HOLOMORPHIC DISCS OF NEGATIVE MASLOV INDEX AND EXTENDED DEFORMATIONS IN MIRROR SYMMETRY

DENIS AUROUX

Abstract. The SYZ approach to mirror symmetry for log Calabi-Yau manifolds starts from a Lagrangian torus fibration on the complement of an anticanonical divisor. A mirror space is constructed by gluing local charts (moduli spaces of local systems on generic torus fibers) via wall-crossing transformations which account for corrections to the analytic structure of moduli spaces of objects of the Fukaya category induced by bubbling of Maslov index 0 holomorphic discs, and made into a Landau-Ginzburg model by equipping it with a regular function (the superpotential) which enumerates Maslov index 2 holomorphic discs.

When they occur, holomorphic discs of negative Maslov index deform this picture by introducing inconsistencies in the wall-crossing transformations, so that the mirror is no longer an analytic space; the geometric features of the corrected mirror can be understood in the language of extended deformations of Landau-Ginzburg models. We illustrate this phenomenon (and show that it actually occurs) by working through the construction for an explicit example (a log Calabi-Yau 4-fold obtained by blowing up a toric variety), and discuss a family Floer approach to the geometry of the corrected mirror in this setting. Along the way, we introduce a Morse-theoretic model for family Floer theory which may be of independent interest.

1. Introduction

1.1. **SYZ mirror symmetry relative to a nef anticanonical divisor.**

The Strominger-Yau-Zaslow (SYZ) approach to mirror symmetry gives a geometric construction of mirror spaces from Lagrangian torus fibrations on Calabi-Yau manifolds: roughly speaking, a mirror Calabi-Yau is obtained as a dual torus fibration, modified by "instanton corrections" in the presence of singular fibers [SYZ96]. A more modern interpretation of the SYZ conjecture describes the mirror as a moduli space of objects of the Fukaya category of X supported on the torus fibers; this viewpoint leads naturally to Fukaya's family Floer program [Fuk02, Abo14, Abo17, Tu14, Yuan20], which produces a rigid analytic mirror space out of a fibration by unobstructed Lagrangian tori of vanishing Maslov class (as well as a functor from the Fukaya category of X to coherent sheaves on the rigid analytic mirror, which one may then try to use to prove homological mirror symmetry).

The SYZ approach was subsequently extended to the setting of log Calabi-Yau pairs (X, D), where X is a smooth Kähler manifold and D is a (reduced, normal crossings) complex hypersurface in X representing the anticanonical class $-K_X$. Given a suitable Lagrangian torus fibration on the complement of D, one first constructs an SYZ mirror to the open Calabi-Yau $X^0 = X \setminus D$, before analyzing the manner in which the divisor D deforms the Lagrangian Floer theory of the torus fibers (and hence the geometry of the mirror). This deformation is typically described by a regular function $W \in \mathcal{O}(X^\vee)$ called *superpotential*, so that the SYZ mirror of X (or more accurately, of the pair (X, D)) is a *Landau-Ginzburg model* $(X^\vee, W)$. The superpotential W records the fact that the torus fibers, while unobstructed in $X \setminus D$, are only *weakly unobstructed* as objects of the Fukaya category of X, i.e. the Floer-theoretic obstruction $\mathfrak{m}_0 \in CF(L, L)$ is a scalar multiple $W \cdot 1_L$ of the identity, where W is a weighted count of Maslov index 2 holomorphic discs with boundary on L. See e.g. [Aur07] for an informal overview, and [AAK16, Yuan20] for a more up-to-date perspective. (We briefly review the main ingredients in §2.4 below.)

The situation is simplest when the Lagrangian torus fibers do not bound any holomorphic discs in $X \setminus D$, and D is numerically effective (nef). The fibers are assumed to have vanishing Maslov class in $X \setminus D$, so the Maslov index of a disc is equal to twice its intersection number with D, and the simplest holomorphic discs (intersecting D just once) have Maslov index 2. The prototypical setting where these assumptions are satisfied is when X is toric Fano and D is the toric anticanonical divisor. The mirror $X^\vee$ is then an algebraic torus (parametrizing rank 1 local systems on the fibers of the toric moment map), and it follows from an explicit classification of Maslov index 2 discs bounded by the fibers that $W \in \mathcal{O}(X^\vee)$ is a Laurent polynomial determined combinatorially by the moment polytope [CO06, Aur07, FOOO10]. The next simplest case is that of semi-Fano toric varieties, when the toric anticanonical divisor D is nef but not necessarily ample. In this case, the coefficients of the Laurent polynomial W are modified by the contributions of nodal configurations consisting of a Maslov index 2 disc in X together with one or more rational curves with $c_1(X) \cdot C = 0$ contained in the toric divisor D. The first example in which these contributions were determined explicitly is the Hirzebruch surface $\mathbb{F}_2$, i.e. the total space of the $\mathbb{CP}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-2))$ over $\mathbb{CP}^1$ [Aur09, FOOO12]. General results were subsequently obtained by Chan et al. using comparisons between open and closed Gromov-Witten invariants; see e.g. [Chan11, CLL12, CL14, CLLT17].

Outside of the toric setting, the Lagrangian torus fibration $\pi : X^0 \to B$ typically has singular fibers, and the geometric picture is complicated by the presence of holomorphic discs of Maslov index 0. (Still assuming D to be nef, these are precisely the discs which do not intersect D). The fibers of π which bound such discs typically lie along (a small neighborhood of) a union of *walls* of codimension 1 in B. There is a discontinuity in the Floer-theoretic behavior of the fibers of π on either side of a wall, due to bubbling of Maslov index 0 discs; nonetheless, it follows from deep results of Fukaya et al. [FOOO09] that, across each wall, the moduli spaces of local systems on the fibers can be glued together via a suitable analytic coordinate change (the *wall-crossing transformation*) to construct a moduli space of objects of the Fukaya category of X^0 supported on the fibers of π, i.e. the mirror $X^\vee$ [Aur07, AAK16, Tu14, Yuan20]. In general there may be an infinite collection of walls, possibly covering a dense subset of B, so that $X^\vee$ cannot be described explicitly but rather arises as the limit of an inductive construction [KS06, GS11].

While the positions of the walls in B depend on the choice of complex structure, near the *large complex structure limit* (also known as *tropical limit*) the whole process can be understood combinatorially in terms of tropical geometry: B carries an integral affine structure (outside of the locus B^{sing} of singular fibers of π), and the *scattering diagram*, i.e. the set of walls and the corresponding wall-crossing transformations, can be determined via an inductive process first proposed by Kontsevich and Soibelman [KS06], based on *consistency* of the scattering diagram, i.e. the requirement that the wall-crossing transformations must satisfy the cocycle property around each codimension 2 locus where walls intersect. This approach allows one to bypass symplectic geometry altogether: the Gross-Siebert approach to mirror symmetry starts from a toric degeneration to construct a tropical manifold B, its scattering diagram, and a mirror; see e.g. [GS11], [GHK15], etc.

Under the assumption that D is nef, discs (or stable discs, i.e. nodal unions of discs and spheres) which intersect D have Maslov index at least two, so X and X^0 have the same scattering diagram, and the mirror of (X, D) is a Landau-Ginzburg model $(X^\vee, W)$ where $X^\vee$ is entirely determined by the geometry of X^0. The main point of this paper is to show that *this generally fails to hold when D is not nef,* even in examples where the geometry of SYZ fibrations is well understood. Namely, if D contains rational curves with $c_1(X) \cdot C < 0$, then:

(1) the scattering diagram for (X, D) may contain additional walls compared to that for X^0, or the wall-crossing transformations for (X, D) may differ from those of X^0;
(2) in the presence of discs of negative Maslov index, the scattering diagram for (X, D) may be *inconsistent*, so that the wall-crossing transformations defining $X^\vee$ no longer satisfy the cocycle condition.

While the example we give below is mostly a proof of concept, this has significant implications. For instance, the construction of Landau-Ginzburg mirrors for general hypersurfaces in toric varieties given in [AAK16] may require modifications when the stated assumptions about Chern numbers of rational curves do not hold; more generally, it is not quite clear which classes of varieties should be expected to admit genuine Landau-Ginzburg B-model mirrors, rather than deformed LG models of the sort we discuss below. By contrast, this issue does not seem to affect the other direction of homological mirror symmetry: forthcoming work of the author with Abouzaid (the sequel to [AA24]) is expected to prove that the (suitably defined) Fukaya categories of the Landau-Ginzburg A-models given by the construction in [AAK16] are indeed equivalent to the derived categories of the corresponding hypersurfaces, without Chern class restrictions.

Remark. Our results do not contradict in any way the recent work of Gross and Siebert [GS22] (see also Keel and Yu [KY23]) constructing a canonical scattering diagram for log Calabi-Yau pairs (X, D) and proving its consistency. Namely, Gross-Siebert's scattering diagram only includes Maslov index 0 discs which are contained in $X \setminus D$, and determines the SYZ mirror of $X \setminus D$; whereas we are studying SYZ mirror symmetry for X, whose scattering diagram also involves Maslov index 0 configurations consisting of a disc in X together with one or more rational curves in D.

1.2. **A log Calabi-Yau 4-fold with an inconsistent scattering diagram.** Our main example is the following. Let $K_{\mathbb{CP}^1} = \mathcal{O}_{\mathbb{CP}^1}(-2)$ be the total space of the canonical bundle of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. The toric mirror Landau-Ginzburg model of $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ is (a domain in) the algebraic torus $(\mathbb{K}^*)^4$ (where $\mathbb{K}$ is the nonarchimedean field over which we define the Fukaya category, say the Novikov field over $\mathbb{C}$ for concreteness) with coordinates $(z_1, \dots, z_4)$, equipped with the superpotential

$$W = z_1 + z_2 + (1 + q^2 + qz_3 + qz_3^{-1})z_4,$$

where $q \in \mathbb{K}^*$ is a constant determined by the choice of Kähler form (namely, q^2 is the Novikov weight of the zero section $C_0 \subset K_{\mathbb{CP}^1}$).

Theorem 1.1. *Let X be the blowup of $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ at $H_0 = \mathbb{C} \times \{1\} \times L_0$ and $H_\infty = \{1\} \times \mathbb{C} \times L_\infty$, where L_0 and L_∞ are the fibers of $K_{\mathbb{CP}^1}$ over 0 and $\infty \in \mathbb{CP}^1$ respectively, and let D be the proper transform of the toric anticanonical divisor of $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$. Equip X with a suitable Kähler form. Then $X \setminus D$ carries a fibration by Lagrangian tori of vanishing Maslov class, whose SYZ mirror consists of four charts which are domains in $(\mathbb{K}^*)^4$, with superpotentials*

$$\begin{aligned} W_{--} &= z_1 + z_2 + (1 + q^2 + qz_3 + qz_3^{-1})z_4, \\ W_{-+} &= z_1 + z_2(1 + qq'z_4 + q'z_3z_4) + (1 + q^2 + qz_3 + qz_3^{-1})z_4, \\ W_{+-} &= z_1(1 + qq''z_4 + q''z_3^{-1}z_4) + z_2 + (1 + q^2 + qz_3 + qz_3^{-1})z_4, \\ W_{++} &= z_1(1 + qq''z_4 + q''z_3^{-1}z_4) + z_2(1 + qq'z_4 + q'z_3z_4) + q'q''z_1z_2z_4 + \\ &\quad + (1 + q^2 + qz_3 + qz_3^{-1})z_4, \end{aligned} \tag{1.1}$$

where $q', q'' \in \mathbb{K}^*$ are suitable constants. These charts are glued pairwise by coordinate transformations which preserve z_3, z_4 and act on z_1, z_2 by

$$\begin{aligned} &\varphi_{-0}(z_1, z_2) = (z_1, z_2(1 + qq'z_4 + q'z_3z_4)), && \varphi_{-0}^*(W_{--}) = W_{-+}, \\ &\varphi_{+0}(z_1, z_2) = (z_1, z_2(1 + qq'z_4 + q'z_3z_4 + q'q''z_1z_4)), && \varphi_{+0}^*(W_{+-}) = W_{++}, \\ &\varphi_{0-}(z_1, z_2) = (z_1(1 + qq''z_4 + q''z_3^{-1}z_4), z_2), && \varphi_{0-}^*(W_{--}) = W_{+-}, \\ &\varphi_{0+}(z_1, z_2) = (z_1(1 + qq''z_4 + q''z_3^{-1}z_4 + q'q''z_2z_4), z_2), && \varphi_{0+}^*(W_{-+}) = W_{++}. \end{aligned} \tag{1.2}$$

The wall-crossing transformations (1.2) are *inconsistent*, in the sense that

$$\varphi_{-0} \circ \varphi_{0+} \neq \varphi_{0-} \circ \varphi_{+0}.$$

This inconsistency arises from the presence of a codimension 2 locus in the base of the SYZ fibration over which the fibers bound stable nodal discs of Maslov index -2. Indeed, the cocycle property for wall-crossing transformations is equivalent to the statement that Maslov index 0 discs can only break into unions of Maslov index 0 discs; whereas in our example they can also degenerate to the union of discs of Maslov indices 2 and -2. We expect this to be a general feature of mirror symmetry in settings where the non-negativity of Maslov index cannot be guaranteed.

The proof of Theorem 1.1 is given in Section 2; the main new ingredient compared to previous calculations on blowups of toric varieties (see in particular [AAK16]) is a study of the contributions of stable nodal configurations consisting of a holomorphic disc in X together with a rational curve in D.

Remark 1.2. By contrast, the construction of the SYZ mirror of $X \setminus D$ involves the same four charts, but the wall-crossing transformations have simpler expressions:

$$\begin{aligned} \varphi^o_{-0} = \varphi^o_{+0} :& \qquad (z_1, z_2) \mapsto (z_1, z_2(1 + q'z_3z_4)) \\ \varphi^o_{0-} = \varphi^o_{0+} :& \qquad (z_1, z_2) \mapsto (z_1(1 + q''z_3^{-1}z_4), z_2). \end{aligned} \tag{1.3}$$

(These are determined by Maslov index 0 discs in $X \setminus D$, whereas the additional terms in (1.2) correspond to Maslov index 0 configurations with sphere components in D.) The formulas (1.3) match the consistent scattering diagram constructed by Gross-Siebert [GS22] for the mirror of $X \setminus D$.

We also give in §2.7 the analogous formulas for the mirror of a compact example, namely the projective log Calabi-Yau $(\bar{X}, \bar{D})$ obtained from (X, D) by compactifying $\mathbb{C}$ to $\mathbb{CP}^1$ and $K_{\mathbb{CP}^1}$ to the Hirzebruch surface $\mathbb{F}_2$: namely $\bar{X}$ is the blowup of $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{F}_2$ at $\bar{H}_0 = \mathbb{CP}^1 \times \{1\} \times \bar{L}_0$ and $\bar{H}_\infty = \{1\} \times \mathbb{CP}^1 \times \bar{L}_\infty$, where $\bar{L}_0$ and $\bar{L}_\infty$ are the fibers of the projection from $\mathbb{F}_2$ to $\mathbb{CP}^1$ over 0 and ∞, and $\bar{D}$ is the proper transform of the toric anticanonical divisor of $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{F}_2$.

The mirror as a deformed Landau-Ginzburg model. Even though the mirror in Theorem 1.1 is no longer a Landau-Ginzburg model, setting $q'q'' = 0$ in the formulas (1.1)–(1.2) (i.e., discarding the terms $q'q''z_1z_2z_4$ in W_{++}, φ_{+0}, and φ_{0+}) cures the inconsistency in (1.2) and gives a well-defined Landau-Ginzburg model $(X^\vee, W)$, of which the mirror in Theorem 1.1 can be viewed as a *deformation*. By a result of Lin and Pomerleano [LP13, Theorem 3.1], the Hochschild cohomology of the category of matrix factorizations of $(X^\vee, W)$ is the hypercohomology of the complex of sheaves $(\Lambda^* T_{X^\vee}, \iota_{dW})$ on $X^\vee$. We claim that the first-order deformation in Theorem 1.1 can be viewed as a class in

$$HH^*(MF(X^\vee, W)) = \mathbb{H}^*(X^\vee, (\Lambda^* T_{X^\vee}, \iota_{dW}))$$

determined by the contributions of holomorphic discs of Maslov index -2 in X.

The Maslov index -2 discs bounded by the Lagrangian torus fibers sweep a complex codimension 2 locus in X, namely $\{1\} \times \{1\} \times K_{\mathbb{CP}^1}$, with a Floer-theoretic weight equal to $q'q''z_4$ mod higher order terms (see §2). Thus, the first-order deformation induced by these discs is a 2-cocycle on the base of the fibration with values in the second cohomology of the fiber, hence dually in $\Lambda^2 T_{X^\vee}$, whose value on the relevant overlap of coordinate charts is

$$q'q''z_4\, \partial_{\log z_1} \wedge \partial_{\log z_2}. \tag{1.4}$$

This element $w^{(2)} \in H^2(X^\vee, \Lambda^2 T_{X^\vee})$ is not closed under ι_{dW}, but it can be completed to a Hochschild cocycle in $HH^{even}(MF(X^\vee, W))$ by adding to it a 1-cochain $w^{(1)}$ with values in $T_{X^\vee}$, whose Čech coboundary cancels out $\iota_{dW}(w^{(2)})$; meaning that the value of $\delta w^{(1)}$ on the overlap of coordinate charts is the vector field

$$\iota_{dW}(q'q''z_4\, \partial_{\log z_1} \wedge \partial_{\log z_2}) = q'q''z_4(z_1\partial_{\log z_2} - z_2\partial_{\log z_1}),$$

which is exactly the inconsistency in (1.2). Specifically, we can match (1.2) by setting

$$w^{(1)}_{-0} = w^{(1)}_{0-} = 0, \quad w^{(1)}_{+0} = q'q''z_1z_4\partial_{\log z_2}, \quad \text{and} \quad w^{(1)}_{0+} = q'q''z_2z_4\partial_{\log z_1}.$$

Finally, cancelling out $\iota_{dW}(w^{(1)})$ in turn forces one to also add a 0-cochain $w^{(0)}$ with values in $\mathcal{O}_{X^\vee}$, namely we take

$$w^{(0)}_{--} = w^{(0)}_{-+} = w^{(0)}_{+-} = 0 \quad \text{and} \quad w^{(0)}_{++} = q'q''z_1z_2z_4.$$

1.3. **A family Floer perspective.** The above example shows that the construction of SYZ mirrors in the presence of discs of negative Maslov index requires a change of perspective from the usual approach. In Section 3 we begin a general (but informal) exploration of the geometry of SYZ mirror symmetry in the setting considered here from the perspective of family Floer homology.

Consider as before a Lagrangian torus fibration $\pi : X^0 \to B$ on $X^0 = X \setminus D$ whose fibers $F_b = \pi^{-1}(b)$ have vanishing Maslov class in X^0 and are weakly unobstructed in X. Denote by $X^{\vee 0}$ the *uncorrected* SYZ mirror of the smooth locus, a rigid analytic space whose points correspond to unitary rank 1 local systems on the smooth fibers of π. More precisely, we restrict ourselves to a simply connected subset B^0 of the smooth locus $B \setminus \mathrm{critval}(\pi)$, so as to ignore the issues of compactification over the singular fibers of π and consistency around the singular fibers, which are largely orthogonal to our discussion.

The pushforward of the sheaf of analytic functions on $X^{\vee 0}$ under the rigid analytic torus fibration $\pi^\vee : X^{\vee 0} \to B^0$ defines a sheaf $\mathcal{O}_{an}$ over B^0, which is a certain completion of a local system over B^0 whose fiber at b is $\mathbb{K}[H_1(F_b)]$. (This is just a fancy way of saying that rigid analytic functions on affinoid domains in $X^{\vee 0}$ are given by Laurent series which satisfy appropriate convergence conditions.)

Moduli spaces of pseudo-holomorphic discs in X with boundary in the fibers F_b (where b is allowed to vary over B^0) determine A_∞-operations $\{\mathfrak{m}_k\}_{k\geq 0}$ not just on Floer cochains of a fixed fiber F_b with coefficients in $\mathbb{K}[H_1(F_b)]$, but also on cochains on $\pi^{-1}(B^0) \subset X^0$ with coefficients in the pullback of $\mathcal{O}_{an}$, or equivalently, via Künneth decomposition, on

$$\mathfrak{C} = \bigoplus_{i,j} \mathfrak{C}^{i,j} := \bigoplus_{i,j} C^i(B^0\,;\, C^j(F_b)\,\hat{\otimes}\,\mathcal{O}_{an}). \tag{1.5}$$

The precise nature of these cochains depends on the chosen model for Lagrangian Floer theory. Under very strong transversality assumptions on evaluation maps, a convenient model consists of an enlargement of differential forms to include currents of integration along smooth submanifolds (cf. §3.1.1). While it is likely that these assumptions can be lifted by working with Kuranishi structures, it seems technically easier to work with a Morse-theoretic model, such as the one we describe in §§3.1.2–3.1.3 (see also Keeley Hoek's thesis [Hoek25] for a more detailed treatment).

Definition 1.3. The Floer complex $\mathfrak{C}$ is *weakly family unobstructed* if $\mathfrak{m}_0$ can be expressed as a sum of degree i cochains with values in degree i cocycles,

$$\alpha^{(i)} \in C^i(B^0\,;\, Z^i(F_b)\,\hat{\otimes}\,\mathcal{O}_{an}), \quad i = 0, 1, \dots \tag{1.6}$$

This definition is quite restrictive (exactly how much depends on the precise model chosen for cochains) and clearly not satisfied by all SYZ fibrations, but we conjecture that weak family unobstructedness should arise in SYZ mirror symmetry from the existence of a degeneration of the complex structure on X to the tropical limit, possibly after correction by a suitably chosen weak family bounding cochain (see below). Indeed, in the tropical limit one expects that the moduli spaces of holomorphic discs which sweep loci of real codimension $2i$ inside X should concentrate along "walls" of codimension i inside B^0.

There is a natural bracket of degree -1 on $H^*(F_b) \otimes \mathbb{K}[H_1(F_b)]$, defined by

$$\{z^\gamma\,\alpha, z^{\gamma'}\alpha'\} = z^{\gamma+\gamma'}\left(\alpha \wedge (\iota_\gamma \alpha') + (-1)^{|\alpha|}(\iota_{\gamma'}\alpha) \wedge \alpha'\right) \tag{1.7}$$

for all $\alpha, \alpha' \in H^*(F_b)$ and $\gamma, \gamma' \in H_1(F_b)$. Extending to the completion $H^*(F_b) \hat{\otimes} \mathcal{O}_{an}$ and combining with the cup-product on B^0, this determines a bracket on $C^*(B^0; H^*(F_b) \hat{\otimes} \mathcal{O}_{an})$, symmetric on even degree elements, which we again denote by $\{\cdot, \cdot\}$.

Conjecture 1.4. *For SYZ fibrations on log Calabi-Yau varieties near the tropical limit, there exists a model of the family Floer complex $\mathfrak{C}$ for which $\mathfrak{m}_0$ can be expressed as an element in $\bigoplus C^i(B^0; H^i(F_b) \hat{\otimes} \mathcal{O}_{an})$ and satisfies, up to sign, the* master equation

$$\delta \mathfrak{m}_0 = \frac{1}{2}\{\mathfrak{m}_0, \mathfrak{m}_0\}, \tag{1.8}$$

where $\{\cdot,\cdot\}$ is the bracket defined by (1.7) *and δ is the differential on cochains on B^0.*

Remark 1.5. We expect (1.8) to hold whenever the moduli spaces of holomorphic discs underlying $\mathfrak{m}_0$ behave like closed manifolds, as a consequence of (a family version of) the master equation in Floer theory with free loop space coefficients [Fuk06, Irie20]; see Section 3.2. So the expectation that this happens for SYZ fibrations near the tropical limit is the geometric content of the conjecture. However, it is quite possible that the conjecture is too strong as stated, and that the A_∞-structure on $\mathfrak{C}$ may need to be deformed by a suitable "weak family bounding cochain" $\mathfrak{b} \in \mathfrak{C}_{>0} \subset \mathfrak{C}$ (the subspace of elements whose components have positive Novikov valuation everywhere), in order for the deformed $\mathfrak{m}_0$ term

$$\mathfrak{m}_0^{\mathfrak{b}} := \mathfrak{m}_0 + \mathfrak{m}_1(\mathfrak{b}) + \mathfrak{m}_2(\mathfrak{b}, \mathfrak{b}) + \cdots \in \mathfrak{C}_{>0}$$

to satisfy the requirements of the conjecture. We note that, even when $\mathfrak{m}_0^{\mathfrak{b}}$ satisfies weak family unobstructedness, there is no geometric reason for the master equation to hold; rather, it needs to be imposed as an extra requirement on $\mathfrak{b}$. It also seems natural to require $\mathfrak{b}$ to vanish outside of a neighborhood of the walls in B^0 (when there are infinitely many walls, this statement should be understood order by order).

On the other hand, we sketch in Section 3.3 a possible approach to the master equation in a Morse-theoretic setup, via a deformation of the moduli space of treed holomorphic discs.

Via the isomorphism $H^i(F_b, \mathbb{R}) \simeq \Lambda^i H^1(F_b, \mathbb{R}) \simeq \Lambda^i T_B$, an element of $H^i(F_b) \hat{\otimes} \mathcal{O}_{an}$ naturally determines a section of $\Lambda^i T_{X^{\vee 0}}$ over $(\pi^\vee)^{-1}(b)$, where we again denote by $X^{\vee 0}$ the *uncorrected* mirror, equipped with the rigid analytic torus fibration $\pi^\vee : X^{\vee 0} \to B^0$ (locally modelled on the valuation map, and dual to π). Hence, under the assumption of weak family unobstructedness, the components $\alpha^{(i)}$ of $\mathfrak{m}_0$ determine elements

$$W^{(i)} \in C^i(X^{\vee 0}, \Lambda^i T_{X^{\vee 0}}), \tag{1.9}$$

which encode the instanton corrections to the geometry of $X^{\vee 0}$; see Section 3.4. We denote by $\mathbb{W} = \sum_{i \geq 0} W^{(i)} \in C^*(X^{\vee 0}, \Lambda^* T_{X^{\vee 0}})$ the sum of these terms. The master equation for $\mathfrak{m}_0$ can be transcribed (by an easy argument, cf. §3.4) into an analogous identity for $\mathbb{W}$:

Proposition 1.6. *If $\mathfrak{m}_0$ satisfies* (1.8)*, then $\mathbb{W} = W^{(0)} + W^{(1)} + \cdots \in C^*(X^{\vee 0}, \Lambda^* T_{X^{\vee 0}})$ satisfies*

$$\delta \mathbb{W} + \frac{1}{2}[\mathbb{W}, \mathbb{W}] = 0, \tag{1.10}$$

where δ is the differential on cochains and $[\cdot,\cdot]$ is the bracket induced by the cup-product and the Schouten-Nijenhuis bracket.

Equation (1.10) is equivalent to the property that the operator $\delta + [\mathbb{W}, \cdot]$ squares to zero; in particular, the components of $\mathbb{W}$ satisfy the equations

$$(\delta + [W^{(1)}, \cdot])\, W^{(0)} \;=\; 0, \tag{1.11}$$

$$(\delta + [W^{(1)}, \cdot])^2 \;=\; [\iota_{dW^{(0)}}(W^{(2)}), \cdot], \tag{1.12}$$

$$(\delta + [W^{(1)}, \cdot])\, W^{(2)} \;=\; \iota_{dW^{(0)}}(W^{(3)}), \tag{1.13}$$

and so on. The geometric interpretation of these equations depends on the chosen model for cochains in the above discussion, though in all cases $W^{(1)}$ can be viewed as a deformation of the analytic structure of $X^{\vee 0}$, and (1.11) states that $W^{(0)}$ is analytic with respect to the deformed structure, even as (1.12) measures the failure of $\delta + W^{(1)}$ to genuinely equip the corrected mirror with an analytic structure.

If we view $\mathfrak{m}_0$ as an element of the de Rham complex $(\Omega^*(B^0, H^*(F_b) \,\hat{\otimes}\, \mathcal{O}_{an}), d)$ of differential forms on B^0 with coefficients in the sheaf $H^*(F_b) \,\hat{\otimes}\, \mathcal{O}_{an}$, then $\mathbb{W}$ ends up being an element of $(\Omega^{0,*}(X^{\vee 0}, \Lambda^* T_{X^{\vee 0}}), d'')$, the tropical Dolbeault complex of differential forms on $X^{\vee 0}$ with coefficients in polyvector fields. (See [CLD12, Jell22] for a general construction; the version we need here is significantly simpler because our forms are pulled back from the fixed tropicalization $\pi^\vee : X^{\vee 0} \to B^0$.)

Assuming convergence, we can view $X^{\vee 0}$ as a family of complex manifolds over a punctured disc, degenerating to the tropical limit. The tropical Dolbeault complex specializes to the usual Dolbeault complex, and we can then view $W^{(1)} \in \Omega^{0,1}(X^{\vee 0}, T_{X^{\vee 0}})$ as a deformation of the complex structure on $X^{\vee 0}$ (deforming $\bar\partial$ to $\bar\partial + W^{(1)}$). The equation (1.11) then states that the function $W^{(0)} : X^{\vee 0} \to \mathbb{C}$ is holomorphic with respect to this deformed complex structure; and (1.12) states that the deformation in general fails to be integrable, i.e. $\bar\partial + W^{(1)}$ is only an *almost-complex structure*, whose Nijenhuis tensor is required to be equal to $\iota_{dW^{(0)}}(W^{(2)}) \in \Omega^{0,2}(X^{\vee 0}, T_{X^{\vee 0}})$.

On the other hand, if we work with Čech cochains rather than differential forms, then we end up with a picture similar to that discussed above for our main example: $W^{(1)}$ can be viewed as a deformation of the gluing transformations used to assemble $X^\vee$ from local affinoid charts, (1.11) states that the expressions for $W^{(0)}$ in these local charts match under the deformed gluing transformations, and (1.12) states that $\iota_{dW^{(0)}}(W^{(2)})$ measures the amount by which the deformed gluing transformations fail to satisfy the cocycle condition.

These two perspectives on deformed Landau-Ginzburg models ought to be equivalent; for example it is readily apparent from both viewpoints that the critical locus of the superpotential $W^{(0)}$ remains an honest analytic space (since the right-hand side of (1.12) vanishes along it), even when the deformed total space fails to be one, so that it still makes sense to try and relate the symplectic geometry of X to the algebraic geometry of $\mathrm{crit}(W^{(0)})$ in order to establish homological mirror symmetry.

Remark 1.7. The A_∞-structure on the family Floer complex $\mathfrak{C}$ is a curved deformation (induced by holomorphic discs) of the classical algebraic structure on $C^*(B^0; C^*(F_b) \hat{\otimes} \mathcal{O}_{an})$, which we have just seen can be compared to the dg-algebra $C^*(X^{\vee 0}; \Lambda^* T_{X^{\vee 0}})$ of cochains with values in polyvector fields on the uncorrected mirror. Even though we expect that the curvature $\mathfrak{m}_0$ of the family Floer complex $\mathfrak{C}$ determines the required instanton corrections to the geometry of $X^{\vee 0}$ (see also Remark 3.18), it is not true (even in the simplest examples) that $\mathfrak{C}$ itself, as constructed in §3.1, describes cochains with values in polyvector fields on the corrected mirror. Indeed, unlike the latter algebra, $\mathfrak{C}$ has nonzero curvature; and its differential $\mathfrak{m}_1$ does not match with the desired expression $\delta + \{\mathfrak{m}_0, \cdot\}$. On the other hand, work in progress of the author with Keeley Hoek suggests that a variant of the construction in §3.3 can be used to define an *uncurved* algebraic structure on the family Floer complex that appears to describe the geometry of the corrected mirror.

Acknowledgements. I am heavily indebted to Ludmil Katzarkov and Maxim Kontsevich for many stimulating discussions about deformations of Landau-Ginzburg models, which directly led to this investigation of their geometric origin in Lagrangian Floer theory. I am also grateful for the hospitality of IHÉS, where most of this work was carried out. This work was partially supported by NSF grant DMS-2202984 and by the Simons Foundation (grant # 385573, Simons Collaboration on Homological Mirror Symmetry).

2. A 4-dimensional example

This section is devoted to the geometric construction of our main example and proof of Theorem 1.1. The geometric setup is similar to [Aur09, Section 3] and [AAK16, Sections 3-5], which also deal with SYZ mirror symmetry for blowups of toric varieties.

2.1. **The geometric setup.** Let $K_{\mathbb{CP}^1} = \mathcal{O}_{\mathbb{CP}^1}(-2)$ be the total space of the canonical bundle of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, and denote by L_0 and L_∞ the fibers of $K_{\mathbb{CP}^1}$ over 0 and ∞ in $\mathbb{CP}^1$. We equip the product $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ with coordinates (x_1, x_2, x_3, x_4), where x_1, x_2 are the standard coordinates of $\mathbb{C}^2$, $x_3 \in \mathbb{C} \cup \{\infty\}$ is a coordinate on $\mathbb{CP}^1$, and x_4 is a coordinate in the fibers of $K_{\mathbb{CP}^1}$ in the trivialization given by the 1-form $d \log x_3$ over $\mathbb{C}^*$. In other terms, the affine chart $\{x_3 \neq \infty\} \subset K_{\mathbb{CP}^1}$ is isomorphic to $\mathbb{C}^2$ with coordinates $(x_3, x_3^{-1}x_4)$, while the affine chart $\{x_3 \neq 0\}$ is isomorphic to $\mathbb{C}^2$ with coordinates (x_3^{-1}, x_3x_4).

We denote by X the blowup of $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ along $H_0 = \mathbb{C} \times \{1\} \times L_0$, i.e. the locus where $x_2 = 1$ and $x_3 = 0$, and along $H_\infty = \{1\} \times \mathbb{C} \times L_\infty$, i.e. the locus where $x_1 = 1$ and $x_3 = \infty$; we denote again by $x_1, \dots, x_4$ the pullbacks of the coordinates of $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ under the blowup map $p : X \to \mathbb{C}^2 \times K_{\mathbb{CP}^1}$. The T^2-action on $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ rotating the x_3 and x_4 coordinates leaves H_0 and H_∞ invariant, and hence lifts to X.

We equip X with a T^2-invariant Kähler form ω constructed as in [AAK16, Section 3.2], symplectomorphic to a toric Kähler form on $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ away from a neighborhood of the exceptional divisors $E_0 = p^{-1}(H_0)$ and $E_\infty = p^{-1}(H_\infty)$. For example, one can take

$$\omega = p^*(\omega_{\mathbb{C}^2} \oplus \omega_{K_{\mathbb{CP}^1}}) + \frac{i\epsilon'}{2\pi}\partial\bar\partial \left(\chi \log(|x_2 - 1|^2 + |x_3|^2)\right) + \frac{i\epsilon''}{2\pi}\partial\bar\partial \left(\chi \log(|x_1 - 1|^2 + |x_3^{-1}|^2)\right),$$

where $\omega_{\mathbb{C}^2} \oplus \omega_{K_{\mathbb{CP}^1}}$ is a product toric Kähler form on $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ (standard along the first factor), $\epsilon', \epsilon'' > 0$ are the areas of the fibers of the exceptional divisors E_0 and E_∞, and $\chi \log : \mathbb{R}_+ \to \mathbb{R}$ is the product of the logarithm with a suitable cut-off function.

We denote by $\mu = (\mu_3, \mu_4)$ the moment map of the T^2-action on X rotating the x_3 and x_4 coordinates; away from $E_0 \cup E_\infty$ it coincides with the pullback of the moment map of the chosen toric Kähler form on $K_{\mathbb{CP}^1}$, and they have the same moment polytope $\Delta \subset \mathbb{R}^2$.

We normalize the moment map so that

$$\Delta = \{(\xi_3, \xi_4) \in \mathbb{R}^2 \,|\, \xi_4 \geq \max(0, |\xi_3| - a)\},$$

where $a > 0$ is half the symplectic area of the zero section of $K_{\mathbb{CP}^1}$.

For every $(\xi_3, \xi_4) \in \Delta$, the reduced space $\mu^{-1}(\xi_3, \xi_4)/T^2$ is canonically identified with $\mathbb{C}^2$ via projection to the x_1 and x_2 coordinates. The reduced Kähler form $\omega_{red,(\xi_3,\xi_4)}$ is a product form in the (x_1, x_2) coordinates, and coincides with the standard Kähler form of $\mathbb{C}^2$ whenever $\mu^{-1}(\xi_3, \xi_4)$ lies sufficiently far away from $E_0 \cup E_\infty$. Near E_0 (which maps to the region of Δ where $\xi_4 \leq -\xi_3 - a + \epsilon'$), the x_2-component of the reduced Kähler form differs from the standard area form near $x_2 = 1$, and similarly near E_∞ (where $\xi_4 \leq \xi_3 - a + \epsilon''$), the x_1-component differs from the standard area form near $x_1 = 1$. The reduced Kähler form is singular along $x_2 = 1$ for $\xi_4 = -\xi_3 - a + \epsilon'$ (this corresponds to a stratum of points with S^1 stabilizers where E_0 meets the proper transform of $\mathbb{C} \times \{1\} \times K_{\mathbb{CP}^1}$), and similarly along $x_1 = 1$ for $\xi_4 = \xi_3 - a + \epsilon''$ (where E_∞ meets the proper transform of $\{1\} \times \mathbb{C} \times K_{\mathbb{CP}^1}$). (The arguments are similar to [AAK16, Section 4.1] and we omit the details.)

Since the reduced Kähler form on $\mu^{-1}(\xi_3, \xi_4)/T^2$ is a product form on $\mathbb{C}^2$, the product tori $\{|x_1| = r_1,\ |x_2| = r_2\}$ are Lagrangian in the reduced space, hence their lifts to $\mu^{-1}(\xi_3, \xi_4) \subset X$ are T^2-invariant Lagrangian submanifolds of X, singular when the lift contains degenerate T^2-orbits and smooth otherwise. Hence, we have:

Definition-Proposition 2.1. *For $(r_1, r_2, \xi_3, \xi_4) \in B := \mathbb{R}^2_+ \times \mathrm{int}(\Delta)$, denote by $F_{(r_1,r_2,\xi_3,\xi_4)}$ the Lagrangian submanifold of X defined by the equations*

$$|x_1| = r_1,\ |x_2| = r_2,\ \mu_3 = \xi_3,\ \mu_4 = \xi_4.$$

Denoting by $D \subset X$ the proper transform of the union of the toric divisors of $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$,

$$\pi = (|x_1|, |x_2|, \mu_3, \mu_4) : X \setminus D \to B$$

defines a Lagrangian torus fibration on $X \setminus D$, with singular fibers over

$$B^{sing} = \{r_2 = 1,\ \xi_4 = -\xi_3 - a + \epsilon'\} \cup \{r_1 = 1,\ \xi_4 = \xi_3 - a + \epsilon''\} \subset B. \tag{2.1}$$

The fibers of π which lie sufficiently far from the exceptional divisors, i.e., away from

$$B^{exc} = \{r_2 = 1,\ \xi_4 \leq -\xi_3 - a + \epsilon'\} \cup \{r_1 = 1,\ \xi_4 \leq \xi_3 - a + \epsilon''\} \subset B, \tag{2.2}$$

are lifts to X of product tori in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$, hence special Lagrangian with respect to the holomorphic volume form $p^*(\prod d\log x_i)$ on $X \setminus D$ with simple poles along D. This implies immediately:

Lemma 2.2. *The fibers of π have vanishing Maslov class in $X \setminus D$, and the Maslov index of a disc in X with boundary on a fiber of π is twice its algebraic intersection number with D.*

2.2. Discs and spheres.

The next few sections are devoted to the enumerative geometry of stable holomorphic discs in X with boundary on the fibers of π. We start with two lemmas describing the relevant discs and spheres.

Lemma 2.3. *Let F be a fiber of π which is the lift to X of a product torus $\{|x_i| = r_i\}$ in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$, and let $u : D^2 \to X$ be a holomorphic disc with boundary on F. Then:*

(1) *The components of* $p \circ u$ *have Blaschke product expansions*

$$x_1(z) = e^{i\theta_1}\, r_1 \prod_{i=1}^{n_1} \frac{z-\alpha_{i,1}}{1-\bar{\alpha}_{i,1} z}, \qquad x_2(z) = e^{i\theta_2}\, r_2 \prod_{i=1}^{n_2} \frac{z-\alpha_{i,2}}{1-\bar{\alpha}_{i,2} z}, \tag{2.3}$$

$$x_3(z) = e^{i\theta_3}\, r_3 \prod_{i=1}^{n_3} \left(\frac{z-\alpha_{i,3}}{1-\bar{\alpha}_{i,3} z}\right)^{\epsilon_{i,3}}, \qquad x_4(z) = e^{i\theta_4}\, r_4 \prod_{i=1}^{n_3} \frac{z-\alpha_{i,3}}{1-\bar{\alpha}_{i,3} z} \prod_{i=1}^{n_4} \frac{z-\alpha_{i,4}}{1-\bar{\alpha}_{i,4} z},$$

where $e^{i\theta_k} \in S^1$, $\alpha_{i,k} \in D^2$, *and* $\epsilon_{i,3} \in \{\pm 1\}$.

(2) u *is regular, except possibly if* $x_1(z)$ *or* $x_2(z)$ *is constant and equal to* 1 *for all* z.

(3) *The Maslov index of* u *is*

$$\mu(u) = 2(n_1 + n_2 + n_3 + n_4 - k_0 - k_\infty), \tag{2.4}$$

where k_0 *and* k_∞ *are the total contact orders of* $p \circ u$ *with* H_0 *and* H_∞*; in the absence of multiple roots* k_0 *is the number of* $i \in \{1, \dots, n_3\}$ *such that* $\epsilon_{i,3} = +1$ *and* $x_2(\alpha_{i,3}) = 1$*, and* k_∞ *is the number of* i *such that* $\epsilon_{i,3} = -1$ *and* $x_1(\alpha_{i,3}) = 1$.

Proof. (1) The Blaschke product expansions follow from the general classification of holomorphic discs with boundary on T^n-orbits in toric manifolds, see e.g. [CO06, Theorem 5.3]; the only specific feature in our case is that, given our choice of coordinates on $K_{\mathbb{CP}^1}$, x_3 is allowed to have poles, and x_4 must vanish at the zeroes and poles of x_3.

(2) Via the projection p, moduli spaces of holomorphic discs in X with boundary on F correspond to moduli spaces of holomorphic discs in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ with prescribed contact orders with H_0 and H_∞. For fixed $x_1(z)$ and $x_2(z)$, requiring $x_3(z)$ to vanish to given order at a certain roots of $x_2(z) - 1$, and/or to have poles of given order at certain roots of $x_1(z) - 1$, cuts out a smooth subvariety of the space of possible Blaschke products of given degree for $x_3(z)$, of the expected codimension except when $x_2(z) - 1$ or $x_1(z) - 1$ vanishes identically. (This is because the conditions amount to independent linear constraints on the coefficients of the polynomials $\prod_{\epsilon_{i,3}=+1}(z - \alpha_{i,3})$ and $\prod_{\epsilon_{i,3}=-1}(z - \alpha_{i,3})$.) The regularity of u then follows from a general regularity result for holomorphic discs in the toric setting [CO06, Theorem 6.1] and from the fact that the prescribed incidence conditions with H_0 and H_∞ define a transversely cut out, smooth submanifold of the expected codimension.

(3) By Lemma 2.2, the Maslov index of u is twice its intersection number with the divisor D; since $D + E_0 + E_\infty$ is the pullback of the toric anticanonical divisor of $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$, the intersection number is given by counting the zeroes and poles of $x_1, \dots, x_4$, and excluding intersections with the exceptional divisors E_0 and E_∞, which correspond to the intersections of $p \circ u$ with H_0 and H_∞. □

Lemma 2.4. *The only simple holomorphic spheres in* X *are* (1) *the spheres* $S_{(x_1,x_2)}$ *given by the product of a point* $\{(x_1, x_2)\} \in \mathbb{C}^2$ *with the zero section of* $K_{\mathbb{CP}^1}$*, or their proper transforms when* $x_1 = 1$ *and/or* $x_2 = 1$*, and* (2) *the fibers of the projection* $p : X \to \mathbb{C}^2 \times K_{\mathbb{CP}^1}$ *above the points of* $H_0 \cup H_\infty$.

Proof. By the maximum principle, x_1, x_2 and x_4 are necessarily constant along any holomorphic map $u : S^2 \to X$, and x_4 is necessarily zero since the nonzero levels of x_4 are biholomorphic to $\mathbb{C}^2 \times \mathbb{C}^*$. If x_3 is nonconstant then we end up with $S_{(x_1,x_2)}$ or a multiple cover; otherwise the image of u is contained in a fiber of p over the blown up locus $H_0 \cup H_\infty$. □

We note that $S_{(x_1,x_2)}$ has normal bundle $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2)$ for $x_1, x_2 \neq 1$, while $S_{(x_1,1)}$ has normal bundle $\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)$, similarly for $S_{(1,x_2)}$, and $S_{(1,1)}$ has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. Since the $\bar\partial$ operator on $\mathcal{O}(-2)$ fails to be surjective, the curves $S_{(x_1,x_2)}$ are not regular. However, we will see that the union of $S_{(x_1,x_2)}$ with a holomorphic disc that meets the zero section of $K_{\mathbb{CP}^1}$ transversely is regular as a stable disc.

2.3. **Regularity of stable nodal discs.** Let C be a nodal Rieman surface with boundary (in our case, C will be a Riemann sphere glued to a disc at an interior point, e.g. the origin), and let $u : C \to X$ be a holomorphic map with boundary on a Lagrangian submanifold L (in our case a fiber of π). The first-order deformations of u and their obstructions can be analyzed by methods of algebraic geometry, following Behrend-Fantechi [Beh97, BF97]. Recall that, when C is smooth and u is an immersion, the deformations and obstructions are governed by H^0 and H^1 of the normal bundle $N_u = u^*TX/TC$, i.e. first-order deformations correspond to holomorphic sections of N_u over C with real boundary conditions given by $N_{\partial u} = u^*TL/T\partial C$, while obstructions live in $H^1(C, N_u)$. In the presence of singularities, the dual of the normal bundle is replaced by a complex of sheaves, and the deformations and obstructions are given by

$$\mathrm{Ext}^i_C(\{u^*\Omega^1_X \xrightarrow{du^*} \Omega^1_C\}, \mathcal{O}_C)$$

for $i = 0$ and $i = 1$; see [PT14, §$1\frac{1}{2}$] and the discussion before Lemma 2.6 in [GHS03].

As noted in [GHS03], things are simpler if we assume that the two branches of u near each node of C are immersed and their tangent lines at the node are distinct: then $\mathcal{H}om_{\mathcal{O}_C}(\{u^*\Omega^1_X \to \Omega^1_C\}, \mathcal{O}_C)$ is isomorphic to a coherent sheaf $\mathcal{N}_u$, the *normal sheaf* of u, whose global sections and first cohomology determine the deformations and obstructions. The sheaf $\mathcal{N}_u$ has an explicit description if we assume moreover that the restriction of u to each component of C is an immersion. In this case, the restriction of $\mathcal{N}_u$ to each component of C is the sheaf of meromorphic sections of the normal bundle with at most a simple pole at each node, whose normal direction must be the tangent space to the other branch of u through the node; the meromorphic sections over the two branches must additionally satisfy a matching condition at the node, which we state below.

Lemma 2.5. *Let $C = C' \cup_p C''$ be a curve with a single node p, and $u : (C, \partial C) \to (X, L)$ a holomorphic map whose restrictions $u' = u_{|C'}$ and $u''_{|C''}$ are immersions; assume moreover that the tangent lines $du'(T_pC'), du''(T_pC'') \subset T_{u(p)}X$ are distinct, and denote by $N_{u'} = u'^*TX/TC'$ and $N_{u''} = u''^*TX/TC''$ the normal bundles to the two components. Denote by z', z'' local coordinates on C', C'' near p.*

Then the first-order deformations of u (resp. the obstruction space) are the global sections over C (resp. the first cohomology group) of the normal sheaf $\mathcal{N}_u$ (with real boundary conditions along ∂C), which are pairs of sections

$$(v', v'') \in H^0(C', N_{u'} \otimes \mathcal{O}_{C'}(p)) \oplus H^0(C'', N_{u''} \otimes \mathcal{O}_{C''}(p))$$

(i.e., meromorphic sections of $N_{u'}$ and $N_{u''}$ with at most simple poles at p) satisfying the following matching conditions:

- *there exists a constant $\lambda \in \mathbb{C}$ such that the polar parts of v' and v'' are respectively*

$$v'(z') \sim \frac{\lambda}{z'} \frac{\partial u''}{\partial z''}(p) \qquad \textit{and} \qquad v''(z'') \sim \frac{\lambda}{z''} \frac{\partial u'}{\partial z'}(p); \tag{2.5}$$

- *the projections of v' and v'' onto $T_pX/(du'(T_pC') + du''(T_pC''))$ coincide at p.*

Lemma 2.5 can also be understood from a differential geometric perspective, since a node-smoothing deformation of u can be viewed as the restriction to the family of curves $C_t = \{z'z'' = \gamma(t)\}$ of a family of maps $\tilde{u}_t$ from $C' \times C''$ (in our case, $\mathbb{CP}^1 \times D^2$) to X. We then have, for $z', z'' \neq 0$,

$$\frac{d}{dt}_{|t=0}(\tilde{u}_t(z', \gamma(t)/z')) = \frac{\partial \tilde{u}_t}{\partial t}_{|t=0}(z', 0) + \frac{\gamma'(0)}{z'} \frac{\partial \tilde{u}_0}{\partial z''}(z', 0), \quad \text{and}$$
$$\frac{d}{dt}_{|t=0}(\tilde{u}_t(\gamma(t)/z'', z'')) = \frac{\partial \tilde{u}_t}{\partial t}_{|t=0}(0, z'') + \frac{\gamma'(0)}{z''} \frac{\partial \tilde{u}_0}{\partial z'}(0, z'').$$

The first term in these expressions, $\partial \tilde{u}_t/\partial t$, is a genuine section of u^*TX over C (i.e., a pair of sections of u'^*TX and u''^*TX whose values at the node coincide); while the second term has a first-order pole at the origin, where the leading order term is exactly as in (2.5) with $\lambda = \gamma'(0)$ (the rate at which the node is getting smoothed by the deformation). Deformations of the map u are then governed by the kernel and cokernel of the $\bar{\partial}$ operator on sections of u^*TX with the appropriate behavior at the node; or, after quotienting out by vector fields on C (reparametrizations by diffeomorphisms), the kernel and cokernel of the $\bar{\partial}$ operator on pairs of sections of the normal bundles $N_{u'}$ and $N_{u''}$ (allowed to have a simple pole at p and satisfying the matching conditions described in Lemma 2.5).

While the above suffices for our purposes, we also refer the reader to [SSZ25, Section 6] for a related discussion in the framework of polyfolds.

We now use Lemma 2.5 to prove the regularity of certain nodal configurations in X with boundary on the fibers of π.

Lemma 2.6. *Given $(x_1, x_2) \in (\mathbb{C}^*)^2$, let $u : C = \mathbb{CP}^1 \cup D^2 \to X$ be a stable map with boundary on $F_{(|x_1|,|x_2|,\xi_3,\xi_4)}$ whose restriction to $\mathbb{CP}^1$ parametrizes the sphere $S_{(x_1,x_2)}$, and whose restriction to D^2 parametrizes a disc of suitable radius in the x_4 coordinate, with constant values of x_1, x_2, x_3.*

(1) *If $x_1, x_2 \neq 1$, then u is regular as a stable disc in X with boundary in $F_{(|x_1|,|x_2|,\xi_3,\xi_4)}$.*

(2) *If $x_1 = 1$ and $x_2 \neq 1$, then u is regular as a stable disc in X with boundary in the family of fibers $F_{(r_1,|x_2|,\xi_3,\xi_4)}$ where r_1 is allowed to vary.*

(2′) *If $x_1 \neq 1$ and $x_2 = 1$, then u is regular as a stable disc in X with boundary in the family of fibers $F_{(|x_1|,r_2,\xi_3,\xi_4)}$ where r_2 is allowed to vary.*

(3) *If $x_1 = x_2 = 1$, then u is regular as a stable disc in X with boundary in the family of fibers $F_{(r_1,r_2,\xi_3,\xi_4)}$ where r_1 and r_2 are allowed to vary.*

Proof. The normal sheaf splits into a direct sum $\mathcal{N}_u = \mathcal{N}_{u,1} \oplus \mathcal{N}_{u,2} \oplus \mathcal{N}_{u,34}$, where the first two summands correspond to the x_1 and x_2 directions and $\mathcal{N}_{u,34}$ corresponds to deformations inside $\{(x_1, x_2)\} \times K_{\mathbb{CP}^1}$ (or its proper transform if x_1 or x_2 is 1). To prove the vanishing of $H^1(C, \mathcal{N}_u)$ (or equivalently the surjectivity of the appropriate $\bar\partial$ operator) we consider each summand separately.

The first summand $\mathcal{N}_{u,1}$ is a holomorphic line bundle over C, whose restriction to $\mathbb{CP}^1$ is $\mathcal{O}$ if $x_1 \neq 1$ and $\mathcal{O}(-1)$ if $x_1 = 1$, and trivial over D^2, with a trivial real line subbundle as boundary condition. The $\bar\partial$ operator on the disc component is surjective, with a real 1-dimensional kernel corresponding to constant sections; meanwhile, the $\bar\partial$ operator on the $\mathbb{CP}^1$ component is surjective, and for $x_1 \neq 1$ it remains surjective if we restrict the domain to sections of $\mathcal{O}$ which have a prescribed value at the node. Thus, $H^1(C, \mathcal{N}_{u,1}) = 0$ when $x_1 \neq 1$. However, for $x_1 = 1$ the $\bar\partial$ operator on the $\mathbb{CP}^1$ component is only surjective if we consider all sections of $\mathcal{O}(-1)$, without imposing a value at the node; and the $\bar\partial$ operator on the disc component is no longer surjective if we restrict its domain to functions which take a prescribed value at the node. Thus, regularity fails if we consider u as a disc with boundary on a fixed fiber of π. Instead, we relax the boundary condition and consider deformations of u among discs with boundary on fibers $F_{(r_1,|x_2|,\xi_3,\xi_4)}$ where r_1 is allowed to vary. Modifying the problem in this way enlarges the domain of the $\bar\partial$ operator on the disc component to the space of complex-valued functions whose imaginary part is constant (rather than zero) at the boundary, so that surjectivity holds even if we restrict to functions that take a prescribed value at the node.

The situation is identical for $\mathcal{N}_{u,2}$: we find that $H^1(C, \mathcal{N}_{u,2}) = 0$ when $x_2 \neq 1$, and for $x_2 = 1$ we achieve regularity by relaxing the boundary condition and allowing r_2 to vary.

Finally, $\mathcal{N}_{u,34}$ is a sheaf of sections of the normal bundles to the components of $u(C)$ inside $K_{\mathbb{CP}^1}$ with at most simple poles at the node and matching residues. The normal bundle to the $\mathbb{CP}^1$ component is $\mathcal{O}(-2)$, so its $\bar\partial$ operator has a one-dimensional cokernel; however, the $\bar\partial$ operator becomes surjective if we enlarge the domain to allow a simple pole at the nodal point. Meanwhile, the normal bundle to the disc component is trivial, with trivial real boundary condition, so the corresponding $\bar\partial$ operator is surjective (on honest sections, and hence also on sections with a fixed polar part). Thus $H^1(C, \mathcal{N}_{u,34}) = 0$. □

Lemma 2.7. *Assume $F_{(r_1,r_2,\xi_3,\xi_4)}$ is the lift to X of a product torus in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$.*

(1) *For $r_2 > 1$, let $u : C = \mathbb{CP}^1 \cup D^2 \to X$ be a stable map with boundary on $F_{(r_1,r_2,\xi_3,\xi_4)}$ such that $u_{|\mathbb{CP}^1}$ parametrizes the sphere $S_{(x_1,1)}$ for some x_1 such that $|x_1| = r_1$, and $u_{|D^2}$ is as in (2.3) with $n_1 = n_3 = 0$ and $n_2 = n_4 = 1$ (i.e., x_1 and x_3 are constant while x_2 and x_4 have degree one), with $x_2(z) = 1$ at the unique point where $x_4(z) = 0$. If $x_1 \neq 1$ then u is regular as a stable disc with boundary on $F_{(r_1,r_2,\xi_3,\xi_4)}$. If $x_1 = 1$ then u is regular as a stable disc with boundary on a family of fibers where r_1 is allowed to vary.*

(1′) *Similarly for a stable map with boundary on $F_{(r_1,r_2,\xi_3,\xi_4)}$ ($r_1 > 1$) which is the union of $S_{(1,x_2)}$ with a disc on which x_2 and x_3 are constant, $x_1(z)$ and $x_4(z)$ have degree 1, and $x_1(z) = 1$ at the unique point where $x_4(z) = 0$.*

(2) *For $r_1, r_2 > 1$, the union of $S_{(1,1)}$ with a disc on which x_3 is constant while $x_1(z)$, $x_2(z)$ and $x_4(z)$ have degree 1, and $x_1(z) = x_2(z) = 1$ at the unique point where $x_4(z) = 0$, is regular as a stable map with boundary on $F_{(r_1,r_2,\xi_3,\xi_4)}$.*

Proof. The normal sheaf $\mathcal{N}_u$ has a subsheaf $\mathcal{N}_{u,1} \oplus \mathcal{N}_{u,2}$ corresponding to deformations which take place purely along the x_1 and x_2 directions. We establish vanishing of the first cohomology separately for $\mathcal{N}_{u,1}$, $\mathcal{N}_{u,2}$, and the quotient $\mathcal{N}_{u,34} := \mathcal{N}_u/(\mathcal{N}_{u,1} \oplus \mathcal{N}_{u,2})$.

When x_1 is constant along the map u, the situation for $\mathcal{N}_{u,1}$ is exactly as in Lemma 2.6, and the same argument proves the vanishing of $H^1(C, \mathcal{N}_{u,1})$ if $x_1 \neq 1$, and the regularity once we allow r_1 to vary if $x_1 = 1$. When x_1 has degree 1 on the disc component of u, the restriction of $\mathcal{N}_{u,1}$ to the disc component is still a trivial holomorphic line bundle, but now the boundary condition is given by a family of real lines which rotates by one turn in the positive direction along the unit circle, namely the line spanned by $i\, x_1(z)$ at every $z \in \partial D^2$. The $\bar\partial$ operator on this space of sections is surjective, and remains surjective even after we restrict the domain to sections which take a prescribed value at the node. (This can be checked e.g. by comparing the index of the operator and the dimension of its kernel, which consists of infinitesimal automorphisms of the disc). This implies the vanishing of $H^1(C, \mathcal{N}_{u,1})$ even when $x_1 = 1$. The argument for $\mathcal{N}_{u,2}$ is identical.

Finally, $\mathcal{N}_{u,34}$ can be identified (via projection to the (x_3, x_4) coordinates) with the normal sheaf of the projection of $u(C)$ to $K_{\mathbb{CP}^1}$, which is exactly as in Lemma 2.6 and whose first cohomology vanishes by the same argument. □

Next, we give some constraints on nodal configurations which contribute to the enumerative geometry of discs in X with boundary on fibers of π.

Consider a stable disc $u : C \to X$ with boundary on a fiber of π which does not meet $E_0 \cup E_\infty$, i.e., $F = \pi^{-1}(b)$ for $b \in B \setminus B^{exc}$. Denote by $N_1(u)$ the total intersection number of $u(C)$ with $p^{-1}(\{0\} \times \mathbb{C} \times K_{\mathbb{CP}^1})$, i.e., the sum of the degrees n_1 in (2.3) for the disc components of u , and by $N_2(u)$ the intersection number with $p^{-1}(\mathbb{C} \times \{0\} \times K_{\mathbb{CP}^1})$, i.e. the sum of the values of n_2 for the disc components. (Note that the x_1 and x_2 components of $p \circ u$ always have Blaschke product expansions, without needing to assume that F is the lift of a product torus in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$.) Let $N_{34}(u)$ be the total intersection number of $u(C)$ with the preimages of the toric divisors of $K_{\mathbb{CP}^1}$; when F is the lift to X of a product torus, $N_{34}(u)$ is the sum of the quantities $n_3 + n_4$ in (2.3) for the disc components of u. Finally, denote by $K_0(u)$ and $K_\infty(u)$ the intersection numbers of $u(C)$ with E_0 and E_∞; i.e., $K_0(u)$ is the sum of the quantities k_0 in (2.4) for the disc components, plus the degrees of the sphere components mapping to the curves $S_{x_1,1}$, minus the degrees of the sphere components mapping to fibers of the projection p contained in E_0; and similarly for $K_\infty(u)$. The Maslov index of u is

$$\mu(u) = 2(N_1(u) + N_2(u) + N_{34}(u) - K_0(u) - K_\infty(u)). \tag{2.6}$$

Proposition 2.8. *There exist arbitrarily small deformations J' of the complex structure on X such that, given any holomorphic stable disc $u : C \to X$ with boundary on a fiber F of π that does not meet $E_0 \cup E_\infty$, if u deforms to a J'-holomorphic stable disc then:*

(1) *The sum of the multiplicities of the spheres S_{x_1,x_2} in $u(C)$ is at most $N_{34}(u)$;*

(2) $K_0(u) \leq N_{34}(u)$ *and* $K_\infty(u) \leq N_{34}(u)$*;*

(3) $K_0(u) \leq N_2(u)$*, except possibly if x_2 is constant and equal to 1 on a disc component of $u(C)$; and $K_\infty(u) \leq N_1(u)$, except possibly if x_1 is constant and equal to 1 everywhere on a disc component of $u(C)$.*

Proof. The x_4 coordinate defines a Lefschetz fibration $f = x_4 : K_{\mathbb{CP}^1} \to \mathbb{C}$, whose two critical points both lie in the fiber $x_4 = 0$ (which is the union of the zero section of $K_{\mathbb{CP}^1}$ and the lines L_0 and L_∞). We deform this Lefschetz fibration slightly so that its two critical values become distinct, and deform the complex structure to some J' so that the deformed fibration $f' : K_{\mathbb{CP}^1} \to \mathbb{C}$ remains J'-holomorphic. (This deformation can be viewed as an open subset of the deformation of the Hirzebruch surface $\mathbb{F}_2$ considered in [Aur09, Section 3.2], so that the deformed total space can be identified with the complement of a curve of bidegree $(1,1)$ inside $\mathbb{CP}^1 \times \mathbb{CP}^1$.) The projection of F to $K_{\mathbb{CP}^1}$ is disjoint from $f^{-1}(0)$, and so by choosing the deformation to be sufficiently small we can ensure that it is also disjoint from the preimage under f' of a small disc containing both critical values. We may additionally assume that L_0 and L_∞ remain components of the singular fibers of f' (now living over the two distinct critical values), or denote by L'_0, L'_∞ the J'-holomorphic deformations of L_0, L_∞ which arise in this manner.

We deform the complex structure on X to the blowup of $(\mathbb{C}^2 \times K_{\mathbb{CP}^1}, J_0 \oplus J')$ along $H'_0 = \mathbb{C} \times \{1\} \times L'_0$ and $H'_\infty = \{1\} \times \mathbb{C} \times L'_\infty$. By abuse of notation, we again denote by J' the deformed complex structure on X, and by $f' : X \to \mathbb{C}$ the pullback of f' under the composition of the blowup map $p' : X \to \mathbb{C}^2 \times K_{\mathbb{CP}^1}$ and projection to the second factor.

Now, assume that a holomorphic stable disc $u : C \to X$ with boundary on F deforms to a J'-holomorphic stable disc $u' : C' \to X$. The only rational curves in (X, J') lie inside the fibers of p' in the exceptional divisors $E'_0 = p'^{-1}(H'_0)$ and $E'_\infty = p'^{-1}(H'_\infty)$, and have nonpositive intersection numbers with E'_0 and E'_∞. Meanwhile, the disc components of u' have total intersection number $N_{34}(u)$ with the fibers of f' near the origin. Hence, by positivity of intersections with the components of the singular fibers of f', the intersection numbers of $p' \circ u'$ with $\mathbb{C}^2 \times L'_0$ and $\mathbb{C}^2 \times L'_\infty$ are bounded by $N_{34}(u)$. This has two consequences. First, the sum of the multiplicities of the spheres S_{x_1,x_2} in $u(C)$ (each of which contributes 1 to the intersection numbers of $p \circ u$ with $\mathbb{C}^2 \times L_0$ and $\mathbb{C}^2 \times L_\infty$) is at most $N_{34}(u)$. Second, the total contact orders of $p' \circ u'$ with H'_0 and H'_∞ are at most $N_{34}(u)$, so, after adding the non-positive contributions of any sphere components, $[u'(C')] \cdot [E'_0] = K_0(u)$ and $[u'(C')] \cdot [E'_\infty] = K_\infty(u)$ are bounded by $N_{34}(u)$.

On the other hand, the disc components of u' project to the x_1 coordinate as a multisection of degree $N_1(u)$ over the disc of radius r_1, which implies that the intersection number of $p' \circ u'$ with $\{x_1 = 1\}$ is bounded by $N_1(u)$. Therefore, the total contact order of $p' \circ u'$ with H'_∞ is at most $N_1(u)$, unless x_1 is constant and equal to 1 on a component of $p' \circ u'$; this in turn implies that $[u'(C')] \cdot [E'_\infty] = K_\infty(u)$ is bounded by $N_1(u)$. The bound $K_0(u) \leq N_2(u)$ (unless x_2 is constant and equal to 1 on a component) is proved similarly by considering the intersection number of $p' \circ u'$ with $\{x_2 = 1\}$. □

Remark 2.9. Using basic methods of complex analysis to classify holomorphic discs in conic bundles (arguing as in [Aur07, Aur15, AAK16]), the deformation considered in the proof of Proposition 2.8 can also be used to give an alternative proof of Lemmas 2.6–2.7 by explicitly finding the discs that the various nodal configurations deform to, as well as another derivation of the superpotential formulas given in Section 2.5.

Corollary 2.10. *For $r_1 \neq 1$ and $r_2 \neq 1$, an arbitrarily small perturbation of the complex structure ensures that all holomorphic stable discs in X with boundary on $F_{(r_1,r_2,\xi_3,\xi_4)}$ have Maslov index at least 2.*

Proof. Let $u : C \to X$ be a non-constant holomorphic stable disc with boundary on $F = F_{(r_1,r_2,\xi_3,\xi_4)}$. Since $r_1, r_2 \neq 1$, F is disjoint from $E_0 \cup E_\infty$, and its projection to $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ is disjoint from the toric divisors. Therefore, either one of x_1, x_2 is non-constant along u, in which case $N_1(u)$ or $N_2(u)$ is positive, or $p \circ u$ is a non-constant disc in $\{(x_1, x_2)\} \times K_{\mathbb{CP}^1}$ with boundary on a product torus, in which case $N_{34}(u)$ must be positive.

By Proposition 2.8 (3), the stable discs which survive the perturbation of the complex structure to J' satisfy $K_0(u) \leq N_2(u)$ and $K_\infty(u) \leq N_1(u)$. (The other possibility, that x_1 or x_2 is constant and equal to 1 on a disc component, is excluded by the assumption that $r_1, r_2 \neq 1$.) Using (2.6), it follows that $\mu(u) \geq 2N_{34}(u)$. If $N_{34}(u) > 0$ the conclusion follows. Otherwise, if $N_{34}(u) = 0$ then Proposition 2.8 (2) implies that $K_0(u) = K_\infty(u) = 0$, so $\mu(u) = 2(N_1(u) + N_2(u)) \geq 2$. □

2.4. **A brief review of SYZ mirror symmetry.** Before proceeding further, we recall the construction of the SYZ mirror of X relative to the anticanonical divisor D; see [AAK16, Section 2 and Appendix A] and [Yuan20] for details.

The construction of the mirror $X^\vee$ starts from a moduli space of objects of the Fukaya category of $X^0 = X \setminus D$ consisting of weakly unobstructed fibers of $\pi : X^0 \to B$ equipped with rank 1 unitary local systems. We work over the Novikov field over a field k, say $k = \mathbb{C}$ for concreteness,

$$\mathbb{K} = \Lambda_k = \left\{ \sum a_i T^{\lambda_i} \,|\, a_i \in k,\ \lambda_i \in \mathbb{R},\ \lambda_i \to +\infty \right\},$$

and recall the unitary subgroup $U_{\mathbb{K}} = \mathrm{val}^{-1}(0) \subset \mathbb{K}^*$, where the valuation map $\mathrm{val} : \mathbb{K}^* \to \mathbb{R}$ is defined by $\mathrm{val}(\sum a_i T^{\lambda_i}) = \min\{\lambda_i \,|\, a_i \neq 0\}$. Unitary rank 1 local systems over a Lagrangian torus $F_b = \pi^{-1}(b)$ are determined by their holonomy $\mathrm{hol} \in \hom(\pi_1(F_b), U_{\mathbb{K}}) = H^1(F_b, U_{\mathbb{K}})$, which enters into the formulas for weighted counts of holomorphic discs in Lagrangian Floer theory; specifically, a disc with boundary on F_b representing the class $\beta \in \pi_2(X, F_b)$ is counted with a weight

$$z^\beta = T^{\omega(\beta)} \mathrm{hol}(\partial\beta) \in \mathbb{K}^*.$$

Over a simply connected subset $P \subset B$ where the fibers of π are smooth and do not bound any holomorphic discs of Maslov index less than 2, using isotopies between the fibers to identify $\pi_2(X, F_b) \simeq \pi_2(X, F_{b'})$ for $b, b' \in P$, there is a natural analytic structure on

$$X_P^\vee := \bigsqcup_{b \in P} H^1(F_b, U_{\mathbb{K}})$$

for which the functions $z^\beta \in \mathcal{O}(X_P^\vee)$ are analytic. (Typically one might take P to be a bounded rational convex polyhedral subset, so that $X_P^\vee$ is an affinoid domain.) $X_P^\vee$ can be identified with a domain in $(\mathbb{K}^*)^n$ by considering the coordinates $z_i = z^{\beta_i}$ for some choice of classes $\beta_1, \dots, \beta_n$ such that $\partial\beta_1, \dots, \partial\beta_n$ are a basis of $H_1(F_b, \mathbb{Z})$; all other z^β are then Laurent monomials in $z_1, \dots, z_n$. Moreover, the non-archimedean torus fibration defined by the natural projection $X_P^\vee \to P$ is modelled on the valuation map in these coordinates, in the sense that the diagram

$$\begin{array}{ccc} X_P^\vee & \xrightarrow{(z_i)_{1\leq i\leq n}} & (\mathbb{K}^*)^n \\ \big\downarrow & & \big\downarrow \mathrm{val} \\ P & \xrightarrow{(\omega(\beta_i))_{1\leq i\leq n}} & \mathbb{R}^n \end{array} \tag{2.7}$$

commutes.

The superpotential $W \in \mathcal{O}(X_P^\vee)$ is the coefficient of identity in the Floer-theoretic obstruction $\mathfrak{m}_0$ for fibers of π equipped with unitary rank 1 local system, i.e. a weighted count of Maslov index 2 holomorphic discs with boundary on F_b passing through a generic point of F_b. Namely,

$$W = \sum_{\mu(\beta)=2} n_\beta\, z^\beta, \tag{2.8}$$

where $n_\beta \in \mathbb{Z}$ is the degree of the evaluation map $ev : \mathcal{M}_1(F_b, \beta) \to F_b$ from the moduli space of holomorphic discs with boundary in F_b representing the class β and one boundary marked point, $\mathcal{M}_1(F_b, \beta)$, to the Lagrangian F_b (after fixing suitable orientations of both spaces, and possibly a perturbation to achieve regularity of the moduli space).

The mirror $X^\vee$ is assembled from the subsets $X_P^\vee$ via suitable gluing maps: the transition functions between the affine coordinates in the bottom row of (2.7) are given by elements of $GL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$, and in the absence of discs of Maslov index less than two the local analytic coordinates (z_i) on the subsets $X_P^\vee \subset X^\vee$ transform by the corresponding monomial automorphisms of $(\mathbb{K}^*)^n$. However, walls over which the fibers of π bound Maslov index 0 discs induce a modification of the transition functions between between the portions of $X^\vee$ which correspond to subsets of B lying on either side of the wall. The existence of analytic (valuation-preserving) coordinate changes which restore the analytic dependence of Floer theory across the wall follows from the work of Fukaya-Oh-Ohta-Ono on the invariance of Floer cohomology for Lagrangians with weak bounding cochains [FOOO09] (in their language, the wall-crossing coordinate transformation arises as an induced map on the moduli space of weak bounding cochains); the new phenomenon we will evidence below, however, is that in the presence of discs of negative Maslov index these coordinate changes need not be path-independent.

Remark 2.11. The above statements about gluing maps between local charts of $X^\vee$ which lie over different subsets of B (and, a fortiori, the composition of such transformations along paths in B) require further explanation, since the local charts corresponding to disjoint subsets of B do not actually overlap in $X^\vee$. The key observation, known as Fukaya's trick [Fuk10], is that when two fibers F, F' of π are close enough to be mapped to each other by an isotopy ψ such that the almost-complex structure $J' = \psi_* J$ is ω-tame, the Floer theory of F' with respect to J' is related to the Floer theory of F with respect to J by analytic continuation. This implies that Floer-theoretic expressions calculated for fibers of π over a given region of B (possibly just a single fiber) can be analytically continued over a slightly larger region of B. We can then form a cover of B by rational convex polyhedral subsets that overlap nontrivially, with the understanding that, on the overlaps, the gluing maps amount to a comparison of the Floer theory of given fibers of π with respect to almost-complex structures that are pulled back along different isotopies. In practice, for sufficiently simple examples, such as the one we consider here, it is often the case that the wall-crossing transformations are given by birational maps, and can be composed freely without worrying about convergence issues. For this reason we do not discuss the details further, and instead refer the reader to [Abo14, Section 3] and [Yuan20, Section 4] for details.

A key property that can be used to determine the wall-crossing coordinate transformations between the coordinate charts $X_P^\vee$ is that the local expressions (2.8) for the superpotential must match and assemble to a global analytic function $W \in \mathcal{O}(X^\vee)$ (to the extent that the corrected mirror $X^\vee$ is globally well-defined). More generally, the same property holds for weighted counts of holomorphic discs in moduli spaces that match under wall-crossing after accounting for disc bubbling. For example, expressions $\sum n_\beta z^\beta$ where the sum ranges over classes β with fixed intersection numbers with certain divisors $D_i \subset X$ are also invariant under wall-crossing, as long as the intersection numbers with D_i are zero for all Maslov index 0 bubbles.

2.5. **The superpotential: discs of Maslov index 2.** We now return to our main example, and prove the first part of Theorem 1.1, namely we determine the superpotential on each chart of the mirror $X^\vee$. By Corollary 2.10, the Lagrangian tori $F_{(r_1,r_2,\xi_3,\xi_4)}$ only bound discs of Maslov index at least 2 as soon as $r_1 \neq 1$ and $r_2 \neq 1$; hence we work separately over each of the four domains $P_{--} = \{r_1 < 1,\ r_2 < 1\}$, $P_{-+} = \{r_1 < 1,\ r_2 > 1\}$, $P_{+-} = \{r_1 > 1,\ r_2 < 1\}$ and $P_{++} = \{r_1 > 1,\ r_2 > 1\}$, calculating the superpotential on each chart and showing that it is given by (1.1).

The very simplest holomorphic discs that we will encounter are the lifts to X of "standard" Maslov index 2 holomorphic discs bounded by product tori in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$. Using the same notations as in Lemma 2.3, these are the discs for which $n_1+n_2+n_3+n_4 = 1$ (i.e., one of the n_i is equal to 1 and the others are zero). For $i \in \{1,2,4\}$, we denote by β_i the class of a disc of the appropriate radius along the x_i coordinate axis, while the other coordinates x_j for $j \neq i$ are constant, and by $z_i = z^{\beta_i}$ the corresponding Floer-theoretic weight. For $i = 3$ there are two different classes of discs with $n_3 = 1$ and $n_1 = n_2 = n_4 = 0$, depending on whether the x_3 coordinate has a zero or a pole (i.e., whether the disc intersects $\mathbb{C}^2 \times L_0$ or $\mathbb{C}^2 \times L_\infty$). We denote by $\beta_{3,\pm}$ their homotopy classes, and define $z_3 = q^{-1} z_4^{-1} z^{\beta_{3,+}}$, where $q = T^a$. Since $\beta_{3,+} + \beta_{3,-} = 2\beta_4 + [S_{(x_1,x_2)}]$, and the Novikov weight of the zero section of $K_{\mathbb{CP}^1}$ is $T^{2a} = q^2$, we find that $z^{\beta_{3,\pm}} = q z_3^{\pm 1} z_4$.

We will use z_1, z_2, z_3, z_4 as coordinates on each of the four charts that make up $X^\vee$; we observe that these are the weights of disc classes $\beta_1, \dots, \beta_4$ (up to a factor of q in the case of z_3) whose boundaries $\partial\beta_i$ correspond to the standard basis of the first homology of product tori in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$. Moreover, using the fact that the symplectic area of a disc which is invariant under a Hamiltonian S^1-action is equal to the difference between the moment map values at its boundary and at its center, one finds that $\mathrm{val}(z_3) = \xi_3$ and $\mathrm{val}(z_4) = \xi_4$.

We are now ready to determine the formulas for the superpotential on each of the four charts. Since the counts n_β do not vary inside each of the four regions $P_{\pm,\pm}$, it suffices to carry out the calculation for fibers of π which are lifts of product tori in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$.

Since we only consider fibers of π for which $r_1 \neq 1$ and $r_2 \neq 1$, Proposition 2.8 implies that all the Maslov index 2 holomorphic stable discs $u : C \to X$ contributing to the sum (2.8) satisfy $K_0(u) \leq \min(N_2(u), N_{34}(u))$ and $K_\infty(u) \leq \min(N_1(u), N_{34}(u))$. As noted in the proof of Corollary 2.10, plugging these bounds into (2.6) it follows that

$$\mu(u) \geq 2\max(N_1(u), N_2(u), N_{34}(u)). \tag{2.9}$$

Hence, we only need to consider stable discs for which each of $N_1(u), N_2(u), N_{34}(u)$ is either 0 or 1. Moreover, since equality must hold in (2.9), necessarily $K_0(u) = \min(N_2(u), N_{34}(u))$ and $K_\infty(u) = \min(N_1(u), N_{34}(u))$. It is apparent from the proof of Proposition 2.8 that for these equalities to hold, u cannot have any sphere components contained in the fibers of π; whereas by Proposition 2.8(1) the total multiplicities of the spheres $S_{(x_1,x_2)}$ add up to at most $N_{34}(u) \leq 1$, i.e. C contains at most one sphere component, and any such component must map to some $S_{(x_1,x_2)}$ with degree one. Furthermore, since each disc component of u must separately satisfy the constraints of Proposition 2.8, the Maslov index 2 configurations we consider have only one disc component.

With these constraints in hand, we can list all the possible homotopy classes which may contribute to the superpotential.

Case 1: $N_{34}(u) = 0$. Then there are no sphere components, $K_0(u) = K_\infty(u) = 0$, and $N_1(u) + N_2(u) = 1$. Hence x_3, x_4 are constant along u, while one of x_1, x_2 is constant and the other parametrizes a disc of radius r_i parallel to the x_i coordinate axis. Thus $[u]$ is either β_1 or β_2, and its weight is either z_1 or z_2. Both of these families of discs are regular and contain exactly one disc through each point of $F = F_{(r_1,r_2,\xi_3,\xi_4)}$. The orientation of the moduli space works out as in the classical toric case, and $n_{\beta_1} = n_{\beta_2} = 1$. Summarizing, the contributions of the discs with $N_{34}(u) = 0$ add up to

$$z_1 + z_2. \tag{2.10}$$

Case 2: $N_{34}(u) = 1$, $n_3 = 0$, $n_4 = 1$. (Recall that $N_{34}(u)$ is the sum of the n_3 and n_4 degrees appearing in (2.3).) Then x_3 is constant along the disc component of u, while x_4 parametrizes a disc of the appropriate radius.

When $N_1(u) = N_2(u) = 0$ (i.e., the disc component of u is parallel to the x_4 coordinate axis and represents the class β_4), we can either have just the disc component, or consider its union with the sphere $S = S_{(x_1,x_2)}$ (for the same constant values taken by x_1, x_2 along the disc); the latter nodal configuration is regular by Lemma 2.6(1). In both cases there is one such configuration through each point of F, and the orientations work out as in the toric case (the contributions of the sphere component to the linearized Cauchy-Riemann problem amount to complex linear operators and do not affect signs). Hence $n_{\beta_4} = n_{\beta_4+[S]} = 1$, contributing $(1+q^2)z_4$ to the superpotential. We will now see that in all other cases (when either $N_1(u)$ or $N_2(u)$ is non-zero) a sphere component *must* be present.

$N_1(u)$ is either zero or one. If it is zero then x_1 is constant along u. If $N_1(u) = 1$, then we must have $K_\infty(u) = 1$ as well; and since the disc component does not meet E_∞ (x_3 has no pole), this forces the presence of a sphere component mapping to $S_{(1,x_2)}$ for some value of x_2. This in turn implies that x_1 must equal 1 at the point of the disc component where x_4 vanishes. Since x_1 takes values in the disc of radius r_1, this is only possible if $r_1 > 1$. After a suitable reparametrization, the x_1 and x_4 coordinates along the disc component of

u can be put in the form

$$x_1(z) = r_1 z, \qquad x_4(z) = e^{i\theta} r_4 \frac{r_1 z - 1}{r_1 - z}$$

for some $e^{i\theta} \in S^1$, and the sphere $S_{(1,x_2)}$ is attached to the disc at $z = 1/r_1$.

Similarly, $N_2(u)$ is either zero or one; if it is zero then x_2 is constant; if $N_2(u) = 1$ then necessarily $r_2 > 1$, the x_2 component takes the value 1 at the point of the disc where x_4 vanishes, and a sphere component $S_{(x_1,1)}$ is attached to the disc at that point.

The case where $N_1(u)$ and $N_2(u)$ are both equal to 1 does occur; this requires the sphere component to map to $S_{(1,1)}$. Hence we must have $(x_1, x_2) = (1, 1)$ at the point of the disc where x_4 vanishes (and necessarily r_1, r_2 are both greater than 1).

The various configurations we have found are precisely those covered by Lemma 2.7; hence they are regular, and one easily checks that there is one disc in each family through each point of F. As before, the incidence constraints and contributions from sphere components modify the linearized Cauchy-Riemann problem by complex linear operators, so that the evaluation maps again have degree 1. Thus,

$$\begin{aligned} n_{\beta_1+\beta_4+[S_{(1,x_2)}]} &= 1 \qquad \text{for } r_1 > 1 \qquad \text{(and 0 otherwise)},\\ n_{\beta_2+\beta_4+[S_{(x_1,1)}]} &= 1 \qquad \text{for } r_2 > 1 \qquad \text{(and 0 otherwise)},\\ n_{\beta_1+\beta_2+\beta_4+[S_{(1,1)}]} &= 1 \qquad \text{for } r_1, r_2 > 1 \quad \text{(and 0 otherwise)}.\end{aligned}$$

Setting $q' = T^{a-\epsilon'}$ and $q'' = T^{a-\epsilon''}$, the Floer weights of $S_{(1,x_2)}, S_{(x_1,1)}, S_{(1,1)}$ are respectively $T^{2a-\epsilon''} = qq''$, $T^{2a-\epsilon'} = qq'$, and $T^{2a-\epsilon'-\epsilon''} = q'q''$. Hence, the contributions of discs with $n_3 = 0$ and $n_4 = 1$ add up to

$$\begin{cases} (1+q^2)z_4 & \text{if } r_1 < 1 \text{ and } r_2 < 1\\ (1+q^2)z_4 + qq'z_2z_4 & \text{if } r_1 < 1 \text{ and } r_2 > 1\\ (1+q^2)z_4 + qq''z_1z_4 & \text{if } r_1 > 1 \text{ and } r_2 < 1\\ (1+q^2)z_4 + qq''z_1z_4 + qq'z_2z_4 + q'q''z_1z_2z_4 & \text{if } r_1 > 1 \text{ and } r_2 > 1 \end{cases} \tag{2.11}$$

Case 3: $N_{34}(u) = 1$, $n_3 = 1$, $n_4 = 0$. Because the disc component of u does not meet the preimage of the zero section of $K_{\mathbb{CP}^1}$, there cannot be any sphere component, and u is as in (2.3). Moreover, x_3 has either a zero or a pole along u, but not both, so u meets at most one of E_0 or E_∞. However, for this to happen, x_1 or x_2 needs to be non-constant and take the value 1 at the point where x_3 has its pole or zero. There are therefore three subcases.

If $N_1(u) = N_2(u) = 0$, then x_1 and x_2 are constant along u, and u represents one of the classes $\beta_{3,\pm}$ discussed above; arguing as in the toric case, $n_{\beta_{3,+}} = n_{\beta_{3,-}} = 1$.

If $N_1(u) = 1$, then $K_\infty(u) = 1$, forcing x_3 to have a pole and not a zero; this in turn forces $K_0(u) = 0$ and $N_2(u) = 0$, i.e. x_2 is constant along u. Moreover, x_1 needs to take the value 1 at the pole of x_3, which can only happen if $r_1 > 1$. Assuming this is the case, after a suitable reparametrization we can write

$$x_1(z) = r_1 z, \quad x_3(z) = e^{i\theta_3} r_3 \frac{r_1 - z}{r_1 z - 1}, \quad x_4(z) = e^{i\theta_4} r_4 \frac{r_1 z - 1}{r_1 - z}$$

for some $e^{i\theta_3}, e^{i\theta_4} \in S^1$. There is one such disc through every point of F.

If $N_2(u) = 1$, then $K_0(u) = 1$, forcing x_3 to have a zero and not a pole; hence $K_\infty(u) = 0$, $N_1(u) = 0$, and x_1 is constant along u. Moreover x_2 takes the value 1 at the zero of x_3. Such discs can only exist if $r_2 > 1$; after a suitable reparametrization they are of the form

$$x_2(z) = r_2 z, \quad x_3(z) = e^{i\theta_3} r_3 \frac{r_2 z - 1}{r_2 - z}, \quad x_4(z) = e^{i\theta_4} r_4 \frac{r_2 z - 1}{r_2 - z},$$

and there is one such disc through each point of F.

Summarizing, the contributions of discs with $n_3 = 1$ and $n_4 = 0$ add up to

$$\begin{cases} qz_3z_4 + qz_3^{-1}z_4 & \text{if } r_1 < 1 \text{ and } r_2 < 1 \\ qz_3z_4 + qz_3^{-1}z_4 + q'z_2z_3z_4 & \text{if } r_1 < 1 \text{ and } r_2 > 1 \\ qz_3z_4 + qz_3^{-1}z_4 + q''z_1z_3^{-1}z_4 & \text{if } r_1 > 1 \text{ and } r_2 < 1 \\ qz_3z_4 + qz_3^{-1}z_4 + q''z_1z_3^{-1}z_4 + q'z_2z_3z_4 & \text{if } r_1 > 1 \text{ and } r_2 > 1. \end{cases} \tag{2.12}$$

Adding (2.10), (2.11) and (2.12), we arrive at the expressions (1.1) for the superpotential on the various charts of $X^\vee$.

2.6. **Wall-crossing: discs of Maslov index** 0 **and** -2**.** In this section we study the wall-crossing transformations along which the coordinate charts $X^\vee_{\pm,\pm}$ corresponding to the domains $P_{\pm,\pm} \subset B$ are glued to each other. Our first observation is that, after a small perturbation of the complex structure as in Proposition 2.8, Maslov index 0 discs only exist along the walls $r_1 = 1$ and $r_2 = 1$, and are entirely contained in the divisors $\{x_1 = 1\}$ and $\{x_2 = 1\}$, while negative Maslov index discs can only exist at $r_1 = r_2 = 1$.

Proposition 2.12. *Every holomorphic stable disc $u : C \to X$ with boundary on a smooth fiber $F = F_{(r_1,r_2,\xi_3,\xi_4)}$ of π which deforms to a stable disc for arbitrarily small perturbations of the complex structure on X chosen as in Proposition 2.8 satisfies the following:*

(1) if u has negative Maslov index, then $r_1 = r_2 = 1$;

(2) if $(r_1, r_2) \neq (1, 1)$ and u has Maslov index zero, then either $r_1 = 1$, in which case $x_1 = 1$ at every point of $u(C)$, or $r_2 = 1$, in which case $x_2 = 1$ at every point of $u(C)$.

Proof. We prove, equivalently, that if r_1 and r_2 are not both equal to 1 then $\mu(u) \geq 0$, and if $\mu(u) = 0$ then the conclusion of (2) holds. There are two cases: either $r_1 \neq 1$ or $r_2 \neq 1$. The argument is the same for both; we give the proof for $r_2 \neq 1$.

As in Proposition 2.8, we deform slightly the Lefschetz fibration $f = x_4 : K_{\mathbb{CP}^1} \to \mathbb{C}$ to $f' : K_{\mathbb{CP}^1} \to \mathbb{C}$ with two distinct singular fibers. Denote by $\Delta = f^{-1}(0)$ the singular fiber of f, which is the union of the toric divisors of $K_{\mathbb{CP}^1}$, and by $\Delta'_0, \Delta'_\infty \subset K_{\mathbb{CP}^1}$ the two singular fibers of f', labelled so that L'_0 (i.e., L_0 or a small deformation thereof) is a component of Δ'_0 and L'_∞ (L_∞ or a small deformation) is a component of Δ'_∞. Recall that we deform X to the blowup of $(\mathbb{C}^2 \times K_{\mathbb{CP}^1}, J_0 \oplus J')$ along $H'_0 = \mathbb{C} \times \{1\} \times L'_0$ and $H'_\infty = \{1\} \times \mathbb{C} \times L'_\infty$.

Because $r_2 \neq 1$, the fiber F is disjoint not only from the anticanonical divisor $D \subset X$ but also from the exceptional divisor E_0. This implies that, for a small enough deformation, it is also disjoint from the proper transform Z'_∞ of $\mathbb{C}^2 \times \Delta'_\infty$ under the blowup at H'_∞. (Indeed, Z'_∞ is a small deformation of the union Z_∞ of the proper transform of $\mathbb{C}^2 \times \Delta$ and the exceptional divisor E_0.)

Furthermore, the anticanonical divisor $D \subset X$ is homologous in the complement of F to the (non-effective) divisor $D'_- = (\{0\} \times \mathbb{C} \times K_{\mathbb{CP}^1}) + (\mathbb{C} \times \{0\} \times K_{\mathbb{CP}^1}) + Z'_\infty - E'_0$ (since the proper transform of $\mathbb{C}^2 \times \Delta$ is homologous in $X \setminus F$ to $Z'_\infty - E'_0$). Hence, the Maslov index of the J'-holomorphic deformation $u' : C' \to X$ of the holomorphic disc u is equal to twice its intersection number with D'_-.

If $r_2 < 1$, then the maximum principle for $|x_2|$ implies that the image of u' is disjoint from E'_0. It then follows from positivity of intersections between the J'-holomorphic curve $u'(C')$ and the other components of D'_- (and the absence of any rational curves intersecting those components negatively) that $\mu(u') = 2[u'(C')] \cdot [D'_-] \geq 0$, and if $\mu(u') = 0$ then $u'(C')$ is disjoint from every component of D'_-.

If $r_2 > 1$, then we can deform D'_- inside the complement of F to an effective divisor D'_+ which is the sum of three components: $\{0\} \times \mathbb{C} \times K_{\mathbb{CP}^1}$, the proper transform of $\mathbb{C} \times \{1\} \times K_{\mathbb{CP}^1}$ under the blowup at H'_0, and Z'_∞. It then follows from positivity of intersections (and the lack of rational curves intersecting D'_+ negatively) that $\mu(u') = 2[u'(C')] \cdot [D'_+] \geq 0$, and if $\mu(u') = 0$ then $u'(C')$ is disjoint from every component of D'_+.

Since $\mu(u') = \mu(u)$, we have proved that $\mu(u)$ is non-negative, and if it is zero then the image of u' is disjoint from the components of D'_+ or D'_- depending on the value of r_2.

From now on we assume that $\mu(u) = \mu(u') = 0$. Since $u'(C') \cap D'_\pm = \emptyset$, the image of u' is disjoint from $\{0\} \times \mathbb{C} \times K_{\mathbb{CP}^1}$ and from Z'_∞; since the deformation from Z_∞ to Z'_∞ does not cross F, the intersection numbers of $u(C)$ with $\{0\} \times \mathbb{C} \times K_{\mathbb{CP}^1}$ and Z_∞ also vanish. A first consequence is that $x_1 \circ u$ and $x_1 \circ u'$ are nowhere vanishing holomorphic functions on C and C', taking values in the circle of radius r_1 at the boundary; this implies that x_1 is constant along $u(C)$ and $u'(C')$.

Now assume, in addition to $\mu(u) = 0$, that the constant value of x_1 along $u(C)$ is not equal to 1. Thus, $u(C)$ is disjoint from E_∞, and its total intersection number with $Z_\infty \cup E_\infty$ (the total transform of $\mathbb{C}^2 \times \Delta$) is zero. Since the boundary of $u(C)$ lies away from $Z_\infty \cup E_\infty = \{x_4 = 0\}$, the intersection number of $u(C)$ with the levels of x_4 near zero is also zero. The nonzero levels of x_4 do not contain any rational curves, so positivity of intersection implies that $u(C)$ is disjoint from those levels of x_4, and hence also from $x_4 = 0$. This in turn implies that $u(C)$ is disjoint from Z_∞, hence from the proper transform of $\mathbb{C}^2 \times \Delta$ and from the exceptional divisor E_0.

We have now shown that $u(C)$ is disjoint from all components of the anticanonical divisor $D \subset X$, except possibly $p^{-1}(\mathbb{C} \times \{0\} \times K_{\mathbb{CP}^1})$. The vanishing of $\mu(u)$ then implies that $u(C)$ is also disjoint from that divisor. (Or, slightly abusing the notation introduced before Proposition 2.8: having shown that $N_1(u) = N_{34}(u) = K_0(u) = K_\infty(u) = 0$, we deduce from $\mu(u) = 0$ that $N_2(u) = 0$ as well.) The non-vanishing of x_2 in turn implies that x_2 is constant on $u(C)$. Arguing as in the proof of Corollary 2.10, we now have that u is a stable disc in $\{(x_1, x_2)\} \times K_{\mathbb{CP}^1} \subset X$ with boundary on a product torus (since F is T^2-invariant), and disjoint from all the toric divisors of $K_{\mathbb{CP}^1}$. Such a disc is necessarily constant.

Summarizing: if $\mu(u) = 0$ and $r_2 \neq 1$ then x_1 is constant along u, and if moreover u is not a constant disc then the value of x_1 along $u(C)$ is necessarily equal to 1 (which also implies that $r_1 = 1$). This completes the proof in the case where $r_2 \neq 1$. The argument for the case $r_1 \neq 1$ is identical (up to exchanging the roles of x_1 and x_2, E_0 and E_∞, etc.). $\square$

Corollary 2.13. *The wall-crossing coordinate transformations $\varphi_{0-} : X^\vee_{+-} \to X^\vee_{--}$ and $\varphi_{0+} : X^\vee_{++} \to X^\vee_{-+}$ across the walls at $r_1 = 1$ preserve the coordinates z_2, z_3, z_4. The wall-crossing coordinate transformations $\varphi_{-0} : X^\vee_{-+} \to X^\vee_{--}$ and $\varphi_{+0} : X^\vee_{++} \to X^\vee_{+-}$ across the walls at $r_2 = 1$ preserve the coordinates z_1, z_3, z_4.*

Proof. The wall-crossing transformations φ_{0-} and φ_{0+} are determined by the bubbling phenomena that occur in moduli spaces of holomorphic discs with boundary on fibers $F_{(r_1,r_2,\xi_3,\xi_4)}$ of π as the value of r_1 passes through 1 (after regularization by a small perturbation of the complex structure as in Proposition 2.8). We focus our attention on Maslov index 2 discs representing the classes β_2, $\beta_{3,\pm}$ and β_4, whose boundary passes through a generic point of $F_{(r_1,r_2,\xi_3,\xi_4)}$. The value of x_1 along each of these discs is constant, and equal to the value of the x_1 coordinate at the chosen boundary point constraint. Thus, as long as the family of point constraints we choose for varying r_1 avoids $x_1 = 1$ as the value of r_1 crosses 1, it follows from Proposition 2.12 that none of these discs can participate in any disc bubbling phenomena. (Indeed, given that all Maslov indices are non-negative, wall-crossing for Maslov index 2 discs only involves Maslov index 0 bubbles, but by Proposition 2.12 those all live inside the divisor $\{x_1 = 1\}$.) It follows that the portions of the superpotential which count those discs must match under the wall-crossing transformations. As noted in Section 2.5, $n_{\beta_2} = n_{\beta_{3,\pm}} = n_{\beta_4} = 1$ in all four coordinate charts. Hence, the terms z_2, $qz_3^{\pm1}z_4$, and z_4 in the expressions for $W_{\pm,\pm}$ must match under $\varphi_{0\pm}$; it follows that φ_{0-} and φ_{0+} preserve each of the coordinates z_2, z_3, z_4.

The argument for φ_{-0} and φ_{+0} is identical: we consider the contributions to the superpotential from Maslov index 2 discs representing the classes β_1, $\beta_{3,\pm}$ and β_4, along which x_2 is constant, so that disc bubbling across $r_2 = 1$ can be excluded by considering a family of point constraints that avoid $x_2 = 1$; this implies the invariance of z_1, z_3, z_4 under the wall-crossing transformations. □

Theorem 1.1 now follows directly from the calculations of the superpotentials $W_{\pm,\pm}$ carried out in Section 2.5, the fact that the expressions (1.1) must match under the wall-crossing coordinate transformations, and Corollary 2.13.

To be more explicit, the "basic" **stable discs of Maslov index 0** that arise along the walls at $r_2 = 1$ for $\xi_4 > -\xi_3 - a + \epsilon'$ (i.e., away from the exceptional divisor E_0) belong to three families, one of which only exists for $r_1 > 1$:

(1) The proper transform of a "standard" disc in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$ with $n_3 = 1$ and $n_1 = n_2 = n_4 = 0$, with $x_2 = 1$, and where x_3 has a zero rather than a pole. These discs have Maslov index 2 in $\mathbb{C}^2 \times K_{\mathbb{CP}^1}$, but intersect the toric divisor $\mathbb{C}^2 \times L_0$ at a point of H_0, so that their lift to X is disjoint from the divisor D and has Maslov index zero. (These are the "typical" Maslov index 0 discs that arise in blowups of toric varieties along codimension 2 subvarieties contained in a toric divisor; compare [AAK16].) These discs represent the class $\beta_{3,+} - [\ell_0]$, where $[\ell_0]$ is the class of the fiber of p above a point of E_0, and their Floer-theoretic weight is $q' z_3 z_4$.

(2) The union of a standard disc along the x_4 coordinate axis (representing the class β_4) at $x_2 = 1$ and a rational curve $S_{(x_1,1)}$. These stable discs are regular by Lemma 2.6(2'), and their weight is $qq'z_4$.

(3) For $r_1 > 1$: the union of $S_{(1,1)}$ with a disc on which x_2 and x_3 are constant, with $x_2 = 1$, while x_1 and x_4 have degree 1, and $x_1 = 1$ at the unique point where x_4 vanishes. The disc component can be parametrized by $x_1(z) = r_1 z$, $x_4(z) = e^{i\theta} r_4(r_1 z - 1)/(r_1 - z)$, and represents the class $\beta_1 + \beta_4$. These stable discs are regular by Lemma 2.7(1'), and their weight is $q'q''z_1z_4$.

There are of course other Maslov index 0 discs, representing classes which are linear combinations (with non-negative integer coefficients) of these three, including multiple covers as well as discs built from unions of the above configurations. The proof of Theorem 1.1 shows that the various Maslov index 0 discs present along the walls at $r_2 = 1$ altogether amount to the wall-crossing transformations φ_{-0} and φ_{+0} described by (1.2). A similar analysis can be carried out for the walls at $r_1 = 1$.

To complete our discussion, we briefly consider the **stable discs of negative Maslov index** which occur at $r_1 = r_2 = 1$; for simplicity we only consider the fibers of π which lie away from the exceptional divisors E_0 and E_∞, and only aim to identify the "basic" negative Maslov index discs from which all others may be constructed.

Assume that a stable disc $u : C \to X$ of negative Maslov index deforms to a J'-holomorphic stable disc $u' : C' \to X$ under arbitrarily small deformations of the complex structure as in Proposition 2.8. Restricting to a subset of the components of u, we may assume that C' has only one disc component. (When decomposing u according to the components of C', at least one of the resulting pieces must still have negative Maslov index.) It then follows from Proposition 2.8 that x_1 and x_2 are constant and equal to 1 along $u(C)$. Indeed, if $x_1 \not\equiv 1$ then Proposition 2.8 gives $K_0(u) \le N_{34}(u)$ and $K_\infty(u) \le N_1(u)$, so using (2.6) we conclude that $\mu(u) \ge 2N_2(u) \ge 0$; and similarly if $x_2 \not\equiv 1$ then $K_0(u) \le N_2(u)$ and $K_\infty(u) \le N_{34}(u)$ so that $\mu(u) \ge 2N_1(u) \ge 0$. This in turn implies that $u(C)$ is a stable disc with boundary on a product torus in $p^{-1}(\{(1,1)\} \times K_{\mathbb{CP}^1})$; we can restrict our attention to the proper transform of $\{(1,1)\} \times K_{\mathbb{CP}^1}$, since sphere components inside E_0 or E_∞ have positive Chern number. We are thus left with a disc in $K_{\mathbb{CP}^1}$, whose x_3 and x_4 components admit Blaschke product expressions with n_3 and $n_3 + n_4$ factors as in (2.3), together with one or more sphere components mapping to $S_{(1,1)}$ with total multiplicity m.

The Maslov index in X of such a stable disc is $\mu(u) = 2n_4 - 4m$. Moreover, positivity of intersection of $u'(C')$ with the divisors Z'_0 and Z'_∞ (and careful consideration of the local contributions to these intersections) implies that $m \le n_4$.[1] Hence, the very simplest configuration with $\mu(u) = -2$ corresponds to the case where $n_3 = 0$ and $n_4 = m = 1$, i.e. the union of a standard disc along the x_4 coordinate axis (representing the class β_4) and the rational curve $S_{(1,1)}$. This configuration is regular by Lemma 2.6 (3) (in the sense described there), and its Floer-theoretic weight is $q'q''z_4$.

[1]The intersection number of $u'(C')$ with Z'_0 (resp. Z'_∞) is the number of poles (resp. zeroes) of x_3 plus n_4 minus m. Considering the local contributions to these intersection numbers over the regions of C' which correspond to clusters of sphere components of C, non-negativity of the local intersection numbers implies that the total multiplicity of the sphere components attached at any point of a disc component of $u(C)$ is at most the order of contact of the disc component with the zero section of $K_{\mathbb{CP}^1}$. Thus, near every point of the domain the local contribution to m is bounded by the local contribution to n_4.

The next case to consider is when $n_3 > 0$ and $n_4 = m = 1$. These configurations arise in families that have excess dimension along the x_3, x_4 factors (as the disc component is the proper transform of a disc of higher Maslov index in $K_{\mathbb{CP}^1}$), but carry nontrivial obstruction bundles along the x_1 and/or x_2 coordinate axes ($\mathcal{N}_{u,1}$ or $\mathcal{N}_{u,2}$ in the terminology of Lemma 2.6) depending on whether x_3 has poles and/or zeroes. We conjecture that these discs do not contribute to the enumerative geometry of X. Specifically, it seems that a suitable deformation of the complex structure on X would ensure that the walls of Maslov index 0 discs with $n_3 = 1$ propagating from the exceptional divisors E_0 and E_∞ live at slightly different values of x_1 and x_2 than the Maslov index -2 discs with $n_4 = 1$ which propagate from $S_{(1,1)}$, preventing the occurrence of configurations representing a linear combination of these classes. More generally, we conjecture that the only stable discs of negative Maslov index relevant to the enumerative geometry of X are those we have discussed above, representing the class $\beta_4 + [S_{(1,1)}]$.

2.7. **A compact example.** Our main example is not very interesting from the perspective of homological mirror symmetry, as the mirror superpotential does not have any critical points in the geometrically relevant range of values of the coordinates z_i ($\mathrm{val}(z_i) \in \mathbb{R}^2_{\geq 0} \times \Delta$), and the wrapped Fukaya category of X is expected to be trivial. In this section we briefly describe the analogous result for a compactified example.

Let $\bar{X}$ be the blowup of $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{F}_2$ at $\bar{H}_0 = \mathbb{CP}^1 \times \{1\} \times \bar{L}_0$ and $\bar{H}_\infty = \{1\} \times \mathbb{CP}^1 \times \bar{L}_\infty$, equipped with a suitable T^2-invariant Kähler form; here $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-2))$ is the second Hirzebruch surface, and $\bar{L}_0$ and $\bar{L}_\infty$ are the fibers of the projection from $\mathbb{F}_2$ to $\mathbb{CP}^1$ over 0 and ∞. The proper transform $\bar{D}$ of the toric anticanonical divisor of $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{F}_2$ is an anticanonical divisor in $\bar{X}$. We construct a Lagrangian torus fibration on $\bar{X} \setminus \bar{D}$ with fibers

$$F_{(r_1,r_2,\xi_3,\xi_4)} = \{|x_1| = r_1,\ |x_2| = r_2,\ \mu_3 = \xi_3,\ \mu_4 = \xi_4\}$$

exactly as in Definition-Proposition 2.1, with the only difference that (ξ_3, ξ_4) now take values in the interior of the moment polytope of $\mathbb{F}_2$, i.e.

$$\bar{\Delta} = \{(\xi_3, \xi_4) \in \mathbb{R}^2 \mid \max(0, |\xi_3| - a) \leq \xi_4 \leq b\}.$$

Here a is again half the symplectic area of the exceptional section of $\mathbb{F}_2$, and b is the symplectic area of the fibers of the projection to $\mathbb{CP}^1$. Let A_1, A_2 be the symplectic areas of the two $\mathbb{CP}^1$ factors, and denote by ϵ' and ϵ'' the sizes of the blowups as previously.

The derivation of the SYZ mirror of the log Calabi-Yau pair $(\bar{X}, \bar{D})$ equipped with this Lagrangian torus fibration runs along the same lines as the argument presented above for (X, D); in particular, it is again the case that Maslov index zero discs only arise along walls at $r_1 = 1$ and $r_2 = 1$, and negative Maslov index discs only arise at $r_1 = r_2 = 1$.

Proposition 2.14. *The SYZ mirror of* $(\bar{X}, \bar{D})$ *is built out of four charts which are domains in* $(\mathbb{K}^*)^4$, *with superpotentials*

$$\begin{aligned} W_{--} &= z_1 + q_1 z_1^{-1}(1 + qq''z_4 + q''z_3^{-1}z_4) + z_2 + q_2 z_2^{-1}(1 + qq'z_4 + q'z_3z_4) \\ &\quad + q_1q_2q'q''z_1^{-1}z_2^{-1}z_4 + (1 + q^2 + qz_3 + qz_3^{-1})z_4 + q_4z_4^{-1}, \\ W_{-+} &= z_1 + q_1 z_1^{-1}(1 + qq''z_4 + q''z_3^{-1}z_4) + z_2(1 + qq'z_4 + q'z_3z_4) + q_2z_2^{-1} \\ &\quad + q_1q'q''z_1^{-1}z_2z_4 + (1 + q^2 + qz_3 + qz_3^{-1})z_4 + q_4z_4^{-1}, \\ W_{+-} &= z_1(1 + qq''z_4 + q''z_3^{-1}z_4) + q_1z_1^{-1} + z_2 + q_2z_2^{-1}(1 + qq'z_4 + q'z_3z_4) \\ &\quad + q_2q'q''z_1z_2^{-1}z_4 + (1 + q^2 + qz_3 + qz_3^{-1})z_4 + q_4z_4^{-1}, \\ W_{++} &= z_1(1 + qq''z_4 + q''z_3^{-1}z_4) + q_1z_1^{-1} + z_2(1 + qq'z_4 + q'z_3z_4) + q_2z_2^{-1} \\ &\quad + q'q''z_1z_2z_4 + (1 + q^2 + qz_3 + qz_3^{-1})z_4 + q_4z_4^{-1}, \end{aligned} \tag{2.13}$$

where $q = T^a$, $q' = T^{a-\epsilon'}$, $q'' = T^{a-\epsilon''}$, $q_1 = T^{A_1}$, $q_2 = T^{A_2}$, *and* $q_4 = T^b$. *These charts are glued pairwise by coordinate transformations which preserve* z_3, z_4 *and act on* z_1, z_2 *by*

$$\begin{aligned} &\varphi_{-0}(z_1, z_2) = (z_1, z_2(1 + qq'z_4 + q'z_3z_4 + q_1q'q''z_1^{-1}z_4)), && \varphi_{-0}^*(W_{--}) = W_{-+}, \\ &\varphi_{+0}(z_1, z_2) = (z_1, z_2(1 + qq'z_4 + q'z_3z_4 + q'q''z_1z_4)), && \varphi_{+0}^*(W_{+-}) = W_{++}, \\ &\varphi_{0-}(z_1, z_2) = (z_1(1 + qq''z_4 + q''z_3^{-1}z_4 + q_2q'q''z_2^{-1}z_4), z_2), && \varphi_{0-}^*(W_{--}) = W_{+-}, \\ &\varphi_{0+}(z_1, z_2) = (z_1(1 + qq''z_4 + q''z_3^{-1}z_4 + q'q''z_2z_4), z_2), && \varphi_{0+}^*(W_{-+}) = W_{++}. \end{aligned} \tag{2.14}$$

The proof is essentially identical to that of Theorem 1.1, except the case analysis is more tedious as the x_1 and x_2 coordinates can now have poles as well as zeroes (as does x_4, though this doesn't matter nearly as much, as the standard discs hitting the section at infinity of $\mathbb{F}_2$, with weight $q_4z_4^{-1}$, do not participate in any of the wall-crossing). It is helpful to note, as a consistency check, that the symmetry $x_1 \leftrightarrow x_1^{-1}$ of $\bar{X}$ induces a symmetry of the mirror, which exchanges z_1 and $q_1z_1^{-1}$ while swapping the chambers with $r_1 < 1$ and those with $r_1 > 1$. Similarly, $x_2 \leftrightarrow x_2^{-1}$ induces a symmetry of the mirror which exchanges z_2 and $q_2z_2^{-1}$ while swapping the chambers with $r_2 < 1$ and those with $r_2 > 1$.

3. Deformed Landau-Ginzburg models from family Floer theory

3.1. **Family Floer theory.** As before, we consider a Lagrangian torus fibration $\pi : X^0 \to B$ on the complement $X^0 = X \setminus D$ of an anticanonical divisor D in a Kähler manifold X, whose fibers $F_b = \pi^{-1}(b)$ have vanishing Maslov class in X^0. Let B^0 be a simply connected open subset of B which is disjoint from the critical values of π. We consider the *uncorrected mirror*

$$X^{\vee 0} = X^{\vee}_{B^0} := \bigsqcup_{b \in B^0} H^1(F_b, U_{\mathbb{K}}),$$

with its natural analytic structure for which the Floer-theoretic weights of disc classes $\beta \in \pi_2(X, F_b)$ define analytic functions $z^\beta \in \mathcal{O}(X^{\vee 0})$; we denote by $\pi^\vee : X^{\vee 0} \to B^0$ the natural projection map.

Fixing a base point $b_0 \in B^0$ and a basis $\gamma_1, \dots, \gamma_n$ of $H_1(F_{b_0}, \mathbb{Z})$ (hence of the first homology of every fiber over B^0), we can consider the Floer-theoretic weights z_i $(1 \le i \le n)$

of cylinders with boundary on $F_{b_0} \cup F_b$, obtained by transporting a loop in the class γ_i in the fibers of π over a path connecting b_0 to b inside B^0. The coordinates $(z_1, \dots, z_n)$ allow us to identify $X^{\vee 0}$ with a domain in $(\mathbb{K}^*)^n$. The functions z^β are then Laurent monomials in $z_1, \dots, z_n$ (with exponents determined by the coefficients of $\partial\beta$ in the basis $(\gamma_1, \dots, \gamma_n)$).

Given a subset P of B^0, analytic functions on $X^\vee_P = (\pi^\vee)^{-1}(P)$ are Laurent series in $z_1, \dots, z_n$ which converge adically at all points of P; these are a certain completion of the ring of Laurent polynomials $\mathbb{K}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \mathbb{K}[H_1(F_b)]$. The collection of these completions as P ranges over suitable subsets of B^0 (e.g. polyhedral subsets whose faces have rational slopes with respect to the natural affine structure of B^0, whose inverse images are affinoid domains in $(\mathbb{K}^*)^n$) then determines a sheaf $\mathcal{O}_{an} = \pi^\vee_*(\mathcal{O}_{X^{\vee 0}})$ on B^0.

Remark 3.1. The main reason why we restrict ourselves to a simply connected subset of B is to be able to treat the uncorrected mirror $X^{\vee 0}$ as a single space, rather than as a collection of local charts to be assembled in a manner that is inconsistent (until appropriately corrected) around the singular fibers due to the monodromy of the affine structure on B. This allows us to view Floer-theoretic corrections as geometric deformations of a single space. Another convenient feature is that, since the abelian groups $\pi_2(X, F_b)$ form a local system over $B \setminus \mathrm{critval}(\pi)$, they can be transported over paths in B^0 to provide distinguished isomorphisms between the groups $\pi_2(X, F_b)$ for all $b \in B^0$; we use this repeatedly in the discussion below in order to treat the classes of discs with boundary in arbitrary fibers of π over B^0 as elements of a single relative homotopy group.

However, by essence our constructions are local over (the smooth part of) B, and the Floer-theoretic structures on cochains with coefficients in $\mathcal{O}_{an}$ we introduce below can be defined over all of $B \setminus \mathrm{critval}(\pi)$. If one works with the Morse-theoretic model of family Floer theory we describe below, the corrections to the mirror geometry naturally come out to be Čech cochains, and it is not particularly difficult to upgrade the construction to work over all of $B \setminus \mathrm{critval}(\pi)$ by reformulating the output in a way that only refers to the local pieces $X^\vee_P$ rather than to the whole of $X^{\vee 0}$.

We consider Floer-theoretic operations induced by moduli spaces of holomorphic discs with boundary on the fibers of π on cochains on $X^{00} = \pi^{-1}(B^0)$ with coefficients in the pullback of $\mathcal{O}_{an}$, giving an A_∞-deformation of the classical differential and cup-product. There are various possible models; we describe two, of which the first one is more intuitive but unlikely to be well-defined without further foundational work, while the second one should be viewed as a more realistic setup to develop the theory. (Note in any case that our main discussion only focuses on $\mathfrak{m}_0$ and its properties.)

3.1.1. *Singular differential forms.* We denote by $C^k(X^{00}, \pi^*\mathcal{O}_{an})$ the space of linear combinations of differential forms of degree j with coefficients in $\pi^*\mathcal{O}_{an}$ on smooth codimension ℓ submanifolds of X^{00}, for all $0 \leq j, \ell \leq k$ such that $j + \ell = k$, i.e., the completion of $\bigoplus_{j+\ell=k} \bigoplus_{\mathrm{codim}\, Y=\ell} \Omega^j(Y) \otimes \pi^*\mathcal{O}_{an}$ with respect to the Novikov valuation. We regard these cochains as an enlargement of differential forms of degree k on X^{00} which includes currents of integration along smooth submanifolds.

Given a nonzero class $\beta \in \pi_2(X, F_b)$ and $d \geq 0$, we denote by

$$\overline{\mathcal{M}}_{d+1}(X^{00}, \beta, J) = \bigcup_{b \in B^0} \overline{\mathcal{M}}_{d+1}(\pi^{-1}(b), \beta, J)$$

the moduli space of J-holomorphic stable maps from nodal discs with $d+1$ boundary marked points $z_0, \dots, z_d$ (in order along the boundary) to X, with boundary contained in some fiber of π over a point of B^0, possibly regularized by some perturbation. (As noted above, we use parallel transport over B^0 to identify the groups $\pi_2(X, F_b)$ with each other for all $b \in B^0$.) This moduli space carries $d+1$ evaluation maps $ev_{\beta,0}, \dots, ev_{\beta,d} : \overline{\mathcal{M}}_{d+1}(X^{00}, \beta, J) \to X^{00}$ (all mapping to the same fiber of π by construction). Assume (rather optimistically) that $\overline{\mathcal{M}}_{d+1}(X^{00}, \beta, J)$ is a smooth manifold with corners of dimension $2n + d - 2 + \mu(\beta)$, with

$$\partial \overline{\mathcal{M}}_{d+1}(X^{00}, \beta, J) = \bigcup_{\substack{\beta_1+\beta_2=\beta \\ d_1+d_2=d+1 \\ 1 \leq i \leq d_1}} \overline{\mathcal{M}}_{d_1+1}(X^{00}, \beta_1, J) \;{}_{ev_{\beta_1,i}}\!\times_{ev_{\beta_2,0}} \overline{\mathcal{M}}_{d_2+1}(X^{00}, \beta_2, J). \tag{3.1}$$

Assume moreover that, for given $\alpha_1, \dots, \alpha_d \in C^*(X^{00}, \pi^*\mathcal{O}_{an})$ supported on submanifolds $Y_1, \dots, Y_d \subset X^{00}$, the evaluation map $ev_{\beta,i}$ is transverse to Y_i for $i = 1, \dots, d$, the submanifolds $ev_{\beta,i}^{-1}(Y_i) \subset \overline{\mathcal{M}}_{d+1}(X^{00}, \beta, J)$ intersect transversely, and the restriction of the evaluation map $ev_{\beta,0}$ to their intersection is a submersion onto a smooth submanifold of X^{00} (or that a consistent perturbation scheme can be used to achieve these properties). Then we define

$$\mathfrak{m}_{d,\beta}(\alpha_1, \dots, \alpha_d) = (ev_{\beta,0})_*(ev_{\beta,1}^*\alpha_1 \wedge \dots \wedge ev_{\beta,d}^*\alpha_d).$$

For $\beta = 0$ we set $\mathfrak{m}_{1,0}(\alpha) = \delta\alpha$, the natural extension to $C^k(X^{00}, \pi^*\mathcal{O}_{an})$ of the de Rham differential (if α is supported on $Y \subset X^{00}$ then $\delta\alpha = d\alpha + \alpha_{|\partial Y}$), and $\mathfrak{m}_{2,0}(\alpha_1, \alpha_2) = \alpha_1 \wedge \alpha_2$ (as a form supported on the intersection of the supporting submanifolds of α_1 and α_2, which are assumed to be transverse); $\mathfrak{m}_{d,0}$ is zero for $d \neq 1, 2$.

Finally, we set

$$\mathfrak{m}_d(\alpha_1, \dots, \alpha_d) = \sum_\beta z^\beta \, \mathfrak{m}_{d,\beta}(\alpha_1, \dots, \alpha_d). \tag{3.2}$$

In particular,

$$\mathfrak{m}_0 = \sum_{\beta \neq 0} z^\beta \, (ev_{\beta,0})_* 1_{\overline{\mathcal{M}}_1(X^{00}, \beta, J)},$$

where given our assumptions the nonzero terms correspond to currents of integration along $ev_{\beta,0}(\overline{\mathcal{M}}_1(X^{00}, \beta, J))$ when these are embedded submanifolds of X^{00} (obviously an extremely restrictive setting). Assuming the restriction of π to each of these submanifolds is a submersion onto a smooth submanifold of B^0, we can further rewrite $\mathfrak{m}_0$ as a sum of cochains on B^0 with coefficients in $\mathcal{O}_{an}$-valued cochains on the fiber tori, i.e. elements of the bigraded complex $\mathfrak{C}$ defined in (1.5).

It seems likely that deformation by a suitable bounding cochain $\mathfrak{b} \in \mathfrak{C}_{>0}$ can be used to "smudge" the support of $\mathfrak{m}_0$ and turn it into a smooth differential form, avoiding many of the pitfalls of working with currents. We will not consider this further, and instead turn our attention to a Morse-theoretic model whose technical foundations are easier to set up.

3.1.2. *Morse cochains and perturbed holomorphic treed discs.* We fix a Morse function f and a Morse-Smale metric on X^{00}, and assume that ∇f is transverse to the boundary of X^{00} over ∂B^0. (See the next section for a particularly convenient class of Morse functions for our purposes.) We now denote by $C^k(X^{00}, \pi^*\mathcal{O}_{an})$ the space of linear combinations of index k critical points of f, with coefficients in $\mathcal{O}_{an}$; the coefficient of $p \in \mathrm{crit}(f)$ is typically expressed as a sum of monomials z^β, $\beta \in \pi_2(X, F_{\pi(p)})$, and lies in a suitable completion of $\mathbb{K}[H_1(F_{\pi(p)})]$ (more on this below). We use a family version of the construction described in [CW22, Chapter 4] for a single Lagrangian (itself an elaboration on the work of Cornea and Lalonde [CL06]), and define Floer operations in terms of counts of perturbed J-holomorphic treed discs. (See [Hoek25] for details.)

Given $\beta \in \pi_2(X, F_b)$, $d \geq 0$, and $p_0, \dots, p_d \in \mathrm{crit}(f)$, we denote by

$$\overline{\mathcal{M}}_{d+1}(p_0, p_1, \dots, p_d; \beta, J)$$

the moduli space of *perturbed J-holomorphic treed discs* with inputs at $p_1, \dots, p_d$ and output at p_0, representing the class β. These consist of:

- an oriented metric ribbon tree T with $d+1$ semi-infinite edges (d inputs and one output);
- for each $d_v + 1$-valent vertex v of T, a stable (perturbed) pseudo-holomorphic map u_v from a (nodal) disc D_v with $d_v + 1$ boundary marked points $z_{v,0}, \dots, z_{v,d_v}$ (and possibly also some interior marked points) to X, with boundary in the fiber of π over some point $b_v \in B^0$;
- for a finite edge e of T connecting the output of a vertex v to the i-th input of a vertex v', a gradient flow line u_e of (a perturbation of) f connecting $u_v(z_{v,0})$ to $u_{v'}(z_{v',i})$;
- for a semi-infinite edge of T connecting the i-th input of the tree to the j-th input of a vertex v (resp. the output of a vertex v to the output of the tree), a gradient flow line connecting the critical point p_i to $u_v(z_{v,j})$ (resp. $u_v(z_{v,0})$ to p_0).

(As a degenerate case, for $d = 1$ and $\beta = 0$ the moduli space consists of gradient flow lines of f connecting two critical points p_1 and p_0.)

Recalling that the abelian groups $\pi_2(X, F_b)$ form a local system over $B \setminus \mathrm{critval}(\pi)$, we use the identifications given by parallel transport along the images under π of the gradient flow lines u_e and define the total class of a treed disc to be the sum of the classes of its components, $\beta = \sum_v \beta_v$, where $\beta_v = [u_v] \in \pi_2(X, F_{b_v})$.

Transversality can be achieved as in [CW22] by considering domain-dependent perturbations of the complex structure and of the Morse function, using interior intersections with Donaldson hypersurfaces to stabilize the domain discs. The latter point requires some adjustment compared to the case of a single Lagrangian, as we cannot arrange for a single stabilizing divisor to be disjoint from all the fibers of π simultaneously. However, for each rational point $b \in B^0$ we can find a stabilizing divisor D_b which is disjoint from $\pi^{-1}(b)$, and hence from $\pi^{-1}(U_b)$ for some neighborhood U_b of b. A finite number of these neighborhoods U_{b_i}, $i = 1, \dots, N$ suffice to cover an arbitrarily large compact subset of B^0 (containing the projections of all the critical points of f and connecting Morse flow trees). Discs with boundary in $\pi^{-1}(b)$ can thus be equipped with several collections of marked points, coming from the intersections with the stabilizing divisors D_{b_i} for all i such that $b \in U_{b_i}$. One then needs to choose consistent domain-dependent perturbation data for discs equipped with several collections of interior marked points, in a manner which depends continuously on b and moreover factors through the forgetful map which erases the marked points coming from intersections with D_{b_i} whenever b gets sufficiently close to ∂U_{b_i}.

With this understood, we define

$$\mathfrak{m}_d(p_1,\dots,p_d) = \sum_{p_0,\beta} \big(\#\overline{\mathcal{M}}_{d+1}(p_0,p_1,\dots,p_d;\beta,J)\big)\, z^{\beta}\, p_0 \tag{3.3}$$

for generators of the Morse complex, where the sum ranges over critical points p_0 and classes β such that the expected dimension of $\overline{\mathcal{M}}_{d+1}(p_0,p_1,\dots,p_d;\beta,J)$ is zero. We then extend the definition of $\mathfrak{m}_d$ to general inputs in $C^*(X^{00},\pi^*\mathcal{O}_{an})$ in an $\mathcal{O}_{an}$-linear manner; in particular,

$$\mathfrak{m}_d(z^{\alpha_1}p_1,\dots,z^{\alpha_d}p_d) := z^{\alpha_1+\dots+\alpha_d}\,\mathfrak{m}_d(p_1,\dots,p_d),$$

where as before we implicitly use parallel transport in the local system $\{\pi_2(X,F_b)\}_{b\in B^0}$ to make sense of the sum $\alpha_1+\dots+\alpha_d$.

As in the case of a single Lagrangian, the A_∞-relations follow from the fact that the boundary of $\overline{\mathcal{M}}_{d+1}(p_0,p_1,\dots,p_d;\beta,J)$ consists of configurations in which a gradient flow lines breaks through a critical point of f, i.e. pairs of perturbed J-holomorphic treed disks.

Because of the manner in which the Floer-theoretic weights of holomorphic discs are transported along Morse gradient flow lines to different fibers of π, the total symplectic areas of the J-holomorphic treed discs in the moduli space $\overline{\mathcal{M}}_{d+1}(p_0,\dots,p_d;\beta,J)$ do not coincide with the symplectic area of the class $\beta\in\pi_2(X,F_{\pi(p_0)})$, which determines the valuation of each term in (3.3); in fact the latter quantity does not even need to be positive in general. The convergence of the sum (3.3) is therefore not automatic. One possible solution is to choose the Morse function f so that its gradient flow trees are guaranteed to remain within subsets of B^0 that are sufficiently small for Fukaya's trick to apply.

Specifically, every point $b\in B^0$ admits a neighborhood V_b such that the fibers of π over points of V_b can be mapped to F_b by diffeomorphisms $\phi_{b'\to b}$ which are C^1-close to identity, ensuring that $\phi^*_{b'\to b}\omega$ tames J and that the symplectic areas of a J-holomorphic disc with boundary on $F_{b'}$ with respect to ω and $\phi^*_{b'\to b}\omega$ differ by at most a bounded multiplicative factor. Thus, given critical points $p_1,\dots,p_d\in\pi^{-1}(V_b)$, and assuming that the gradient flow lines appearing in any treed disc with inputs $p_1,\dots,p_d$ are guaranteed to remain within $\pi^{-1}(V_b)$, the symplectic area of such a treed disc and the valuation of its contribution to $\mathfrak{m}_d(p_1,\dots,p_d)$ differ by at most a bounded factor; hence the sum (3.3) converges by the same Gromov compactness argument as in the case of a single Lagrangian. Moreover, convergence also holds for linear combinations of critical points in $\pi^{-1}(V_b)$ with coefficients given by Laurent series which converge adically at every point of V_b. With this understood, we cover an arbitrarily large compact subset of B^0 by finitely many of the neighborhoods $V_{b_i},\ i=1\dots,M$, and choose the Morse function f in such a way that the gradient flow lines appearing in any treed disc are guaranteed to be entirely contained within a single V_{b_i}.

3.1.3. *Adapted Morse functions.* While the above construction can be carried out for fairly general Morse functions (with the restrictions noted), the connection to family Floer theory becomes clearer for specific classes of Morse functions, constructed as follows.

Start from a simplicial decomposition $\mathcal{P}$ of a large compact subset onto which B^0 retracts, with every cell of $\mathcal{P}$ contained in a single open subset V_{b_i}. Pick a Morse function $h : B^0 \to \mathbb{R}$ and a Morse-Smale metric on B^0, such that for every k-cell $\sigma \in \mathcal{P}^{[k]}$ the function h has a unique critical point b_σ in the interior of σ, of index k, whose descending manifold is σ itself. (Such a function and metric can be constructed e.g. from a barycentric subdivision of $\mathcal{P}$.) Then construct the Morse function $f : X^{00} \to \mathbb{R}$ by combining the pullback of h under the projection π with Morse functions on the fibers of π, as well as a Morse-Smale metric on X^{00}, in such a way that:

- all the critical points of f project to critical points of h;
- for each cell $\sigma \in \mathcal{P}^{[k]}$, the restriction of f to $\pi^{-1}(b_\sigma)$ is a standard Morse function on the n-torus (i.e., it has 2^n critical points, whose ascending and descending submanifolds represent dual standard bases of $H_*(T^n)$), and every index j critical point of $f_{|\pi^{-1}(b_\sigma)}$ is also a critical point of f, of index $k + j$;
- for each cell σ, the gradient flow of f is tangent to $\pi^{-1}(b_\sigma)$, and the union of the descending submanifolds of the critical points of f which lie in $\pi^{-1}(b_\sigma)$ is $\pi^{-1}(\sigma)$.

Definition 3.2. We call a Morse function $f : X^{00} \to \mathbb{R}$ with these properties *adapted* to the simplicial decomposition $\mathcal{P}$.

(The assumption that $f_{|\pi^{-1}(b_\sigma)}$ has only 2^n critical points and vanishing Morse differential is extraneous and might be best left out of the definition, but it is convenient for the rest of our discussion.)

The main advantage of adapted Morse functions for our purposes is that Morse cochains can be expressed as Morse cochains for the function h on B^0 with coefficients in the Morse complexes of the functions $f_{|\pi^{-1}(b_\sigma)}$. In this sense, for adapted f we have

$$C^*(X^{00}, \pi^*\mathcal{O}_{an}) = C^*(B^0; C^*(F_b) \hat{\otimes} \mathcal{O}_{an});$$

denoting $\mathfrak{C}^{i,j} = C^i(B^0; C^j(F_b) \hat{\otimes} \mathcal{O}_{an})$, this recovers the setting considered in (1.5). Moreover, the assumption made on the restrictions of f to the critical fibers implies that the fiberwise Morse differential vanishes, so in fact we have

$$\mathfrak{C}^{i,j} = C^i(B^0; H^j(F_b) \hat{\otimes} \mathcal{O}_{an}).$$

By construction the Morse differential δ and the Floer differential $\mathfrak{m}_1$ on this complex are filtered, in the sense that the Morse index i on B^0 is non-decreasing; and the only terms which preserve i are the (trivial) Morse differential and the Floer differential on $C^*(F_{b_\sigma}) = H^*(F_{b_\sigma})$ for each σ. Meanwhile, the terms which increase i by one correspond to Morse, resp. Floer-theoretic continuation maps from $C^*(F_{b_\sigma})$ to $C^*(F_{b_{\sigma'}})$ over a gradient flow line of h from b_σ to $b_{\sigma'}$; and those which increase i by more than one correspond to homotopies between different compositions of such continuation maps.

Remark 3.3. It is typically possible to arrange for the latter homotopies to vanish in Morse theory (e.g., since we have assumed B^0 to be simply connected and one also typically has $\pi_2(B^0) = 0$, by trivializing π over B^0 and taking f to be the sum of the pullback of h and a fixed Morse function on T^n). The Morse complex $(\mathfrak{C}, \delta)$ is then identified with the Čech complex $\check{C}^*(B^0; H^*(F_b) \hat{\otimes} \mathcal{O}_{an})$ for the polyhedral cover of (a retract of) B^0 given by the stars of the vertices of $\mathcal{P}$. We will use this fact below to recast $\mathfrak{m}_0$ as a Čech cochain (with values in polyvector fields) on the uncorrected mirror $X^{\vee 0}$.

We finish this section by noting the manner in which the Floer-theoretic obstruction $\mathfrak{m}_0 \in \mathfrak{C}$ encodes information not only about the holomorphic discs bounded by individual fibers of π but also about those bounded by families of fibers over the simplices of $\mathcal{P}$. Namely, the part of $\mathfrak{m}_0$ which lies over an index 0 critical point of h at a vertex of $\mathcal{P}$ counts (treed) holomorphic discs bounded by the fibers of π over that point, in the sense of Floer theory for a single Lagrangian; whereas the portion of $\mathfrak{m}_0$ which lives over an index i critical point b_σ of h corresponds to (treed) family counts of holomorphic discs bounded by the fibers of π over the i-dimensional cell $\sigma \subset B^0$.

In this sense, the component of $\mathfrak{m}_0$ in $\oplus_j \mathfrak{C}^{i,j}$ counts families of holomorphic discs that occur along (possibly thickened) codimension i walls in B^0, i.e. those which can be meaningfully counted along i-dimensional families of fibers of π. The notion of weak family unobstructedness (Definition 1.3) expresses the requirement that all non-zero counts should live in fiberwise cohomological degree $j = i$, i.e. correspond to discs of Maslov index $2 - 2i$.

Remark 3.4. Besides fleshing out the details of the construction of the curved A_∞-algebra $\mathfrak{C}$ via perturbed J-holomorphic treed discs, Hoek's thesis [Hoek25] also implements a key step of the family Floer program in this setting by constructing a functor from the Fukaya category of Lagrangian sections of the fibration π to the category of A_∞-modules over $\mathfrak{C}$.

3.2. **A heuristic derivation of the master equation.** The algebraic properties of $\mathfrak{m}_0$ generally follow from the fact that the boundary strata of moduli spaces of holomorphic discs are fibered products of moduli spaces of discs, as expressed in (3.1). Most immediately, this yields the identity $\mathfrak{m}_1(\mathfrak{m}_0) = 0$, which is part of the A_∞-equations. Our goal, however, is to find (when possible) a constraint involving only $\mathfrak{m}_0$: the master equation (1.8).

In this section we give a heuristic derivation of this equation under the assumption that the moduli spaces of holomorphic discs entering into the definition of $\mathfrak{m}_0$ are *fiberwise closed*, in order to provide motivation for Conjecture 1.4. (It seems difficult, or in any case well beyond the scope of this paper, to make the argument rigorous under realistic assumptions.)

One particularly convenient way to understand the origin of the master equation in Lagrangian Floer theory is at the level of loop spaces, as first proposed by Fukaya [Fuk06], and further studied by Irie [Irie20], even though the technical details are daunting. (Working in families however does not bring much additional complexity.) A very informal account is as follows. The moduli space $\mathcal{M}_1(X^{00}, \beta, J)$ carries an evaluation map not only to X^{00}, but also to its free loop space $\mathcal{L}X^{00}$ (in fact, to free loops contained in the fibers of π). (This requires preferred parametrizations of the boundary loops, which can be done e.g. by stabilizing the domains or by using arc length in X^{00}). Denote by $\mathfrak{m}^{\mathcal{L}}_{0,\beta} \in C_{2n-2+\mu(\beta)}(\mathcal{L}X^{00})$ the evaluation pushforward of the fundamental chain of $\mathcal{M}_1(X^{00}, \beta, J)$ (after a suitable regularization). Summing over relative classes, we set $\mathfrak{m}^{\mathcal{L}}_0 = \sum_\beta \mathfrak{m}^{\mathcal{L}}_{0,\beta}\, z^\beta \in C_*(\mathcal{L}X^{00}; \pi^*\mathcal{O}_{an})$. By analogy with [Fuk06, Irie20], one expects that (up to sign)

$$\partial \mathfrak{m}^{\mathcal{L}}_0 = \frac{1}{2}\{\mathfrak{m}^{\mathcal{L}}_0, \mathfrak{m}^{\mathcal{L}}_0\}, \tag{3.4}$$

where $\{\cdot,\cdot\}$ denotes a chain-level refinement of the Chas-Sullivan bracket on $H_*(\mathcal{L}X^{00})$; or rather, as shown by Irie, the chain-level master equation also involves higher order terms due to the chain-level loop bracket actually being only part of a homotopy Lie (L_∞) structure on chains on the loop space [Irie20]. To avoid the inherent difficulties of chain-level string topology, we focus on the main case of interest to us, and assume that the moduli spaces of holomorphic discs under consideration are fiberwise closed manifolds. To further avoid the need to regularize the moduli spaces, we make the following (unrealistic) assumptions about $\overline{\mathcal{M}}_1(X^{00},\beta,J)$:

- (regularity) $\overline{\mathcal{M}}_1(X^{00},\beta,J)$ is a smooth manifold with corners, of the expected dimension, whose boundary is as in (3.1);
- (transversality) the projections $\pi_*:\overline{\mathcal{M}}_1(X^{00},\beta,J)\to B^0$ induced by π are submersions onto smooth submanifolds of B^0 with boundary and corners which meet transversely;
- (fiberwise closed) the fibers of $\pi_*:\overline{\mathcal{M}}_1(X^{00},\beta,J)\to B^0$ are closed manifolds, i.e.

$$\pi_*(\partial\overline{\mathcal{M}}_1(X^{00},\beta,J))\subset\partial(\pi_*(\overline{\mathcal{M}}_1(X^{00},\beta,J)). \tag{3.5}$$

Then we can view $\mathfrak{m}_0^{\mathcal{L}}$ as a chain on B^0 with coefficients in $H_*(\mathcal{L}F_b)\,\hat{\otimes}\,\mathcal{O}_{an}$, and the master equation (3.4) expresses the boundary of $\mathfrak{m}_0^{\mathcal{L}}$ (as a chain on B^0) in terms of the bracket induced by the classical cup-product on B^0 and the Chas-Sullivan loop bracket [CS99, Definition 4.1] on $H_*(\mathcal{L}F_b)$.

Lemma 3.5. *Under these assumptions,* (3.1) *implies that* $\mathfrak{m}_0^{\mathcal{L}}\in C_*(B^0;H_*(\mathcal{L}F_b)\,\hat{\otimes}\,\mathcal{O}_{an})$ *satisfies the master equation* (3.4)*.*

Sketch of proof. On one hand, $\partial\mathfrak{m}_{0,\beta}^{\mathcal{L}}$ is the image of the boundary of $\overline{\mathcal{M}}_1(X^{00},\beta,J)$ under the loop space-valued evaluation map. On the other hand, the evaluation image of $\overline{\mathcal{M}}_2(X^{00},\beta_1,J)\;{}_{ev_{\beta_1,1}}\!\times_{ev_{\beta_2,0}}\overline{\mathcal{M}}_1(X^{00},\beta_2,J)$ is the chain formed by inserting the loops that appear in $\mathfrak{m}_{0,\beta_2}^{\mathcal{L}}$ into the loops that make up $\mathfrak{m}_{0,\beta_1}^{\mathcal{L}}$ whenever the latter pass through the base points of the former, i.e. $\mathfrak{m}_{0,\beta_2}^{\mathcal{L}}*\mathfrak{m}_{0,\beta_1}^{\mathcal{L}}$ in the notation of [CS99, §3]. Summing over all β_1,β_2 such that $\beta_1+\beta_2=\beta$, we find that the evaluation image of the right-hand side of (3.1) is equal to the coefficient of z^β in $\mathfrak{m}_0^{\mathcal{L}}*\mathfrak{m}_0^{\mathcal{L}}=\frac{1}{2}\{\mathfrak{m}_0^{\mathcal{L}},\mathfrak{m}_0^{\mathcal{L}}\}$. □

Now we observe that each term $\mathfrak{m}_{0,\beta}^{\mathcal{L}}$ consists of loops representing the class $\partial\beta\in H_1(F_b)$, and recall that each component of $\mathcal{L}F_b$ is homotopy equivalent to F_b itself, via evaluation at the base point. A simple calculation shows:

Lemma 3.6. *Denoting by $\mathcal{L}_\gamma T^n$ the component of $\mathcal{L}T^n$ which consists of loops in the class $\gamma\in H_1(T^n)$, and using evaluation at the base point and Poincaré duality to identify $H_*(\mathcal{L}_\gamma T^n)$ with $H_*(T^n)\simeq H^{n-*}(T^n)\simeq\bigwedge^{n-*}H^1(T^n)$, up to sign the Chas-Sullivan bracket $\{\cdot,\cdot\}:H_*(\mathcal{L}_\gamma T^n)\otimes H_*(\mathcal{L}_{\gamma'}T^n)\to H_*(\mathcal{L}_{\gamma+\gamma'}T^n)$ is given by*

$$\{\alpha,\alpha'\}=\alpha\wedge(\iota_\gamma\alpha')+(-1)^{|\alpha|}(\iota_{\gamma'}\alpha)\wedge\alpha'.$$

Proof. We can represent the classes α,α' by cycles consisting of straight line loops on a flat torus, with tangent vectors given by γ and γ' respectively (under the identification of the first homology of a flat torus with the lattice of integer tangent vectors). The element $\iota_\gamma\alpha'$ is (Poincaré dual to) the cycle on T^n obtained by spreading the evaluation image of

α' by translation along γ, i.e. the set of points $p \in T^n$ such that the straight loop in the direction of γ based at p hits the base point of one of the loops in the cycle α'. Thus, $\alpha \wedge (\iota_\gamma \alpha')$ corresponds to the cycle formed by the base points of loops in the chain α which hit the base points of the loops in the chain α'. This is, up to sign, the operation denoted $\alpha' * \alpha$ in [CS99, Section 3], whose skew-symmetrization is the Chas-Sullivan bracket [CS99, Definition 4.1]. □

Using Lemma 3.6 to rewrite the Chas-Sullivan bracket $\{\mathfrak{m}_0^{\mathcal{L}}, \mathfrak{m}_0^{\mathcal{L}}\}$ in terms of the bracket defined by (1.7) on $H^*(T^n) \otimes \mathbb{K}[H_1(T^n)]$, we arrive at:

Corollary 3.7. *Still assuming regularity of moduli spaces and transversality of evaluation maps, if* $\mathfrak{m}_0^{\mathcal{L}} \in C_*(B^0; H_*(\mathcal{L}F_b) \hat{\otimes} \mathcal{O}_{an})$ *satisfies* (3.4) *then* $\mathfrak{m}_0 \in C^*(B^0; H^*(F_b) \hat{\otimes} \mathcal{O}_{an})$ *satisfies* (1.8)*.*

We can in fact give a more direct derivation of (1.8) without involving loop spaces:

Proposition 3.8. *Assuming that family Floer theory can be set up using the singular differential forms model of Section 3.1.1 and that the moduli spaces of holomorphic discs are fiberwise closed in the sense of* (3.5)*, the cochain* $\mathfrak{m}_0 \in C^*(B^0; H^*(F_b) \hat{\otimes} \mathcal{O}_{an})$ *satisfies* (1.8) *up to sign.*

Proof. Fixing a family of flat metrics on the fibers of π over B^0, we can deform by simultaneous homotopies the boundary loops of all holomorphic stable discs in $\overline{\mathcal{M}}_1(F_b, \beta, J)$ (parametrized e.g. by arc length) into straight line geodesics representing the class $[\partial\beta] \in H_1(F_b)$, for all $b \in B^0$. This produces a homotopy between the chains represented by the evaluation map

$$(ev_{\beta,0}, ev_{\beta,1}) : \overline{\mathcal{M}}_2(X^{00}, \beta, J) \to X^{00} \times X^{00}$$

and by

$$(ev_{\beta,0}, t_{[\partial\beta]} \circ ev_{\beta,0}) : \overline{\mathcal{M}}_1(X^{00}, \beta, J) \times S^1 \to X^{00} \times X^{00},$$

where $t_{[\partial\beta]} : X^{00} \times S^1 \to X^{00}$ denotes translation along the straight line geodesics in the class $[\partial\beta]$ inside the fibers of π. This implies that

$$(ev_{\beta_1,0})_* \left[\overline{\mathcal{M}}_2(X^{00}, \beta_1, J) \ {}_{ev_{\beta_1,1}}\!\times_{ev_{\beta_2,0}} \overline{\mathcal{M}}_1(X^{00}, \beta_2, J)\right]$$

and

$$(ev_{\beta_1,0})_* \left[\overline{\mathcal{M}}_1(X^{00}, \beta_1, J) \times S^1 \ {}_{t_{[\partial\beta_1]} \circ ev_{\beta_1,0}}\!\times_{ev_{\beta_2,0}} \overline{\mathcal{M}}_1(X^{00}, \beta_2, J)\right]$$

are equal as cochains on B^0 with coefficients in $H^*(F_b)$. Since the fiber product expresses the condition that the output marked points of the two discs line up along a straight line geodesic in the class $[\partial\beta_1]$, the latter cochain can be expressed as

$$\begin{aligned}(ev_{\beta_1,0})_* &\left[\overline{\mathcal{M}}_1(X^{00}, \beta_1, J) \ {}_{ev_{\beta_1,0}}\!\times_{t_{[\partial\beta_1]} \circ ev_{\beta_2,0}} (\overline{\mathcal{M}}_1(X^{00}, \beta_2, J) \times S^1)\right] \\ &= (ev_{\beta_1,0})_* \left[\overline{\mathcal{M}}_1(X^{00}, \beta_1, J)\right] \cap t_{[\partial\beta_1]} \left((ev_{\beta_2,0})_*[\overline{\mathcal{M}}_1(X^{00}, \beta_2, J)] \times S^1\right).\end{aligned}$$

Since spreading a homology class along $[\partial\beta_1]$ corresponds under Poincaré duality to interior product with $[\partial\beta_1]$, this expression can be rewritten more concisely as

$$\mathfrak{m}_{0,\beta_1} \wedge \iota_{[\partial\beta_1]}(\mathfrak{m}_{0,\beta_2}).$$

It then follows from (3.1) that $z^\beta\,\delta\mathfrak{m}_{0,\beta}$ is, up to sign, equal to

$$\sum_{\beta_1+\beta_2=\beta} z^{\beta_1+\beta_2}\,\mathfrak{m}_{0,\beta_1}\wedge\iota_{[\partial\beta_1]}\mathfrak{m}_{0,\beta_2} = \frac{1}{2}\sum_{\beta_1+\beta_2=\beta}\{z^{\beta_1}\mathfrak{m}_{0,\beta_1}, z^{\beta_2}\mathfrak{m}_{0,\beta_2}\}.$$

Summing over β then gives (1.8). □

3.3. **Spliced treed J-holomorphic discs and the master equation.** With some care, it seems likely that the argument of Proposition 3.8 can be transcribed into the language of Morse cochains and perturbed holomorphic treed discs, to arrive at a similar result in that setup, still subject to very strong assumptions about moduli spaces of discs. However, it is more appealing to try to modify the model of Section 3.1.2 to arrive at a setup where the master equation holds in full generality. In this section, we sketch such an approach.

Despite the fairly detailed outline, the description we give here is by no means complete: we skip over various limiting cases, and do not attempt to check consistency, discuss orientations, or prove the existence of suitable perturbation data. The details of the construction will appear elsewhere.

Remark 3.9. The approach we describe here using "standard loops" has some advantages but also some notable drawbacks, chief among them the need to choose and keep track of a number of homotopies between various types of loops. As of this writing it is likely that the construction will eventually be modified to rely on a suitable geometric flow and evolve families of fiberwise loops along the tree portions of spliced treed discs, rather than homotoping them to standard loops.

The boundary of the usual moduli space of treed holomorphic discs $\overline{\mathcal{M}}_1(p_0;\beta,J)$ consists of configurations where the length of an internal edge becomes infinite; these can be viewed as pairs of treed discs where the output of one treed disc serves as an *input* for the other, giving rise to the identity $\mathfrak{m}_1(\mathfrak{m}_0)=0$. Our aim is to modify the moduli space so that its boundary consists of pairs of configurations whose *outputs* are matched to each other via the bracket $\{\cdot,\cdot\}$ defined in (1.7). We do this by allowing the matching condition at the ends of broken (infinite length) gradient flow lines to deform towards the output p_0. More precisely, once the length of a gradient flow line in a treed disc becomes infinite (i.e., the flow line breaks through a Morse critical point), we first allow the incidence condition for the end point of the flow line to deform along a homotopy from the boundary loop of the appropriate disc component to a "standard" loop in the same homotopy class, and then we allow the standard loop to slide along the gradient flow tree towards the output of the treed disc. (Standard loops, defined below, are a class of loops which are well-behaved with respect to the action of $H_1(F_b)$ on the Morse cohomology of f by interior product.) For simplicity, we assume that we work with an adapted Morse function for some simplicial decomposition $\mathcal{P}$ of B^0 in the sense of Definition 3.2.

3.3.1. *Standard loops.* By Definition 3.2, the critical points of an adapted Morse function $f: X^{00}\to\mathbb{R}$ lie in fibers $F_{b_\sigma}=\pi^{-1}(b_\sigma)$ indexed by the cells $\sigma\in\mathcal{P}^{[k]}$ of $\mathcal{P}$, and the restriction of f to F_{b_σ} is a standard Morse function on T^n, i.e. there is a basis $e_{\sigma,1},\dots,e_{\sigma,n}$ of $H_1(F_{b_\sigma})$ such that the ascending and descending submanifolds of the critical points of $f_{|F_{b_\sigma}}$ represent exterior products of elements of the basis. For $I\subseteq\{1,\dots,n\}$ we denote by

$p_{\sigma,I}$ the critical point of index $|I|$ whose descending (resp. ascending) submanifold within F_{b_σ} represents the homology class $e_{\sigma,I} = \bigwedge_{i\in I} e_{\sigma,i}$ (resp. $e_{\sigma,\overline{I}}$, where $\overline{I} = \{1,\dots,n\} - I$).

Given a homology class $[\gamma] = \sum n_i e_{\sigma,i} \in H_1(F_{b_\sigma}, \mathbb{Z})$, interior product with $[\gamma]$ defines an operator of degree -1 on $H^*(F_{b,\sigma}, \mathbb{Z}) \simeq CM^*(f_{|F_{b_\sigma}}, \mathbb{Z})$,

$$\iota_{[\gamma]} : CM^*(f_{|F_{b_\sigma}}) \to CM^{*-1}(f_{|F_{b_\sigma}}),$$

which maps $p_{\sigma,I}$ to $\iota_{[\gamma]}(p_{\sigma,I}) = \sum_{i\in I} (-1)^{|I\cap\{1,\dots,i-1\}|}\, n_i\, p_{\sigma,I-\{i\}}$. For $p_{\sigma,I} \in \mathrm{crit}(f)$, we denote by $[\overline{W}^+(p_{\sigma,I})]$ the fundamental chain of the (closure of the) ascending manifold of $p_{\sigma,I}$ inside X^{00}, and for $[\gamma] \in H_1(F_{b_\sigma})$ we define $[\overline{W}^+(\iota_{[\gamma]}(p_{\sigma,I}))]$ to be the appropriate linear combination of the ascending manifolds $\overline{W}^+(p_{\sigma,I-\{i\}})$, $i \in I$.

Definition 3.10. Let $\widetilde{X}^{00}$ be the space of pairs $([\gamma], x)$ where $x \in X^{00}$ and $[\gamma] \in H_1(F_{\pi(x)}, \mathbb{Z})$. A *system of standard loops* for f is a smooth submersive map from $\widetilde{X}^{00} \times S^1$ to X^{00}, $([\gamma], x, t) \mapsto s_{[\gamma]}(x,t) = s_{[\gamma],x}(t)$, such that:

1. for all $[\gamma]$ and x, $s_{[\gamma],x} : S^1 \to X^{00}$ is a loop in $F_{\pi(x)}$ based at x, representing the homology class $[\gamma]$;
2. for every critical point $p_{\sigma,I}$ of f, and for every $[\gamma] \in H_1(F_{b_\sigma})$,
$$p_* s_{[\gamma]}^{-1}([\overline{W}^+(p_{\sigma,I})]) = [\overline{W}^+(\iota_{[\gamma]}(p_{\sigma,I}))] \tag{3.6}$$
as chains in X^{00} modulo degenerate chains supported on the lower-dimensional submanifold $\overline{W}^+(p_{\sigma,I})$; here $p : X^{00} \times S^1 \to X^{00}$ is the projection to the first factor, and we implicitly identify $H_1(F_b) \simeq H_1(F_{b_\sigma})$ for b near b_σ.

For $[\gamma] = e_{\sigma,i}$, $i \in I$, condition (2) states that the loop $s_{[\gamma],x}$ passes through the ascending manifold of $p_{\sigma,I}$ if and only if x lies in the ascending manifold of $p_{\sigma,I-\{i\}}$, and in that case it does so just once (counting with appropriate signs). Likewise for general $[\gamma]$ and linear combinations of ascending manifolds of the critical points appearing in $\iota_{[\gamma]}(p_{\sigma,I})$.

As will be clear from the arguments below, it would in fact suffice for the two sides of (3.6) to be equivalent from the perspective of Morse theory, i.e. that they intersect in the same manner with the ascending and descending submanifolds of other critical points of f.

Lemma 3.11. *When B^0 is simply connected, there exists an adapted Morse function f which admits a system of standard loops.*

Proof. Since $\pi_1(B^0) = 1$, the structure group of the fibration $\pi : X^{00} \to B^0$ reduces to translations of the n-torus $T^n = (S^1)^n$, i.e. we have well-defined fiberwise coordinates up to translation on the fibers of π. We can then choose the admissible Morse function f so that its restriction to each fiber of π is the sum of standard Morse functions on the S^1 factors and a Morse function on the base B^0, and choose the metric in a suitable manner, so that the ascending manifold $\overline{W}^+(p_{\sigma,I})$ is invariant under translation along the i-th S^1 factor in the fibers of π whenever $i \notin I$, and translating it along the i-th S^1 factor for $i \in I$ yields exactly $\overline{W}^+(p_{\sigma,I-\{i\}})$.

Denoting by e_i the homology class of the i-th S^1 factor, and given a class $[\gamma] = \sum n_i e_i$ and a point $x \in X^{00}$, we define the loop $s_{[\gamma],x}$ to be the concatenation of loops based at x which run successively n_i times along each S^1 factor of $F_{\pi(x)}$ (with vanishing derivatives at the end points so that the concatenation is a smooth loop). The identity (3.6) then follows from the observation that a loop based at x and running along the i-th S^1 factor intersects $\overline{W}^+(p_{\sigma,I})$ if and only if x lies in the image of $\overline{W}^+(p_{\sigma,I})$ under translation along the i-th S^1 factor, i.e. $\overline{W}^+(p_{\sigma,I-\{i\}})$ if $i \in I$ and $\overline{W}^+(p_{\sigma,I})$ itself otherwise. □

3.3.2. *Spliced treed discs.* A *spliced treed disc* consists of a collection of $k+1$ treed discs $\mathbb{T}_\alpha = (T_\alpha, \{D_v\}_{v\in \mathrm{Vert}(T_\alpha)})$, $\alpha \in \{0, \dots, k\}$, inductively attached onto each other by k semi-infinite edges e_α^{spl}, $\alpha \in \{1, \dots, k\}$ (the *splicings*). Each splicing e_α^{spl} connects the output of the treed disc $\mathbb{T}_\alpha$ to some point t_α^{spl} (the *target* of the splicing) in $T^+_{<\alpha} := \bigcup_{\alpha'<\alpha} T_{\alpha'} \cup \bigcup_{\alpha'<\alpha} e_{\alpha'}^{spl}$, the underlying tree of the configuration obtained by splicing the treed discs $\mathbb{T}_{\alpha'}$ for $\alpha' < \alpha$.

The end result of this process differs from a stable treed disc in that there are some broken (infinite length) internal edges, formed by the output edges of the tree discs $\mathbb{T}_\alpha$ together with the splicing edges e_α^{spl}, and these broken edges do not attach to the boundary of a disc, but rather onto the underlying tree

$$T^+ = \bigcup_\alpha T_\alpha \cup \bigcup_\alpha e_\alpha^{spl}$$

of the spliced treed disc; the manner in which this translates into an incidence condition for the end point of a gradient flow line is governed by *splicing data* which we describe below. (Note that T^+ is not a ribbon tree, as the splicing data does not specify how the splicing edge fits into a cyclic ordering at its target.)

The splicing data for a given splicing depends on whether its target lies on an edge of T^+ or at a vertex, and on the number of splicings which share the same target $t = t_\alpha^{spl}$. The set of incidence conditions we impose on the ends of the splicings with target t is parametrized by a manifold with corners $S^{spl}(t)$; when all the splicings have distinct targets, $S^{spl}(t)$ is S^1 if t lies on an edge of T^+, or $[0,1]\times S^1$ if it lies at a vertex. The incidence conditions we impose on the ends of the splicing edges with target t are described by maps $\sigma_\alpha : S^{spl}(t) \to X^{00}$ for all α such that $t_\alpha^{spl} = t$, defined below.

Definition 3.12. A *spliced treed J-holomorphic disc* $u : \mathbb{T}^+ \to X$ *with domain* $\mathbb{T}^+ = \bigcup \mathbb{T}_\alpha \cup \bigcup e_\alpha^{spl}$ consists of:

- for each α, a (perturbed) treed J-holomorphic disc $u_\alpha : \mathbb{T}_\alpha \to X$, i.e., (perturbed) stable J-holomorphic discs $u_v : D_v \to X$ with boundary in some fiber F_{b_v} of π for every vertex v of T_α, connected to each other and to the output critical point $p_{0,\alpha} \in \mathrm{crit}(f)$ by (perturbed) gradient flow lines u_e of f for every edge e of T_α;
- for each point t of T^+ which is the target of one or more splicings, a choice of splicing data $\theta_t \in S^{spl}(t)$;
- for each splicing edge e_α^{spl}, a semi-infinite (perturbed) gradient flow line of f whose negative end converges to the critical point $p_{0,\alpha}$, and whose positive end maps to the point $\sigma_\alpha(\theta_{t_\alpha^{spl}}) \in X^{00}$.

The description of the maps $\sigma_\alpha : S^{spl}(t) \to X^{00}$ involves the standard loops introduced in the previous section, as well as the following definition:

Definition 3.13. The *weight* $\beta_{t\in e}$ of an edge e of the tree T^+ underlying a spliced treed holomorphic disc $u : \mathbb{T}^+ \to X$ at a point $t \in e$ is the sum of the homotopy classes $\beta_v = [u_v]$ of all the disc components of u which correspond to vertices $v \in \bigcup \mathrm{Vert}(T_\alpha)$ such that the path in T^+ from v to the output of T^+ passes through t, and reaches t via the edge e.

As in §3.1.2, we use the identifications between the abelian groups $\pi_2(X, F_b)$, $b \in B^0$ along the images under π of the gradient flow lines u_e to define the sum of the homotopy classes β_v and view the weight $\beta_{t\in e}$ as an element of $\pi_2(X, F_b)$ for $b = \pi(u_e(t))$. We also introduce the homology class

$$[\gamma_{t\in e}] = \partial\beta_{t\in e} \in H_1(F_b). \tag{3.7}$$

With this understood, let e_α^{spl} be a splicing edge, with target $t = t_\alpha^{spl} \in T^+$.

Case 1. Assume t is an interior point of an edge e of T^+, mapping to $x = u_e(t) \in X^{00}$, and no other splicing has the same target t. Then we require the end point of the splicing to lie on $s_{[\gamma_{t\in e}],x}$, the standard loop at x in the homology class $[\gamma_{t\in e}] = \partial\beta_{t\in e}$. Namely, we set $S^{spl}(t) = S^1$, and require the end point of e_α^{spl} to map to $\sigma_\alpha(\theta_t) := s_{[\gamma_{t\in e}],x}(\theta_t)$.

Case 2. Assume t is a vertex v of one of the trees T_α, corresponding to a stable J-holomorphic disc $u_v : D_v \to X$ with boundary in F_{b_v}, and no other splicing has the same target. Denote by e_i the edges of T_α that attach to the input boundary marked points $z_{v,i} \in \partial D_v$, by $\beta_{v,i} = \beta_{v\in e_i}$ the weights of these edges at their end points, and by $x_i = u_v(z_{v,i})$ the end points of the gradient flow lines $u(e_i)$. Finally, let $\beta_{v,tot} = \beta_v + \sum \beta_{v,i} \in \pi_2(X, F_{b_v})$, where $\beta_v = [u_v(D_v)]$ is the class of the stable disc $u_v : D_v \to X$.

We use this data to define two loops in F_{b_v}, both based at the image of the output marked point of D_v, $x = u_v(z_{v,0})$. On one hand, let $\sigma_1 = s_{[\partial\beta_{v,tot}],x}$ be the standard loop at x in the homology class $\partial\beta_{v,tot}$. On the other hand, let σ_0 be the loop obtained by inserting the standard loop $s_{[\partial\beta_{v,i}],x_i}$ at each input marked point $z_{v,i}$ into the boundary loop $u_{v|\partial D_v}$ of the disc u_v. This loop does not have a canonical parametrization by S^1, but we can choose one in a consistent manner, using the fact that the domain D_v is stable (possibly after adding interior marked points corresponding to intersections with stabilizing divisors).

Denote by $\sigma : [0,1] \times S^1 \to F_{b_v}$ a homotopy between σ_0 and σ_1 produced by some consistent method of interpolation between based loops in the fibers of π; for example, after identifying all the fibers with flat tori we can just use straight line interpolation. We set $S^{spl}(t) = [0,1] \times S^1$, $\sigma_\alpha = \sigma$, and require the end point of e_α^{spl} to map to $\sigma(\theta_t)$.

This choice is motivated by the observation that the boundary of the homotopy σ precisely accounts for the various ways in which a splicing with target v can deform: the target can move into the output edge, whence the required incidence condition becomes a standard loop in the class $\partial\beta_{v,tot}$ (cf. Case 1 above), or it can move into one of the input edges e_i, and the incidence condition becomes a standard loop in the class $\partial\beta_{v,i}$; or the splicing can disappear altogether by deforming to an honest gradient flow line attached to $u_v(\partial D_v)$.

Things become more complicated when two or more splicings share the same target. We describe the splicing data and incidence condition in the next simplest case, to illustrate the general construction, which will appear elsewhere.

Case 3. An interior point t of an edge e of T^+ is the target of exactly two splicings $e^{spl}_{\alpha_1}$ and $e^{spl}_{\alpha_2}$. Denote by $\beta = \beta_{t\in e}$, $\beta_1 = \beta_{t\in e^{spl}_{\alpha_1}}$ and $\beta_2 = \beta_{t\in e^{spl}_{\alpha_2}}$ the weights of the different edges which attach together at t, and let $x = u_e(t)$.

As in Case 2 above, the space $S^{spl}(t)$ and the maps $\sigma_{\alpha_1}, \sigma_{\alpha_2} : S^{spl}(t) \to X^{00}$ should describe a homotopy between the incidence conditions imposed on the ends of the splicing edges $e^{spl}_{\alpha_1}$ and $e^{spl}_{\alpha_2}$ after small deformations which make their targets t_1 and t_2 distinct. There are four manners in which such a configuration can deform to one where the two splicings have distinct targets t_1 and t_2:

- Type I: t_1 lies before t_2 along e (farther from the output),
- Type I': t_1 lies after t_2 along e (closer to the output),
- Type II: t_1 can move to the edge $e^{spl}_{\alpha_2}$,
- Type II': t_2 can move to $e^{spl}_{\alpha_1}$.

When t_1 lies before t_2 along e (Type I), the incidence condition at t_1 is given by a standard loop in the class $\partial\beta$, while at t_2 it is a standard loop in the class $\partial(\beta+\beta_1)$ (both based at points close to x). When t_1 lies on $e^{spl}_{\alpha_2}$ instead (Type II), the incidence condition at t_2 is a standard loop in the class $\partial\beta$ (based near x), while at t_1 it is a standard loop in the class $\partial\beta_2$ (now based near the end point of the second splicing, rather than x). Similarly for Types I' and II', exchanging the indices 1 and 2.

We set $S^{spl}(t)$ to be the disjoint union of two copies of $[0,1]\times S^1\times S^1$. On the first one, we define $\sigma_{\alpha_1}(\tau,\theta_1,\theta_2) = s_{[\partial\beta],x}(\theta_1)$, i.e. the incidence condition for $e^{spl}_{\alpha_1}$ is independent of $\tau\in[0,1]$ and lies along the standard loop at x in the class $\partial\beta$. Meanwhile, we pick for each value of θ_1 a homotopy (chosen by some consistent process, e.g. straight line interpolation after identifying $F_{\pi(x)}$ with a flat torus)

$$\varsigma_{[\partial\beta],[\partial\beta_1],x,\theta_1} : [0,1]\times S^1 \to F_{\pi(x)}$$

between the standard loop $s_{[\partial\beta]+[\partial\beta_1],x}$ (for $\tau=0$) and the loop obtained by inserting $s_{[\partial\beta_1],y}$ into $s_{[\partial\beta],x}$ at the point $y = s_{[\partial\beta],x}(\theta_1)$ (for $\tau=1$); and we set

$$\sigma_{\alpha_2}(\tau,\theta_1,\theta_2) = \varsigma_{[\partial\beta],[\partial\beta_1],x,\theta_1}(\tau,\theta_2).$$

This parametrizes a homotopy between the incidence conditions associated to Type I deformations (for $\tau=0$) and the union of the incidence conditions for Type II' deformations together with the "symmetric" incidence condition where both splicings map to the standard loop $s_{[\partial\beta],x}$. On the second copy of $[0,1]\times S^1\times S^1$, we set instead

$$\sigma_{\alpha_1}(\tau,\theta_1,\theta_2) = \varsigma_{[\partial\beta],[\partial\beta_2],x,\theta_2}(\tau,\theta_1) \quad\text{and}\quad \sigma_{\alpha_2}(\tau,\theta_1,\theta_2) = s_{[\partial\beta],x}(\theta_2).$$

This yields a homotopy between the incidence conditions for the remaining types of deformation (Types I' and II) and the symmetric incidence condition $s_{[\partial\beta],x}\times s_{[\partial\beta],x}$, so that the two components of $S^{spl}(t)$ taken together provide the desired homotopy between incidence conditions.

We expect that a similar construction of homotopies between different incidence conditions can be used to deal with the remaining cases (when two splicings have a vertex as common target, or when more than two splicings have the same target). A detailed treatment will appear elsewhere.

3.3.3. *The master equation.* Returning to the notation of §§3.1.2–3.1.3, we define

$$\mathfrak{m}_0^{spl} \in C^*(X^{00}, \pi^*\mathcal{O}_{an}) = C^*(B^0; H^*(F_b) \hat{\otimes} \mathcal{O}_{an})$$

to be the same weighted sum as in (3.3) (with $d = 0$), except we use moduli spaces of spliced treed holomorphic discs instead of their ordinary counterparts. (As usual, $\mathfrak{m}_0^{spl}$ is a weighted count of *rigid* spliced treed holomorphic discs, i.e. those which arise in zero-dimensional moduli spaces, while the master equation comes from considering the boundaries of one-dimensional moduli spaces.) The master equation is now expected to arise from the behavior of spliced treed holomorphic discs at the boundary of the moduli space.

Boundary configurations where the output edge of the tree breaks through a critical point of the Morse function f contribute $\delta\mathfrak{m}_0^{spl}$, where δ is the Morse differential.

Otherwise, once the length of an internal edge e of a treed holomorphic disc becomes infinite, it turns into a splicing edge, whose target can move up along the remaining part of the tree, all the way to the output edge e_{out}. The boundary of the moduli space of spliced discs is reached once the target of the splicing has moved "to infinity" along e_{out}, at which point the gradient flow line corresponding to e_{out} must also break through a critical point of f below the target of the splicing. Thus, in the limit we have a pair of rigid treed holomorphic discs $u_\pm : \mathbb{T}_\pm \to X$ with outputs $p_\pm \in \mathrm{crit}(f)$, together with gradient flow lines of f from $p_\pm$ to a pair of points $x_\pm$ with the property that x_- lies on the standard loop $s_{[\gamma_+],x_+}$ through x_+ representing the homology class $[\gamma_+] = [\partial\beta_+]$ of the boundary of the treed disc u_+ (and then upward from x_+ to a critical point of f which is the overall output of the limit configuration). The conditions involving $p_\pm$ and $x_\pm$ can be rewritten as:

$$x_+ \in [\overline{W}^+(p_+)] \cap p_* s^{-1}_{[\gamma_+]}([\overline{W}^+(p_-)]),$$

which by (3.6) is the same as $[\overline{W}^+(p_+)] \cap \big[\overline{W}^+(\iota_{[\gamma_+]}(p_-))\big]$. In other terms, the gradient flow line from p_- to x_- can be replaced by a gradient flow line from one of the critical points appearing in the linear combination $\iota_{[\gamma_+]}(p_-)$ to x_+, and after this modification the top-most portion of the limiting configuration amounts to a gradient flow tree computing the cup-product $p_+ \wedge \iota_{[\gamma_+]}(p_-)$.

Summing over all such configurations, with appropriate weights, we therefore arrive at

$$\mathfrak{m}_0^{spl} * \mathfrak{m}_0^{spl} = \frac{1}{2}\{\mathfrak{m}_0^{spl}, \mathfrak{m}_0^{spl}\},$$

where $*$ denotes the operation of degree -1 on $C^*(X^{00}, \pi^*\mathcal{O}_{an}) = C^*(B^0; H^*(F_b) \hat{\otimes} \mathcal{O}_{an})$ defined by

$$(z^{\gamma_+}p_+) * (z^{\gamma_-}p_-) = z^{\gamma_+ + \gamma_-}\, p_+ \wedge \iota_{\gamma_+}(p_-), \tag{3.8}$$

whose skew-symmetrization is the bracket (1.7).

Unlike the case of ordinary treed holomorphic discs, configurations where the targets of several spliced edges simultaneously escape to infinity can also contribute to the codimension 1 boundary of the moduli space of spliced treed discs.

When the targets of two different splicings e_1 and e_2 both escape towards the output of the spliced treed disc in such a way that the distance between the two targets also goes to infinity, we arrive at codimension 2 strata consisting of three treed discs u_1, u_2, u_+ with outputs p_1, p_2, p_+, together with gradient flow lines of f which attach onto each other via standard loops. Using (3.6), these can be recast by the same trick as above as a broken gradient flow tree computing one of $p_+ \wedge \iota_{[\gamma_+]}(p_1 \wedge \iota_{[\gamma_1]}(p_2))$, $p_+ \wedge \iota_{[\gamma_+]}(p_2 \wedge \iota_{[\gamma_2]}(p_1))$, $(p_+ \wedge \iota_{[\gamma_+]}(p_1)) \wedge \iota_{[\gamma_+ + \gamma_1]}(p_2)$, or $(p_+ \wedge \iota_{[\gamma_+]}(p_2)) \wedge \iota_{[\gamma_+ + \gamma_2]}(p_1)$. (Here $[\gamma_1], [\gamma_2], [\gamma_+]$ are the homology classes associated to the boundary loops of the treed discs u_1, u_2, u_+.)

However, when the targets of e_1 and e_2 remain a finite distance apart, the configuration of gradient flow lines gets recast as a gradient flow tree of the sort used to define higher A_∞-operations in Morse theory, and/or the incidence conditions we impose on the end points of the gradient flow lines involve homotopies between the various products of standard loops appearing in the above expressions (see the discussion of Case 3 in §3.3.2 above). Due to the extra degree of freedom afforded by the various homotopies, these strata have codimension 1 rather than 2.

Algebraically, these homotopies define an operation of degree -3 on $C^*(X^{00}, \pi^*\mathcal{O}_{an})^{\otimes 3}$, whose skew-symmetrization ℓ_3 is the next term in a shifted L_∞-structure whose first two operations are the Morse differential δ and (up to sign) the bracket $\{\cdot,\cdot\}$. Configurations in which two splicing targets escape to infinity then contribute an additional term $\frac{1}{6}\ell_3(\mathfrak{m}_0^{spl}, \mathfrak{m}_0^{spl}, \mathfrak{m}_0^{spl})$ to the master equation; and so on with higher homotopies when more than two edge lengths simultaneously become infinite. To summarize:

Conjecture 3.14. *(1) The Morse complex $C^*(X^{00}, \pi^*\mathcal{O}_{an})$ carries operations*

$$\ell_m : C^*(X^{00}, \pi^*\mathcal{O}_{an})^{\otimes m} \to C^*(X^{00}, \pi^*\mathcal{O}_{an})[3-2m], \quad m \geq 1,$$

defined in terms of counts of spliced configurations consisting of $m-1$ splicing edges and one infinite gradient flow line, without any disc components; in particular ℓ_1 is the Morse differential and ℓ_2 is (up to sign) the bracket $\{\cdot,\cdot\}$. These operations define a shifted L_∞-structure on $C^*(X^{00}, \pi^*\mathcal{O}_{an})$.

(2) Weighted counts of rigid spliced treed holomorphic discs define an element $\mathfrak{m}_0^{spl} \in C^*(X^{00}, \pi^*\mathcal{O}_{an})$ which satisfies the L_∞ master equation

$$\sum_{m\geq 1} \frac{1}{m!}\ell_m((\mathfrak{m}_0^{spl})^{\otimes m}) = \delta\mathfrak{m}_0^{spl} \pm \frac{1}{2}\{\mathfrak{m}_0^{spl}, \mathfrak{m}_0^{spl}\} + \frac{1}{6}\ell_3(\mathfrak{m}_0^{spl}, \mathfrak{m}_0^{spl}, \mathfrak{m}_0^{spl}) + \cdots = 0. \tag{3.9}$$

Remark 3.15. Since the product on the Morse complex is only homotopy associative, it shouldn't be surprising that in general the bracket $\{\cdot,\cdot\}$ should only be the leading term of an L_∞ structure, and hence that the master equation should also involve higher order terms. As noted in §3.2 above, Irie's result on the chain-level master equation with loop space coefficients [Irie20] also involves higher terms, due to the chain-level string bracket being part of an L_∞ structure on the space of chains on the free loop space.

However, we expect that, for suitable choices of the adapted Morse function f and of the system of standard loops $(s_{[\gamma],x})$, one can arrange for the higher terms of the L_∞-structure to vanish, reducing (3.9) to the ordinary master equation (1.8). The reason for this expectation is that, choosing f as in Remark 3.3, the Morse complex we consider can be recast as the Čech complex of a suitable cover of B^0 with coefficients in $H^*(F_b)\,\hat{\otimes}\mathcal{O}_{an}$, which is a dg-algebra; and, as we shall see below, the bracket $\{\cdot,\cdot\}$ corresponds under mirror symmetry to the Schouten-Nijenhuis bracket for Čech cochains on the uncorrected mirror $X^{\vee 0}$ with coefficients in polyvector fields, which satisfies the Jacobi identity at chain level.

Remark 3.16. Spliced treed discs can be used to produce not only the element $\mathfrak{m}_0^{spl}$ but also a wealth of algebraic operations on $C^*(X^{00},\pi^*\mathcal{O}_{an})$. Work in progress of the author with Keeley Hoek suggests that, by counting spliced treed holomorphic discs whose inputs all lie on the tree that contains the output (i.e., the treed discs that feed into splicings do not carry any inputs), one can define operations $\tilde{\mathfrak{m}}_k^{spl}$ for $k\geq 1$ which make $C^*(X^{00},\pi^*\mathcal{O}_{an})$ into an *uncurved* A_∞-algebra. The details will appear elsewhere. Beyond this, one might hope that more general moduli spaces of spliced treed discs also endow $C^*(X^{00},\pi^*\mathcal{O}_{an})$ with the structure of a framed E_2-algebra (or homotopy BV algebra); compare with [AGV24].

3.4. **From $\mathfrak{m}_0$ to the geometry of the corrected mirror.** We finally turn our attention to the passage from family Floer theory to the geometry of instanton corrections on the mirror. As explained at the beginning of §3.1, the uncorrected mirror $X^{\vee 0}$ comes equipped with a rigid analytic torus fibration $\pi^\vee : X^{\vee 0}\to B^0$, locally modelled on the valuation map $H^1(F_b,\mathbb{K}^*)\to H^1(F_b,\mathbb{R})$, or more explicitly after choosing a basis $(\gamma_1,\dots,\gamma_n)$ of $H_1(F_b,\mathbb{Z})$, the valuation map $(\mathbb{K}^*)^n\to\mathbb{R}^n$.

Under the assumption of weak family unobstructedness (Definition 1.3), family Floer theory as described in the preceding sections determines an element $\mathfrak{m}_0\in C^*(B^0,H^*(F_b)\,\hat{\otimes}\,\mathcal{O}_{an})$, where $\mathcal{O}_{an}=\pi^\vee_*(\mathcal{O}_{X^{\vee 0}})$ is a completion of the ring $\mathbb{K}[H_1(F_b)]$ of Laurent polynomials in the local coordinates of $X^{\vee 0}$. Moreover, $\mathfrak{m}_0$ can be expressed as a sum of elements

$$\alpha^{(i)}\in C^i(B^0,H^i(F_b)\,\hat{\otimes}\,\mathcal{O}_{an})$$

which encode counts of holomorphic discs of Maslov index $2-2i$ in X bounded by i-dimensional families of fibers of the Lagrangian torus fibration $\pi : X^{00}\to B^0$.

The natural isomorphism $H^1(F_b,\mathbb{R})\simeq T_bB$ allows us to map elements of $H^1(F_b)$ (resp. $H^*(F_b)=\Lambda^*H^1(F_b)$) to vector fields (resp. poly vector fields) on $X^{\vee 0}$. Namely, denoting by $(z_1,\dots,z_n)$ the local coordinates on $X^{\vee 0}$ induced by a choice of basis $(\gamma_1,\dots,\gamma_n)$ of $H_1(F_b)$, and by $(\gamma_1^*,\dots,\gamma_n^*)$ the dual basis of $H^1(F_b)$, we map each basis element γ_j^* to the vector field $\partial_{\log z_j}=z_j\partial_{z_j}$, and extend this map to $H^*(F_b)\,\hat{\otimes}\,\mathcal{O}_{an}$ by setting

$$z^\gamma\,\gamma_{j_1}^*\wedge\cdots\wedge\gamma_{j_k}^*\mapsto z^\gamma\,\partial_{\log z_{j_1}}\wedge\cdots\wedge\partial_{\log z_{j_k}}. \tag{3.10}$$

Combining this with pullback of cochains under the projection $\pi^\vee : X^{\vee 0}\to B^0$, we obtain a (bigraded) map

$$C^*(B^0,H^*(F_b)\,\hat{\otimes}\,\mathcal{O}_{an})\to C^*(X^{\vee 0},\Lambda^*T_{X^{\vee 0}}). \tag{3.11}$$

If we use the Morse-theoretic model of §3.1.2 for an adapted Morse function in the sense of §3.1.3, then by Remark 3.3 the Morse cochains on the left-hand side of (3.11) can be recast as Čech cochains for a certain polyhedral cover of B^0 (by the stars $\mathcal{U}_v$ of the vertices of the simplicial decomposition $\mathcal{P}$). The right-hand side of (3.11) should then be interpreted as Čech cochains for a cover of $X^{\vee 0}$ by affinoid domains approximating the preimages $(\pi^\vee)^{-1}(\mathcal{U}_v)$, $v \in \mathrm{vert}(\mathcal{P})$.

On the other hand, if we assume the existence of a model for family Floer theory in which $\mathfrak{m}_0$ is expressed as a differential form on B^0 with values in $H^*(F_b)\,\hat{\otimes}\,\mathcal{O}_{an}$, then the right-hand side of (3.11) should be interpreted in terms of *tropical differential forms* (also known as *superforms*) on $X^{\vee 0}$ (see e.g. [CLD12, Jell22]). Choosing a basis of $H_1(F_b)$ as above, and denoting by $(x_1,\dots,x_n)$ and $(z_1,\dots,z_n)$ the corresponding local coordinates on B^0 and on $X^{\vee 0}$ (with $\mathrm{val}(z_j)=x_j$), we define

$$(\pi^\vee)^*(dx_j) = d'' \log|z_j|,$$

a superform of type $(0,1)$; and similarly for exterior products. By definition, the pullback of differential forms intertwines the de Rham differential d on $\Omega^*(B^0)$ and the tropical Dolbeault differential d'' on $\Omega^{0,*}(X^{\vee 0})$. Consequently, the map (3.11) also intertwines the de Rham differential d on $\Omega^*(B^0, H^*(F_b)\,\hat{\otimes}\,\mathcal{O}_{an})$ and the tropical Dolbeault differential d'' on $\Omega^{0,*}(X^{\vee 0}, \Lambda^* T_{X^{\vee 0}})$.

The key to the proof of Proposition 1.6 is the following lemma:

Lemma 3.17. *The map* (3.11) *intertwines the bracket* $\{\cdot,\cdot\}$ *on* $C^*(B^0; H^*(F_b)\,\hat{\otimes}\,\mathcal{O}_{an})$ *defined by* (1.7) *and the negative of the Schouten-Nijenhuis bracket* $-[\cdot,\cdot]$ *on* $C^*(X^{\vee 0}, \Lambda^* T_{X^{\vee 0}})$.

Proof. Since the map (3.11) is compatible with the cup-product of cochains, it suffices to compare the brackets on $H^*(F_b)\,\hat{\otimes}\,\mathcal{O}_{an}$ and on $\Lambda^* T_{X^{\vee 0}}$.

Recall that the Schouten-Nijenhuis bracket is a bracket of degree -1 on polyvector fields, characterized by the following properties [Mar97]: given a smooth function f, vector fields X, Y, and polyvector fields P, Q, R of degrees p, q, r,

(1) $[X,f] = L_X f = \iota_{df}(X)$;
(2) $[X,Y] = L_X Y$;
(3) $[P,Q] = -(-1)^{(p-1)(q-1)}[Q,P]$;
(4) $[P, Q\wedge R] = [P,Q]\wedge R + (-1)^{(p-1)q} Q\wedge [P,R]$;
(5) $[P\wedge R, Q] = P\wedge [R,Q] + (-1)^{(q-1)r}[P,Q]\wedge R$.

Assume now that the polyvector fields P and Q have constant components in some local coordinate system, and let f, g be two smooth functions. Then properties (2), (4) and (5) imply that $[P,Q]=0$, whereas (1) and (5) imply that $[P,g] = (-1)^{p-1}\iota_{dg}(P)$. Hence, by (4) we have $[P, gQ] = (-1)^{p-1}\iota_{dg}(P)\wedge Q$. Meanwhile, (3) and (4) imply that $[f, gQ] = g[f,Q] = -g\iota_{df}(Q)$. Finally, using (5) once more we arrive at

$$\begin{aligned}[fP, gQ] &= (-1)^{p-1} f\iota_{dg}(P)\wedge Q - (-1)^{(q-1)p} g\iota_{df}(Q)\wedge P \\ &= (-1)^{p-1} f\,\iota_{dg}(P)\wedge Q - g\,P\wedge \iota_{df}(Q).\end{aligned} \tag{3.12}$$

Recall that, for $\gamma \in H_1(F_b)$, $\alpha, \alpha' \in H^*(F_b)$, we define

$$\{z^\gamma\,\alpha, z^{\gamma'}\alpha'\} = z^{\gamma+\gamma'}\big(\alpha\wedge(\iota_\gamma \alpha') + (-1)^{|\alpha|}(\iota_{\gamma'}\alpha)\wedge\alpha'\big). \tag{3.13}$$

Denote by $V, V' \in \Lambda^* T_{X^{\vee 0}}$ the images of α, α' under (3.10), which are polyvector fields with constant coefficients in terms of the basis formed by exterior products of $\partial_{\log z_i}$. By (3.12),

$$-[z^\gamma V, z^{\gamma'} V'] = z^{\gamma'} V \wedge \iota_{d(z^\gamma)}(V') + (-1)^{|\alpha|} z^\gamma \iota_{d(z^{\gamma'})}(V) \wedge V'. \tag{3.14}$$

Thus, in order to complete our comparison of the two brackets it suffices to show that (3.10) maps $z^\gamma \iota_\gamma \alpha'$ to $\iota_{d(z^\gamma)}(V')$, and similarly for the other interior product appearing in (3.13).

Distributing the interior products into the expressions of α' and V' in the chosen bases, it is in fact sufficient to compare $z^\gamma \iota_\gamma \alpha'$ to $\iota_{d(z^\gamma)}(V')$ in the specific case where $\alpha' = \gamma_j^*$ is an element of the chosen basis of $H^1(F_b)$, and $V' = \partial_{\log z_j}$. Expressing γ in the chosen basis of $H_1(F_b)$ as $\gamma = a_1\gamma_1 + \cdots + a_n\gamma_n$, we find that

$$z^\gamma \iota_\gamma(\gamma_j^*) = a_j z^\gamma = \partial_{\log z_j}(z^\gamma) = \iota_{d(z^\gamma)}(\partial_{\log z_j}),$$

which completes the proof. □

Proof of Proposition 1.6. Recall that $\mathbb{W}$ (resp. $W^{(i)}$) is by definition the image of $\mathfrak{m}_0$ (resp. its components $\alpha^{(i)}$) under the map (3.11). The compatibility of (3.11) with the differentials implies that it maps $\delta\mathfrak{m}_0$ to $\delta\mathbb{W}$; while Lemma 3.17 implies that it maps $\{\mathfrak{m}_0, \mathfrak{m}_0\}$ to $-[\mathbb{W}, \mathbb{W}]$. The master equation (1.8) for $\mathfrak{m}_0$ thus maps to the equation (1.10) for $\mathbb{W}$. □

We also give a derivation of equations (1.11)–(1.13) for completeness. Recall that the Schouten-Nijenhuis bracket satisfies the Jacobi identity

$$[[P,Q],R] = [P,[Q,R]] - (-1)^{(p-1)(q-1)}[Q,[P,R]]$$

(where $p = \deg P$, $q = \deg Q$). Since $\mathbb{W}$ is even, this yields

$$[[\mathbb{W},\mathbb{W}],\cdot] = 2[\mathbb{W},[\mathbb{W},\cdot]].$$

Moreover, the differential δ on cochains does not interact with the Schouten-Nijenhuis bracket (this is manifest in the case of Čech cochains, and for tropical differential forms it follows from the fact that δ is the Dolbeault differential d'' on $(0,*)$-forms while the Schouten-Nijenhuis bracket involves differentiation along analytic vector fields). Hence,

$$\delta([\mathbb{W},\cdot]) = [\delta\mathbb{W},\cdot] - [\mathbb{W},\delta(\cdot)].$$

Thus, assuming (1.10), we have

$$(\delta + [\mathbb{W},\cdot])^2 = \delta[\mathbb{W},\cdot] + [\mathbb{W},\delta(\cdot)] + [\mathbb{W},[\mathbb{W},\cdot]] = [\delta\mathbb{W},\cdot] + \frac{1}{2}[[\mathbb{W},\mathbb{W}],\cdot] = 0. \tag{3.15}$$

Writing $\mathbb{W} = W^{(0)} + W^{(1)} + \dots$ with $W^{(i)} \in C^i(X^{\vee 0}, \Lambda^i T_{X^{\vee 0}})$, the component of (1.10) in bidegree $(1,0)$ (i.e., in $C^1(X^{\vee 0}, \mathcal{O}_{X^{\vee 0}})$) is

$$\delta W^{(0)} + [W^{(1)}, W^{(0)}] = 0,$$

which gives (1.11). The component of (3.15) in bidegree $(2,1)$ is

$$(\delta + [W^{(1)},\cdot])^2 + [W^{(2)},[W^{(0)},\cdot]] + [W^{(0)},[W^{(2)},\cdot]] = 0,$$

which can be rewritten using the Jacobi identity as

$$(\delta + [W^{(1)},\cdot])^2 + [[W^{(2)},W^{(0)}],\cdot] = 0.$$

Since $[W^{(2)}, W^{(0)}] = -\iota_{dW^{(0)}}(W^{(2)})$, this yields (1.12). Next, the component of (1.10) in bidegree $(3,2)$ is

$$\delta W^{(2)} + [W^{(1)}, W^{(2)}] + [W^{(0)}, W^{(3)}] = 0,$$

which yields (1.13); and so on.

Remark 3.18. The components of the Čech cochain groups $C^*(B^0; H^*(F_b)\hat{\otimes}\mathcal{O}_{an})$ obtained from a simplicial decomposition of B^0 are certain completions of $H^*(F_b)\otimes\mathbb{K}[H_1(F_b)]$, which can be viewed as the symplectic cohomologies (in the classical limit, before any instanton corrections) of the local pieces of the corresponding decomposition of X^{00}. (See also Groman and Varolgunes' work introducing the relative symplectic cohomology sheaf of an SYZ fibration [GV22].) In this sense, the map (3.11) can be viewed as a local version of the expected isomorphism between the Hochschild cohomologies of the Fukaya category of X^{00} (without instanton corrections) and the derived category of the uncorrected mirror $X^{\vee 0}$, with the latter recast in terms of polyvector fields via the Hochschild-Kostant-Rosenberg isomorphism. In this language, Lemma 3.17 expresses the fact that homological mirror symmetry intertwines the BV-structures on these local Hochschild cohomologies.

The claim that the element $\mathbb{W} \in C^*(X^{\vee 0}, \Lambda^* T_{X^{\vee 0}})$ obtained by applying (3.11) to $\mathfrak{m}_0$ represents the instanton corrections to be applied to the geometry of the uncorrected mirror $X^{\vee 0}$ also follows naturally from this perspective. Namely, we expect that the Fukaya category of X can be recovered from the local relative wrapped Fukaya categories of the pieces of X^{00} by a pullback diagram. Prior to any instanton corrections, the "classical limit" of this pullback diagram (using the "classical" local wrapped categories rather than their relative deformations inside X) matches the description of the derived category of the uncorrected mirror $X^{\vee 0}$ in terms of local pieces. The components of the Floer-theoretic obstruction $\mathfrak{m}_0 \in C^*(B^0; H^*(F_b)\hat{\otimes}\mathcal{O}_{an})$ describe the deformations of the classical local wrapped categories and of their gluing data needed to account for holomorphic discs in X and arrive at the relative wrapped Fukaya categories and the diagram via which they recover the Fukaya category of X. Thus, the corresponding element $\mathbb{W} \in C^*(X^{\vee 0}, \Lambda^* T_{X^{\vee 0}})$ should similarly be used to deform the local pieces of the uncorrected mirror $X^{\vee 0}$ and their gluing data in order to arrive at the correct mirror.

(In the Calabi-Yau setting, the above ideas are also likely related to Chan, Leung and Ma's work on the construction of mirror spaces by using Maurer-Cartan elements to deform gluings of BV algebras [CLM25].)

References

[Abo14] M. Abouzaid, *Family Floer cohomology and mirror symmetry*, Proc. 2014 ICM (Seoul), Vol. II, pp. 813–836.

[Abo17] M. Abouzaid, *The family Floer functor is faithful*, J. Eur. Math. Soc. **19** (2017), 2139–2217.

[AA24] M. Abouzaid, D. Auroux, *Homological mirror symmetry for hypersurfaces in* $(\mathbb{C}^*)^n$, Geom. Topol. **28** (2024), 2825–2914.

[AAK16] M. Abouzaid, D. Auroux, L. Katzarkov, *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*, Publ. Math. IHÉS **123** (2016), 199–282.

[AGV24] M. Abouzaid, Y. Groman, U. Varolgunes, *Framed* E_2 *structures in Floer theory*, Adv. Math. **450** (2024), 109755.

[Aur07] D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. **1** (2007), 51–91.

[Aur09] D. Auroux, *Special Lagrangian fibrations, wall-crossing, and mirror symmetry*, Surveys in Differential Geometry, Vol. 13, H.D. Cao and S.T. Yau Eds., International Press, 2009, 1–47.

[Aur15] D. Auroux, *Infinitely many monotone Lagrangian tori in* $\mathbb{R}^6$, Inventiones Math. **201** (2015), 909–924.

[Beh97] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Inventiones Math. **127** (1997), 601–617.

[BF97] K. Behrend, B. Fantechi, *The intrinsic normal cone*, Inventiones Math. **128** (1997), 45–88.

[CLD12] A. Chambert-Loir, A. Ducros, *Formes différentielles réelles et courants sur les espaces de Berkovich*, preprint, arXiv:1204.6277.

[Chan11] K. Chan, *A formula equating open and closed Gromov-Witten invariants and its applications to mirror symmetry*, Pacific J. Math. **254** (2011), 275–293.

[CLL12] K. Chan, S.-C. Lau, N. C. Leung, *SYZ mirror symmetry for toric Calabi-Yau manifolds*, J. Differential Geom. **90** (2012), 177–250.

[CL14] K. Chan, S.-C. Lau, *Open Gromov–Witten invariants and superpotentials for semi-Fano toric surfaces*, Int. Math. Res. Not. IMRN **2014**, no. 14, 3759–3789.

[CLLT17] K. Chan, S.-C. Lau, N. C. Leung, H.-H. Tseng, *Open Gromov-Witten invariants, mirror maps, and Seidel representations for toric manifolds*, Duke Math. J. **166** (2017), 1405–1462.

[CLM25] K. Chan, N. C. Leung, Z. M. Ma, *Smoothing, scattering, and a conjecture of Fukaya*, Forum Math. Pi **13** (2025), art. no. e7, 65 pp.

[CW22] F. Charest, C. Woodward, *Floer cohomology and flips*, Mem. Amer. Math. Soc. **279** (2022), no. 1372, 1–166.

[CS99] M. Chas, D. Sullivan, *String topology*, preprint, arXiv:math/9911159.

[CO06] C.-H. Cho, Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), 773–814.

[CL06] O. Cornea, F. Lalonde, *Cluster homology: an overview of the construction and results*, Electron. Res. Announc. Amer. Math. Soc. **12** (2006), 1–12.

[Fuk02] K. Fukaya, *Floer homology for families – a progress report*, Integrable Systems, Topology, and Physics (Tokyo, 2000), Amer. Math. Soc., 2002, pp. 33–68.

[Fuk06] K. Fukaya, *Application of Floer homology of Lagrangian submanifolds to symplectic topology*, Morse theoretic methods in nonlinear analysis and in symplectic topology, NATO Sci. Ser. II Math. Phys. Chem., **217**, Springer, 2006, 231–276.

[Fuk10] K. Fukaya, *Cyclic symmetry and adic convergence in Lagrangian Floer theory*, Kyoto J. Math. **50** (2010),521–590.

[FOOO09] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction I and II*, AMS/IP Studies in Advanced Math., vol. 46, Amer. Math. Soc. and International Press, 2009.

[FOOO10] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, *Lagrangian Floer theory on compact toric manifolds I*, Duke Math. J. **151** (2010), 23–174.

[FOOO12] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, *Toric degeneration and nondisplaceable Lagrangian tori in* $S^2 \times S^2$, Int. Math. Res. Not. IMRN **2012**, no. 13, 2942–2993.

[GHS03] T. Graber, J. Harris, J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), 57–67.

[GHK15] M. Gross, P. Hacking, S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, Publ. Math. IHÉS **122** (2015), 65–168.

[GS11] M. Gross, B. Siebert, *From real affine geometry to complex geometry*, Ann. of Math. **174** (2011), 1301–1428.

[GS22] M. Gross, B. Siebert, *The canonical wall structure and intrinsic mirror symmetry*, Inventiones Math. **229** (2022), 1101–1202.

[GV22] Y. Groman, U. Varolgunes, *Closed string mirrors of symplectic cluster manifolds*, preprint, arXiv:2211.07523.

[Hoek25] K. Hoek, *A Morse-theoretic approach to family Floer homology*, Ph.D. Thesis, Harvard University, 2025.

[Irie20] K. Irie, *Chain level loop bracket and pseudo-holomorphic discs*, J. Topol. **13** (2020), 870–938.

[Jell22] P. Jell, *Tropical cohomology with integral coefficients for analytic spaces*, Facets of algebraic geometry, vol. II, London Math. Soc. Lect. Note Ser. **473**, Cambridge Univ. Press, 2022, 26–62.
[KKP17] L. Katzarkov, M. Kontsevich, T. Pantev, *Bogomolov-Tian-Todorov theorems for Landau-Ginzburg models*, J. Differential Geom. **105** (2017), 55–117.
[KY23] S. Keel, T. Yue Yu, *The Frobenius structure theorem for affine log Calabi-Yau varieties containing a torus*, Ann. of Math. **198** (2023), 419–536.
[KS06] M. Kontsevich, Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, The unity of mathematics, Progr. Math. **244**, Birkhäuser, 2006, pp. 321–385.
[LP13] K. Lin, D. Pomerleano, *Global matrix factorizations*, Math. Res. Lett. **20** (2013), 91–106.
[Mar97] C.-M. Marle, *The Schouten-Nijenhuis bracket and interior products*, J. Geom. Physics **23** (1997), 350–359.
[McS04] D. McDuff, D. Salamon, *J-holomorphic curves in symplectic topology*, Amer. Math. Soc. Colloq. Publ. vol. 52, Amer. Math. Soc., 2004.
[PT14] R. Pandharipande, R. P. Thomas, *13/2 ways of counting curves*, Moduli spaces, London Math. Soc. Lecture Note Ser. **411**, Cambridge Univ. Press, 2014, pp. 282–333.
[SSZ25] W. Schmaltz, S. Suhr, K. Zehmisch, *Non-fillability of overtwisted contact manifolds via polyfolds*, Arnold Math. J. **11** (2025), in press (arXiv:2011.02249).
[SYZ96] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), 243–259.
[Tu14] J. Tu, *On the reconstruction problem in mirror symmetry*, Adv. Math. **256** (2014), 449–478.
[Yuan20] H. Yuan, *Family Floer program and non-archimedean SYZ mirror construction*, preprint, arXiv:2003.06106.

Harvard University, Department of Mathematics, 1 Oxford St., Cambridge MA 02138, USA
Email address: auroux@math.harvard.edu