{\sigma}, M^{\tau}). \tag{10.9}$$

Objects of $\mathbf{preDISP}_n(R)$ (resp. $\mathrm{DISP}_n(R)$) are called *n-truncated higher predisplays* (resp. *displays*) over R.

Let $\mathrm{DISP}_n^{+}(R) \subset \mathrm{DISP}_n(R)$ be the full subcategory defined by the condition $M \in \mathrm{Vec}_n^{+}(R)$. The categories $\mathrm{DISP}_n^{[0,a]}(R), \mathrm{preDISP}_n^{+}(R), \mathrm{preDISP}_n^{[0,a]}(R)$ are defined similarly.

[20]This notation is motivated by the following. A graded S-module is the same as an $\mathcal{O}$-module on the quotient stack $(\operatorname{Spec} S)/\mathbb{G}_m$. If R is perfect this stack is the Nygaard-filtered prismatization $R^{\mathcal{N}}$ (see [Bh]), and our notation $M\{i\}$ agrees with the notation for Breuil-Kisin twists.

The functors $M \mapsto M^\sigma$ and $M \mapsto M^\tau$ are tensor functors. So $\mathrm{preDISP}_n(R)$, $\mathrm{preDISP}_n^+(R)$, $\mathrm{DISP}_n(R)$, $\mathrm{DISP}_n^+(R)$ are tensor categories.

10.5.2. *On M^σ and M^τ.* Given a graded $W_n(R)^\oplus$-module M, set

$$M_{-\infty} := M/(t-1)M = \varinjlim(M_0 \xrightarrow{t} M_{-1} \xrightarrow{t} \ldots),$$

$$M_\infty := M/(u-1)M = \varinjlim(M_0 \xrightarrow{u} M_1 \xrightarrow{u} \ldots).$$

Define a $W_n(R)$-structure on $M_{\pm\infty}$ via the embedding $W_n(R) \hookrightarrow W_n(R)^\oplus$. Then $M_{-\infty} = M^\tau$, and M_∞ is obtained from M^σ by restriction of scalars via $F : W_n(R) \to W_n(R)$.

$\mathrm{Hom}(M^\sigma, M^\tau)$ identifies with the group of σ-linear maps $M \to M_{-\infty}$. Since $\sigma(u) = 1$ and $M_\infty := M/(u-1)M$, the latter identifies with the group of F-linear maps $M_\infty \to M_{-\infty}$.

10.6. **The tensor functor $\mathrm{preDISP}_n^+(R) \to \mathrm{sDisp}_n(R)$.** The definition of $\mathrm{sDisp}_n(R)$ was given in §6.1. As before, we will use the notation $S := W_n(R)^\oplus$ and the notation S_i for the i-th graded component of S, so $S_0 = W_n(R)$.

10.6.1. *The functor* $\mathrm{preDISP}_n^+(R) \to \mathrm{sDisp}_n(R)$. Given $(M, f) \in \mathrm{preDISP}_n^+(R)$, we will define an object $(P, Q, F, F_1) \in \mathrm{sDisp}_n(R)$.

Since $M \in \mathrm{Vec}_n^+(R)$, we have a decomposition

$$M = \bigoplus_{i \geq 0} L_i \otimes_{S_0} S\{-i\}, \tag{10.10}$$

where each L_i is a finitely generated S_0-module and $L_i = 0$ for i big enough. Set

$$P := M_0, \quad Q := tM_1 \subset P.$$

By (10.10), P is a finitely generated S_0-module and P/Q is a finitely generated projective module over $S_0/tS_1 = R$.

By §10.5.2, we can think of $f \in \mathrm{Hom}(M^\sigma, M^\tau)$ as a σ-linear map $M \to M_{-\infty} = M_0 = P$. Restricting it to M_0 and M_1, we get F-linear maps $F : P \to P$ and $\tilde{F}_1 : M_1 \to P$. We will show that

$$\tilde{F}_1(\mathrm{Ker}(M_1 \xrightarrow{t} M_0)) \subset J_{n,R} \cdot P, \quad \text{where } J_{n,R} := \mathrm{Ker}(W_n(R) \xrightarrow{V} W_n(R)) \tag{10.11}$$

and therefore $\tilde{F}_1$ induces an F-linear map $F_1 : Q \to P/J_{n,R} \cdot P$.

By (10.10), $\mathrm{Ker}(M_1 \xrightarrow{t} M_0) = S_1' \cdot M_0$, where $S_1' := \mathrm{Ker}(S_1 \xrightarrow{t} S_0)$. So to prove (10.11), it suffices to show that $\sigma(S_1') = J_{n,R}$. To see this, apply (10.7) for $i = 1$.

One checks that $(P, Q, F, F_1) \in \mathrm{sDisp}_n(R)$.

10.6.2. *Tensor structure.* Both $\mathrm{preDISP}_n^+(R)$ and $\mathrm{sDisp}_n(R)$ are tensor categories (in the case of sDisp_n, see §6.3). Let us upgrade the functor $\mathrm{preDISP}_n^+(R) \to \mathrm{sDisp}_n(R)$ constructed in §10.6.1 to a tensor functor.

If $M, M' \in \mathrm{Vec}_n(R)$ and $M'' = M \otimes M'$ then we have a morphism

$$\beta : M_0 \otimes M_0' \to M_0''. \tag{10.12}$$

If $M, M' \in \mathrm{Vec}_n^+(R)$ then (10.12) is an isomorphism and

$$\beta((tM_1) \otimes M_0' + M_0 \otimes tM_1') = tM_1''$$

(it suffices to check these statements if $M = S\{-i\}, M' = S\{-i'\}$, where $i, i' \geq 0$).

Now let $(M,f)\in \mathrm{preDISP}_n^+(R)$ and $(M',f')\in \mathrm{preDISP}_n^+(R)$. Let

$$(M'',f''):=(M,f)\otimes(M',f').$$

Then we have F-linear maps $F: M_0\to M_0$ and $F_1: tM_1\to M_0/J_{n,R}\cdot M_0$ defined in §10.6.1. We also have similar maps

$$F': M'_0\to M'_0,\quad F'_1: tM'_1\to M'_0/J_{n,R}\cdot M'_0,\quad F'': M''_0\to M''_0,\quad F''_1: tM''_1\to M''_0/J_{n,R}\cdot M''_0.$$

It remains to check the following properties of the map (10.12):

$$F''(\beta(x\otimes x'))=\beta(F(x)\otimes F(x'))\quad \text{ for } x\in M_0,\ x'\in M'_0,$$
$$F''_1(\beta(x\otimes x'))=\beta(F(x)\otimes F'_1(x'))\quad \text{ for } x\in M_0,\ x'\in tM'_1,$$
$$F''_1(\beta(x\otimes x'))=\beta(F_1(x)\otimes F'(x'))\quad \text{ for } x\in tM_1,\ x'\in M'_0.$$

This is straightforward.

10.7. **The equivalence** $\mathrm{DISP}_n^{[0,1]}(R)\xrightarrow{\sim}\mathrm{Disp}_n(R)$**.** The categories

$$\mathrm{Disp}_n(R) \text{ and } \mathrm{sDisp}_n^{\mathrm{strong}}(R)$$

were discussed in §6.2.

10.7.1. In §10.6.1 we defined a functor $\mathrm{preDISP}_n^+(R)\to \mathrm{sDisp}_n(R)$. One upgrades it to a functor

$$\mathrm{preDISP}_n^+(R)\to \mathrm{sDisp}_n^{\mathrm{strong}}(R) \tag{10.13}$$

by making the following modifications. First, instead of setting $Q:=tM_1\subset M_0$ we set $Q:=M_1$. Second, the map $\varepsilon: I_{n+1,R}\otimes_{W_n(R)}P\to Q$ mentioned in formula (6.3) is defined by identifying $I_{n+1,R}$ with S_1.

10.7.2. One checks that the functor (10.13) induces equivalences

$$\mathrm{preDISP}_n^{[0,1]}(R)\xrightarrow{\sim}\mathrm{sDisp}_n^{\mathrm{strong}}(R), \tag{10.14}$$

$$\mathrm{DISP}_n^{[0,1]}(R)\xrightarrow{\sim}\mathrm{Disp}_n(R), \tag{10.15}$$

and similarly, $\mathrm{Vec}_n^{[0,1]}(R)$ identifies with the category of "truncated pairs" from [L13, Def. 3.2]. This is known to the experts. The equivalence (10.15) is constructed in [Da, Lemma 2.25] in the case $n=\infty$.

10.7.3. *Tensor structure on* $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$*.* In §10.10 we will see that $\mathrm{preDISP}_n^{[0,a]}(R)$ is a tensor category. So (10.14) yields a structure of tensor category on $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$.

10.8. **The morphism** $\mathrm{Disp}_n^{d,d'}\to \mathrm{sDisp}_n^{d^2,d'(d-d')}$**.**

10.8.1. In §9.1.3 we defined a morphism $\mathrm{Disp}_n^{d,d'}\to \mathrm{sDisp}_n^{d^2,d'(d-d')}$ by associating to a display $\mathscr{P}\in \mathrm{Disp}_n^{d,d'}(R)$ the tensor product of the semidisplays corresponding to $\mathscr{P}$ and $\mathscr{P}^t$, where $\mathscr{P}^t$ is the dual display.

By §10.7.2, we can think of $\mathscr{P}$ as an object of $\mathrm{DISP}_n^{[0,1]}(R)$. Then the construction of §9.1.3 can be written as

$$\mathscr{P}\mapsto \Phi(\mathscr{P})\otimes\Phi(\mathscr{P}^t), \tag{10.16}$$

where $\Phi: \mathrm{preDISP}_n^+(R)\to \mathrm{sDisp}_n(R)$ is the functor from §10.6.1.

10.8.2. By §10.6.2, Φ is a tensor functor. We have

$$\mathscr{P}^t = \mathscr{P}^*\{-1\}, \tag{10.17}$$

where $\mathscr{P}^* := \operatorname{Hom}_S(\mathscr{P}, S)$ and $S := W_n(R)^\oplus$. So formula (10.16) can be rewritten as

$$\mathscr{P} \mapsto \Phi(\mathscr{P} \otimes \mathscr{P}^*\{-1\}). \tag{10.18}$$

10.8.3. Suppose that the graded projective $W_n(R)^\oplus$-module underlying $\mathscr{P}$ has rank d. Such a module is the same as a $\mathbb{G}_m$-equivariant $GL(d)$-torsor on $\operatorname{Spec} W_n(R)^\oplus$. Thus $\mathscr{P}$ can be viewed as a $\mathbb{G}_m$-equivariant $GL(d)$-torsor on $\operatorname{Spec} W_n(R)^\oplus$ equipped with an additional structure (the latter corresponds to f from formula (10.9)). $\mathscr{P} \otimes \mathscr{P}^*$ is just the vector bundle corresponding to this $GL(d)$-torsor and the adjoint representation of $GL(d)$.

10.9. **Tensor structure on $\mathrm{Vec}_n^{[0,a]}(R)$.** The reader may prefer to skip the remaining part of §10. The goal of §10.9-10.10 is to define a tensor structure on the category $\mathrm{preDISP}_n^{[0,a]}(R)$ (and therefore on $\mathrm{sDisp}_n^{\mathrm{strong}}(R)$), and §10.11 will be used in a single sentence in §C.5.

Lemma 10.9.1. *Let $f : M' \to M$ be a morphism in $\mathrm{Vec}_n^+(R)$. The following properties of f are equivalent:*

(i) f induces an isomorphism $\operatorname{Hom}(M'', M') \xrightarrow{\sim} \operatorname{Hom}(M'', M)$ for all $M'' \in \mathrm{Vec}_n^{[0,a]}(R)$;

(ii) f induces an isomorphism $\operatorname{Hom}(S\{-i\}, M') \xrightarrow{\sim} \operatorname{Hom}(S\{-i\}, M)$ for $i = 0, 1, \dots, a$;

(iii) f induces an isomorphism $M'_i \to M_i$ for all $i \le a$. ☐

Lemma 10.9.2. *(i) The inclusion $\mathrm{Vec}_n^{[0,a]}(R) \hookrightarrow \mathrm{Vec}_n^+(R)$ has a right adjoint*

$$\Pi_a : \mathrm{Vec}_n^+(R) \to \mathrm{Vec}_n^{[0,a]}(R).$$

(ii) The functor Π_a identifies $\mathrm{Vec}_n^{[0,a]}(R)$ with the localization of $\mathrm{Vec}_n^+(R)$ by the morphisms $M' \to M$ inducing an isomorphism $M'_i \to M_i$ for each $i \le a$.

Proof. Let us prove (i). Given $M \in \mathrm{Vec}_n^+(R)$, we have to construct a pair (M', f), where $M' \in \mathrm{Vec}_n^{[0,a]}(R)$ and $f \in \operatorname{Hom}(M', M)$ satisfies the equivalent conditions of Lemma 10.9.1. We can assume that $M = S\{-i\}$, $i > a$. In this case let $M' = S\{-a\}$ and let f be equal to $t^{i-a} \in S_{a-i} = \operatorname{Hom}(M', M)$.

Statement (ii) follows from Lemma 10.9.1 and the fact that Π_a is right adjoint to the inclusion $\mathrm{Vec}_n^{[0,a]}(R) \hookrightarrow \mathrm{Vec}_n^+(R)$. ☐

Lemma 10.9.3. *Let $M, M', N \in \mathrm{Vec}_n^+(R)$. Let $\tilde{M} := M \otimes N$, $\tilde{M}' := M' \otimes N$. If $f : M \to M'$ induces an isomorphism $M_i \to M'_i$ for all $i \le a$ then the morphism $\tilde{M} \to \tilde{M}'$ corresponding to f induces an isomorphism $\tilde{M}_i \to \tilde{M}'_i$ for all $i \le a$.*

Proof. This is clear if $N = S\{-j\}$, $j \ge 0$. The general case follows. ☐

10.9.4. *Tensor structure on $\mathrm{Vec}_n^{[0,a]}(R)$.* Recall that $\mathrm{Vec}_n^+(R)$ is a tensor category. The subcategory $\mathrm{Vec}_n^{[0,a]}(R) \subset \mathrm{Vec}_n^+(R)$ is not closed under tensor product if $a > 0$ and $R \ne 0$. However, Lemma 10.9.2(ii) combined with Lemma 10.9.3 provides a structure of tensor category on $\mathrm{Vec}_n^{[0,a]}(R)$ and a structure of tensor functor on the functor $\Pi_a : \mathrm{Vec}_n^+(R) \to \mathrm{Vec}_n^{[0,a]}$. Explicitly, for $M_1, M_2 \in \mathrm{Vec}_n^{[0,a]}(R)$ one has

$$M_1 \underset{a}{\otimes} M_2 = \Pi_a(M_1 \otimes M_2), \tag{10.19}$$

where $\underset{a}{\otimes}$ (resp. $\otimes$) denotes the tensor product in $\mathrm{Vec}_n^{[0,a]}(R)$ (resp. in $\mathrm{Vec}_n^+(R)$).

10.10. **Tensor structure on** $\mathrm{preDISP}_n^{[0,a]}(R)$**.**

10.10.1. Π_a *as a functor* $\mathrm{preDISP}_n^+(R) \to \mathrm{preDISP}_n^{[0,a]}(R)$. Let $M \in \mathrm{preDISP}_n^+(R)$. Define $M' \in \mathrm{Vec}_n^{[0,a]}(R) \subset \mathrm{Vec}_n^+(R)$ by $M' := \Pi_a(M)$, where Π_a is as in Lemma 10.9.2. In $\mathrm{Vec}_n^+(R)$ we have a canonical morphism $M' \to M$. It induces isomorphisms $M'_i \xrightarrow{\sim} M_i$ for $i \le a$ and therefore an isomorphism $(M')^\tau = M'_{-\infty} \xrightarrow{\sim} M_{-\infty} = M^\tau$. The composite map

$$(M')^\sigma \to M^\sigma \to M^\tau \xrightarrow{\sim} (M')^\tau$$

makes M' into an object of $\mathrm{preDISP}_n^{[0,a]}(R)$. Thus we have defined Π_a as a functor

$$\mathrm{preDISP}_n^+(R) \to \mathrm{preDISP}_n^{[0,a]}(R).$$

The functor $\Pi_a : \mathrm{preDISP}_n^+(R) \to \mathrm{preDISP}_n^{[0,a]}(R)$ is right adjoint to the inclusion

$$\mathrm{preDISP}_n^{[0,a]}(R) \hookrightarrow \mathrm{preDISP}_n^+(R).$$

10.10.2. *Tensor structure on* $\mathrm{preDISP}_n^{[0,a]}(R)$. Now one gets a structure of tensor category on $\mathrm{preDISP}_n^{[0,a]}(R)$ and a structure of tensor functor on the functor

$$\Pi_a : \mathrm{preDISP}_n^+(R) \to \mathrm{preDISP}_n^{[0,a]}$$

just as in §10.9.4, see formula (10.19).

10.11. **The Zink complex of an object of** $\mathrm{preDISP}_n(R)$**.** To any $M \in \mathrm{preDISP}_n(R)$ we will associate a complex $C_M^{\cdot}$ of commutative group ind-schemes over R. If M is in $\mathrm{preDISP}_n^+(R)$ then $C_M^{\cdot} = C_{\mathscr{P}}^{\cdot}$, where $\mathscr{P}$ is the semidisplay corresponding to M (in the sense of §10.6) and $C_{\mathscr{P}}^{\cdot}$ is given by formula (7.3). We will use the notation of §10.5.1.

10.11.1. *The complex* $K_N^{\cdot}$. Let $(N, f) \in \mathbf{preDISP}_n(R)$; according to §10.5.1, this means that N is a graded $W_n(R)^\oplus$-module and $f \in \mathrm{Hom}(N^\sigma, N^\tau)$. We have canonical maps $\alpha : N_0 \to N^\tau$ and $\beta : N_0 \to N^\sigma$, where α is $W_n(R)$-linear and β is F-linear. Define $K_N^{\cdot}$ to be the following complex of abelian groups:

(i) $K_N^i = 0$ for $i \ne 0, -1$, $K_N^{-1} = N_0$, $K_N^0 = N^\tau$,

(ii) the differential $d : K_N^{-1} \to K_N^0$ is the additive map $\alpha - f \circ \beta : N_0 \to N^\tau$.

10.11.2. *Remark.* Suppose that $(N, f) \in \mathbf{preDISP}_n(R)$ satisfies the following conditions:

(i) f is an isomorphism;

(ii) $\mathrm{Ker}(N \xrightarrow{t} N) \cap \mathrm{Ker}(N \xrightarrow{u} N) = 0$.

Then the map $(\alpha, f \circ \beta) : N_0 \to N^\tau \times N^\tau$ is injective, so it identifies $K_N^{-1} = N_0$ with a subgroup of $K_N^0 \times K_N^0$. After this identification, $d : K_N^{-1} \to K_N^0$ is just the difference between the two projections $K_N^{-1} \to K_N^0$.

10.11.3. *Cohomological interpretation.* If $f : N^\sigma \to N^\tau$ is an isomorphism then (N, f) can be interpreted as an $\mathcal{O}$-module on a certain stack $\mathscr{Y}_R$, and $K_N^{\cdot}[-1]$ computes the cohomology of this $\mathcal{O}$-module. The stack $\mathscr{Y}_R$ is obtained from the quotient stack $(\mathrm{Spec}\, W_n(R)^\oplus)/\mathbb{G}_m$ by gluing together the disjoint open substacks $(\mathrm{Spec}\, W_n(R)^\oplus[t^{-1}])/\mathbb{G}_m$ and $(\mathrm{Spec}\, W_n(R)^\oplus[u^{-1}])/\mathbb{G}_m$, which are both equal to $\mathrm{Spec}\, W_n(R)$ by §10.3.4. If R is perfect then $\mathscr{Y}_R = R^{\mathrm{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, where R^{Syn} is the *syntomification* stack from [Bh].

10.11.4. *A canonical object of* $\mathbf{preDISP}_n(R)$. Let $W(R)^{\oplus}$ be the non-truncated Witt frame, see §10.2.6. To describe $W(R)^{\oplus}$ explicitly, one just replaces W_n by W in all formulas from §10.3.1.

As a group, each graded component of $W(R)^{\oplus}$ is isomorphic to $W(R)$. Replacing each $W(R)$ by $\hat{W}^{(F^n)}(R)$ (where $\hat{W}^{(F^n)} \subset W$ is as in §7.1.2), one gets a graded ideal of $W(R)^{\oplus}$, which we denote by $\hat{W}^{(F^n)}(R)^{\oplus}$. The $W(R)^{\oplus}$-module $\hat{W}^{(F^n)}(R)^{\oplus}$ is, in fact, a $W_n(R)^{\oplus}$-module. Moreover, one has a canonical isomorphism

$$(\hat{W}^{(F^n)}(R)^{\oplus})^{\sigma} \xrightarrow{\sim} (\hat{W}^{(F^n)}(R)^{\oplus})^{\tau},$$

so $\hat{W}^{(F^n)}(R)^{\oplus}$ is an object of $\mathbf{preDISP}_n(R)$. (Usually, it is not in $\mathrm{preDISP}_n(R)$.)

10.11.5. *The Zink complex.* Let $M \in \mathbf{preDISP}_n(R)$. For any R-algebra $\tilde{R}$, we set

$$C^{\cdot}_M(\tilde{R}) := K^{\cdot}_{N(\tilde{R})},$$

where $N(\tilde{R})$ is the tensor product of $M\{1\}$ and $\hat{W}^{(F^n)}(\tilde{R})^{\oplus}$ over $W_n(R)^{\oplus}$ (the goal of shifting the grading of M in the definition of $N(\tilde{R})$ is to simplify the formulation of Lemma 10.11.7). The complex $C^{\cdot}_M$ will be called the *Zink complex* of M. Each of its terms is a functor from R-algebras to abelian groups. According to the next lemma, the functors C^i_M are nice if $M \in \mathrm{preDISP}_n(R)$ (i.e., if M is fintely generated and projective as a $W_n(R)^{\oplus}$-module).

Lemma 10.11.6. *Let* $M \in \mathrm{preDISP}_n(R)$. *Let* r *be the rank of* M *viewed as a finitely generated projective* $W_n(R)^{\oplus}$*-module.*

(i) For each i*, the functor* C^i_M *is an ind-scheme over* R.

(ii) If $i \in \{0, -1\}$ *then Zariski-locally on* $\operatorname{Spec} R$*, the group ind-scheme* C^i_M *is isomorphic to the direct sum of* r *copies of* $\hat{W}_R^{(F^n)}$. *If* $i \notin \{0, -1\}$ *then* $C^i_M = 0$.

Proof. Use the decomposition (10.8). ☐

Lemma 10.11.7. *Let* $M \in \mathrm{preDISP}_n^+(R)$. *Then one has a canonical isomorphism* $C^{\cdot}_M \xrightarrow{\sim} C^{\cdot}_{\mathscr{P}}$*, where* $\mathscr{P}$ *is the semidisplay corresponding to* M *(in the sense of §10.6) and* $C^{\cdot}_{\mathscr{P}}$ *is given by formula* (7.3). ☐

Corollary 10.11.8. *Let* $M \in \mathrm{preDISP}_n(R)$.

(i) If $M \in \mathrm{preDISP}_n^+(R)$ *then* $C^{\cdot}_M$ *is quasi-isomorphic to* $\mathfrak{Z}_{\mathscr{P}}$*, where* $\mathscr{P}$ *is the semidisplay corresponding to* M *and* $\mathfrak{Z}_{\mathscr{P}} \in \mathrm{Sm}_n(R)$ *is defined by formula* (7.2).

(ii) If $M\{1\} \in \mathrm{preDISP}_n^+(R)$ *then* $C^{\cdot}_M$ *is acyclic.*

Here the words "acyclic" and "quasi-isomorphic" are understood in the sense of complexes of functors $\{R\text{-algebras}\} \to \mathrm{Ab}$.

Proof. Statement (i) immediately follows from Lemma 10.11.7 and the definition of $\mathfrak{Z}_{\mathscr{P}}$. In the situation of (ii), the semidisplay $\mathscr{P} = (P, Q, F, F_1)$ satisfies the condition $Q = P$. This condition implies that $\mathfrak{Z}_{\mathscr{P}} = 0$ by Lemma 7.1.4(ii). ☐

10.11.9. *Remarks.* (i) If $M \notin \mathrm{preDISP}_n^+(R)$ then $H^{-1}(C^{\cdot}_M)$ can be nonzero, see formula (10.22) below.

(ii) Let $N(\tilde{R})$ be as in §10.11.5. If $M \in \mathrm{DISP}_n(R)$ then $N(\tilde{R})$ satisfies the two conditions[21] from §10.11.2. So we can identify the group K_N^{-1} with a subgroup of $K_N^0 \times K_N^0$, after which $d : K_N^{-1} \to K_N^0$ is just the difference between the two projections $K_N^{-1} \to K_N^0$.

10.11.10. *The case* $n = 1$. By [L21, Example 3.6.4], $\mathrm{DISP}_1(R)$ is canonically equivalent[22] to the category of *F-zips* over R. An F-zip over R is an R-module L equipped with a descending filtration $\mathrm{Fil}^{\cdot} L$, an ascending filtration $\mathrm{Fil}_{\cdot} L$, and an isomorphism

$$\mathrm{gr}_{\cdot} L \xrightarrow{\sim} \mathrm{Fr}^* \mathrm{gr}^{\cdot} L; \tag{10.20}$$

it is assumed that $\mathrm{gr}^{\cdot} L$ is a finitely generated projective R-module and

$$\mathrm{Fil}^i L = L \text{ for } i \ll 0, \quad \mathrm{Fil}^i L = 0 \text{ for } i \gg 0, \quad \mathrm{Fil}_i L = 0 \text{ for } i \ll 0, \quad \mathrm{Fil}_i L = L \text{ for } i \gg 0.$$

Let $M \in \mathrm{DISP}_1(R)$ correspond to an F-zip L. Then one checks that the Zink complex $C_M^{\cdot}$ is as follows. First, $C_M^0 = L^{\hat{\sharp}}$, where $L^{\hat{\sharp}} := L \otimes_R (\hat{W}_R^{(F)})$. So by §10.11.9(ii), we can think of C_M^{-1} as a subgroup of $L^{\hat{\sharp}} \times_{\mathrm{Spec}\, R} L^{\hat{\sharp}}$, and then $d : C_M^{-1} \to C_M^0 = L^{\hat{\sharp}}$ is just the difference between the two projections $C_M^{-1} \to L^{\hat{\sharp}}$. Second, this subgroup is the fiber product of $(\mathrm{Fil}^0 L)^{\hat{\sharp}}$ and $(\mathrm{Fil}_0 L)^{\hat{\sharp}}$ over $(\mathrm{gr}^0 L)^{\hat{\sharp}}$, where the map $(\mathrm{Fil}_0 L)^{\hat{\sharp}} \to (\mathrm{gr}^0 L)^{\hat{\sharp}}$ is the composition

$$(\mathrm{Fil}_0 L)^{\hat{\sharp}} \twoheadrightarrow (\mathrm{gr}_0 L)^{\hat{\sharp}} \xrightarrow{\sim} \mathrm{Fr}^*(\mathrm{gr}^0 L)^{\hat{\sharp}} \xrightarrow{V} (\mathrm{gr}^0 L)^{\hat{\sharp}}. \tag{10.21}$$

Note that $C_M^{-1} \supset (\mathrm{Fil}^1 L)^{\hat{\sharp}} \times_{\mathrm{Spec}\, R} (\mathrm{Fil}_{-1} L)^{\hat{\sharp}}$, so

$$H^{-1}(C_M^{\cdot}) \supset (\mathrm{Fil}^1 L)^{\hat{\sharp}} \cap (\mathrm{Fil}_{-1} L)^{\hat{\sharp}}. \tag{10.22}$$

In particular, $H^{-1}(C_M^{\cdot})$ can be nonzero.

Let $\tilde{C}_M^{\cdot}$ be the quotient of $C_M^{\cdot}$ by the acyclic subcomplex $0 \to (\mathrm{Fil}^1 L)^{\hat{\sharp}} \xrightarrow{\mathrm{id}} (\mathrm{Fil}^1 L)^{\hat{\sharp}} \to 0$ (so the map $C_M^{\cdot} \to \tilde{C}_M^{\cdot}$ is a quasi-isomorphism). To describe $\tilde{C}_M^{\cdot}$ more explicitly, note that $\tilde{C}_M^{-1} \subset (\mathrm{gr}^0 L)^{\hat{\sharp}} \times (\mathrm{Fil}_0 L)^{\hat{\sharp}}$ is just the graph of the map (10.21), so $\tilde{C}_M^{-1}$ identifies with $(\mathrm{Fil}_0 L)^{\hat{\sharp}}$ via the projection $(\mathrm{gr}^0 L)^{\hat{\sharp}} \times (\mathrm{Fil}_0 L)^{\hat{\sharp}} \to (\mathrm{Fil}_0 L)^{\hat{\sharp}}$. After this identification, $\tilde{C}_M^{\cdot}$ becomes the complex

$$0 \to (\mathrm{Fil}_0 L)^{\hat{\sharp}} \xrightarrow{\gamma - \delta} (L/\mathrm{Fil}^1 L)^{\hat{\sharp}} \to 0, \tag{10.23}$$

where γ is the composite map

$$(\mathrm{Fil}_0 L)^{\hat{\sharp}} \twoheadrightarrow (\mathrm{gr}_0 L)^{\hat{\sharp}} \xrightarrow{\sim} \mathrm{Fr}^*(\mathrm{gr}^0 L)^{\hat{\sharp}} \xrightarrow{V} (\mathrm{gr}^0 L)^{\hat{\sharp}} \hookrightarrow L/\mathrm{Fil}^1 L \tag{10.24}$$

and δ is the composite map $(\mathrm{Fil}_0 L)^{\hat{\sharp}} \hookrightarrow L^{\hat{\sharp}} \twoheadrightarrow (L/\mathrm{Fil}^1 L)^{\hat{\sharp}}$.

Note that if $M \in \mathrm{DISP}_n^+(R)$ then the first and last arrow of (10.24) are isomorphisms, so (10.23) is isomorphic to the complex

$$0 \to \mathrm{Fr}^*(\mathrm{gr}^0 L)^{\hat{\sharp}} \xrightarrow{V - A^{\hat{\sharp}}} (\mathrm{gr}^0 L)^{\hat{\sharp}} \to 0,$$

where $A : \mathrm{Fr}^* \mathrm{gr}^0 L \to \mathrm{gr}^0 L$ is the composition of the following linear maps:

$$\mathrm{Fr}^* \mathrm{gr}^0 L \xrightarrow{\sim} \mathrm{gr}_0 L \hookrightarrow L \twoheadrightarrow \mathrm{gr}^0 L.$$

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[21] To check condition (ii), note that $\mathrm{Ker}(L \xrightarrow{t} L) \cap \mathrm{Ker}(L \xrightarrow{u} L) = 0$ if $L = (\hat{W}^{(F^n)})^{\oplus}$.

[22] Warning: in [L21, Example 3.6.4] the filtration C^* is always descending and the filtration D_* is always ascending, even if stated otherwise! (The confusion in loc. cit. is purely terminological.)

Appendix A. Recollections on $\hat{W}$

A.1. **The group ind-schemes $\hat{W}, \hat{W}^{\mathrm{big}}$.** For a ring R, let $\hat{W}(R)$ be the set of all $x \in W(R)$ such that all components of the Witt vector x are nilpotent and almost all of them are zero. Then $\hat{W}(R)$ is an ideal in $W(R)$ preserved by F and V. Quite similarly, one defines an ideal $\hat{W}^{\mathrm{big}}(R)$ in the ring $W^{\mathrm{big}}(R)$ of big Witt vectors, which is preserved by F_n and V_n for all $n \in \mathbb{N}$.

The functors $\hat{W}$ and $\hat{W}^{\mathrm{big}}$ are group ind-schemes over $\mathbb{Z}$. It is known that $\hat{W}^{\mathrm{big}}$ is Cartier dual to W^{big}, and $\hat{W}_{\mathbb{Z}_{(p)}}$ is dual to $W_{\mathbb{Z}_{(p)}}$ (here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p, and $\hat{W}_{\mathbb{Z}_{(p)}}$ is the base change of $\hat{W}$). Details are explained below.

A.2. **Duality between W^{big} and $\hat{W}^{\mathrm{big}}$.**

A.2.1. *The canonical character of $\hat{W}^{\mathrm{big}}$.* One has

$$W^{\mathrm{big}}(R) = \mathrm{Ker}(R[[t]]^\times \twoheadrightarrow R^\times), \quad \hat{W}^{\mathrm{big}}(R) = \mathrm{Ker}(R[t]^\times \twoheadrightarrow R^\times),$$

where the maps $R[[t]]^\times \twoheadrightarrow R^\times$ and $R[t]^\times \twoheadrightarrow R^\times$ are given by evaluation at $t = 0$. Define $\lambda : \hat{W}^{\mathrm{big}}(R) \to R^\times$ to be the map that takes $f \in \mathrm{Ker}(R[t]^\times \twoheadrightarrow R^\times)$ to $f(1)$.

One has

$$\lambda \circ V_n = \lambda \quad \text{for all } n \in \mathbb{N}. \tag{A.1}$$

A.2.2. *The pairing.* We have a pairing

$$W^{\mathrm{big}}(R) \times \hat{W}^{\mathrm{big}}(R) \to R^\times, \quad (x, y) \mapsto \langle x, y \rangle := \lambda(xy). \tag{A.2}$$

Formula (A.1) implies that

$$\langle F_n(x), y \rangle = \langle x, V_n(y) \rangle, \quad \langle V_n(x), y \rangle = \langle x, F_n(y) \rangle. \tag{A.3}$$

One can view (A.2) as a paring

$$W^{\mathrm{big}} \times \hat{W}^{\mathrm{big}} \to \mathbb{G}_m\,. \tag{A.4}$$

A.2.3. *Nondegeneracy of the pairing.* P. Cartier proved that (A.4) induces an isomorphism

$$W^{\mathrm{big}} \xrightarrow{\sim} \underline{\mathrm{Hom}}(\hat{W}^{\mathrm{big}}, \mathbb{G}_m), \tag{A.5}$$

see [Ca, Thm. 2]. A detailed exposition is given in [H, §37.5]; according to [H, §E.6.2], it is based on some lecture notes of Cartier. Key idea: $\hat{W}^{\mathrm{big}}$ identifies with the inductive limit of $\mathrm{Sym}^n \hat{\mathbb{A}}^1$, where the transition map $\mathrm{Sym}^n \hat{\mathbb{A}}^1 \to \mathrm{Sym}^{n+1} \hat{\mathbb{A}}^1$ comes from $0 \in \hat{\mathbb{A}}^1$.

A.2.4. *Relation to Contou-Carrère's symbol.* The pairing (A.4) is closely related to Contou-Carrère's tame symbol, see [Co], [De, §2.9], and [BBE, Prop. 3.3(ii)(c)].

A.3. **Duality between $W_{\mathbb{Z}_{(p)}}$ and $\hat{W}_{\mathbb{Z}_{(p)}}$.**

A.3.1. *The pairing.* Let R be a $\mathbb{Z}_{(p)}$-algebra. Then one has a canonical embedding

$$W(R) \hookrightarrow W^{\mathrm{big}}(R).$$

So (A.2) induces a pairing

$$W(R) \times \hat{W}(R) \to R^{\times}, \quad (x, y) \mapsto \langle x, y \rangle, \tag{A.6}$$

which can be viewed as a paring

$$W_{\mathbb{Z}_{(p)}} \times \hat{W}_{\mathbb{Z}_{(p)}} \to (\mathbb{G}_m)_{\mathbb{Z}_{(p)}}. \tag{A.7}$$

By (A.3), we have

$$\langle Fx, y \rangle = \langle x, Vy \rangle, \quad \langle Vx, y \rangle = \langle x, Fy \rangle. \tag{A.8}$$

A.3.2. *Nondegeneracy of the pairing.* It is known that the pairing (A.7) induces an isomorphism

$$W_{\mathbb{Z}_{(p)}} \xrightarrow{\sim} \underline{\mathrm{Hom}}(\hat{W}_{\mathbb{Z}_{(p)}}, (\mathbb{G}_m)_{\mathbb{Z}_{(p)}}). \tag{A.9}$$

This follows from Cartier's theorem mentioned in §A.2.3: indeed, $\hat{W}_{\mathbb{Z}_{(p)}}$ identifies with $\hat{W}^{\mathrm{big}}_{\mathbb{Z}_{(p)}} / \sum\limits_{(n,p)=1} V_n(\hat{W}^{\mathrm{big}}_{\mathbb{Z}_{(p)}})$, so (A.5) implies that $\underline{\mathrm{Hom}}(\hat{W}_{\mathbb{Z}_{(p)}}, (\mathbb{G}_m)_{\mathbb{Z}_{(p)}})$ identifies with

$$\bigcap_{(n,p)=1} \mathrm{Ker}(F_n : W^{\mathrm{big}}_{\mathbb{Z}_{(p)}} \to W^{\mathrm{big}}_{\mathbb{Z}_{(p)}}) = W_{\mathbb{Z}_{(p)}}.$$

Let us note that in this article we only need nondegeneracy of the pairing

$$W_{\mathbb{F}_p} \times \hat{W}_{\mathbb{F}_p} \to (\mathbb{G}_m)_{\mathbb{F}_p},$$

which is proved in [DG, Ch. V, §4.5]. A related fact is proved in §4 of Chapter III of [Dem].

Appendix B. Explicit presentations of the stacks $\mathrm{sDisp}_n^{d,d'}$, $\mathrm{sDisp}_n^{d,d',\mathrm{weak}}$, $\mathrm{sDisp}_n^{d,d',\mathrm{strong}}$, and $\mathrm{Disp}_n^{d,d'}$

B.0.1. *The goal.* Given integers d and d' such that $0 \le d' \le d$ and an integer $n \ge 0$, we defined in §6.5 the stacks $\mathrm{sDisp}_n^{d,d'}$, $\mathrm{sDisp}_n^{d,d',\mathrm{weak}}$, $\mathrm{sDisp}_n^{d,d',\mathrm{strong}}$, and $\mathrm{Disp}_n^{d,d'}$. We are going to describe each of them as a quotient of an explicit scheme by an explicit group action.

Let R be an $\mathbb{F}_p$-algebra. Recall that $\mathrm{sDisp}_n^{d,d'}(R)$ is the full subgroupoid of the underlying groupoid of $\mathrm{sDisp}_n(R)$ whose objects are quadruples $(P, Q, F, F_1) \in \mathrm{sDisp}_n(R)$ such that $\mathrm{rank}\, P = d$ and $\mathrm{rank}(P/Q) = d'$. The groupoids $\mathrm{sDisp}_n^{d,d',\mathrm{weak}}(R)$, $\mathrm{sDisp}_n^{d,d',\mathrm{strong}}(R)$, and $\mathrm{Disp}_n^{d,d'}(R)$ are defined similarly but with the following changes:

(i) in the case of $\mathrm{Disp}_n^{d,d'}(R)$ and $\mathrm{sDisp}_n^{d,d',\mathrm{strong}}(R)$ replace P/Q by $\mathrm{Coker}(Q \to P)$;

(ii) in the case of $\mathrm{sDisp}_n^{d,d',\mathrm{weak}}(R)$ the condition for P is that $\mathrm{rank}(P/I_{n,R}P) = d$.

Note that $\mathrm{sDisp}_1^{d,d'} \neq \emptyset$ only if $d' = d$.

B.0.2. *Rigidifications.* Let $T_0 := W_n(R)^{d'}$, $L_0 := W_n(R)^{d-d'}$.

By a rigidification of an object of $(P, Q, F, F_1) \in \mathrm{sDisp}_n^{d,d}(R)$ we mean an isomorphism $(P, Q) \xrightarrow{\sim} (T_0 \oplus L_0, I_{n,R} \cdot T_0 \oplus L_0)$.

By a rigidification of an object of $(M, \mathcal{Q}, F, F_1) \in \mathrm{sDisp}_n^{d,d',\mathrm{weak}}(R)$ we mean an isomorphism $(M, \mathcal{Q}) \xrightarrow{\sim} (T_0 \oplus \bar{L}_0, I_{n,R} \cdot T_0 \oplus \bar{L}_0)$, where $\bar{L}_0 := L_0/J_{n,R} \cdot L_0$.

By a rigidification of an object of $\mathrm{Disp}_n^{d,d'}(R)$ or $\mathrm{sDisp}_n^{d,d',\mathrm{strong}}(R)$ we mean a normal decomposition (in the sense of [L13, §3.2]) plus an isomorphism between the corresponding pair (T, L) and the pair (T_0, L_0).

Let $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d'}(R)$ be the set of isomorphism classes of pairs consisting of an object of $\mathrm{sDisp}_n^{d,d'}(R)$ and a rigidification of it. Similarly, define functors $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{weak}}$, $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{strong}}$, and $\mathrm{Disp}_{n,\mathrm{rig}}^{d,d'}$. Each of the four functors can be described in terms of matrices (see §B.0.3 below). This description shows that each of the functors is representable by an affine scheme.

B.0.3. *Rigidfications in terms of matrices.* One has

$$\mathrm{Disp}_{n,\mathrm{rig}}^{d,d'}(R) = GL(d, W_n(R)), \quad \mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{strong}}(R) = \mathrm{Mat}(d, W_n(R)).$$

The first equality is explained in [LZ, §1.2]; see also [BP, §2.3], where only the case $n = \infty$ is considered. The second equality is quite similar to the first one.

$\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d'}(R)$ is the set of block matrices of the following shape:

$$\begin{pmatrix} W_n & W_{n-1} \\ W_n & W_{n-1} \end{pmatrix}$$

By this we mean that $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d'}(R)$ is the set of block matrices

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \tag{B.1}$$

where x_{11} (resp. x_{22}) is a square matrix of size d' (resp. $d - d'$) and the entries of the matrix x_{ij} are in $W_n(R)$ if $j = 1$ and in $W_{n-1}(R)$ if $j = 2$. The first d columns of (B.1) are the images of the basis vectors of T_0 under $F : P \to P$; the other columns are the images of the basis vectors of L_0 under $F_1 : Q \to P/J_{n,R} \cdot P$.

Similarly, $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{weak}}(R)$ is the set of block matrices of the following shape:

$$\begin{pmatrix} W_n & W_{n-1} \\ W_{n-1} & W_{n-1} \end{pmatrix}$$

B.0.4. $\mathrm{sDisp}_n^{d,d'}$ *as a quotient stack.* Let $H_n^{d,d'}(R) := \mathrm{Aut}(P_0, Q_0)$, where $P_0 := T_0 \oplus L_0$, $Q_0 := I_{n,R} \cdot T_0 \oplus L_0$. Then $H_n^{d,d'}$ is a group scheme acting on the scheme $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d'}$, and

$$\mathrm{sDisp}_n^{d,d'} = \mathrm{sDisp}_{n,\mathrm{rig}}^{d,d'} / H_n^{d,d'}.$$

In the language of §B.0.3, $H_n^{d,d'}$ is the group of invertible block matrices of the following shape:

$$\begin{pmatrix} W_n & I_n \\ W_n & W_n \end{pmatrix} \tag{B.2}$$

(here I_n denotes the functor $R \mapsto I_{n,R}$). An element

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \in H_n^{d,d'}(R)$$

acts on $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d'}(R)$ by

$$x \mapsto hx\Phi(h)^{-1}, \quad \text{where } \Phi(h) := \begin{pmatrix} F(h_{11}) & V^{-1}(h_{12}) \\ pF(h_{21}) & F(h_{22}) \end{pmatrix}. \tag{B.3}$$

Informally,

$$\Phi(h) = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} F(h) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

B.0.5. $\mathrm{sDisp}_n^{d,d',\mathrm{weak}}$ *as a quotient stack.* This is parallel to §B.0.4, but instead of (B.2) one has to consider invertible matrices of the following shape:

$$\begin{pmatrix} W_n & I_n \\ W_{n-1} & W_{n-1} \end{pmatrix}$$

B.0.6. $\mathrm{Disp}_n^{d,d'}$ *and* $\mathrm{sDisp}_n^{d,d',\mathrm{strong}}$ *as quotient stacks.* One has

$$\mathrm{Disp}_n^{d,d'} = \mathrm{Disp}_{n,\mathrm{rig}}^{d,d'} / \mathrm{BP}_n^{d,d'}, \quad \mathrm{sDisp}_n^{d,d',\mathrm{strong}} = \mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{strong}} / \mathrm{BP}_n^{d,d'},$$

where $\mathrm{BP}_n^{d,d'}$ is a certain group scheme acting on $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{strong}}$ and preserving $\mathrm{Disp}_{n,\mathrm{rig}}^{d,d'}$ (which is an open subscheme of $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{strong}}$). The group scheme $\mathrm{BP}_n^{d,d'}$ was introduced by O. Bültel and G. Pappas[23] in the case $n = \infty$ and then used (for arbitrary n) in [LZ, §1.2] to describe $\mathrm{Disp}_n^{d,d'}$. The definition of $\mathrm{BP}_n^{d,d'}$ and its action on $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{strong}}$ is recalled below.

Let $GL(d, W_n)$ be the group scheme over $\mathbb{F}_p$ representing the functor $R \mapsto GL(d, W_n(R))$. The group $\mathrm{BP}_n^{d,d'}$ is a certain subgroup of $GL(d, W_n) \times GL(d, W_n)$, and it acts on $\mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{strong}}$ by two-sided translations; more precisely, a pair

$$(g, h) \in \mathrm{BP}_n^{d,d'}(R) \subset GL(d, W_n(R)) \times GL(d, W_n(R))$$

acts by

$$x \mapsto hxg^{-1}, \quad \text{where } x \in \mathrm{Mat}(d, W_n(R)) = \mathrm{sDisp}_{n,\mathrm{rig}}^{d,d',\mathrm{strong}}(R).$$

The subgroup $\mathrm{BP}_n^{d,d'}(R) \subset GL(d, W_n(R))) \times GL(d, W_n(R))$ consists of pairs (g, h) such that

$$g_{ij} = p^{i-j} F(h_{ij}) \text{ for } 1 \leq j \leq i \leq 2, \tag{B.4}$$

$$h_{12} = V(g_{12}), \tag{B.5}$$

where g_{ij}, h_{ij} are the blocks of the block matrices g, h (as before, the blocks g_{11}, h_{11} have size d'). Note that (B.5) is an "avatar" of (B.4) because $FV = VF = p$.

Informally, $\mathrm{BP}_n^{d,d'}(R)$ is the graph of a *multivalued* map $\Phi : H_n^{d,d'}(R) \to GL(d, W_n(R))$, where $H_n^{d,d'} \subset GL(d, W_n)$ is the group of invertible block matrices of the shape

$$\begin{pmatrix} W_n & I_n \\ W_n & W_n \end{pmatrix}$$

[23]See [BP, §2.3]. Let us note that the authors of [BP] refer (after their Definition 1.0.1) to other works in which the subgroup was introduced and used; one of them is a 2008 e-print by Bültel.

and Φ is essentially as in formula (B.3) (except that Φ is multi-valued now).

Appendix C. $\mathrm{Disp}_n^{G,\mu}$ and $\mathrm{Lau}_n^{G,\mu}$

C.1. **The setting and the plan.**

C.1.1. *The setting.* Let G be a smooth affine group scheme over $\mathbb{Z}/p^n\mathbb{Z}$ equipped with a homomorphism $\mu : \mathbb{G}_m \to G$.

C.1.2. *Key example.* Let $d \geq d' \geq 0$. Let $G = GL(d)$ and

$$\mu(\lambda) = \mathrm{diag}(\lambda, \dots, \lambda, 1, \dots, 1), \tag{C.1}$$

where λ appears d' times and 1 appears $d - d'$ times.

C.1.3. *Plan.* In the setting of §C.1.1, there is a stack $\mathrm{Disp}_n^{G,\mu}$ over $\mathbb{F}_p$ such that in the situation of §C.1.2 one has $\mathrm{Disp}_n^{G,\mu} = \mathrm{Disp}_n^{d,d'}$. We will recall the definition of $\mathrm{Disp}_n^{G,\mu}$ in §C.2. Assuming that μ is *1-bounded* in the sense of §C.4.1, we will define in §C.4-C.5 a group scheme $\mathrm{Lau}_n^{G,\mu}$ over $\mathrm{Disp}_n^{G,\mu}$ such that in the situation of §C.1.2 one has $\mathrm{Lau}_n^{G,\mu} = \mathrm{Lau}_n^{d,d'}$. If μ is 1-bounded then according to [GM], one has an algebraic stack $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{F}_p$. We conjecture that it is a gerbe over $\mathrm{Disp}_n^{G,\mu}$ banded by $\mathrm{Lau}_n^{G,\mu}$ (see §C.5). In §C.7 we describe the Cartier dual of $\mathrm{Lau}_n^{G,\mu}$ very explicitly.

C.1.4. *On the setting of §C.1.1.* The setting of [BP, L21, GM] is more general than that of §C.1.1: μ is there a cocharacter of $G \otimes O/p^nO$, where O is the ring of integers of a finite unramified extension of $\mathbb{Q}_p$. This generalization is important.

C.2. **The stack $\mathrm{Disp}_n^{G,\mu}$.**

Let us recall the definition of $\mathrm{Disp}_n^{G,\mu}$ from [L21] (in the earlier article [BP] this was a description rather than the definition). It is quite parallel to the description of $\mathrm{Disp}_n^{d,d'}$ given in §B.0.6 of Appendix B.

C.2.1. *Outline.* Let $G(W_n)$ be the affine group scheme over $\mathbb{F}_p$ representing the functor $R \mapsto G(W_n(R))$ on the category of $\mathbb{F}_p$-algebras. The group scheme $G(W_n) \times G(W_n)$ acts on the scheme $G(W_n)$ as follows: a pair (g, h) in $G(W_n(R)) \times G(W_n(R))$ acts by

$$U \mapsto hUg^{-1}, \quad \text{where } U \in G(W_n(R)). \tag{C.2}$$

The stack $\mathrm{Disp}_n^{G,\mu}$ is defined to be the quotient of the scheme $G(W_n)$ by the action of a certain subgroup[24] $\mathrm{BP}_n^{G,\mu} \subset G(W_n) \times G(W_n)$. The subgroup is defined below.

C.2.2. *Definition of* $\mathrm{BP}_n^{G,\mu}$. $\mathbb{G}_m$ acts on G and $H^0(G, \mathcal{O}_G)$: namely, $\lambda \in \mathbb{G}_m$ acts on G by $g \mapsto \mu(\lambda) g \mu(\lambda)^{-1}$, and it acts on $H^0(G, \mathcal{O}_G)$ by taking $\varphi \in H^0(G, \mathcal{O}_G)$ to the function

$$g \mapsto \varphi(\mu(\lambda) g \mu(\lambda)^{-1}).$$

The action of $\mathbb{G}_m$ on $H^0(G, \mathcal{O}_G)$ induces a grading

$$H^0(G, \mathcal{O}_G) = \bigoplus_{k \in \mathbb{Z}} H^0(G, \mathcal{O}_G)_k\,. \tag{C.3}$$

[24] In [L21, §5.1] it is called the *display group*.

For a $\mathbb{Z}$-graded ring A, let $G(A)^{\mathbb{G}_m} \subset G(A)$ denote the subgroup of *graded* ring homomorphisms $H^0(G, \mathcal{O}_G) \to A$; this is indeed a subgroup because $G(A)^{\mathbb{G}_m} = \mathrm{Mor}^{\mathbb{G}_m}(\mathrm{Spec}\, A, G)$. Finally,

$$\mathrm{BP}_n^{G,\mu}(R) := G(W_n(R)^{\oplus})^{\mathbb{G}_m},$$

where $W_n(R)^{\oplus}$ is the n-truncated Witt frame, see §10.3.1. Since $W_n(R)^{\oplus}$ is a graded subring of $W_n(R)[u, u^{-1}] \times W_n(R)[t, t^{-1}]$, we see that $\mathrm{BP}_n^{G,\mu}(R)$ is a subgroup of the group

$$G(W_n(R)[u, u^{-1}] \times W_n(R)[t, t^{-1}])^{\mathbb{G}_m} = G(W_n(R) \times W_n(R)) = G(W_n(R)) \times G(W_n(R))$$

(we have used the homomorphisms $W_n(R)[u, u^{-1}] \to W_n(R)$ and $W_n(R)[t, t^{-1}] \to W_n(R)$ given by evaluation at $u = 1$ and $t = 1$).

The following lemma describes $\mathrm{BP}_n^{G,\mu}$ in matrix terms.

Lemma C.2.3. *Let $\rho : G \to GL(r)$ be a homomorphism such that*

$$\rho(\mu(\lambda)) = \mathrm{diag}(\lambda^{m_1}, \dots, \lambda^{m_r}).$$

Let $\rho_{ij} \in H^0(G, \mathcal{O}_G)$ be its matrix elements.

(i) If $(g, h) \in \mathrm{BP}_n^{G,\mu}(R) \subset G(W_n(R)) \times G(W_n(R))$ then

$$\rho_{ij}(g) = p^{m_j - m_i} F(\rho_{ij}(h)) \quad \text{if } m_i \le m_j, \tag{C.4}$$

$$\rho_{ij}(h) = p^{m_i - m_j - 1} V(\rho_{ij}(g)) \quad \text{if } m_i > m_j. \tag{C.5}$$

(ii) If $\rho : G \to GL(r)$ is a closed immersion and $g, h \in G(W_n(R))$ satisfy (C.4)-(C.5) *then $(g, h) \in \mathrm{BP}_n^{G,\mu}(R)$.*

Proof. (i) Note that in terms of the grading (C.3), we have $\rho_{ij} \in H^0(G, \mathcal{O}_G)_{m_i - m_j}$.

Let $(g, h) \in \mathrm{BP}_n^{G,\mu}(R)$. According to the definition of $\mathrm{BP}_n^{G,\mu}$ (see §C.2.2), this implies that the pair $(\rho_{ij}(g), \rho_{ij}(h)) \in W_n(R) \times W_n(R)$ belongs to $f(S_{m_i - m_j})$, where

$$S := W_n(R)^{\oplus} \subset W_n(R)[u, u^{-1}] \times W_n(R)[t, t^{-1}]$$

and the map $f : W_n(R)[u, u^{-1}] \times W_n(R)[t, t^{-1}] \to W_n(R) \times W_n(R)$ is given by evaluation at $u = t = 1$. So formulas (C.4)-(C.5) follow from (10.6)-(10.7).

(ii) If $\rho : G \to GL(r)$ is a closed immersion then the ring $H^0(G, \mathcal{O}_G)$ is generated by the functions ρ_{ij}, so the above argument can be reversed. □

Corollary C.2.4. *In the situation of §C.1.2 one has* $\mathrm{BP}_n^{G,\mu} = \mathrm{BP}_n^{d,d'}$, $\mathrm{Lau}_n^{G,\mu} = \mathrm{Lau}_n^{d,d'}$.

Proof. Compare equations (C.4)-(C.5) with (B.4)-(B.5). □

C.3. **$\mathrm{BP}_n^{G,\mu}$ and $\mathrm{Disp}_n^{G,\mu}$ in terms of G-torsors.** For an $\mathbb{F}_p$-algebra R, let

$$X_R := \mathrm{Spec}\, W_n(R)^{\oplus}.$$

C.3.1. *The G-torsor $\mathcal{P}_R^{\mu}$.* The group $\mathbb{G}_m$ acts on X_R. On the other hand, $\mathbb{G}_m$ acts on G by left translations via $\mu : \mathbb{G}_m \to G$. Thus we get an action of $\mathbb{G}_m$ on $X_R \times G$. It commutes with the action of G on $X_R \times G$ by right translations. So $X_R \times G$ is a $\mathbb{G}_m$-equivariant G-torsor over X_R. The corresponding G-torsor over the stack $X_R/\mathbb{G}_m$ is denoted by $\mathcal{P}_R^{\mu}$. (In other words, $\mathcal{P}_R^{\mu}$ is the G-torsor over $X_R/\mathbb{G}_m$ induced via μ by the $\mathbb{G}_m$-torsor $X_R \to X_R/\mathbb{G}_m$.)

One checks that

$$\mathrm{Aut}\, \mathcal{P}_R^{\mu} = \mathrm{BP}_n^{G,\mu}(R). \tag{C.6}$$

C.3.2. *G-torsors of type μ.* We say that a G-torsor over $X_R/\mathbb{G}_m$ has *type* μ if it becomes isomorphic to $\mathcal{P}_R^\mu$ after etale localization with respect to R. By (C.6), *a G-torsor of type μ over $X_R/\mathbb{G}_m$ is the same as an etale* $\mathrm{BP}_n^{G,\mu}$*-torsor over* $\operatorname{Spec} R$. In fact, "etale" can be replaced by "fpqc" because $\mathrm{BP}_n^{G,\mu}$ is smooth by [L21, Lemma 2.3.8].

C.3.3. *Remarks.* (i) If μ' and μ are conjugate then $\mathcal{P}_R^{\mu'} \simeq \mathcal{P}_R^\mu$. Conversely, if $R \neq 0$ and $\mathcal{P}_R^{\mu'} \simeq \mathcal{P}_R^\mu$ then μ' and μ are conjugate; to see this, use the homomorphism $W_n(R)^\oplus \to R$ from §10.3.5.

(ii) It follows from [GM, Prop. 5.5.2] that for every G-torsor $\mathcal{F}$ on $X_R/\mathbb{G}_m$ and every $x \in \operatorname{Spec} R$ there exist a homomorphism $\mu : \mathbb{G}_m \to G$ and an open subset $\operatorname{Spec} R' \subset \operatorname{Spec} R$ containing x such that the pullback of $\mathcal{F}$ to $X_{R'}/\mathbb{G}_m$ has type μ.

C.3.4. $\mathrm{Disp}_n^{G,\mu}$ *in terms of G-torsors.* By §C.3.2, one can reformulate [L21, Lemma 5.3.8] as follows: $\mathrm{Disp}_n^{G,\mu}(R)$ identifies with the groupoid of G-torsors $\mathcal{F}$ of type μ over $X_R/\mathbb{G}_m$ equipped with an isomorphism $f : \mathcal{F}^\sigma \xrightarrow{\sim} \mathcal{F}^\tau$. Here $\mathcal{F}^\sigma, \mathcal{F}^\tau$ are the pullbacks of $\mathcal{F}$ via the morphisms $\operatorname{Spec} W_n(R) \to X_R/\mathbb{G}_m$ corresponding to the homomorphisms

$$\sigma : W_n(R)^\oplus \to W_n(R), \quad \tau : W_n(R)^\oplus \to W_n(R)$$

from §10.3.6. In this language, the canonical map $G(W_n(R)) \to \mathrm{Disp}_n^{G,\mu}(R)$ takes an element $U \in G(W_n(R))$ to $(\mathcal{F}, f)$, where $\mathcal{F} = \mathcal{P}_R^\mu$ and $f : \mathcal{F}^\sigma \xrightarrow{\sim} \mathcal{F}^\tau$ is given by U.

C.4. **A morphism** $\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n$.

C.4.1. *The 1-boundedness condition.* Let $\mathfrak{g} := \mathrm{Lie}(G)$. Composing the adjoint representation of G with $\mu : \mathbb{G}_m \to G$, one gets an action of $\mathbb{G}_m$ on $\mathfrak{g}$ or equivalently, a grading

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i \tag{C.7}$$

compatible with the Lie bracket. From now on, we assume that the $\mathbb{G}_m$-action is *1-bounded*, by which we mean that

$$\mathfrak{g}_i = 0 \quad \text{for all } i > 1. \tag{C.8}$$

Under this assumption, we will define in §C.4.3 a morphism $\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n$. In §C.6 we will describe it more explicitly.

C.4.2. *Twists of $\mathfrak{g}$.* Let $\mathcal{F}$ be a G-torsor of type μ over $X_R/\mathbb{G}_m$. Let $\mathfrak{g}_\mathcal{F}$ be the $\mathcal{F}$-twist of the G-module $\mathfrak{g}$; this is a vector bundle on $X_R/\mathbb{G}_m$. Pulling it back to X_R, we get a finitely generated projective graded S-module, where $S := W_n(R)^\oplus$; by abuse of notation, we still denote it by $\mathfrak{g}_\mathcal{F}$. We claim that

$$\mathfrak{g}_\mathcal{F}\{-1\} \in \mathrm{Vec}_n^+(R), \tag{C.9}$$

where $\{-1\}$ denotes the shift of grading (see §10.4.1) and $\mathrm{Vec}_n^+(R)$ is as in §10.4.2. It suffices to check this if $\mathcal{F} \simeq \mathcal{P}_R^\mu$. In this case $\mathfrak{g}_\mathcal{F} \simeq \bigoplus_i \mathfrak{g}_i \otimes S\{i\}$, where $\{i\}$ denotes the shift of grading and the $\mathfrak{g}_i$'s are as in (C.7). So (C.9) follows from the 1-boundedness assumption (C.7).

C.4.3. *A morphism* $\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n$. By §C.3.4, we can think of an object of $\mathrm{Disp}_n^{G,\mu}(R)$ as a pair $(\mathcal{F}, f : \mathcal{F}^\sigma \xrightarrow{\sim} \mathcal{F}^\tau)$, where $\mathcal{F}$ is a G-torsor of type μ over $X_R/\mathbb{G}_m$. By §C.4.2, we have an object $\mathfrak{g}_{\mathcal{F}}\{-1\} \in \mathrm{Vec}_n^+(R)$. Using $f : \mathcal{F}^\sigma \xrightarrow{\sim} \mathcal{F}^\tau$, one upgrades it to an object of the category $\mathrm{DISP}_n^+(R)$ from §10.5. Thus we have constructed a functor

$$\mathrm{Disp}_n^{G,\mu}(R) \to \mathrm{DISP}_n^+(R). \tag{C.10}$$

Composing it with the functor $\mathrm{DISP}_n^+(R) \to \mathrm{sDisp}_n(R)$ from §10.6, one gets a functor

$$\mathrm{Disp}_n^{G,\mu}(R) \to \mathrm{sDisp}_n(R). \tag{C.11}$$

C.5. The group scheme $\mathrm{Lau}_n^{G,\mu}$.

C.5.1. Let $\mathrm{Lau}_n^{G,\mu}$ be the commutative n-smooth group scheme over $\mathrm{Disp}_n^{G,\mu}$ corresponding to the composite morphism

$$\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n \xrightarrow{\mathfrak{Z}} \mathrm{Sm}_n, \tag{C.12}$$

where the first arrow is (C.11) and the second one is given by Zink's functor (see §7.1). By Lemma 10.11.7, the functor $\mathrm{Disp}_n^{G,\mu}(R) \to \mathrm{Sm}_n(R)$ can also be described as follows[25]: it takes a pair $(\mathcal{F}, f : \mathcal{F}^\sigma \xrightarrow{\sim} \mathcal{F}^\tau)$ to the 0-th cohomology of the Zink complex $C^{\cdot}_{\mathfrak{g}_{\mathcal{F}}\{-1\}}$ (the latter is defined in §10.11.5).

In the situation of §C.1.2 one has

$$\mathrm{Lau}_n^{G,\mu} = \mathrm{Lau}_n^{d,d'}; \tag{C.13}$$

this follows from Theorem 9.1.5 combined with §10.8.

C.5.2. On the other hand, let $\mathrm{BT}_n^{G,\mu}$ be the stack defined in [GM]; the definition of $\mathrm{BT}_n^{G,\mu}$ uses the *syntomification* functor $R \mapsto R^{\mathrm{Syn}}$ from [Bh]. By [GM, Theorem D], if μ is 1-bounded then the stack $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{Z}/p^m\mathbb{Z}$ is algebraic for every m; moreover, in the situation of §C.1.2 one has $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{Z}/p^m\mathbb{Z} = \mathscr{B}\mathscr{T}_n^{d,d'} \otimes \mathbb{Z}/p^m\mathbb{Z}$.

One has a canonical morphism $\phi_n : \mathrm{BT}_n^{G,\mu} \otimes \mathbb{F}_p \to \mathrm{Disp}_n^{G,\mu}$, see [GM, Rem. 9.1.3] (it is given by pullback with respect to a certain morphism $X_R/\mathbb{G}_m \to R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, where X_R is as in §C.3 and $R^{\mathcal{N}}$ is the Nygaard-filtered prismatization[26] of $\operatorname{Spec} R$). In the situation of §C.1.2 this is the morphism mentioned in Theorem 1.1.1.

Conjecture C.5.3. *The morphism* $\phi_n : \mathrm{BT}_n^{G,\mu} \otimes \mathbb{F}_p \to \mathrm{Disp}_n^{G,\mu}$ *is a gerbe banded by* $\mathrm{Lau}_n^{G,\mu}$.

C.5.4. *Remarks.* (i) Conjecture C.5.3 holds for $n = 1$ (see [GM, Rem. 9.3.4] and the related part of the proof of [GM, Thm. 9.3.2]).

(ii) In the situation of §C.1.2, Conjecture C.5.3 holds for all n. This follows from Lau's Theorem 1.1.1 combined with formula (C.13).

(iii) As far as I understand, E. Lau [L25] has proved Conjecture C.5.3 in general.

[25] This description shows that even without assuming μ to be 1-bounded, $\mathrm{Lau}_n^{G,\mu}$ is defined as a *complex of group ind-schemes* concentrated in degrees $0, -1$, but 1-boundedness ensures that this complex is a group scheme.

[26] See [Bh, Def. 5.3.10]. By [Bh, §5.4], if R is smooth over $\mathbb{F}_p$ one can use [Bh, Def. 3.3.13].

C.6. **Explicit description of the morphism** $\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n$**.** In §C.6.2-C.6.3 we give an explicit description of the morphism $\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n$ defined in §C.4. This description is obtained by unraveling the definitions in a straightforward way, so we just formulate the answer.

C.6.1. *A preliminary remark.* The composition $\mathrm{Disp}_n^{G,\mu} \to \mathrm{DISP}_n^+ \to \mathrm{Vec}_n^+$ was described in §C.4.2. Combining this description with §C.3.4, we see that the composite map

$$G(W_n) \to \mathrm{Disp}_n^{G,\mu} \to \mathrm{DISP}_n^+ \to \mathrm{Vec}_n^+ \tag{C.14}$$

is *constant.* More precisely, the map (C.14) takes every $U \in G(W_n(R))$ to the object

$$M = \bigoplus_i \mathfrak{g}_i \otimes S\{i-1\} \in \mathrm{Vec}_n^+(R), \quad \text{ where } S := W_n(R)^{\oplus}.$$

Thus $M_j = \bigoplus\limits_i S_{i+j-1} \otimes \mathfrak{g}_i$. Since $\mathfrak{g}_i = 0$ for $i > 1$, we get

$$M_0 = W_n(R) \otimes \mathfrak{g}, \quad M_1 := (W_n(R) \otimes \mathfrak{g}_{\le 0}) \oplus (I_{n+1,R} \otimes \mathfrak{g}_1),$$

and the map $t : M_1 \to M_0$ comes from the canonical map $I_{n+1,R} \to W_n(R)$.

C.6.2. *The morphism* $G(W_n) \to \mathrm{sDisp}_n$ *in explicit terms.* The composite morphism

$$G(W_n) \to \mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n$$

takes an element $U \in G(W_n(R))$ to the quadruple[27] $(P_R, Q_R, \Phi_U, \Phi'_U) \in \mathrm{sDisp}_n(R)$, where the $W_n(R)$-modules P_R, Q_R, and the F-linear maps $\Phi_U : P_U \to P_U$, $\Phi'_U : Q_U \to P_U/J_{n,R}P_U$ are as follows:

(i) in terms of §C.6.1, $P_R = M_0$ and $Q_R = \mathrm{Im}(M_1 \xrightarrow{t} M_0)$; explicitly,

$$P_R = W_n(R) \otimes \mathfrak{g}, \quad Q_R = (W_n(R) \otimes \mathfrak{g}_{\le 0}) \oplus (I_{n,R} \otimes \mathfrak{g}_1) \subset P_R\,.$$

(ii) $\Phi_U : P_R \to P_R$ and $\Phi'_U : Q_R \to P_R/J_{n,R} \cdot P_R$ are the F-linear maps such that

$$\Phi_U(a \otimes x) = p^{1-i}\,\mathrm{Ad}_U(F(a) \otimes x) \quad \text{ for } a \in W_n(R), x \in \mathfrak{g}_i, \tag{C.15}$$

$$\Phi'_U(a \otimes x) = p^{-i}\,\mathrm{Ad}_U(\bar{F}(a) \otimes x) \quad \text{ for } a \in W_n(R), x \in \mathfrak{g}_i \text{ if } i \le 0, \tag{C.16}$$

$$\Phi'_U(a \otimes x) = \mathrm{Ad}_U(V^{-1}(a) \otimes x) \quad \text{ for } a \in I_{n,R}, x \in \mathfrak{g}_1. \tag{C.17}$$

(here $\bar{F}(a) \in W_{n-1}(R)$ is the image of $F(a)$).

Thus $\Phi_U = \mathrm{Ad}_U \circ \Phi_1$, $\Phi'_U = \mathrm{Ad}_U \circ \Phi'_1$. Note that (C.17) is an "avatar" of (C.16).

C.6.3. *The morphism* $\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n$ *in explicit terms.* Recall that the stack $\mathrm{Disp}_n^{G,\mu}$ is the quotient of the scheme $G(W_n)$ by the following action of $\mathrm{BP}_n^{G,\mu}$: an element

$$(g,h) \in \mathrm{BP}_n^{G,\mu}(R) \subset G(W_n(R)) \times G(W_n(R)$$

takes $U \in G(W_n(R))$ to hUg^{-1}. So descending the morphism $G(W_n) \to \mathrm{sDisp}_n$ described in §C.6.2 to a morphism $\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n$ amounts to specifying an isomorpism

$$(P_R, Q_R, \Phi_U, \Phi'_U) \xrightarrow{\sim} (P_R, Q_R, \Phi_{hUg^{-1}}, \Phi'_{hUg^{-1}}),$$

where $(g,h) \in \mathrm{BP}_n^{G,\mu}(R)$. This isomorphism is $\mathrm{Ad}_h : P_R \xrightarrow{\sim} P_R$.

[27]In this quadruple $\Phi_U : P_U \to P_U$ and $\Phi'_U : Q_U \to P_U$ play the role of F and F_1 from §6.1.2

C.7. **Explicit description of** $(\mathrm{Lau}_n^{G,\mu})^*$**.** Let $(\mathrm{Lau}_n^{G,\mu})^*$ be the Cartier dual of $\mathrm{Lau}_n^{G,\mu}$. We will describe $(\mathrm{Lau}_n^{G,\mu})^*$ as a subgroup of a very simple group scheme $\tilde{A}$ over $\mathrm{Disp}_n^{G,\mu}$.

C.7.1. *The group scheme* $\tilde{A}$. Let A be the group scheme over $\mathbb{F}_p$ whose R-points are additive maps $\mathfrak{g} \to W_n(R)$ or equivalently, $W_n(R)$-linear maps $W_n(R) \otimes \mathfrak{g} \to W_n(R)$. We have the coadjoint action of $G(W_n(R))$ on $A(R)$. So $G(W_n)$ acts on A. Precomposing this action with the homomorphism $\mathrm{BP}_n^{G,\mu} \hookrightarrow G(W_n) \times G(W_n) \xrightarrow{\mathrm{pr}_2} G(W_n)$, we get an action of $\mathrm{BP}_n^{G,\mu}$ on A. On the other hand, we have the $\mathrm{BP}_n^{G,\mu}$-torsor $G(W_n) \to \mathrm{Disp}_n^{G,\mu}$. Define $\tilde{A}$ to be the twist of A by this torsor. Thus $\tilde{A}$ is a smooth commutative group scheme over $\mathrm{Disp}_n^{G,\mu}$.

C.7.2. *Explicit description of* $(\mathrm{Lau}_n^{G,\mu})^*$. Recall that $\mathrm{Lau}_n^{G,\mu}$ corresponds to the composition $\mathrm{Disp}_n^{G,\mu} \to \mathrm{sDisp}_n \xrightarrow{\mathfrak{Z}} \mathrm{Sm}_n$. If $\mathscr{P} = (P, Q, F, F_1) \in \mathrm{sDisp}_n(R)$ then by §7.2, $\mathfrak{Z}^*_{\mathscr{P}}$ is the subgroup of the group scheme $\mathrm{Hom}_{W_n(R)}(P, W_{n,R})$ defined by equations (7.9)-(7.10). Combining this with §C.6, we see that $(\mathrm{Lau}_n^{G,\mu})^*$ is a closed subgroup of $\tilde{A}$. It remains to describe the closed subscheme

$$(\mathrm{Lau}_n^{G,\mu})^* \times_{\mathrm{Disp}_n^{G,\mu}} G(W_n) \subset A \times G(W_n). \tag{C.18}$$

Combining equations (7.9)-(7.10) with (C.15)-(C.17), we get the following description of the subscheme (C.18).

Let (η, U) be an R-point of $A \times G(W_n)$, so $\eta \in \mathrm{Hom}(\mathfrak{g}, W_n(R))$, $U \in G(W_n(R))$. Then (η, U) *is an R-point of* $(\mathrm{Lau}_n^{G,\mu})^* \times_{\mathrm{Disp}_n^{G,\mu}} G(W_n)$ *if and only if η and U satisfy the equations*

$$F(\eta(x)) = \eta(\mathrm{Ad}_U(x)) \quad \text{ if } x \in \mathfrak{g}_1, \tag{C.19}$$

$$\eta(x) = p^{-i} V(\eta(\mathrm{Ad}_U(x))) \quad \text{ if } x \in \mathfrak{g}_i \text{ and } i \le 0. \tag{C.20}$$

Note that (C.19) is an "avatar" of (C.20).

In the next remark we describe a way to transform the system of equations (C.19)-(C.20) by eliminating some of the unknowns (although it is not clear whether this is worth doing).

C.7.3. *Remark.* Recall that $A := \mathrm{Hom}(\mathfrak{g}, W_n)$. Let $A_1 := \mathrm{Hom}(\mathfrak{g}, W_n)$; this is a direct summand of A. Proposition 7.4.7 implies that the composite map

$$(\mathrm{Lau}_n^{G,\mu})^* \times_{\mathrm{Disp}_n^{G,\mu}} G(W_n) \hookrightarrow A \times G(W_n) \twoheadrightarrow A_1 \times G(W_n)$$

is still a closed immersion. It also describes its image by rather explicit equations.

Appendix D. A conjectural description of $\mathrm{BT}_n^{G,\mu}$

Just as in Appendix C, let G be a smooth affine group scheme over $\mathbb{Z}/p^n\mathbb{Z}$ equipped with a homomorphism $\mu : \mathbb{G}_m \to G$. Let $\mathrm{BT}_n^{G,\mu}$ be the stack defined in [GM] along the lines of [D23, Appendix C].

D.1. **The goal.** Suppose that μ is 1-bounded. Then by [GM, Theorem D], the stack $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{Z}/p^m\mathbb{Z}$ is algebraic for every m, i.e., it can be represented as a quotient of a scheme by an action of a flat groupoid. One would like to have an *explicit* presentation of $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{Z}/p^m\mathbb{Z}$ as such a quotient. The goal of this Appendix is to formulate a conjecture in this direction, see Conjecture D.8.4.

D.2. **Format of the conjecture.** The formulation of the conjecture uses ideas from the theory of prismatization [Bh, D20] and "sheared prismatization" [BMVZ, BKMVZ].

D.2.1. *The format.* We will define a stack of $\mathbb{Z}/p^n\mathbb{Z}$-algebras denoted by ${}^s\mathscr{R}_n$. Using ${}^s\mathscr{R}_n$, we will define a group stack $G({}^s\mathscr{R}_n)$. We will also define another group stack, denoted by $G({}^s\mathscr{R}_n^{\oplus})^{\mathbb{G}_m}$; its definition involves the homomorphism $\mu : \mathbb{G}_m \to G$. One has a canonical homomorphism $G({}^s\mathscr{R}_n^{\oplus})^{\mathbb{G}_m} \to G({}^s\mathscr{R}_n) \times G({}^s\mathscr{R}_n)$. It defines an action of $G({}^s\mathscr{R}_n^{\oplus})^{\mathbb{G}_m}$ on the stack $G({}^s\mathscr{R}_n)$ by two-sided translations. We conjecture that the quotient with respect to this action is canonically isomorphic to $\mathrm{BT}_n^{G,\mu}$ (assuming that μ is 1-bounded).

D.2.2. *Analogy with the definition of* $\mathrm{Disp}_n^{G,\mu}$*.* The format from §D.2.1 is similar to the format of the definition of $\mathrm{Disp}_n^{G,\mu}$ given in §C.2: namely, $G({}^s\mathscr{R}_n)$ and $G({}^s\mathscr{R}_n^{\oplus})^{\mathbb{G}_m}$ from §D.2.1 are analogs of $G(W_n)$ and $\mathrm{BP}_n^{G,\mu}$ from §C.2.

D.2.3. *Remarks.* (i) Let $\mathrm{BT}_{n,\mathbb{F}_p}^{G,\mu} := \mathrm{BT}_n^{G,\mu} \times \operatorname{Spec} \mathbb{F}_p$, $W_{n,\mathbb{F}_p} := W_n \times \operatorname{Spec} \mathbb{F}_p$. Since $W_{n,\mathbb{F}_p}$ is a scheme of $(\mathbb{Z}/p^n\mathbb{Z})$-algebras, we have the group scheme $G(W_{n,\mathbb{F}_p})$. The $(\mathbb{Z}/p^n\mathbb{Z})$-algebra stack ${}^s\mathscr{R}_{n,\mathbb{F}_p}$ is a quotient of $W_{n,\mathbb{F}_p}$ (see §D.7.2 below), so the conjecture outlined in §D.2.1 ultimately represents $\mathrm{BT}_{n,\mathbb{F}_p}^{G,\mu}$ as a quotient of the scheme $G(W_{n,\mathbb{F}_p})$ by a certain groupoid.

(ii) The situation in mixed characteristic is almost the same. The stack ${}^s\mathscr{R}_n$ can still be represented[28] rather naturally as a quotient of W_n, but the ring scheme W_n is not a scheme of $(\mathbb{Z}/p^n\mathbb{Z})$-algebras. However, if G is lifted to a scheme over $\mathbb{Z}_p$ then $G(W_n)$ is defined, and the conjecture ultimately represents $\mathrm{BT}_n^{G,\mu}$ as a quotient of the scheme $G(W_n)$ by a certain groupoid.

(iii) The idea of representing $\mathrm{BT}_{n,\mathbb{F}_p}^{G,\mu}$ as a quotient of the scheme $G(W_{n,\mathbb{F}_p})$ is natural in view of Conjecture C.5.3, which says that $\mathrm{BT}_{n,\mathbb{F}_p}^{G,\mu}$ is a gerbe over $\mathrm{Disp}_n^{G,\mu}$ banded by a group scheme killed by Fr^n. The pullback of such a gerbe via the map $\mathrm{Fr}^n : \mathrm{Disp}_n^{G,\mu} \to \mathrm{Disp}_n^{G,\mu}$ is trivial, so we get a morphism $\mathrm{Disp}_n^{G,\mu} \to \mathrm{BT}_{n,\mathbb{F}_p}^{G,\mu}$ such that the composition $\mathrm{Disp}_n^{G,\mu} \to \mathrm{BT}_{n,\mathbb{F}_p}^{G,\mu} \to \mathrm{Disp}_n^{G,\mu}$ equals Fr^n. The composition $G(W_{n,\mathbb{F}_p}) \to \mathrm{Disp}_n^{G,\mu} \to \mathrm{BT}_{n,\mathbb{F}_p}$ is faithfully flat (but not smooth), so it represents $\mathrm{BT}_{n,\mathbb{F}_p}^{G,\mu}$ as a quotient of the scheme $G(W_{n,\mathbb{F}_p})$ by some groupoid.

D.3. **Conventions.** A ring R is said to be p-nilpotent if the element $p \in R$ is nilpotent.

The word "stack" will mean a fpqc-stack on the category opposite to that of p-nilpotent rings. The final object in the category of such stacks is denoted by $\operatorname{Spf} \mathbb{Z}_p$; this is the functor that takes each p-nilpotent ring to a one-element set.

Ind-schemes and schemes over $\operatorname{Spf} \mathbb{Z}_p$ are particular classes of stacks. The words "scheme over $\operatorname{Spf} \mathbb{Z}_p$" are understood in the *relative* sense (e.g., $\operatorname{Spf} \mathbb{Z}_p$ itself is a scheme over $\operatorname{Spf} \mathbb{Z}_p$).

W will denote the functor $R \mapsto W(R)$, where R is p-nilpotent. So W is a ring scheme over $\operatorname{Spf} \mathbb{Z}_p$. Same for W_n.

D.4. **Plan.** To formulate the conjecture, we need the ring stacks ${}^s\mathscr{R}_n, {}^s\mathscr{R}_n^{\oplus}$. In §D.5-D.6 we discuss the easier ring stacks $\mathscr{R}_n, \mathscr{R}_n^{\oplus}$. In §D.7.1 we explain the idea of the definition of ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$ (the details are explained in [D25b]). In §D.7.2 we give a simple description of their reductions modulo p; this description can be used as a definition. In §D.8 we formulate the conjecture.

[28] See [D25b, §8]. The presentation of ${}^s\mathscr{R}_n$ as a quotient of W_n given in [D25b, §8] depends on the choice of an integer $m \geq \delta_p$, where $\delta_p = 0$ for $p > 2$ and $\delta_2 = 1$; it is natural to set $m = \delta_p$.

D.5. **The stacks $\mathscr{R}_n$ and $G(\mathscr{R}_n)$.** It is convenient to define various ring stacks as *cones of quasi-ideals,* see [D20, §1.3]. We will be using this approach.

D.5.1. *The case $n = 1$.* Let $n = 1$ (so G is a group scheme over $\mathbb{F}_p$). Then $\mathscr{R}_1 = (\mathbb{A}^1_{\mathbb{F}_p})^{\Delta}$ and $G(\mathscr{R}_1) = G^{\Delta}$, where the superscript Δ stands for *prismatization* in the sense of [Bh, D20]. Thus $\mathscr{R}_1 = \mathrm{Cone}(W \xrightarrow{p} W)$ (this is a stack of $\mathbb{F}_p$-algebras), and $G(\mathscr{R}_1)$ is defined via the procedure called *transmutation* in [Bh]. Namely, $G(\mathscr{R}_1)$ is the functor that associates to a p-nilpotent ring A the groupoid $G(\mathscr{R}_1(A))$; here $\mathscr{R}_1(A)$ is an animated $\mathbb{F}_p$-algebra, so the expression $G(\mathscr{R}_1(A))$ makes sense.

D.5.2. *The stacks $\mathscr{R}_n$ and $G(\mathscr{R}_n)$.* Now let n be any positive integer. Similarly to §D.5.1, $\mathscr{R}_n$ is the stack of $\mathbb{Z}/p^n\mathbb{Z}$-algebras defined by

$$\mathscr{R}_n := \mathrm{Cone}(W \xrightarrow{p^n} W) \tag{D.1}$$

and $G(\mathscr{R}_n)$ is the functor that associates to a p-nilpotent ring A the groupoid $G(\mathscr{R}_n(A))$. Since $\mathscr{R}_n(A)$ is an animated $\mathbb{Z}/p^n\mathbb{Z}$-algebra and G is a scheme over $\mathbb{Z}/p^n\mathbb{Z}$, the expression $G(\mathscr{R}_n(A))$ makes sense. Since G is a group, $G(\mathscr{R}_n)$ is a group stack.

D.5.3. *Digression on n-prismatization.* Similarly to the case $n = 1$ considered in [Bh, D20], one can deform the ring stack (D.1) by replacing p^n with ξ, where ξ is a primitive Witt vector of degree n, which matters only up to multiplication by $W^\times$. Then one can define the functor of n-prismatization denoted by $X \mapsto X^{\Delta_n}$ (where X is a p-adic formal scheme) so that if X is a scheme over $\mathbb{Z}/p^n\mathbb{Z}$ then $X^{\Delta_n} = X(\mathscr{R}_n)$. We will not follow this path. The following property of n-prismatization is somewhat strange: if X is a scheme over $\mathbb{Z}/p^{n-1}\mathbb{Z}$ then $X^{\Delta_n} = \emptyset$.

D.5.4. *An economic presentation of $\mathscr{R}_n$.* The ring stack (D.1) has the following "economic" description:

$$\mathscr{R}_n = \mathrm{Cone}(W^{(F^n)} \to W_n), \tag{D.2}$$

where $W^{(F^n)} := \mathrm{Ker}(F^n : W \to W)$ and the map $W^{(F^n)} \to W_n$ is the tautological one. In the case $n = 1$ this is [D20, Prop. 3.5.1] or [Bh, Cor. 2.6.8]. The argument for any n is similar:

$$\mathrm{Cone}(W^{(F^n)} \to W/V^nW) = \mathrm{Cone}(V^nW \to W/W^{(F^n)}) = \mathrm{Cone}(V^nW \xrightarrow{F^n} W),$$

and $\mathrm{Cone}(V^nW \xrightarrow{F^n} W) = \mathrm{Cone}(W \xrightarrow{F^nV^n} W) = \mathrm{Cone}(W \xrightarrow{p^n} W)$.

D.6. **The stacks $\mathscr{R}_n^\oplus$ and $G(\mathscr{R}_n^\oplus)^{\mathbb{G}_m}$.**

D.6.1. *The graded ring scheme $W^\oplus$.* Let $W^\oplus$ be the functor

$$\{p\text{-nilpotent rings}\} \to \{\text{graded rings}\}, \quad R \mapsto W(R)^\oplus,$$

where $W(R)^\oplus$ is the Witt frame (see §10.2.6). The functor $W^\oplus$ is a $\mathbb{Z}$-graded ring ind-scheme over $\mathrm{Spf}\,\mathbb{Z}_p$. Each graded component of $W^\oplus$ is an affine scheme over $\mathrm{Spf}\,\mathbb{Z}_p$, so by abuse of language we usually call $W^\oplus$ a ring scheme (rather than a ring ind-scheme).

By definition, $W^\oplus$ is a graded ring subscheme of $W[u, u^{-1}] \times W[t, t^{-1}]$; this subscheme is defined by equations (10.6)-(10.7).

D.6.2. *Remark.* Let $F, V : W[u, u^{-1}] \to W[u, u^{-1}]$ be the $\mathbb{Z}_p[u, u^{-1}]$-linear maps extending $F, V : W \to W$. Similarly, one has $F, V : W[t, t^{-1}] \to W[t, t^{-1}]$ and

$$F, V : W[u, u^{-1}] \times W[t, t^{-1}] \to W[u, u^{-1}] \times W[t, t^{-1}].$$

Warning: the subscheme $W^{\oplus} \subset W[u, u^{-1}] \times W[t, t^{-1}]$ is *not preserved* by F or by V. To see this, look at equations (10.6)-(10.7) and recall that the maps $F, V : W \to W$ do not commute.

D.6.3. *The graded ring stack $\mathscr{R}_n^{\oplus}$.* Let $\mathscr{R}_n^{\oplus}$ be the stack of $\mathbb{Z}$-graded $(\mathbb{Z}/p^n\mathbb{Z})$-algebras defined by

$$\mathscr{R}_n^{\oplus} := \mathrm{Cone}(W^{\oplus} \xrightarrow{p^n} W^{\oplus}). \tag{D.3}$$

The embedding $W^{\oplus} \hookrightarrow W[u, u^{-1}] \times W[t, t^{-1}]$ induces a homomorphism of $\mathbb{Z}$-graded $\mathbb{Z}/p^n\mathbb{Z}$-algebra stacks

$$\mathscr{R}_n^{\oplus} \to \mathscr{R}_n[u, u^{-1}] \times \mathscr{R}_n[t, t^{-1}]. \tag{D.4}$$

D.6.4. *An economic presentation of $\mathscr{R}_{n,\mathbb{F}_p}^{\oplus}$.* Let $\mathscr{R}_{n,\mathbb{F}_p}^{\oplus} := \mathscr{R}_n^{\oplus} \times \mathrm{Spec}\,\mathbb{F}_p$ (i.e., $\mathscr{R}_{n,\mathbb{F}_p}^{\oplus}$ is the base change of $\mathscr{R}_n^{\oplus}$ to $\mathbb{F}_p$). Similarly, let $W_{\mathbb{F}_p} := W \times \mathrm{Spec}\,\mathbb{F}_p$, $W_{n,\mathbb{F}_p} := W_n \times \mathrm{Spec}\,\mathbb{F}_p$. Good news: *the subscheme $W_{\mathbb{F}_p}^{\oplus} \subset W_{\mathbb{F}_p}[u, u^{-1}] \times W_{\mathbb{F}_p}[t, t^{-1}]$ is preserved by F and V* (the warning from §D.6.2 does not apply because in characteristic p one has $FV = VF = p$). Thus we have the maps $F, V : W_{\mathbb{F}_p}^{\oplus} \to W_{\mathbb{F}_p}^{\oplus}$.

Let $W_{n,\mathbb{F}_p}^{\oplus}$ be the n-truncated Witt frame, i.e., the functor

$$\{\mathbb{F}_p\text{-algebras}\} \to \{\text{graded } \mathbb{Z}/p^n\mathbb{Z}\text{-algebras}\}, \quad R \mapsto W_n(R)^{\oplus},$$

where $W_n(R)^{\oplus}$ is as in §10.3.1 or §10.2.7. We claim that

$$\mathscr{R}_{n,\mathbb{F}_p}^{\oplus} = \mathrm{Cone}((W_{\mathbb{F}_p}^{\oplus})^{(F^n)} \to W_{n,\mathbb{F}_p}^{\oplus}). \tag{D.5}$$

The proof of (D.5) is similar to that of (D.2); it uses the maps $F, V : W_{\mathbb{F}_p}^{\oplus} \to W_{\mathbb{F}_p}^{\oplus}$.

D.7. The $\mathbb{Z}/p^n\mathbb{Z}$-algebra stacks ${}^s\mathscr{R}_n, {}^s\mathscr{R}_n^{\oplus}$ and the related group stacks.

D.7.1. *Idea of the definition of ${}^s\mathscr{R}_n, {}^s\mathscr{R}_n^{\oplus}$.* One can define ${}^s\mathscr{R}_n, {}^s\mathscr{R}_n^{\oplus}$ similarly to the definitions of $\mathscr{R}_n, \mathscr{R}_n^{\oplus}$ given in §D.5-D.6 but replacing $W, W^{\oplus}$ by certain ring "spaces" ${}^sW, {}^sW^{\oplus}$ over $\mathrm{Spf}\,\mathbb{Z}_p$. (${}^sW$ is called the *space of sheared Witt vectors,* whence the superscript s in the notation for it.) One has

$${}^sW := \varprojlim_n W/\hat{W}^{(F^n)}$$

(the transition maps are equal to F). The quotient $W/\hat{W}^{(F^n)}$ is understood as an fpqc sheaf (this sheaf is not a scheme). There is an obvious homomorphism $F : {}^sW \to {}^sW$ and a less obvious operator $\tilde{V} : {}^sW \to {}^sW$. These data have the properties from §10.2.2, and ${}^sW^{\oplus}$ is obtained from $({}^sW, F, \tilde{V})$ by applying the functor $\mathfrak{L}$ from Proposition 10.2.4. Finally, let

$${}^s\mathscr{R}_n := \mathrm{Cone}({}^sW \xrightarrow{p^n} {}^sW), \quad {}^s\mathscr{R}_n^{\oplus} := \mathrm{Cone}({}^sW^{\oplus} \xrightarrow{p^n} {}^sW^{\oplus});$$

then ${}^s\mathscr{R}_n$ is an $\mathbb{Z}/p^n\mathbb{Z}$-algebra stack, and ${}^s\mathscr{R}_n^{\oplus}$ is a stack of $\mathbb{Z}$-graded $\mathbb{Z}/p^n\mathbb{Z}$-algebras. Similarly to (D.4), we have a homomorphism of $\mathbb{Z}$-graded $\mathbb{Z}/p^n\mathbb{Z}$-algebra stacks

$${}^s\mathscr{R}_n^{\oplus} \to {}^s\mathscr{R}_n[u, u^{-1}] \times {}^s\mathscr{R}_n[t, t^{-1}]. \tag{D.6}$$

The details are explained in [D25b].

D.7.2. *The ring stacks* ${}^s\mathscr{R}_{n,\mathbb{F}_p}, {}^s\mathscr{R}^{\oplus}_{n,\mathbb{F}_p}$. The reductions of ${}^s\mathscr{R}_n, {}^s\mathscr{R}^{\oplus}_n$ modulo p have the following simple descriptions, which can be used as definitions:

$$ {}^s\mathscr{R}_{n,\mathbb{F}_p} = \operatorname{Cone}(\hat{W}^{(F^n)}_{\mathbb{F}_p} \to W_{n,\mathbb{F}_p}), \tag{D.7} $$

$$ {}^s\mathscr{R}^{\oplus}_{n,\mathbb{F}_p} = \operatorname{Cone}((\hat{W}^{\oplus}_{\mathbb{F}_p})^{(F^n)} \to W^{\oplus}_{n,\mathbb{F}_p}). \tag{D.8} $$

Formulas (D.7)-(D.8) are parallel to (D.2) and (D.5). Let us note that $(\hat{W}^{\oplus}_{\mathbb{F}_p})^{(F^n)}$ already appeared in §10.11.4: for any $\mathbb{F}_p$-algebra R, the group of R-points of $(\hat{W}^{\oplus}_{\mathbb{F}_p})^{(F^n)}$ was denoted there by $\hat{W}^{(F^n)}(R)^{\oplus}$.

D.7.3. *The group stack* $G({}^s\mathscr{R}_n)$. As before, let G be a smooth affine group scheme over $\mathbb{Z}/p^n\mathbb{Z}$. Similarly to §D.5.2, we define $G({}^s\mathscr{R}_n)$ to be the functor that associates to a p-nilpotent ring A the groupoid $G({}^s\mathscr{R}_n(A))$. Since ${}^s\mathscr{R}_n(A)$ is an animated $\mathbb{Z}/p^n\mathbb{Z}$-algebra and G is a scheme over $\mathbb{Z}/p^n\mathbb{Z}$, the expression $G({}^s\mathscr{R}_n(A))$ makes sense. Since G is a group, $G({}^s\mathscr{R}_n)$ is a group stack. Note that

$$ G({}^s\mathscr{R}_n(A)) = \operatorname{Hom}(H^0(G,\mathcal{O}_G), {}^s\mathscr{R}_n(A)), \tag{D.9} $$

where Hom is understood in the sense of animated $\mathbb{Z}/p^n\mathbb{Z}$-algebras.

D.7.4. *The group stack* $G({}^s\mathscr{R}^{\oplus}_n)^{\mathbb{G}_m}$. Similarly to (D.9), we define the group stack $G({}^s\mathscr{R}^{\oplus}_n)^{\mathbb{G}_m}$ to be the functor that takes a p-nilpotent ring A to $\operatorname{Hom}^{\mathbb{G}_m}(H^0(G,\mathcal{O}_G), {}^s\mathscr{R}_n(A))$, where $\operatorname{Hom}^{\mathbb{G}_m}$ stands for homomorphisms of $\mathbb{Z}$-graded (animated) $\mathbb{Z}/p^n\mathbb{Z}$-algebras. The grading on $H^0(G,\mathcal{O}_G)$ is defined using $\mu : \mathbb{G}_m \to G$ just as in §C.2.2.

The map (D.6) induces a group homomorphism

$$ G({}^s\mathscr{R}^{\oplus}_n)^{\mathbb{G}_m} \to G({}^s\mathscr{R}_n[u,u^{-1}])^{\mathbb{G}_m} \times G({}^s\mathscr{R}_n[t,t^{-1}])^{\mathbb{G}_m} = G({}^s\mathscr{R}_n) \times G({}^s\mathscr{R}_n). \tag{D.10} $$

D.8. **The conjecture.**

D.8.1. *The quotient of a groupoid by an action of a 2-group.* A *2-group* is a monoidal category in which all objects and morphisms are invertible. Equivalently, a 2-group is a 2-groupoid with a single object.

Recall that if a 2-group $\mathscr{G}$ acts on a groupoid $\mathscr{X}$ then one defines the *quotient 2-groupoid* $\tilde{\mathscr{X}} = \mathscr{X}/\mathscr{G}$ as follows:

(i) $\operatorname{Ob}\tilde{\mathscr{X}} := \operatorname{Ob}\mathscr{X}$;

(ii) for $x_1, x_2 \in \mathscr{X}$ let $\underline{\operatorname{Mor}}_{\tilde{\mathscr{X}}}(x_1,x_2)$ be the following groupoid: its objects are pairs

$$ (g,f), \text{ where } g \in \mathscr{G}, \quad f \in \operatorname{Isom}(x_2, gx_1), $$

and a morphism $(g,f) \to (g',f')$ is a morphism $g \to g'$ such that the corresponding morphism $gx_1 \to g'x_1$ equals $f'f^{-1}$;

(iii) the composition functor $\underline{\operatorname{Mor}}_{\tilde{\mathscr{X}}}(x_1,x_2) \times \underline{\operatorname{Mor}}_{\tilde{\mathscr{X}}}(x_2,x_3) \to \underline{\operatorname{Mor}}_{\tilde{\mathscr{X}}}(x_1,x_3)$ comes from the product in $\mathscr{G}$.

D.8.2. *The quotient of a stack by an action of a group stack.* Apply the construction from §D.8.1 at the level of presheaves, then sheafify the result.

D.8.3. *The 2-stack* $\mathrm{BT}_n^{G,\mu,?}$. The group stack $G({}^s\mathscr{R}_n)\times G({}^s\mathscr{R}_n)$ acts on the stack $G({}^s\mathscr{R}_n)$ by two-sided translations; our convention is that the first copy of $G({}^s\mathscr{R}_n)$ acts by right translations and the second one by left translations, just as in formula (C.2). So the group stack $G({}^s\mathscr{R}_n^{\oplus})^{\mathbb{G}_m}$ (which depends on μ, see §D.7.4) acts on the stack $G({}^s\mathscr{R}_n)$ via the homomorphism (D.10). Let $\mathrm{BT}_n^{G,\mu,?}$ be the quotient 2-stack, see §D.8.2.

The following conjecture is motivated by Conjecture C.5.3.

Conjecture D.8.4. *Suppose that* $\mu:\mathbb{G}_m\to G$ *is 1-bounded in the sense of §C.4.1. Then there is a canonical isomorphism* $\mathrm{BT}_n^{G,\mu}\xrightarrow{\sim}\mathrm{BT}_n^{G,\mu,?}$*, where* $\mathrm{BT}_n^{G,\mu}$ *is the 1-stack defined in* [GM].

D.8.5. *Remarks.* (i) Conjecture D.8.4 implies that if μ is 1-bounded then $\mathrm{BT}_n^{G,\mu,?}$ is a 1-stack.

(ii) By [GM, Thm. D], if μ is 1-bounded then for every $m\in\mathbb{N}$ the restriction of $\mathrm{BT}_n^{G,\mu}$ to the category of $(\mathbb{Z}/p^m\mathbb{Z})$-algebras is a smooth algebraic stack over $\mathbb{Z}/p^m\mathbb{Z}$. So Conjecture D.8.4 implies a similar statement for $\mathrm{BT}_n^{G,\mu,?}$. This statement is somewhat surprising since the definition of $\mathrm{BT}_n^{G,\mu,?}$ involves the ind-scheme $\hat{W}$. It becomes less surprising once you think about formula (7.2) or about the simpler formula $\mathrm{Coker}(\hat{W}_{\mathbb{F}_p}^{(F)}\xrightarrow{V}\hat{W}_{\mathbb{F}_p}^{(F)})=\alpha_p$.

(iii) The above formulation of Conjecture D.8.4 appeared as a result of my conversations with D. Arinkin and N. Rozenblyum. My original formulation was more "elementary" (in the spirit of §D.8.6) but not elegant enough.

(iv) Two variants of Conjecture D.8.4 for $n=\infty$ are formulated on slides 12 and 13 of the talk [D25c].

D.8.6. *How explicit is* $\mathrm{BT}_n^{G,\mu,?}$*?* Short answer: rather explicit (but not quite explicit). Here are some details.

(i) As already said in §D.2.3(i), $\mathrm{BT}_{n,\mathbb{F}_p}^{G,\mu,?}$ can be represented as a quotient of the scheme $G(W_{n,\mathbb{F}_p})$ by a certain groupoid Γ. More precisely, one has an explicit action of an explicit group ind-scheme on the scheme $G(W_{n,\mathbb{F}_p})$ such that Γ is a certain quotient of the groupoid corresponding to this action.

The interested reader can reconstruct the details using [D25a]. The point is that as explained in the Appendix of [D25a], the group stacks $G({}^s\mathscr{R}_{n,\mathbb{F}_p})$, $G({}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus})^{\mathbb{G}_m}$ naturally come from certain crossed modules in the category of ind-schemes, and the map (D.10) naturally comes from a certain homomorphism of crossed modules. One can apply §2 of [D25a] to this homomorphism.

(ii) Assuming[29] that G is lifted to a group scheme over $\mathbb{Z}_p$, one has a similar situation in mixed characteristic, i.e., a presentation of $\mathrm{BT}_n^{G,\mu,?}$ as a quotient of the scheme $G(W_n)$ by a rather explicit groupoid Γ. To get such a presentation, represent ${}^s\mathscr{R}_n$ as a quotient of the ring scheme $C=W_n$ (see [D25b, §8]) and define $\tilde{C}$ as in [D25b, §9.2.1]. Then the group ind-scheme $G(\tilde{C})^{\mathbb{G}_m}$ acts on $G(W_n)$ by 2-sided translations, and Γ is a certain quotient of the groupoid corresponding to this action.

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