RING STACKS CONJECTURALLY RELATED TO THE STACKS $\mathrm{BT}_n^{G,\mu}$

VLADIMIR DRINFELD

ABSTRACT. For $n \in \mathbb{N}$, we define certain ring stacks ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$ using the ring space of sheared Witt vectors. We suggest several models for the ring stacks. Motivation: there is a conjectural description of the stack of n-truncated Barsotti-Tate groups and its Shimurian analogs in terms of ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$.

CONTENTS

2020 *Mathematics Subject Classification.* 14L05, 14F30.
Key words and phrases. Barsotti-Tate group, p-divisible group, Witt vectors, ring stack, prismatization.

1. Introduction

Throughout this article, we fix a prime p.

1.1. **Conventions.** A ring R is said to be p-nilpotent if the element $p\in R$ is nilpotent. Let p-Nilp denote the category of p-nilpotent rings.

We equip $\text{p-Nilp}^{\mathrm{op}}$ with the fpqc topology. The word “stack” will mean a stack on $\text{p-Nilp}^{\mathrm{op}}$. The final object in the category of such stacks is denoted by $\mathrm{Spf}\,\mathbb{Z}_p$; this is the functor that takes each p-nilpotent ring to a one-element set.

Ind-schemes and schemes over $\mathrm{Spf}\,\mathbb{Z}_p$ are particular classes of stacks. The words “scheme over $\mathrm{Spf}\,\mathbb{Z}_p$” are understood in the *relative* sense (e.g., $\mathrm{Spf}\,\mathbb{Z}_p$ itself is a scheme over $\mathrm{Spf}\,\mathbb{Z}_p$).

W will denote the functor $R\mapsto W(R)$, where $R\in\text{p-Nilp}$. So W is a ring scheme over $\mathrm{Spf}\,\mathbb{Z}_p$. Same for W_n.

1.2. **Subject of the paper.**

1.2.1. *The subject.* We will define and discuss certain ring stacks ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$, where $n \in \mathbb{N}$ (more precisely, ${}^s\mathscr{R}_n$ is a $(\mathbb{Z}/p^n\mathbb{Z})$-algebra stack and ${}^s\mathscr{R}_n^{\oplus}$ is a stack of $\mathbb{Z}$-graded $(\mathbb{Z}/p^n\mathbb{Z})$-algebras). Our motivation for introducing these ring stacks is a conjectural relation between them and the stacks $\mathrm{BT}_n^{G,\mu}$ from [GM].

1.2.2. *The stacks* $\mathrm{BT}_n^{G,\mu}$. Let G be a smooth affine group scheme over $\mathbb{Z}/p^n\mathbb{Z}$ and

$$\mu : \mathbb{G}_m \to G$$

a cocharacter satisfying the following 1-*boundedness* condition: all weights of the action of $\mathbb{G}_m$ on $\mathrm{Lie}(G)$ are ≤ 1. Let $\mathrm{BT}_n^{G,\mu}$ be the stack defined in [GM, §9], so if $R \in$ p-Nilp then $\mathrm{BT}_n^{G,\mu}(R)$ is the groupoid of G-bundles on $R^{\mathrm{Syn}} \otimes (\mathbb{Z}/p^n\mathbb{Z})$ satisfying a certain condition, which depends on μ. Here R^{Syn} is the syntomification of R.

By Theorem D from [GM], $\mathrm{BT}_n^{G,\mu}$ is a smooth algebraic stack over $\mathrm{Spf}\,\mathbb{Z}_p$; in other words, for every $m \in \mathbb{N}$ the restriction of $\mathrm{BT}_n^{G,\mu}$ to the category of $\mathbb{Z}/p^m\mathbb{Z}$-algebras is a smooth algebraic stack over $\mathbb{Z}/p^m\mathbb{Z}$. By Theorem A from [GM], if $G = GL(d)$ then $\mathrm{BT}_n^{G,\mu}$ identifies with the stack of n-truncated Barsotti-Tate groups of height d and dimension d', where d' depends on μ.

1.2.3. *Relation between* $\mathrm{BT}_n^{G,\mu}$ *and the rings stacks* ${}^s\mathscr{R}_n, {}^s\mathscr{R}_n^{\oplus}$. Conjecture D.8.4 from [Dr25a] expresses $\mathrm{BT}_n^{G,\mu}$ in terms of the ring stacks ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$. This conjecture and some variants of it were also discussed in [Dr25b] in a rather non-technical way.

The article [Dr25a] contains only a sketch of the definition of ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$. In this paper we give the actual definition of these ring stacks and describe several models for them.

1.3. **Sketch of the definition of** ${}^s\mathscr{R}_n$ **and** ${}^s\mathscr{R}_n^{\oplus}$**.**

1.3.1. *The ideal* $\hat{W} \subset W$. For $R \in$ p-Nilp, let $\hat{W}(R)$ be the set of all $x \in W(R)$ such that all components of the Witt vector x are nilpotent and all but finitely many of them are zero. Then $\hat{W}$ is an ind-subscheme of W; moreover, $\hat{W}$ is an ideal in W preserved by the operators $F, V : W \to W$. For $n \in \mathbb{N}$ we set

$$\hat{W}^{(F^n)} := \mathrm{Ker}(F^n : \hat{W} \to \hat{W}), \quad W^{(F^n)} := \mathrm{Ker}(F^n : W \to \hat{W}).$$

1.3.2. ${}^s\mathscr{R}_n$ *via the ring space* ${}^sW$. Let W^{perf} be the projective limit of the diagram

$$\ldots \xrightarrow{F} W \xrightarrow{F} W.$$

Let

$$ {}^sW = W^{\mathrm{perf}} / \lim_{\substack{\longleftarrow \\ n}} \hat{W}^{(F^n)} = \lim_{\substack{\longleftarrow \\ n}} (W/\hat{W}^{(F^n)}), \tag{1.1}$$

where the transition maps in each of the limits equal F and the quotients are understood in the sense of fpqc sheaves on $\text{p-Nilp}^{\mathrm{op}}$. Thus ${}^sW$ is a ring space (by which we mean an fpqc sheaf of rings on $\text{p-Nilp}^{\mathrm{op}}$); it is called the *ring of sheared Witt vectors*[1]. Note

[1]The name is due to the relation between ${}^sW$ and the theory of sheared prismatization from [M1, M2] and [BKMVZ]. On the other hand, ${}^sW$ can be regared as a "decompletion" of W, see §1.4.3(ii) and the end of §3.6.1.

that W is a quotient of ${}^sW$: indeed, the map $W^{\mathrm{perf}} \to W$ is surjective, and its kernel is $\varprojlim_n W^{(F^n)} \supset \varprojlim_n \hat{W}^{(F^n)}$.

Now define the ring stack ${}^s\mathscr{R}_n$ by

$$ {}^s\mathscr{R}_n := \mathrm{Cone}({}^sW \xrightarrow{p^n} {}^sW). \tag{1.2} $$

1.3.3. *${}^sW^{\oplus}$ and ${}^s\mathscr{R}_n^{\oplus}$.* The homomorphism $F : W \to W$ induces a homomorphism

$$ F : {}^sW \to {}^sW. $$

One also has an important additive homomorphism $\tilde{V} : {}^sW \to {}^sW$; it is defined using the operator $V : W \to W$ *in a nontrivial way* (see §3.2, in which we follow [M2, BMVZ]). Applying to $({}^sW, F, \tilde{V})$ a certain general algebraic construction (which we call the *Lau equivalence*), one gets a $\mathbb{Z}$-graded ring space ${}^sW^{\oplus}$, see §4.2.1. Finally, one sets

$$ {}^s\mathscr{R}_n^{\oplus} := \mathrm{Cone}({}^sW^{\oplus} \xrightarrow{p^n} {}^sW^{\oplus}). \tag{1.3} $$

1.4. **Remarks on ${}^sW$ and $\tilde{V}$.**

1.4.1. The definition of $\tilde{V} : {}^sW \to {}^sW$ is not obvious from (1.1) because in mixed characteristic we have $FV \neq VF$. On the other hand, in characteristic p one has $FV = VF$, so the operator $\tilde{V} : {}^sW_{\mathbb{F}_p} \to {}^sW_{\mathbb{F}_p}$ is clear from (1.1) (here ${}^sW_{\mathbb{F}_p} := {}^sW \times \mathrm{Spec}\,\mathbb{F}_p$).

1.4.2. To define $\tilde{V} : {}^sW \to {}^sW$, it is convenient to replace (1.1) by the equivalent formula (3.1).

1.4.3. *Some history.* (i) I suggested the definition of ${}^sW$ while thinking about [Vo, BKMVZ] and about the stacks $\mathrm{BT}_n^{G,\mu}$. Simultaneously and independently, E. Lau introduced ${}^sW(R)$ in the case where R is a semiperfect $\mathbb{F}_p$-algebra; in this case he defined ${}^sW(R)$ to be the right-hand side of formula (A.5) from our Appendix A.

(ii) In [L14] E. Lau defined a ring $\mathbb{W}(R)$ for a certain class of p-nilpotent rings R, which he calls admissible (see §A.4.1 of our Appendix A); for a smaller class it had been defined in 2001 by Th. Zink [Zi01]. If R is admissible then $\mathbb{W}(R) = {}^sW(R)$ (see Proposition A.4.2 of Appendix A). For admissible R, the operator $\tilde{V} : \mathbb{W}(R) \to \mathbb{W}(R)$ was defined in [Zi01] assuming that $p > 2$; this assumption was removed in [L14].

1.5. **Models for ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$.** By a *model* for a ring stack we mean its realization as a Cone of a quasi-ideal. The models for ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$ provided by (1.2) and (1.3) are far from being economic. However, there are more economic models.

1.5.1. *The situation over $\mathbb{F}_p$.* Let $W_{\mathbb{F}_p} := W \times \mathrm{Spec}\,\mathbb{F}_p$, $\hat{W}_{\mathbb{F}_p} := \hat{W} \times \mathrm{Spec}\,\mathbb{F}_p$, etc. It turns out that

$$ {}^s\mathscr{R}_{n,\mathbb{F}_p} = \mathrm{Cone}(\hat{W}_{\mathbb{F}_p}^{(F^n)} \to W_{n,\mathbb{F}_p}), \tag{1.4} $$

$$ {}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus} = \mathrm{Cone}((\hat{W}_{\mathbb{F}_p}^{(F^n)})^{\oplus} \to W_{n,\mathbb{F}_p}^{\oplus}), \tag{1.5} $$

where $(\hat{W}_{\mathbb{F}_p}^{(F^n)})^{\oplus}$ is obtained by applying the Lau equivalence to the triple

$$ (\hat{W}_{\mathbb{F}_p}^{(F^n)},\ F : \hat{W}_{\mathbb{F}_p}^{(F^n)} \to \hat{W}_{\mathbb{F}_p}^{(F^n)},\ V : \hat{W}_{\mathbb{F}_p}^{(F^n)} \to \hat{W}_{\mathbb{F}_p}^{(F^n)}) $$

and $W^{\oplus}_{n,\mathbb{F}_p}$ is obtained similarly from the triple

$$(W_{n,\mathbb{F}_p},\ F: W_{n,\mathbb{F}_p} \to W_{n,\mathbb{F}_p},\ V: W_{n,\mathbb{F}_p} \to W_{n,\mathbb{F}_p}).$$

The reason why $V: \hat{W}^{(F^n)}_{\mathbb{F}_p} \to \hat{W}^{(F^n)}_{\mathbb{F}_p}$ and $F: W_{n,\mathbb{F}_p} \to W_{n,\mathbb{F}_p}$ are defined is that in characteristic p one has $FV = VF$.

1.5.2. *Mixed characteristic.* In mixed characteristic the situation is more complicated. It turns out that if $p > 2$ then similarly to (1.4), one has

$$ {}^s\mathscr{R}_n = \operatorname{Cone}(\hat{W}^{(F^n)} \to W_n), \tag{1.6}$$

and if $p = 2$ then ${}^s\mathscr{R}_n$ has a slightly more complicated realization as a quotient of W_n.

However, even for $p > 2$ the model (1.6) does not exhibit the operator $\tilde{V}: {}^s\mathscr{R}_n \to {}^s\mathscr{R}_n$. Related fact: in mixed characteristic there is no direct analog of (1.5).

1.5.3. *Other models.* (i) ${}^s\mathscr{R}_n$ has some models which are more economic than (1.2) but less economic than (1.6), see §5.3, §7, and §8.3.6. In §6 we describe a variant of the model for ${}^s\mathscr{R}_n$ from §5.3, which is self-dual up to a cohomological shift, just as the model (1.6).

(ii) The models for ${}^s\mathscr{R}_n$ from §5.3 and §6 exhibit both F and $\tilde{V}$, so they can easily be used to construct models for ${}^s\mathscr{R}^{\oplus}_n$, see §5.4 and the end of §6.1. The economic models for ${}^s\mathscr{R}_n$ from §8.3.6 do not exhibit F and $\tilde{V}$; still, one can use them to construct models for ${}^s\mathscr{R}^{\oplus}_n$ (see §9.2).

1.5.4. *The limit $n = \infty$.* One has

$$\lim_{\substack{\longleftarrow \\ n}} {}^s\mathscr{R}_n = {}^sW. \tag{1.7}$$

So one could consider the ring stacks ${}^s\mathscr{R}_n$ as primary objects and then define ${}^sW$ by (1.7).

1.6. Digression on (sheared) n-prismatization.

1.6.1. *The ring stacks $\mathscr{R}_n$.* Similarly to (1.2), let

$$\mathscr{R}_n := \operatorname{Cone}(W \xrightarrow{p^n} W). \tag{1.8}$$

One can also set $\mathscr{R}^{\oplus}_n := \operatorname{Cone}(W^{\oplus} \xrightarrow{p^n} W^{\oplus})$, where $W^{\oplus}$ is as in §4.1.7(i).

1.6.2. *Recollections on prismatization.* If X is an $\mathbb{F}_p$-scheme then its prismatization X^{Δ} is the functor

$$\text{p-Nilp} \to \text{Groupoids}, \quad A \mapsto X(\mathscr{R}_1(A)), \tag{1.9}$$

where as before, $\mathscr{R}_1 := \operatorname{Cone}(W \xrightarrow{p} W)$; the expression $X(\mathscr{R}_1(A))$ makes sense because $\mathscr{R}_1(A)$ is an animated $\mathbb{F}_p$-algebra. In particular, $(\mathbb{A}^1_{\mathbb{F}_p})^{\Delta} = \mathscr{R}_1$.

The definition of X^{Δ} for any p-adic formal scheme X can be found in [Bh, Dr24]. The idea is to deform the ring stack $\mathscr{R}_1 = \operatorname{Cone}(W \xrightarrow{p} W)$ by replacing p with ξ, where ξ is a primitive Witt vector of degree 1, which matters only up to multiplication by $W^{\times}$.

1.6.3. *Sheared prismatization.* The functor of sheared prismatization, denoted by $X \mapsto X^{\hat{\Delta}}$ is defined in [BKMVZ]; see also [M1, M2]. The idea is to use ${}^s\mathscr{R}_1$ instead of $\mathscr{R}_1$ and to require ξ to be a "strictly primitive" Witt vector of degree 1. One has $(\mathbb{A}^1_{\mathbb{F}_p})^{\hat{\Delta}} = {}^s\mathscr{R}_1$.

1.6.4. *(Sheared) n-prismatization.* For any $n \in \mathbb{N}$, one could define the functor of n-prismatization $X \mapsto X^{\Delta_n}$ and its sheared version $X \mapsto X^{\hat{\Delta}_n}$ so that for $n = 1$ one gets the functors from §1.6.2-1.6.3. To do this, replace $\mathscr{R}_1$, ${}^s\mathscr{R}_1$, $\mathbb{F}_p$ by $\mathscr{R}_n$, ${}^s\mathscr{R}_n$, $\mathbb{Z}/p^n\mathbb{Z}$, and require ξ to be (strictly) primitive of degree n.

To formulate Conjecture D.8.4 from [Dr25a], one only needs $X^{\hat{\Delta}_n}$ in the case where X is a scheme over $\mathbb{Z}/p^n\mathbb{Z}$. In this case we usually write $X({}^s\mathscr{R}_n)$ instead of $X^{\hat{\Delta}_n}$; this is the functor

$$\text{p-Nilp} \to \text{Groupoids}, \quad A \mapsto X({}^s\mathscr{R}_n(A)).$$

Let us note that if X is a scheme over $\mathbb{Z}/p^{n-1}\mathbb{Z}$ then $X^{\Delta_n} = X^{\hat{\Delta}_n} = \emptyset$ (it suffices to check this if $X = \operatorname{Spec} \mathbb{Z}/p^{n-1}\mathbb{Z}$, which is easy).

1.7. **Relation between this paper and** [BKMVZ, BMVZ]**.** (i) Most of this paper[2] is either contained in [BKMVZ, BMVZ] or is a straightforward modification of the material from [BKMVZ, BMVZ] obtained by replacing the Witt vector ξ from §1.6.3 by p^n. But unlike [BKMVZ, BMVZ], we define and discuss the graded ring space ${}^sW^{\oplus}$ and the related stacks ${}^s\mathscr{R}_n^{\oplus}$.

(ii) The work [BMVZ] develops a detailed theory of ${}^sW$. In this paper we give a relatively self-contained exposition of *the more elementary part* of this theory (e.g., unlike [BMVZ], we do not discuss cohomology with coefficients in ${}^sW$).

(iii) Let $\text{p-Nilp}_{\text{good}}$ be the full subcategory of rings $R \in \text{p-Nilp}$ such that the quotient of R by its nilradical is perfect; it is known that $\text{p-Nilp}^{\text{op}}_{\text{good}}$ is a *base* for the fpqc topology on $\text{p-Nilp}^{\text{op}}$ (see §2.1.4). According to [BMVZ], if $R \in \text{p-Nilp}_{\text{good}}$ then $H^i(\operatorname{Spec} R, {}^sW) = 0$ for $i > 0$.

Unlike this paper, the authors of [BMVZ] prefer to consider ${}^sW$ as a sheaf on $\text{p-Nilp}^{\text{op}}_{\text{good}}$ rather than on $\text{p-Nilp}^{\text{op}}$. Reason: from their point of view, if $R \notin \text{p-Nilp}_{\text{good}}$ then the natural object is $R\Gamma(\operatorname{Spec} R, {}^sW)$ rather than ${}^sW(R) = H^0(\operatorname{Spec} R, {}^sW)$.

1.8. **Organization.** In §2-3 we recall the material from [BMVZ, M1, M2] about the ring spaces $Q := W/\hat{W}$ and ${}^sW$. We also formulate Conjecture 3.7.3.

In §4 we discuss the *Lau equivalence* and use it to define the ring space ${}^sW^{\oplus}$.

In §5 we define the ring stacks ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$. Following [BKMVZ, M2], we construct a model for each of them, which is more economic than (1.2) and (1.3).

In §6 we slightly modify the models from §5. The new model for ${}^s\mathscr{R}_n$ is self-dual up to a cohomological shift, just as the model (1.6).

Following [BKMVZ, M2], we describe in §7 a model for ${}^s\mathscr{R}_n$, which represents ${}^s\mathscr{R}_n$ as a quotient of W.

In §8 we represent ${}^s\mathscr{R}_n$ as a quotient of W_n; here we follow [BKMVZ, Vo].

In §9 we discuss a class of models for ${}^s\mathscr{R}_n^{\oplus}$.

[2]Possible exceptions: §3.6-3.7, §4, §6, §9, a part of Appendix §A.

In Appendix A we describe ${}^sW(R)$ assuming that the $\mathbb{F}_p$-alegbra $A := R/pR$ is *weakly semiperfect*, i.e., $\mathrm{Fr}^n(A) = \mathrm{Fr}^{n+1}(A)$ for some $n \geq 0$.

In Appendix B we recall the notion of derived p-completeness.

1.9. **Acknowledgements.** I thank B. Bhatt, E. Lau, A. Mathew, V. Vologodsky, and M. Zhang for explaining their works, sharing their drafts, and numerous discussions.

The author's work on this project was partially supported by NSF grant DMS-2001425.

2. The ring space Q

In this section we recall some material from [BMVZ, M1]. As before, we use the notation and conventions of §1.1.

2.1. The ideal $\hat{W} \subset W$.

2.1.1. *Definition of* $\hat{W}$. For $R \in$ p-Nilp, let $\hat{W}(R)$ be the set of all $x \in W(R)$ such that all components of the Witt vector x are nilpotent and all but finitely many of them are zero. Then $\hat{W}(R)$ is an ideal in $W(R)$ preserved by F and V. Moreover, the preimage of $\hat{W}(R)$ with respect to $V : W(R) \to W(R)$ is *equal* to $\hat{W}(R)$, so

$$\mathrm{Ker}(V : W(R)/\hat{W}(R) \to W(R)/\hat{W}(R)) = 0 \tag{2.1}$$

Recall that W is an affine scheme over $\mathrm{Spf}\,\mathbb{Z}_p$. Clearly, $\hat{W}$ is an ind-subscheme of W which is ind-finite over $\mathrm{Spf}\,\mathbb{Z}_p$.

2.1.2. *Surjectivity of* F. It is known that $F : W \to W$ is surjective as a morphism of fpqc sheaves (e.g., see [Dr24, §3.4]). The same is true for $F : \hat{W} \to \hat{W}$; moreover, $F : \hat{W} \to \hat{W}$ is surjective in the fppf sense, see [BMVZ].

2.1.3. $\hat{W}$*-torsors and seminormality.* According to [Sw], a ring A is said to be seminormal if it is reduced[3] and every homomorphism $f : \mathbb{Z}[x^2, x^3] \to A$ extends to a homomorphism $\tilde{f} : \mathbb{Z}[x] \to A$. Any perfect $\mathbb{F}_p$-algebra is seminormal: in this case $\tilde{f}$ can be defined by setting $\tilde{f}(x) = f(x^p)^{1/p}$.

The quotient of $R \in$ p-Nilp by its nilradical is denoted by R_{red}; note that R_{red} is an $\mathbb{F}_p$-algebra. It is proved in [BMVZ] that

(i) if $R \in$ p-Nilp is such that R_{red} is seminormal then every $\hat{W}$-torsor on the fpqc site of $\mathrm{Spec}\, R$ is trivial (in particular, this is true if R_{red} is perfect);

(ii) for *any* $R \in$ p-Nilp, every $\hat{W}$-torsor on the fpqc site of $\mathrm{Spec}\, R$ is fppf-locally trivial.

2.1.4. *Remark.* The class of p-nilpotent rings R such that R_{red} is perfect plays a central role in [BMVZ]. Such rings form a *base* of the fpqc topology on $\text{p-Nilp}^{\mathrm{op}}$; in other words, for every $R \in$ p-Nilp there exists a faithfully flat R-algebra R' such that R'_{red} is perfect. Moreover, there exists a faithfully flat R-algebra R' such that R'/pR' is semiperfect (this is stronger than perfectness of R_{red}). The proof is reduced to the case where R is the ring of polynomials over $\mathbb{Z}/p^n\mathbb{Z}$ in the variables x_i, $i \in I$; in this case take R' to be the inductive limit of the rings R'_m, where R'_m is the ring of polynomials over $\mathbb{Z}/p^n\mathbb{Z}$ in the variables $x_i^{p^{-m}}$.

[3]In fact, reduceness follows from the other condition, see [Sta, Tag 0EUK].

2.1.5. *$\hat{W}(R)$, where R is p-complete.* An abelian group is said to be *p-complete* if it is complete and separated with respect to the p-adic topology. For a p-complete ring R, we define[4] $\hat{W}(R)$ to be the projective limit of $\hat{W}(R/p^nR)$. If R is p-complete then the projective limit of $W(R/p^nR)$ equals $W(R)$, so $\hat{W}(R) \subset W(R)$.

2.2. **The ring space Q.** By a ring space we mean an fpqc sheaf of rings on p-Nilp$^{\mathrm{op}}$.

2.2.1. *Definition.* Following [BMVZ], we set $Q := W/\hat{W}$ (quotient in the sense of fpqc sheaves or equivalently, in the sense of fppf sheaves; the equivalence follows from §2.1.3(ii)).

If $R \in$ p-Nilp is such that R_{red} is seminormal (e.g., perfect) then $Q(R) = W(R)/\hat{W}(R)$ by §2.1.3.

2.2.2. *Pieces of structure on Q.* It is clear that Q is a ring space[5]. Moreover, the maps $F, V : W \to W$ induce maps $F, V : Q \to Q$. By (2.1) and §2.1.2, one has

$$\operatorname{Ker}(Q \xrightarrow{V} Q) = 0, \tag{2.2}$$

$$\operatorname{Coker}(Q \xrightarrow{F} Q) = 0 \tag{2.3}$$

2.3. **The operator $\tilde{V} : Q \to Q$.**

2.3.1. *The story in a few words.* It turns out that if $p > 2$ then the maps $F, V : Q \to Q$ satisfy the relation $FV = VF = p$. If $p = 2$ this is false, but there is a better map[6] $\tilde{V} : Q \to Q$ such that $F\tilde{V} = \tilde{V}F = p$. The reader may prefer to disregard the case $p = 2$ and assume that $\tilde{V} = V$.

2.3.2. *The element $\bar{\mathbf{u}}$.* It is easy to see that there is a unique $\bar{\mathbf{u}} \in W(\mathbb{Z}_p)/\hat{W}(\mathbb{Z}_p)$ such that

$$V(\bar{\mathbf{u}}) = p; \tag{2.4}$$

moreover, $\bar{\mathbf{u}}$ is invertible. According to [BMVZ],

$$\bar{\mathbf{u}} = 1 \Leftrightarrow p > 2. \tag{2.5}$$

Applying F to (2.4), we get

$$p\bar{\mathbf{u}} = p. \tag{2.6}$$

2.3.3. *The operator $\tilde{V} : Q \to Q$.* Define $\tilde{V} : Q \to Q$ by $\tilde{V} := V \circ \bar{\mathbf{u}}$; in other words, for $R \in$ p-Nilp and $x \in Q(R)$, we have $\tilde{V}(x) := V(ux)$. By (2.4)-(2.6), we have

$$F\tilde{V} = \tilde{V}F = p. \tag{2.7}$$

Of course, $\tilde{V}$ is additive and satisfies the usual identity

$$\tilde{V}(x)y = \tilde{V}(xF(y)). \tag{2.8}$$

By (2.2), we have

$$\operatorname{Ker}(Q \xrightarrow{\tilde{V}} Q) = 0. \tag{2.9}$$

[4]In practice, we will apply this definition to p-complete rings with bounded p^∞-torsion.

[5]Moreover, as explained in [BMVZ], Q is a δ-ring space, and there is a certain compatibility between δ and V.

[6]In a slightly different context, this map was introduced by E. Lau [L14]; then it was rediscovered in [BMVZ].

Proposition 2.3.4. *(i) The map*

$$F : Q/\tilde{V}(Q) \to Q/\tilde{V}(Q) \tag{2.10}$$

is an isomorphism.

(ii) Moreover, $Q/\tilde{V}(Q)$ *is a sheaf of perfect* $\mathbb{F}_p$*-algebras, and* (2.10) *is its Frobenius endomorphism.*

The proof given below is taken from [M1] and [BMVZ]. Moreover, it is proved in [M1] that for any $R \in$ p-Nilp the ring $(Q/\tilde{V}(Q))(R)$ is the colimit perfection of R_{red} (i.e., the colimit of the diagram $R_{\text{red}} \xrightarrow{F} R_{\text{red}} \xrightarrow{F} \ldots$).

Proof of Proposition 2.3.4. $Q := W/\hat{W}$, so $Q/\tilde{V}(Q) = W/(V(W) + \hat{W})$. Therefore $Q/\tilde{V}(Q)$ is the fpqc-sheafification of the presheaf

$$R \mapsto R_{\text{red}}, \quad R \in \text{p-Nilp}\,.$$

Moreover, the map (2.10) comes from $\text{Fr} : R_{\text{red}} \to R_{\text{red}}$. This proves (ii) and injectivity of (2.10). Surjectivity of (2.10) follows from (2.3). □

Corollary 2.3.5. *The complex corresponding to the bicomplex*

$$\begin{array}{ccc} Q & \xrightarrow{\tilde{V}} & Q \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ Q & \xrightarrow{\tilde{V}} & Q \end{array}$$

is acyclic.

Proof. Follows from Proposition 2.3.4(i) and formula (2.9). □

For $i \in \mathbb{N}$ let $Q^{(F^i)} := \text{Ker}(Q \xrightarrow{F^i} Q)$.

Corollary 2.3.6. *For every* $i \in \mathbb{N}$ *the map* $\tilde{V} : Q^{(F^i)} \to Q^{(F^i)}$ *is an isomorphism.*

Proof. Follows from Corollary 2.3.5 and formula (2.3). □

3. The ring space ${}^sW$

In this section we recall some material from [BMVZ, M2] and formulate Conjecture 3.7.3.

3.1. **Definition of ${}^sW$.**

3.1.1. *Definition.* Just as in §2, let $Q := W/\hat{W}$. Let

$${}^sW := W \times_Q Q^{\text{perf}}, \tag{3.1}$$

where Q^{perf} is the projective limit of the diagram

$$\ldots \xrightarrow{F} Q \xrightarrow{F} Q. \tag{3.2}$$

The ring space ${}^sW$ is called the *ring of sheared Witt vectors.*

If $R \in$ p-Nilp is such that R_{red} is perfect then $Q(R) = W(R)/\hat{W}(R)$ by §2.2.1, so ${}^sW(R)$ is rather explicit. In Appendix A we give an even more explicit description of ${}^sW(R)$ if R satisfies a certain condition which is stronger than perfectness of R_{red}.

3.1.2. *Reformulation.* Let W^{perf} be the projective limit of the diagram

$$\ldots \xrightarrow{F} W \xrightarrow{F} W.$$

The canonical homomorphisms $W^{\text{perf}} \to W$ and $W^{\text{perf}} \to Q^{\text{perf}}$ define a homomorphism

$$W^{\text{perf}} \to {}^sW. \tag{3.3}$$

Surjectivity of $F : \hat{W} \to \hat{W}$ (see §2.1.2) implies that (3.3) is surjective as a map of fpqc sheaves, so

$${}^sW = W^{\text{perf}} / \varprojlim_n \hat{W}^{(F^n)} = \varprojlim_n (W/\hat{W}^{(F^n)}), \tag{3.4}$$

where $\hat{W}^{(F^n)} := \operatorname{Ker}(F^n : \hat{W} \to \hat{W})$, the transition maps in each of the limits equal F, and the quotients are understood in the fpqc sense. Note that W^{perf} is an affine scheme over $\operatorname{Spf} \mathbb{Z}_p$ and $\varprojlim_n \hat{W}^{(F^n)}$ is an ind-subscheme of W^{perf} which is also an ideal.

3.1.3. *Pieces of structure on ${}^sW$.* Recall that a ring space is an fpqc sheaf of rings on p-Nilp^{op}. Both (3.1) and (3.4) exhibit ${}^sW$ as a ring space equipped with an endomorphism F. There is also an important additive map $\tilde{V} : {}^sW \to {}^sW$, see §3.2 below. Moreover, ${}^sW$ as a δ-ring space, see §3.4 for more details.

3.1.4. *Some exact sequences.* The map $W \to Q$ is clearly surjective. The map $Q^{\text{perf}} \to Q$ is surjective by (2.3). So we get exact sequences of fpqc sheaves

$$0 \to \hat{W} \to {}^sW \to Q^{\text{perf}} \to 0, \tag{3.5}$$

$$0 \to T_F(Q) \to {}^sW \to W \to 0. \tag{3.6}$$

Here $T_F(Q)$ is the "F-adic Tate module" of Q, i.e.,

$$T_F(Q) := \varprojlim(\ldots \xrightarrow{F} Q^{(F^2)} \xrightarrow{F} Q^{(F)}), \quad \text{where } Q^{(F^n)} := \operatorname{Ker}(F^n : Q \to Q). \tag{3.7}$$

We also have a rather tautological exact sequence

$$0 \to \hat{W}^{\text{perf}} \to W \times_Q W^{\text{perf}} \to {}^sW \to 0, \quad \hat{W}^{\text{perf}} := \varprojlim(\ldots \xrightarrow{F} \hat{W} \xrightarrow{F} \hat{W}), \tag{3.8}$$

where the map $\hat{W}^{\text{perf}} \to W \times_Q W^{\text{perf}}$ is given by $x \mapsto (0, x)$ and the map

$$W \times_Q W^{\text{perf}} \to W \times_Q Q^{\text{perf}} =: {}^sW$$

comes from the canonical map $W^{\text{perf}} \to Q^{\text{perf}}$. In Appendix A we will use (3.8) to give an explicit description of ${}^sW(R)$ for a certain class of rings $R \in$ p-Nilp.

Note that the epimorphism $W \times_Q W^{\text{perf}} \twoheadrightarrow W^{\text{perf}}$ has a canonical splitting

$$W^{\text{perf}} \hookrightarrow W \times_Q W^{\text{perf}}, \quad x \mapsto (\pi(x), x),$$

where $\pi : W^{\text{perf}} \to W$ is the canonical map. So $W \times_Q W^{\text{perf}}$ identifies with the semidirect product $W^{\text{perf}} \ltimes \hat{W}$, where W^{perf} acts on $\hat{W}$ via π. After this identification, (3.8) becomes the exact sequence

$$0 \to \hat{W}^{\text{perf}} \xrightarrow{(1,-1)} W^{\text{perf}} \ltimes \hat{W} \to {}^sW \to 0,$$

where the restriction of the map $W^{\mathrm{perf}} \ltimes \hat{W} \to {}^sW$ to W^{perf} is the epimorphism (3.3) and the restriction to $\hat{W}$ is the embedding $\hat{W} \hookrightarrow {}^sW$ from (3.5).

3.1.5. *The ring* ${}^sW(\mathbb{Z}_p)$. Similarly to §2.1.5, we define ${}^sW(\mathbb{Z}_p)$ to be the projective limit of the rings ${}^sW(\mathbb{Z}/p^n\mathbb{Z})$. The ring homomorphism $\mathbb{Z}_p \to W(\mathbb{Z}_p)$ induces ring homomorphisms $\mathbb{Z}_p \to Q(\mathbb{Z}_p)$, $\mathbb{Z}_p \to Q^{\mathrm{perf}}(\mathbb{Z}_p)$, and finally, $\mathbb{Z}_p \to {}^sW(\mathbb{Z}_p)$. Combining the latter with the map $\hat{W} \hookrightarrow {}^sW$ from (3.5), we get a ring homomorphism $\mathbb{Z}_p \oplus \hat{W}(\mathbb{Z}_p) \to {}^sW(\mathbb{Z}_p)$.

Proposition 3.1.6. *This map $\mathbb{Z}_p \oplus \hat{W}(\mathbb{Z}_p) \to {}^sW(\mathbb{Z}_p)$ is an isomorphism.*

Proof. Apply Proposition A.4.2 of Appendix A to the ring $\mathbb{Z}/p^n\mathbb{Z}$. □

3.2. The map $\tilde{V} : {}^sW \to {}^sW$.

3.2.1. *The Witt vector* $\mathbf{u}$. In §2.3.2 we defined an invertible element $\bar{\mathbf{u}} \in W(\mathbb{Z}_p)/\hat{W}(\mathbb{Z}_p)$. Once and for all, *we fix an element* $\mathbf{u} \in W(\mathbb{Z}_p)$ *such that* $\mathbf{u} \mapsto \bar{\mathbf{u}}$; then $\mathbf{u}$ is automatically invertible. The element $\mathbf{u}$ is unique up to multiplication by an element of $1+\hat{W}(\mathbb{Z}_p) \subset W(\mathbb{Z}_p)$.

A particular choice of $\mathbf{u}$ does not really matter for us: it is straightforward to pass from one choice to another, see §4.2.2. For any p, one can set $\mathbf{u} := V^{-1}(p-[p])$, where $[p] \in W(\mathbb{Z}_p)$ is the Teichmüller element. If $p > 2$ then $\bar{\mathbf{u}} = 1$, so one can set $\mathbf{u} = 1$. According to [BMVZ], in the case $p = 2$ one can take $\mathbf{u}$ to be the Teichmüller element $[-1]$.

For any choice of $\mathbf{u}$, one has

$$\mathbf{u} \in \mathrm{Ker}(W(\mathbb{Z}_p)^\times \to W(\mathbb{F}_p)^\times) \tag{3.9}$$

because $V\mathbf{u} - p \in \hat{W}(\mathbb{Z}_p)$ by the definition of $\mathbf{u}$.

3.2.2. *The map $\tilde{V} : W \to W$.* Similarly to §2.3.3, define an additive map $\tilde{V} : W \to W$ by

$$\tilde{V} := V \circ \mathbf{u}; \tag{3.10}$$

in other words, $\tilde{V}(x) := V(\mathbf{u}x)$. Then

$$\tilde{V}F = V(\mathbf{u}) = \tilde{V}(1), \quad F\tilde{V} = \mathbf{p}, \text{ where } \mathbf{p} := p\mathbf{u}. \tag{3.11}$$

By (2.4) and (2.6), we have

$$V(\mathbf{u}) \in p + \hat{W}(\mathbb{Z}_p), \quad \mathbf{p} \in p + \hat{W}(\mathbb{Z}_p). \tag{3.12}$$

3.2.3. *The map $\tilde{V} : Q^{\mathrm{perf}} \to Q^{\mathrm{perf}}$.* The map $\tilde{V} : Q \to Q$ from §2.3.3 commutes with $F : Q \to Q$. So $\tilde{V}$ acts on the projective system (3.2). Therefore we get an additive map $\tilde{V} : Q^{\mathrm{perf}} \to Q^{\mathrm{perf}}$.

3.2.4. *The map $\tilde{V} : {}^sW \to {}^sW$.* Recall that ${}^sW := W \times_Q Q^{\mathrm{perf}}$. So combining the maps $\tilde{V} : W \to W$ and $\tilde{V} : Q^{\mathrm{perf}} \to Q^{\mathrm{perf}}$ from §3.2.2-3.2.3, we get an additive map $\tilde{V} : {}^sW \to {}^sW$. The equalities (3.11) still hold; note that $V(\mathbf{u}), \mathbf{p} \in {}^sW(\mathbb{Z}_p)$ by (3.12) and §3.1.5, so $V(\mathbf{u})$ and $\mathbf{p}$ make sense as additive endomorphisms of ${}^sW$.

The maps $\tilde{V}, F : {}^sW \to {}^sW$ satisfy the usual identity (2.8). This is also true for the maps $\tilde{V}, F : W \to W$ and $\tilde{V}, F : Q^{\mathrm{perf}} \to Q^{\mathrm{perf}}$.

Lemma 3.2.5. *For every $n \in \mathbb{N}$ one has exact sequences*

$$0 \to {}^sW \xrightarrow{\tilde{V}^n} {}^sW \to W_n \to 0, \tag{3.13}$$

$$0 \to \hat{W}^{(F^n)} \to {}^sW \xrightarrow{F^n} {}^sW \to 0, \tag{3.14}$$

in which the maps ${}^sW \to W_n$ and $\hat{W}^{(F^n)} \to {}^sW$ come from the canonical maps ${}^sW \twoheadrightarrow W$ and $\hat{W} \hookrightarrow {}^sW$ from (3.5)-(3.6).

Proof. Exactness of (3.13) follows from (3.6) and the fact that $\tilde{V} : T_F(Q) \to T_F(Q)$ is an isomorphism by Corollary 2.3.6. Exactness of (3.14) follows from (3.5) and the fact that $F : Q^{\mathrm{perf}} \to Q^{\mathrm{perf}}$ is an isomorphism (by the definition of Q^{perf}); it also follows directly from (3.4). ∎

3.2.6. *Remarks.* (i) By (3.11), for every $n \in \mathbb{N}$ one has

$$F^n\tilde{V}^n = p^n\mathbf{u}_n, \quad \text{where } \mathbf{u}_n := \prod_{i=0}^{n-1} F^i(\mathbf{u}).$$

(ii) As said in §3.2.1, one can set $\mathbf{u}$ to be 1 if $p > 2$ and $[-1]$ if $p = 2$. For this choice of $\mathbf{u}$ one has $F(\mathbf{u}) = 1$ and $\mathbf{u}_n = \mathbf{u}$. So for *any* choice of $\mathbf{u}$ one has $\mathbf{u}_n/\mathbf{u} \in 1 + \hat{W}(\mathbb{Z}_p)$.

3.3. **The operators $F, \tilde{V}$ on $W \times_Q W^{\mathrm{perf}}$.** In this subsection (which can be skipped by the reader) we explain a way to think about $\tilde{V} : {}^sW \to {}^sW$ (see §3.3.4).

3.3.1. The map $F : W^{\mathrm{perf}} \to W^{\mathrm{perf}}$ is clear; it is invertible. We *define* $\tilde{V} : W^{\mathrm{perf}} \to W^{\mathrm{perf}}$ by $\tilde{V} := pF^{-1}$. So the map $W^{\mathrm{perf}} \to W$ is *not* $\tilde{V}$-equivariant.

On the other hand, the map $W^{\mathrm{perf}} \to Q^{\mathrm{perf}}$ is $\tilde{V}$-equivariant: indeed, the map

$$\tilde{V} : Q^{\mathrm{perf}} \to Q^{\mathrm{perf}}$$

equals pF^{-1} by (2.7). So the composite map $W^{\mathrm{perf}} \to Q^{\mathrm{perf}} \to Q$ is $\tilde{V}$-equivariant.

3.3.2. Combining the operators $F, \tilde{V}$ from §3.3.1 with the operators $F, \tilde{V}$ on W, we get a ring homomorphism

$$F : W \times_Q W^{\mathrm{perf}} \to W \times_Q W^{\mathrm{perf}}$$

and an additive homomorphism

$$\tilde{V} : W \times_Q W^{\mathrm{perf}} \to W \times_Q W^{\mathrm{perf}};$$

these operators satisfy the usual relation (2.8). Moreover, $F\tilde{V} = \mathbf{p}'$, where

$$\mathbf{p}' := (\mathbf{p}, p) \in (W \times_Q W^{\mathrm{perf}})(\mathbb{Z}_p).$$

The map $W \times_Q W^{\mathrm{perf}} \to W \times_Q Q^{\mathrm{perf}} = {}^sW$ commutes with F and $\tilde{V}$.

3.3.3. *Remark.* Using the isomorphism $W^{\mathrm{perf}} \ltimes \hat{W} \xrightarrow{\sim} W \times_Q W^{\mathrm{perf}}$ defined at the end of §3.1.4, one can rewrite the maps $F, \tilde{V}$ from §3.3.2 as operators acting on $W^{\mathrm{perf}} \ltimes \hat{W}$, but the formula for the operator $\tilde{V}$ on $W^{\mathrm{perf}} \ltimes \hat{W}$ is a bit ugly.

3.3.4. *A way to avoid Q.* Possibly some readers would like to have a definition of ${}^sW$ and the operators $F, \tilde{V}$ on ${}^sW$ which does not use the space $Q := W/\hat{W}$. Formula (3.4) for ${}^sW$ does not involve Q, but it is inconvenient for defining $\tilde{V} : {}^sW \to {}^sW$. However, one can define ${}^sW$ by the exact sequence

$$0 \to \hat{W}^{\mathrm{perf}} \to W \times_Q W^{\mathrm{perf}} \to {}^sW \to 0$$

(which already appeared in §3.1.4, see (3.8)) and think of $W \times_Q W^{\mathrm{perf}}$ as the ind-scheme

$$\{(x, y) \in W \times W^{\mathrm{perf}} \,|\, \pi(y) - x \in \hat{W}\},$$

where $\pi : W^{\mathrm{perf}} \to W$ is the canonical map. Then the operator $\tilde{V} : {}^sW \to {}^sW$ comes from the operator $\tilde{V}$ on $W \times_Q W^{\mathrm{perf}}$, and the latter is very explicit: it takes $(x, y) \in W \times_Q W^{\mathrm{perf}}$ to $(\tilde{V}(x), pF^{-1}(y))$.

3.4. **${}^sW$ as a δ-ring space.** Recall that ${}^sW := W \times_Q Q^{\mathrm{perf}}$. Each of the spaces W, Q, Q^{perf} is a δ-ring space (this is stronger than being a ring space with a ring endomorphism F). So ${}^sW$ is a δ-ring space. The operators δ and $\tilde{V}$ acting on ${}^sW$ (or on W, Q, Q^{perf}) satisfy the identity

$$\delta(\tilde{V}(x)) = \tilde{V}(1)^{p-1} \cdot \tilde{V}(\delta(x)) + x \cdot \delta(\tilde{V}(1)).$$

More details can be found in [BMVZ], where it is assumed that the element $\mathbf{u} \in W(\mathbb{Z}_p)$ from §3.2.1 is chosen in a particular way (namely, $\mathbf{u} = 1$ if $p > 2$, $\mathbf{u} = [-1]$ if $p = 2$).

3.5. **Derived p-completeness of ${}^sW$.** The general notion of derived p-completeness is recalled in Appendix B.

Lemma 3.5.1. *${}^sW$ is derived p-complete.*

Before giving the proof from [BMVZ], let us make the following remark.

Remark 3.5.2. ${}^sW$ is a sheaf on p-Nilp$^{\mathrm{op}}$ equipped with the fpqc topology. A product of exact sequences of fpqc sheaves is exact, so by Lemma B.2.2 from Appendix B, derived p-completeness of a sheaf $\mathcal{F}$ on p-Nilp$^{\mathrm{op}}$ just means that $\mathcal{F}(R)$ is derived p-complete for each $R \in$ p-Nilp.

3.5.3. *Proof of Lemma 3.5.1.* We follow [BMVZ]. The exact sequence (3.6) shows that it suffices to prove derived p-completeness of W and $T_F(Q)$. By Remark 3.5.2, this amounts to proving derived p-completeness of $W(R)$ and $(T_F(Q))(R)$ for each $R \in$ p-Nilp$^{\mathrm{op}}$. For $W(R)$, this follows from §B.1.2(iii). By definition, $T_F(Q)$ is the projective limit of the sheaves $Q^{(F^n)}$. By (2.7), $p^n Q^{(F^n)} = 0$. So $(T_F(Q))(R)$ is derived p-complete by §B.1.2(i-ii). ∎

3.6. **The operator $1 - \tilde{V} : {}^sW \to {}^sW$.** The remaining part of §3 can be skipped by the reader.

3.6.1. *The goal.* As before, W and ${}^sW$ are considered as sheaves on p-Nilp$^{\mathrm{op}}$. The map $1 - \tilde{V} : W \to W$ is invertible: its inverse is $\sum\limits_{n=0}^{\infty} \tilde{V}^n$. So

$$\mathrm{Ker}(W \xrightarrow{1-\tilde{V}} W) = \mathrm{Coker}(W \xrightarrow{1-\tilde{V}} W) = 0. \tag{3.15}$$

On the other hand, in §3.6.4 we will show that

$$\mathrm{Ker}({}^sW \xrightarrow{1-\tilde{V}} {}^sW) = \mathbb{Z}_p(1), \tag{3.16}$$

$$\text{Coker}({}^sW \xrightarrow{1-\tilde{V}} {}^sW) = 0. \tag{3.17}$$

Let us note that $\text{Ker}({}^sW \xrightarrow{1-F} {}^sW) = \mathbb{Z}_p$, see §3.6.5 below.

Formula (3.16) shows that ${}^sW$ is *not* $\tilde{V}$*-complete* (unlike W). This is *good*: the kernel of $1-\tilde{V} : {}^sW \to {}^sW$ is quite meaningful.

Lemma 3.6.2. *One has a canonical isomorphism*

$$\text{Cone}(\hat{W} \xrightarrow{1-\tilde{V}} \hat{W}) \xrightarrow{\sim} \hat{\mathbb{G}}_m, \tag{3.18}$$

where $\hat{\mathbb{G}}_m$ *is the formal multiplicative group.*

Proof. Recall that $\tilde{V} := V \circ \mathbf{u}$, where $\mathbf{u} \in \text{Ker}(W(\mathbb{Z}_p)^\times \to W(\mathbb{F}_p)^\times)$, see (3.9). There is a unique $\beta \in \text{Ker}(W(\mathbb{Z}_p)^\times \to W(\mathbb{F}_p)^\times)$ such that $\beta/F(\beta) = \mathbf{u}$: namely,

$$\beta = \prod_{i=0}^{\infty} F^i(\mathbf{u}) \tag{3.19}$$

(the infinite product converges). The commutative diagram

$$\begin{array}{ccc} \hat{W} & \xrightarrow{1-\tilde{V}} & \hat{W} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ \hat{W} & \xrightarrow{1-V} & \hat{W} \end{array}$$

defines an isomorphism between $\text{Cone}(\hat{W} \xrightarrow{1-\tilde{V}} \hat{W})$ and the complex $\text{Cone}(\hat{W} \xrightarrow{1-V} \hat{W})$. It is clear that $\text{Ker}(\hat{W} \xrightarrow{1-V} \hat{W}) = 0$. It is well known that the map

$$\lambda : \hat{W} \to \hat{\mathbb{G}}_m, \quad \lambda(\sum_{i=0}^{\infty} V^i[x_i]) := \prod_{i=0}^{\infty} \exp(x_i + \frac{x_i^p}{p} + \frac{x_i^{p^2}}{p^2} + \ldots) \tag{3.20}$$

induces an isomorphism $\text{Coker}(\hat{W} \xrightarrow{1-V} \hat{W}) \xrightarrow{\sim} \hat{\mathbb{G}}_m$. □

Let us note that the isomorphism (3.18) constructed in the proof of Lemma 3.6.2 comes from the map

$$\tilde{\lambda} : \hat{W} \to \hat{\mathbb{G}}_m, \quad \tilde{\lambda}(x) := \lambda(\beta x), \tag{3.21}$$

where λ is given by (3.20) and $\beta \in W(\mathbb{Z}_p)^\times$ is given by (3.19).

Lemma 3.6.3. *One has canonical exact sequences*

$$0 \to \hat{\mathbb{G}}_m \to Q \xrightarrow{1-\tilde{V}} Q \to 0, \tag{3.22}$$

$$0 \to \mu_{p^n} \to Q^{(F^n)} \xrightarrow{1-\tilde{V}} Q^{(F^n)} \to 0, \tag{3.23}$$

$$0 \to \mathbb{Z}_p(1) \to T_F(Q) \xrightarrow{1-\tilde{V}} T_F(Q) \to 0. \tag{3.24}$$

Proof. $Q = W/\hat{W}$, so (3.22) follows from Lemma 3.6.2 and formula (3.15). The morphism $F : Q \to Q$ is surjective, and its restriction to $\text{Ker}(Q \xrightarrow{1-\tilde{V}} Q)$ equals $F\tilde{V} = p$. So (3.23) and (3.24) follow from (3.22). □

3.6.4. *Proof of* (3.16)-(3.17). ${}^sW$ is an extension of W by $T_F(Q)$, so (3.16) and (3.17) follow from (3.15) and the exact sequence (3.24). $\square$

3.6.5. *Remark.* The map $1-F : T_F(Q) \to T_F(Q)$ is invertible because $F : Q^{(F^n)} \to Q^{(F^n)}$ is nilpotent. So

$$\mathrm{Cone}({}^sW \xrightarrow{1-F} {}^sW) = \mathrm{Cone}(W \xrightarrow{1-F} W) = \mathbb{Z}_p[1]. \tag{3.25}$$

3.7. **An autoduality conjecture.**

3.7.1. Let $\alpha \in \mathrm{Ext}({}^sW, \mathbb{Z}_p(1))$ be the class of the extension

$$0 \to \mathbb{Z}_p(1) \to {}^sW \xrightarrow{1-\tilde{V}} {}^sW \to 0 \tag{3.26}$$

provided by formulas (3.16)-(3.17). Clearly

$$(1-\tilde{V}^*)(\alpha) = 0, \tag{3.27}$$

where $\tilde{V}^* : \mathrm{Ext}({}^sW, \mathbb{Z}_p(1)) \to \mathrm{Ext}({}^sW, \mathbb{Z}_p(1))$ is induced by $\tilde{V} : {}^sW \to {}^sW$.

3.7.2. Let $R \in$ p-Nilp. Let ${}^sW_R := {}^sW \times \operatorname{Spec} R$ (i.e., ${}^sW_R$ is the base change of ${}^sW$ to R). Let $\alpha_R \in \mathrm{Ext}({}^sW_R, \mathbb{Z}_p(1)_R)$ be the image of $\alpha \in \mathrm{Ext}({}^sW, \mathbb{Z}_p(1))$. Let

$$f_R : {}^sW(R) \to \mathrm{Ext}({}^sW_R, \mathbb{Z}_p(1)_R) \tag{3.28}$$

be the unique $W(R)$-linear map such that $f_R(1) = \alpha_R$.

Conjecture 3.7.3. *The map* (3.28) *is an isomorphsim.*

3.7.4. *Motivation of the conjecture.* In §8 we will show that if $p > 2$ then

$$\mathrm{Cone}({}^sW \xrightarrow{p^n} {}^sW) \simeq \mathrm{Cone}(\hat{W}^{(F^n)} \longrightarrow W_n), \tag{3.29}$$

see §8.3.7(i). Conjecture 3.7.3 is motivated by this formula and by Cartier duality between $\hat{W}^{(F^n)}$ and W_n.

If p is *any* prime then $\mathrm{Cone}({}^sW \xrightarrow{p^n} {}^sW)$ has a description (see §6), which is more complicated then (3.29) but still self-dual in some sense.

3.7.5. *Remarks.* (i) By Proposition 3.7.7 below, $\mathrm{Hom}({}^sW_R, \mathbb{Z}_p(1)_R) = 0$.

(ii) The following lemma shows that the map (3.28) interchanges F and $\tilde{V}$.

Lemma 3.7.6. $f_R \circ F = \tilde{V}^* \circ f_R$*, and* $f_R \circ \tilde{V} = F^* \circ f_R$.

The proof is based on (3.27).

Proof. (i) If $a \in {}^sW(R)$ then $f_R(F(a)) = F(a)^*\alpha_R = F(a)^*\tilde{V}^*\alpha_R = \tilde{V}^*(a^*\alpha_R) = \tilde{V}^*(f_R(a))$.

(ii) $f_R(\tilde{V}(a)) = \tilde{V}(a)^*\alpha_R = (\tilde{V}aF)^*\alpha_R = F^*a^*\tilde{V}^*\alpha_R = F^*(a^*\alpha_R) = F^*(f_R(a))$. $\square$

Proposition 3.7.7. $\underline{\mathrm{Hom}}({}^sW, \mathbb{Z}_p(1)) = 0$*, where* $\underline{\mathrm{Hom}}$ *stands for the sheaf of* Hom*'s on* p-Nilp$^{\mathrm{op}}$.

Proof. We will use well known facts about Cartier duality between W and $\hat{W}$ (e.g., see [Dr25a, Appendix A] and references therein).

The exact sequence (3.6) shows that it suffices to prove that

$$\underline{\mathrm{Hom}}(T_F(Q), \mathbb{Z}_p(1)) = 0, \tag{3.30}$$

$$\underline{\mathrm{Hom}}(W, \mathbb{Z}_p(1)) = 0. \tag{3.31}$$

Since $F : \hat{W} \to \hat{W}$ is surjective, $Q^{(F^n)} = W^{(F^n)}/\hat{W}^{(F^n)}$. Since $\underline{\mathrm{Hom}}(\hat{W}^{(F^n)}, \mathbb{G}_m) = W_n$ and $\underline{\mathrm{Hom}}(W^{(F^n)}, \mathbb{G}_m) = \hat{W}_n$, we get $\underline{\mathrm{Hom}}(Q^{(F^n)}, \mathbb{G}_m) = \mathrm{Ker}(\hat{W}_n \to W_n) = 0$, which implies (3.30).

Let us prove (3.31). By Cartier duality between W and $\hat{W}$, we have

$$\underline{\mathrm{Hom}}(W, \mathbb{Z}_p(1)) = \underline{\mathrm{Hom}}(\mathbb{Q}_p/\mathbb{Z}_p, \hat{W}).$$

But $\underline{\mathrm{Hom}}(\mathbb{Q}_p/\mathbb{Z}_p, \hat{W}) = 0$ because if $p^n = 0$ in R then $p^n \hat{W}_R \subset V(\hat{W}_R)$. $\square$

4. The $\mathbb{Z}$-graded ring space ${}^sW^{\oplus}$

In §3 we defined a ring space ${}^sW$ and maps $F, \tilde{V} : {}^sW \to {}^sW$. The graded ring space ${}^sW^{\oplus}$ will be obtained from the triple $({}^sW, F, \tilde{V})$ by applying a general algebraic construction descirbed in the next subsection.

4.1. **The Lau equivalence.** In this subsection we retell a part of E. Lau's paper [L21] (but not quite literally).

4.1.1. *The category* $\mathcal{C}$. Let $\mathcal{C}$ be the category of triples (A, t, u), where $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded ring and $t \in A_{-1}$, $u \in A_1$ are such that

(i) multiplication by u induces an isomorphism $A_i \xrightarrow{\sim} A_{i+1}$ for $i \geq 1$;

(ii) multiplication by t induces an isomorphism $A_i \xrightarrow{\sim} A_{i-1}$ for $i \leq 0$.

Because of (i) and (ii), $\mathcal{C}$ has an "economic" description. To formulate it, we will define a category $\mathcal{C}^{\mathrm{ec}}$ (where "ec" stands for "economic") and construct an equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{C}^{\mathrm{ec}}$.

4.1.2. *The category* $\mathcal{C}^{\mathrm{ec}}$. Let $\mathcal{C}^{\mathrm{ec}}$ be the category of diagrams

$$A_0 \underset{V}{\overset{F}{\rightleftarrows}} A_1, \tag{4.1}$$

where A_0 and A_1 are rings, F is a ring homomorphism, and V is an additive map such that

$$a \cdot V(a') = V(F(a)a') \quad \text{for } a \in A_0,\ a' \in A_1 \tag{4.2}$$

and for $a' \in A_1$ we have

$$F(V(a')) = \mathbf{p}a', \quad \text{where } \mathbf{p} := F(V(1)) \in A_1. \tag{4.3}$$

Note that by (4.2) we have $VF = V(1)$, which implies (4.3) if $a' \in F(A_0)$ (but not in general).

4.1.3. *The functor* $\mathcal{C} \to \mathcal{C}^{\mathrm{ec}}$. Given a triple $(A, t, u) \in \mathcal{C}$, we construct a diagram (4.1) as follows:

(i) A_0 is the 0-th graded component of A;

(ii) A_1 is the first graded component of A, and the product of $x, y \in A_1$ is as follows: first multiply x by y in A, then apply the isomorphism $A_2 \xrightarrow{\sim} A_1$ inverse to $u : A_1 \xrightarrow{\sim} A_2$; equivalently, the product in A_1 comes from the product in $A/(u-1)A$ and the natural map $A_1 \to A/(u-1)A$, which is an isomorphism by virtue of §4.1.1(i);

(iii) $F : A_0 \to A_1$ is multiplication by u, and $V : A_1 \to A_0$ is multiplication by t.

Proposition 4.1.4. *The above functor $\mathcal{C} \to \mathcal{C}^{\mathrm{ec}}$ is an equivalence. The inverse functor $\mathfrak{L} : \mathcal{C}^{\mathrm{ec}} \to \mathcal{C}$ takes a diagram $A_0 \underset{V}{\overset{F}{\rightleftarrows}} A_1$ to a certain graded subring of the graded ring*

$$A_0[t, t^{-1}] \times A_1[u, u^{-1}], \quad \deg t := -1, \ \deg u = 1;$$

namely, the i-th graded component of the subring is the set of pairs $(at^{-i}, a'u^i)$, where $a \in A_0$ and $a' \in A_1$ satisfy the relation

$$a' = \mathbf{p}^{-i} F(a) \text{ if } i \leq 0, \quad a = V(\mathbf{p}^{i-1} a') \text{ if } i > 0. \tag{4.4}$$

(As before, $\mathbf{p} := F(V(1)) \in A_1$.) □

The functor $\mathfrak{L} : \mathcal{C}^{\mathrm{ec}} \to \mathcal{C}$ will be called the *Lau equivalence.*

The proof of the proposition is left to the reader. However, let us make some remarks.

4.1.5. *Remarks.* (i) The description of $\mathfrak{L}$ from Proposition 4.1.4 is motivated by the following observation: if $(A, t, u) \in \mathcal{C}$ then the natural map $A \to A[1/t] \times A[1/u]$ is injective, $A[1/t] = A_0[t, t^{-1}]$, and $A[1/u] = A_1[u, u^{-1}]$, where the ring structure on A_1 is as in §4.1.3(ii).

(ii) If $(A, t, u) \in \mathcal{C}$ then the nonpositively graded part of A identifies with $A_0[t]$ and the positively graded one identifies with $uA_1[u]$, where the ring structure on A_1 is as in §4.1.3(ii). So instead of describing A as a subring of $A_0[t, t^{-1}] \times A_1[u, u^{-1}]$, one could describe A as the group $A[t] \oplus uA_1[u]$ equipped with a "tricky" multiplication operation.

4.1.6. *Remarks (to be used in §4.2.2).* (i) The functor $\mathcal{C} \to \mathcal{C}^{\mathrm{ec}}$ from §4.1.3 can also be described as follows: it takes $(A, t, u) \in \mathcal{C}$ to $A_0 \underset{V}{\overset{F}{\rightleftarrows}} R$, where R is the 0-th graded component of the localization $A[u^{-1}]$, the map $F : A_0 \to R$ comes from the map $A \to A[u^{-1}]$, and $V : R \to A_0$ is the composition of $u : A_0 \xrightarrow{\sim} R_1$ and $t : R \to A_0$.

(ii) Let $A_0 \underset{V}{\overset{F}{\rightleftarrows}} R$ be an object of $\mathcal{C}^{\mathrm{ec}}$, and let $(A, t, u) \in \mathcal{C}$ be its image under $\mathfrak{L}$. Let $\alpha \in R^\times$ and $V' = V \circ \alpha$, i.e., $V'(a') = V(\alpha a')$ for all $a' \in R$. Then $A_0 \underset{V'}{\overset{F}{\rightleftarrows}} R$ is an object of $\mathcal{C}^{\mathrm{ec}}$, and its image under $\mathfrak{L}$ is canonically isomorphic to $(A, t, \alpha u)$. The latter follows from the description of the functor $\mathcal{C} \to \mathcal{C}^{\mathrm{ec}}$ given in (i).

4.1.7. *Some examples from* [L21]. (i) For any ring R the maps $F, V : W(R) \to W(R)$ satisfy the properties from §4.1.2 (with $\mathbf{p} = p$). Applying the Lau equivalence to the diagram $W(R) \underset{V}{\overset{F}{\rightleftarrows}} W(R)$, one gets an object of $\mathcal{C}$. Following [L21], we call it the *Witt frame.* Following [D21], we denote it by $W(R)^\oplus$ (in [L21, Example 2.1.3] it is denoted by $\underline{W}(R)$).

(ii) Let $n \in \mathbb{N}$ and let R be an $\mathbb{F}_p$-algebra. Then we have a map $F : W_n(R) \to W_n(R)$ (in addition to $V : W_n(R) \to W_n(R)$). Applying the Lau equivalence to the diagram $W_n(R) \underset{V}{\overset{F}{\rightleftarrows}} W_n(R)$, one gets an object of $\mathcal{C}$. Following [L21], we call it the *n-truncated Witt frame.* Following [D21], we denote it by $W_n(R)^\oplus$ (in Example 2.1.6 of [L21] it is denoted by $\underline{W_n}(R)$).

Let us note that $W(R)^\oplus$ and $W_n(R)^\oplus$ are particular examples of "higher frames" in the sense of [L21, §2].

4.2. **Definition of ${}^sW^{\oplus}$.**

4.2.1. *Definition.* For any $R \in$ p-Nilp, we defined in §3 a diagram ${}^sW(R) \underset{\tilde{V}}{\overset{F}{\rightleftarrows}} {}^sW(R)$, which is an object of $\mathcal{C}^{\mathrm{ec}}$. The image of this diagram under the functor $\mathfrak{L}: \mathcal{C}^{\mathrm{ec}} \to \mathcal{C}$ from Proposition 4.1.4 is denoted by ${}^sW^{\oplus}(R)$. Thus ${}^sW^{\oplus}(R)$ is a $\mathbb{Z}$-graded algebra over the $\mathbb{Z}$-graded ring $\mathbb{Z}_p[t,u]$, where $\deg t = -1$, $\deg u = 1$. The description of $\mathfrak{L}$ given in Proposition 4.1.4 yields a canonical monomorphism of $\mathbb{Z}$-graded rings

$$ {}^sW^{\oplus} \hookrightarrow {}^sW[u,u^{-1}] \times {}^sW[t,t^{-1}] \tag{4.5}$$

such that the map ${}^sW^{\oplus} \to {}^sW[u,u^{-1}]$ induced by (4.5) is an isomorphism in positive degrees and the map ${}^sW^{\oplus} \to {}^sW[t,t^{-1}]$ is an isomorphism in non-positive degrees. In particular, each graded component of ${}^sW^{\oplus}$ is isomorphic to ${}^sW$ as a sheaf of abelian groups.

4.2.2. *Independence of the choice of* $\mathbf{u}$. Recall that the operator $\tilde{V}$ from §3.2 depends on the choice of $\mathbf{u} \in W(\mathbb{Z}_p)$. By §4.1.6(ii), ${}^sW^{\oplus}(R)$ does not depend on this choice up to canonical isomorphism of $\mathbb{Z}$-graded $\mathbb{Z}_p[t]$-algebras (rather than of $\mathbb{Z}_p[t,u]$-algebras).

5. The ring stacks ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^{\oplus}$

5.1. **Ring groupoid generalities.** The goal of this subsection is to give references to the basic definitions from the elementary survey [Dr21] and to introduce the somewhat nonstandard notation related to cones (see §5.1.2 below).

5.1.1. *Ring groupoids.* A definition of the $(2,1)$-category RGrpds of ring groupoids can be found in [Dr21, §2.2.2]. Section 3 of [Dr21] contains several equivalent definitions (or incarnations) of the 1-category of ring groupoids and the definition of the functor from it to the $(2,1)$-category RGrpds. Here are some of the incarnations[7] of the 1-category of ring groupoids discussed in [Dr21]:

(i) groupoids internal to the category of rings (see [Dr21, §3.2.2]);

(ii) DG rings $A^{\boldsymbol{\cdot}}$ with $A^i = 0$ for $i \neq 0, -1$ (see [Dr21, §3.3.3]);

(iii) quasi-ideal pairs (see [Dr21, §3.3.1]), i.e., diagrams $I \xrightarrow{d} A$, where A is a ring, I is an A-module, and $d: I \to A$ is an A-linear map such that $d(x) \cdot y = d(y) \cdot x$ for all $x, y \in I$ (in this situation one says that (I,d) is a quasi-ideal in A).

5.1.2. *Notation:* Cone *and* cone. The object of the $(2,1)$-category of ring groupoids corresponding to a quasi-ideal $I \xrightarrow{d} A$ is denoted by $\mathrm{Cone}(I \xrightarrow{d} A)$. On the other hand, $\mathrm{cone}(I \xrightarrow{d} A)$ will denote a certain object of the *1-category* of DG rings from §5.1.1(ii); namely, the DG ring is the ring $A \oplus I$, where A is in degree 0, I is in degree -1, the diffferential is $d: I \to A$, and the multiplication operation is the obvious one.

5.1.3. *Ring stacks.* Let S be a site (usually, $S = \text{p-Nilp}^{\mathrm{op}}$). A ring stack on S is a prestack on S with values in RGrpds which happens to be a stack; equivalently, a ring stack on S is a ring object in the $(2,1)$-category of stacks of groupoids on S (the meaning of these words is explained in [Dr21, §2.4.1]).

If A is a sheaf of rings on S and (I,d) is a quasi-ideal in A then the meaning of the notation $\mathrm{Cone}(I \xrightarrow{d} A)$ and $\mathrm{cone}(I \xrightarrow{d} A)$ is similar to §5.1.2, in which S was a point.

[7] In each case we describe the objects. Morphisms are defined in the most naive way.

5.2. **Defining ${}^s\mathscr{R}_n$ and ${}^s\mathscr{R}_n^\oplus$.** Let $n \in \mathbb{N}$. We set

$$ {}^s\mathscr{R}_n := \mathrm{Cone}({}^sW \xrightarrow{p^n} {}^sW), \tag{5.1} $$

$$ {}^s\mathscr{R}_n^\oplus := \mathrm{Cone}({}^sW^\oplus \xrightarrow{p^n} {}^sW^\oplus), \tag{5.2} $$

where ${}^sW$ is as in §3.1 and ${}^sW^\oplus$ is as in §4.2.1. Thus ${}^s\mathscr{R}_n$ is a $\mathbb{Z}/p^n\mathbb{Z}$-algebra stack, and ${}^s\mathscr{R}_n^\oplus$ is a stack of $\mathbb{Z}$-graded $\mathbb{Z}/p^n\mathbb{Z}$-algebras and even a stack of $\mathbb{Z}$-graded algebras over the $\mathbb{Z}$-graded ring $(\mathbb{Z}/p^n\mathbb{Z})[t,u]$, where $\deg t = -1$ and $\deg u = 1$. The map (4.5) induces a canonical homomorphism of $\mathbb{Z}$-graded $\mathbb{Z}/p^n\mathbb{Z}$-algebras

$$ {}^s\mathscr{R}_n^\oplus \to {}^s\mathscr{R}_n[u,u^{-1}] \times {}^s\mathscr{R}_n[t,t^{-1}]. \tag{5.3} $$

In the following proposition the word "degree" refers to the graded ring structure (we are *not* talking about cohomological degrees).

Proposition 5.2.1. *The map ${}^s\mathscr{R}_n^\oplus \to {}^s\mathscr{R}_n[u,u^{-1}]$ induced by* (5.3) *is an isomorphism in positive degrees. The map ${}^s\mathscr{R}_n^\oplus \to {}^s\mathscr{R}_n[t,t^{-1}]$ induced by* (5.3) *is an isomorphism in non-positive degrees.*

Proof. Follows from similar properties of the homomorphism (4.5). □

Proposition 5.2.2. *(i) The functor*

$$ R \mapsto {}^s\mathscr{R}_n(R), \quad R \in \text{p-Nilp} $$

commutes with filtered colimits.

(ii) The same is true for the functor $R \mapsto {}^s\mathscr{R}_n^\oplus(R)$, $R \in$ p-Nilp.

Proof. Statement (i) is proved in [BMVZ]. On the other hand, in §8 we will give a description of ${}^s\mathscr{R}_n$ which makes statement (i) obvious: namely, we will represent ${}^s\mathscr{R}_n$ as $\mathrm{Cone}(H \to W_n)$, where H is a group ind-scheme which is ind-finite over $\mathrm{Spf}\,\mathbb{Z}_p$, see formula (8.10) and Lemma 8.3.4.

Statement (ii) follows from (i) and Proposition 5.2.1. □

The ring stacks ${}^s\mathscr{R}_n$ form a projective system. The same is true for ${}^s\mathscr{R}_n^\oplus$.

Proposition 5.2.3. *The projective limit of the ring stacks ${}^s\mathscr{R}_n$ equals ${}^sW$.*

Proof. Combine Lemma 3.5.1 with Lemma B.2.4 from Appendix B. □

5.2.4. *Remark.* The projective limit of the stacks ${}^s\mathscr{R}_n^\oplus$ is a *sheaf*, which *strictly* contains ${}^sW^\oplus$. This easily follows from derived p-completeness of ${}^sW$ and the fact that as a sheaf of abelian groups, ${}^sW^\oplus$ is isomorphic to the direct sum of countably many copies of ${}^sW$ (see the end of §4.2.1).

5.3. **A more economic model for ${}^s\mathscr{R}_n$.** By a *model* for a ring stack we mean its realization as a Cone of a quasi-ideal. In this subsection we discuss a model for ${}^s\mathscr{R}_n$, which is more economic than the one provided by (1.2). We follow [BKMVZ, M2], where the case $n = 1$ is considered.

5.3.1. *The ring space* $W \times_{Q,F^n} Q$. Let $W \times_{Q,F^n} Q$ denote the fiber product in the Cartesian square

$$\begin{array}{ccc} W \times_{Q,F^n} Q & \longrightarrow & Q \\ \downarrow & & \downarrow F^n \\ W & \longrightarrow & Q \end{array} \tag{5.4}$$

(the lower horizontal arrow is the projection $W \twoheadrightarrow W/\hat{W} = Q$). We have a canonical homomorphism $W \times_{Q,F^{n+1}} Q \xrightarrow{(\mathrm{id},F)} W \times_{Q,F^n} Q$, and

$$\lim_{\overleftarrow{n}} (W \times_{Q,F^n} Q) = {}^sW. \tag{5.5}$$

5.3.2. *Remark.* Surjectivity of $F : \hat{W} \to \hat{W}$ implies that the map

$$W \to W \times_{Q,F^n} Q, \quad w \mapsto (F^n w, \bar{w}) \tag{5.6}$$

is surjective. It induces an isomorphism $W/\hat{W}^{(F^n)} \xrightarrow{\sim} W \times_{Q,F^n} Q$, so (5.5) is a reformulation of (3.4)

5.3.3. *A model for* ${}^s\mathscr{R}_n$. The map $W \times_{Q,F^n} Q \xrightarrow{p^n} W \times_{Q,F^n} Q$ factors as

$$W \times_{Q,F^n} Q \overset{\pi}{\twoheadrightarrow} W \to W \times_{Q,F^n} Q,$$

where π is the projection and the second map is

$$W \xrightarrow{(p^n,\tilde{V}^n)} W \times_{Q,F^n} Q. \tag{5.7}$$

Consider W as a module over $W \times_{Q,F^n} Q$ via $\pi : W \times_{Q,F^n} Q \twoheadrightarrow W$, then (5.7) is a quasi-ideal. Let

$$A_n := \mathrm{cone}(W \xrightarrow{(p^n,\tilde{V}^n)} W \times_{Q,F^n} Q).$$

Let us show that the DG ring A_n is a model for ${}^s\mathscr{R}_n$. By construction, A_n is a quotient of the DG ring $\mathrm{cone}(W \times_{Q,F^n} Q \xrightarrow{p^n} W \times_{Q,F^n} Q)$; the latter is a quotient of $\mathrm{cone}({}^sW \xrightarrow{p^n} {}^sW)$ by (5.5). Thus the DG ring $\mathrm{cone}({}^sW \xrightarrow{p^n} {}^sW)$ maps onto A_n.

Proposition 5.3.4. *This map is a quasi-isomorphism, so it induces an isomorphism*

$${}^s\mathscr{R}_n := \mathrm{Cone}({}^sW \xrightarrow{p^n} {}^sW) \xrightarrow{\sim} \mathrm{Cone}(W \xrightarrow{(p^n,\tilde{V}^n)} W \times_{Q,F^n} Q) \tag{5.8}$$

Proof. The kernel of the map from $\mathrm{cone}({}^sW \xrightarrow{p^n} {}^sW)$ to A_n equals

$$\mathrm{cone}(T_F(Q) \xrightarrow{p^n} B), \tag{5.9}$$

where $T_F(Q)$ is given by (3.7) and $B \subset T_F(Q)$ is as follows. A section of $T_F(Q)$ over $R \in$ p-Nilp is a collection of elements $q_i \in Q(R)$, $i \in \mathbb{Z}$, such that $q_0 = 0$ and $F(q_i) = q_{i-1}$ for all i; in these terms, $B \subset T_F(Q)$ is defined by the condition $q_n = 0$.

The complex (5.9) is isomorphic to $\mathrm{cone}(T_F(Q) \xrightarrow{\tilde{V}^n} T_F(Q))$; the latter complex is acyclic because the map $\tilde{V} : T_F(Q) \to T_F(Q)$ is an isomorphism by Corollary 2.3.6. ∎

5.3.5. *Remarks.* (i) Recall that $A_n := \mathrm{cone}(W \xrightarrow{(p^n,\tilde V^n)} W\times_{Q,F^n}Q)$. As n varies, the DG rings A_n and $\mathrm{cone}({}^sW \xrightarrow{p^n} {}^sW)$ form projective systems: the transition maps are given by the commutative diagrams

$$\begin{array}{ccc} W & \xrightarrow{(p^{n+1},\tilde V^{n+1})} & W\times_{Q,F^{n+1}}Q \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle (\mathrm{id},F)} \\ W & \xrightarrow{(p^n,\tilde V^n)} & W\times_{Q,F^n}Q \end{array} \qquad\qquad \begin{array}{ccc} {}^sW & \xrightarrow{p^{n+1}} & {}^sW \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \mathrm{id}} \\ {}^sW & \xrightarrow{p^n} & {}^sW \end{array}$$

The map from $\mathrm{cone}({}^sW \xrightarrow{p^n} {}^sW)$ to A_n defined at the end of §5.3.3 is a homomorphism of projective systems of DG rings.

(ii) The naive projective limit of the DG rings $A_n := \mathrm{cone}(W \xrightarrow{(p^n,\tilde V^n)} W\times_{Q,F^n}Q)$ equals ${}^sW$ by (5.5). By Lemma 3.5.1 (or by a direct argument), the same is true for the derived projective limit.

5.4. **A more economic model for ${}^s\mathscr{R}_n^{\oplus}$.** The DG ring A_n from §5.3.3 is equipped with an endomorphism F and an operator $\tilde V$ satisfying the identities from §3.2 (these maps come from the maps $F,\tilde V : W\to W$ and $F,\tilde V : Q\to Q$). Applying to $(A_n,F,\tilde V)$ a DG version of the Lau equivalence $\mathfrak{L}$ from Proposition 4.1.4, one gets a DG ring $A_n^{\oplus}$ equipped with an additional $\mathbb{Z}$-grading. The corresponding $\mathbb{Z}$-graded ring stack identifies with ${}^s\mathscr{R}_n^{\oplus}$ by Proposition 5.3.4.

6. A self-dual model for ${}^s\mathscr{R}_n$

6.1. **Subject of this section.** In §5.3.3 we constructed a model for ${}^s\mathscr{R}_n$, which was denoted by A_n. In this section (which can be skipped by the reader) we define a DG ring ind-scheme $\tilde A_n$ equipped with a surjective quasi-isomorphism $\tilde A_n \to A_n$. Thus $\tilde A_n$ is another model for ${}^s\mathscr{R}_n$. It turns out that $\tilde A_n$ *is self-dual up to cohomological shift*, see §6.4 below.

$\tilde A_n$ is equipped with operators F and $\tilde V$ (so one can use $\tilde A_n$ to construct a model $\tilde A_n^{\oplus}$ for ${}^s\mathscr{R}_n^{\oplus}$ similarly to §5.4). In §6.5 we describe $\mathrm{Coker}(1-\tilde V : \tilde A_n\to\tilde A_n)$; this is related to the description of $\mathrm{Ker}(1-\tilde V : {}^sW\to{}^sW)$ given in §3.6.

6.2. **The DG ring $\tilde A_n$.**

6.2.1. *Definition.* We define $\tilde A_n$ by the following diagram whose squares are Cartesian:

$$\begin{array}{ccccc} \tilde A_n^{-1} & \xrightarrow{d} & \tilde A_n^0 & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ W & \xrightarrow{(p^n,\tilde V^n)} & W\times_{Q,F^n}Q & \xrightarrow{\mathrm{pr}} & Q \end{array} \tag{6.1}$$

(here the map $W\to W/\hat W = Q$ is the canonical homomorphism). Diagram (6.1) gives a map from $\tilde A_n$ to $A_n := \mathrm{cone}(W \xrightarrow{(p^n,\tilde V^n)} W\times_{Q,F^n}Q)$; it is a surjective quasi-isomorphism.

6.2.2. *Explicit description.* We have the ring scheme $W^2 = W \times W$ over $\operatorname{Spf}\mathbb{Z}_p$ and the quasi-ideal

$$W^2 \xrightarrow{d} W^2, \quad \text{where } d(y_1, y_2) := (p^n y_1, y_2). \tag{6.2}$$

$\tilde{A}_n$ is a DG subring of $\operatorname{cone}(W^2 \xrightarrow{d} W^2)$, namely

$$\tilde{A}_n^0 = \{(x_1, x_2) \in W^2 \,|\, x_1 \equiv F^n x_2\}, \tag{6.3}$$

$$\tilde{A}_n^{-1} = \{(y_1, y_2) \in W^2 \,|\, y_2 \equiv \tilde{V}^n y_1\}. \tag{6.4}$$

Let us explain that (6.3) is just a short way of saying that for every $R \in$ p-Nilp one has

$$\tilde{A}_n^0(R) := \{(x_1, x_2) \in W^2(R) \,|\, x_1 - F^n x_2 \in \hat{W}(R)\}.$$

By (6.3)-(6.4), $\tilde{A}_n^0$ and $\tilde{A}_n^{-1}$ are additively isomorphic to $W \oplus \hat{W}$; in particular, $\tilde{A}_n^0$ and $\tilde{A}_n^{-1}$ are *ind-schemes* (not merely fpqc sheaves).

6.2.3. *The maps $F, \tilde{V} : \tilde{A}_n \to \tilde{A}_n$.* The DG ring $\tilde{A}_n$ is equipped with an endomorphism F and an operator $\tilde{V}$ satisfying the identities from §3.2 (these maps come from $F, \tilde{V} : W \to W$). The maps $F, \tilde{V} : \tilde{A}_n \to \tilde{A}_n$ agree with $F, \tilde{V} : A_n \to A_n$. Similarly to §5.4, one gets a model $\tilde{A}_n^{\oplus}$ for ${}^s\mathscr{R}_n^{\oplus}$ by applying the Lau equivalence to the triple $(\tilde{A}_n, F, \tilde{V})$.

6.2.4. The element $\gamma := (1, p^n) \in W(\mathbb{Z}_p)^2$ belongs to $\tilde{A}_n^{-1}(\mathbb{Z}_p)$ and satisfies the relations

$$d(\gamma) = (p^n, p^n) = p^n, \quad F(\gamma) = \gamma.$$

These relations mean that γ defines an F-equivariant[8] homomorphism from $\operatorname{cone}(\mathbb{Z} \xrightarrow{p^n} \mathbb{Z})$ to $\tilde{A}_n$. The corresponding homomorphism from $\operatorname{cone}(\mathbb{Z} \xrightarrow{p^n} \mathbb{Z})$ to A_n is equal to the one coming from A_n being a quotient of $\operatorname{cone}(W \times_{Q,F^n} Q \xrightarrow{p^n} W \times_{Q,F^n} Q)$, see the end of §5.3.3.

6.3. The projective system $\{\tilde{A}_n\}$.

6.3.1. *The transition maps.* We have maps

$$\tilde{A}_{n+1}^0 \to \tilde{A}_n^0, \quad (x_1, x_2) \mapsto (x_1, Fx_2), \tag{6.5}$$

$$\tilde{A}_{n+1}^{-1} \to \tilde{A}_n^{-1}, \quad (y_1, y_2) \mapsto (py_1, Fy_2). \tag{6.6}$$

These maps define a homomorphism of DG rings

$$\tilde{A}_{n+1} \to \tilde{A}_n, \tag{6.7}$$

which commutes with F and agrees with the homomorphism $A_{n+1} \to A_n$ from §5.3.5(i). *However,* (6.7) *does not commute with $\tilde{V}$ on the nose.*

[8] We are assuming that F acts on $\operatorname{cone}(\mathbb{Z} \xrightarrow{p^n} \mathbb{Z})$ as the identity.

6.3.2. *The limit.* Let $\tilde{A}^i_\infty$ (resp. A^i_∞) be the projective limit of $\tilde{A}^i_n$ (resp. A^i_n). By §5.3.5(ii), $A^0_\infty = {}^sW$, $A^{-1}_\infty = 0$. Let us describe $\tilde{A}^i_\infty$.

We have $\tilde{A}^0_\infty = W \times_Q W^{\mathrm{perf}}$, and the homomorphism $\tilde{A}^0_\infty \to A^0_\infty = {}^sW$ is the canonical map $W \times_Q W^{\mathrm{perf}} \to W \times_Q Q^{\mathrm{perf}} = {}^sW$. It is easy to show that $\tilde{A}^{-1}_\infty$ identifies with $\hat{W}^{\mathrm{perf}}$ so that $d : \tilde{A}^{-1}_\infty \to \tilde{A}^0_\infty$ becomes the map $\hat{W}^{\mathrm{perf}} \hookrightarrow W \times_Q W^{\mathrm{perf}}$ given by $x \mapsto (0, x)$; this map already appeared in (3.8).

This description of $\tilde{A}_\infty$ shows that $\tilde{A}_\infty$ is equipped with operators F and $\tilde{V}$ (they were defined in §3.3.2). The homomorphism of DG rings $\tilde{A}_\infty \to \tilde{A}_n$ commutes with F; *however, it does not commute with* $\tilde{V}$ *on the nose.*

6.3.3. *Remarks.* (i) the maps (6.5) are surjective;

(ii) the maps (6.6) are not surjective, but $R^1 \varprojlim_n \tilde{A}^{-1}_n = 0$.

6.4. **Autoduality of $\tilde{A}_n$.** The group ind-schemes $\tilde{A}^0_n$ and $\tilde{A}^{-1}_n$ defined by (6.3)-(6.4) are Cartier-dual to each other because they are both isomorphic to $W \oplus \hat{W}$. Here is a more precise statement.

Proposition 6.4.1. *Define a group homomorphism* $\xi : \tilde{A}^{-1}_n \to \hat{\mathbb{G}}_m$ *by*

$$\xi(y_1, y_2) := \tilde{\lambda}(y_2 - \tilde{V}^n y_1), \tag{6.8}$$

where $\tilde{\lambda} : \hat{W} \to \hat{\mathbb{G}}_m$ *is given by* (3.21)*. Then*

(i) the pairing

$$\tilde{A}^0_n \times \tilde{A}^{-1}_n \to \mathbb{G}_m, \quad (x, y) \mapsto \langle x, y \rangle := \xi(xy) \tag{6.9}$$

identifies $\tilde{A}^{-1}_n$ *with the Cartier dual of* $\tilde{A}^0_n$*;*

(ii) for $x \in \tilde{A}^0_n$ *and* $y \in \tilde{A}^{-1}_n$ *one has* $\langle Fx, y \rangle = \langle x, \tilde{V}y \rangle$ *and* $\langle \tilde{V}x, y \rangle = \langle x, Fy \rangle$*.*

Proof. We have

$$\tilde{\lambda} \circ \tilde{V} = \tilde{\lambda} \tag{6.10}$$

(see the proof of Lemma 3.6.2). So $\xi \circ \tilde{V} = \xi$. This implies (ii).

To prove (i), let us rewrite (6.9) as a pairing

$$(W \oplus \hat{W}) \times (W \oplus \hat{W}) \to \mathbb{G}_m.$$

Let $x = (x_1, x_2) \in \tilde{A}^0_n$, $y = (y_1, y_2) \in \tilde{A}^{-1}_n$. Then $x_1 = F^n x_2 + \alpha$, $y_2 = \tilde{V}^n y_1 + \beta$, where $\alpha, \beta \in \hat{W}$. We have

$$\langle x, y \rangle = \tilde{\lambda}(x_2 y_2 - \tilde{V}^n(x_1 y_1)), \quad x_2 y_2 - \tilde{V}^n(x_1 y_1) = x_2\beta - \tilde{V}^n(\alpha y_1).$$

So using (6.10), we get $\langle x, y \rangle = \tilde{\lambda}(x_2\beta) \cdot \tilde{\lambda}(\alpha y_1)^{-1}$. It remains to show that the pairing

$$W \times \hat{W} \to \mathbb{G}_m, \quad (u, v) \mapsto \tilde{\lambda}(uv)$$

identifies $\hat{W}$ with the Cartier dual of W. This follows from a similar property of the pairing $(u, v) \mapsto \lambda(uv)$, which is well known (see [Dr25a, Appendix A] and references therein). ∎

6.5. **The operator** $1-\tilde{V}:\tilde{A}_n\to\tilde{A}_n$**.** As before, let $\tilde{\lambda}:\hat{W}\to\hat{\mathbb{G}}_m$ be given by (3.21).

Proposition 6.5.1. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{A}_n^{-1} & \xrightarrow{1-\tilde{V}} & \tilde{A}_n^{-1} & \xrightarrow{\xi} & \hat{\mathbb{G}}_m & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & \tilde{A}_n^{0} & \xrightarrow{1-\tilde{V}} & \tilde{A}_n^{0} & \xrightarrow{\nu} & \hat{\mathbb{G}}_m & \longrightarrow & 0 \end{array} \tag{6.11}$$

where ξ *is defined by* (6.8) *and*

$$\nu(x_1,x_2):=\tilde{\lambda}(p^n x_2-\tilde{V}^n x_1). \tag{6.12}$$

6.5.2. *Remarks.* (i) Formula (6.12) makes sense because $p^n x_2-\tilde{V}^n x_1\in\hat{W}$: this follows from the fact that $F^n x_2-x_1\in\hat{W}$ by (6.3).

(ii) One can rewrite (6.12) as $\nu(a)=\xi(a\gamma)$, where $\gamma\in\tilde{A}_n^{-1}(\mathbb{Z}_p)$ is as in §6.2.4.

6.5.3. *Proof of Proposition 6.5.1.* Commutativity of the right square of (6.11) is checked straightforwardly using (6.2). The lower row of (6.11) is a complex because $\tilde{\lambda}\circ(1-\tilde{V})=0$. To prove its exactness, use the $\tilde{V}$-equivariant exact sequence

$$0\to\hat{W}\to\tilde{A}_n^0\xrightarrow{\pi_2}W\to 0,$$

where $\pi_2(x_1,x_2):=x_2$; one also uses Lemma 3.6.2 and the fact that the map $1-\tilde{V}:W\to W$ is an isomorphism. □

Corollary 6.5.4. *The map* $1-\tilde{V}:{}^s\mathscr{R}_n\to{}^s\mathscr{R}_n$ *is surjective, and its fiber over 0 identifies with* μ_{p^n}*.*

Proof. Follows either from Proposition 6.5.1 or from the exact sequence (3.26). □

7. ${}^s\mathscr{R}_n$ as a quotient of W

In this section we describe a model for ${}^s\mathscr{R}_n$, which represents ${}^s\mathscr{R}_n$ as a quotient of W. This model goes back to [BKMVZ, M2].

7.1. **The model.**

7.1.1. Define a quasi-ideal $I_n\xrightarrow{d}W$ by the Cartesian square

$$\begin{array}{ccc} I_n & \xrightarrow{d} & W \\ \downarrow & & \downarrow{\scriptstyle x\mapsto(F^n x,\bar{x})} \\ W & \xrightarrow{(p^n,\tilde{V}^n)} & W\times_{Q,F^n}Q \end{array} \tag{7.1}$$

whose lower row is (5.7). Let

$$B_n:=\operatorname{cone}(I_n\xrightarrow{d}W).$$

Recall that $A_n:=\operatorname{cone}(W\xrightarrow{(p^n,\tilde{V}^n)}W\times_{Q,F^n}Q)$.

Proposition 7.1.2. *The map* $B_n\to A_n$ *corresponding to diagram* (7.1) *is a quasi-isomorphism.*

Proof. Follows from surjectivity of the right vertical arrow of (7.1), see §5.3.2. □

Combining Propositions 5.3.4 and 7.1.2, we get the following

Corollary 7.1.3. *B_n is a model for ${}^s\mathscr{R}_n$.* □

7.1.4. *Explicit description of B_n.* By §7.1.1, we have

$$I_n = \{(x,y) \in W^2 \mid F^n x = p^n y,\ x - \tilde{V}^n y \in \hat{W}\}, \tag{7.2}$$

$d : I_n \to W$ is given by $d(x,y) = x$, and the W-module structure on I_n is given by

$$a \cdot (x,y) = (ax, F^n(a)y), \qquad \text{where } a \in W,\ (x,y) \in I_n.$$

7.2. **More about B_n.**

7.2.1. *The homomorphism $F : B_n \to B_n$.* Define a DG ring homomorphism $F : B_n \to B_n$ as follows:

(i) the map $B_n^0 \to B_n^0$ is $F : W \to W$;

(ii) in terms of (7.2), the endomorphism of $I_n = B_n^{-1}$ is given by $(x,y) \mapsto (Fx, Fy)$.

Then the homomorphism $B_n \to A_n$ commutes with F.

7.2.2. *A drawback of B_n.* The advantage of the model B_n is that one can use it to construct *economic* models for ${}^s\mathscr{R}_n$, see §8 below. But B_n has the following drawback: the map $\tilde{V} : A_n \to A_n$ does not lift to an additive endomorphism of B_n. Moreover, since there is no operator $\tilde{V}$ acting on B_n, it is hard to use B_n to construct a model for ${}^s\mathscr{R}_n^{\oplus}$.

7.2.3. *The operators $V : B_{n,\mathbb{F}_p} \to B_{n,\mathbb{F}_p}$ and $V : A_{n,\mathbb{F}_p} \to A_{n,\mathbb{F}_p}$.* The drawback of B_n mentioned in §7.2.2 disappears after base change to $\mathbb{F}_p$ (because in characteristic p one has $FV = VF$). Here are more details.

Let $B_{n,\mathbb{F}_p} := B_n \times \operatorname{Spec} \mathbb{F}_p$. Define an additive map $V : B_{n,\mathbb{F}_p} \to B_{n,\mathbb{F}_p}$ as follows:

(i) the map $B_n^0 \to B_n^0$ is $V : W \to W$;

(ii) in terms of (7.2), the endomorphism of $I_n = B_n^{-1}$ is given by $(x,y) \mapsto (Vx, Vy)$.

Let $A_{n,\mathbb{F}_p} := A_n \times \operatorname{Spec} \mathbb{F}_p$. Let $V : A_{n,\mathbb{F}_p} \to A_{n,\mathbb{F}_p}$ be induced by $\tilde{V} : A_n \to A_n$. Unlike $\tilde{V}$, the operator $V : A_{n,\mathbb{F}_p} \to A_{n,\mathbb{F}_p}$ does not depend on the choice of the Witt vector $\mathbf{u} \in W(\mathbb{Z}_p)$ from §3.2.1; this follows from (3.9). Moreover, $V : A_{n,\mathbb{F}_p} \to A_{n,\mathbb{F}_p}$ is described by the most naive formulas similar to those from the definition of $V : B_{n,\mathbb{F}_p} \to B_{n,\mathbb{F}_p}$.

The homomorphism $B_{n,\mathbb{F}_p} \to A_{n,\mathbb{F}_p}$ commutes with V.

7.2.4. *The map $B_{n+1} \to B_n$.* Define a DG ring homomorphism $F : B_{n+1} \to B_n$ as follows:

(i) the map $B_{n+1}^0 \to B_n^0$ is $F : W \to W$;

(ii) in terms of (7.2), the map from $I_{n+1} = B_{n+1}^{-1}$ to $I_n = B_n^{-1}$ is given by $(x,y) \mapsto (Fx, py)$.

Then the diagram

$$\begin{array}{ccc} B_{n+1} & \longrightarrow & B_n \\ \big\downarrow & & \big\downarrow \\ A_{n+1} & \longrightarrow & A_n \end{array}$$

commutes. Recall that the vertical arrows of this diagram are surjective quasi-isomorphisms.

Let B_∞^i be the projective limit of B_n^i. One checks that $d : B_\infty^{-1} \to B_\infty^0 = W^{\text{perf}}$ is injective and $d(B_\infty^{-1}) = \varprojlim_n \hat{W}^{(F^n)}$. This projective limit already appeared in formula (3.4).

7.3. **A model for ${}^s\mathscr{R}^{\oplus}_{n,\mathbb{F}_p}$.** By §7.2.1 and §7.2.3, the DG ring $B_{n,\mathbb{F}_p}$ is equipped with operators F, V satisfying the usual identities. So we can apply to $B_{n,\mathbb{F}_p}$ the Lau equivalence $\mathfrak{L}$ (see Proposition 4.1.4) and get a $\mathbb{Z}$-graded[9] DG ring ind-scheme, which we denote by $B^{\oplus}_{n,\mathbb{F}_p}$.

We have a canonical surjective quasi-isomorphism $B^{\oplus}_{n,\mathbb{F}_p} \twoheadrightarrow A^{\oplus}_{n,\mathbb{F}_p}$, where

$$A^{\oplus}_{n,\mathbb{F}_p} := A^{\oplus}_n \times \operatorname{Spec} \mathbb{F}_p$$

and $A^{\oplus}_n$ is the model for ${}^s\mathscr{R}^{\oplus}_n$ constructed in §5.4. So $B^{\oplus}_{n,\mathbb{F}_p}$ is a model for ${}^s\mathscr{R}^{\oplus}_{n,\mathbb{F}_p}$.

8. ${}^s\mathscr{R}_n$ as a quotient of W_n

In this section we follow [BKMVZ, Vo].

8.1. **The goal.** One has $V^n(\hat{W}^{(F^m)}) \subset W^{(F^{m+n})}$. We will show that $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$ is an ind-scheme, see Lemma 8.3.4 and §8.4. Note that if $m = 0$ then $\hat{W}^{(F^m)} = 0$, so $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)}) = \hat{W}^{(F^n)}$.

Define a number δ_p by

$$\delta_p := 0 \quad \text{for } p > 2, \quad \delta_2 := 1. \tag{8.1}$$

In §8.3.6 we will show that for any integer $m \geq \delta_p$, the ring stack ${}^s\mathscr{R}_n$ is canonically isomorphic to

$$\operatorname{Cone}(\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)}) \to W_n).$$

Moreover, we will see there that ${}^s\mathscr{R}_n \times \operatorname{Spec} \mathbb{F}_p$ canonically identifies with

$$\operatorname{Cone}(\hat{W}^{(F^{m+n})}_{\mathbb{F}_p}/V^n(\hat{W}^{(F^m)}_{\mathbb{F}_p}) \to W_{n,\mathbb{F}_p})$$

for *any* integer $m \geq 0$.

The reader may prefer to disregard the case of mixed characteristic 2 and assume that $m = 0$.

In addition to the above-mentioned models for ${}^s\mathscr{R}_n$, in §8.5 we will construct an economic model for ${}^s\mathscr{R}^{\oplus}_{n,\mathbb{F}_p}$ (this is straightforward because the models for ${}^s\mathscr{R}_{n,\mathbb{F}_p}$ are equipped with operators F and V).

8.2. **A variant of the model from §7.1.** Let $d : I_n \to W$ be the quasi-ideal from §7.1. For each non-negative $m \in \mathbb{Z}$ define $I_{n,m}$ and $d : I_{n,m} \to W$ by the pullback square

$$\begin{array}{ccc} I_{n,m} & \xrightarrow{d} & W \\ \downarrow & & \downarrow{\scriptstyle F^m} \\ I_n & \xrightarrow{d} & W \end{array}$$

Since $F : W \to W$ is surjective, Corollary 7.1.3 implies that for each m one has

$${}^s\mathscr{R}_n = \operatorname{Cone}(I_{n,m} \xrightarrow{d} W). \tag{8.2}$$

By (7.2), we have

$$I_{n,m} = \{(x, y) \in W^2 \mid F^{m+n}x = p^n y,\ F^m x - \tilde{V}^n y \in \hat{W}\}, \tag{8.3}$$

[9]By a $\mathbb{Z}$-graded DG ring we mean a DG ring equipped with an *additional* $\mathbb{Z}$-grading.

$d : I_{n,m} \to W$ is given by $d(x,y) = x$, and the W-module structure on $I_{n,m}$ is given by

$$a \cdot (x,y) = (F^{m+n}(a)x, y), \qquad \text{where } a \in W,\ (x,y) \in I_{n,m}.$$

8.3. **Economic models for ${}^s\mathscr{R}_n$.**

8.3.1. We keep the notation of §8.2. Let

$$I'_{n,m} := \{(x,y) \in W^2 \mid x \in V^n(W),\ y = F^m V^{-n} x\}. \tag{8.4}$$

Lemma 8.3.2. *(i) Let $m \geq \delta_p$, where δ_p is as in* (8.1)*. Then $I'_{n,m} \subset I_{n,m}$, where $I_{n,m}$ and $I'_{n,m}$ are given by formulas* (8.3)-(8.4)*.*

(ii) $I'_{n,m,\mathbb{F}_p} \subset I_{n,m,\mathbb{F}_p}$ for **all** *m. Here $I_{n,m,\mathbb{F}_p}$ is the base change of $I_{n,m}$ to* $\operatorname{Spec} \mathbb{F}_p$*.*

Proof. To prove (ii), it suffices to note that by (8.3) and (3.9), one has the formula

$$I_{n,m,\mathbb{F}_p} = \{(x,y) \in W^2_{\mathbb{F}_p} \mid F^{m+n}x = p^n y,\ F^m x - V^n y \in \hat{W}_{\mathbb{F}_p}\}, \tag{8.5}$$

in which $\tilde{V}$ does not appear.

To prove (i), one has to check that $(F^m V^n - \tilde{V}^n F^m)(W) \subset \hat{W}$. Equivalently, the problem is to show that $F^m V^n = \tilde{V}^n F^m$ if $F, V, \tilde{V}$ are considered as operators in $Q = W/\hat{W}$. By formula (2.7), $\tilde{V}^n F^m = F^m \tilde{V}^n$. Recall that $\tilde{V} := V\bar{\mathbf{u}}$, so $F^m \tilde{V}^n = F^m V^n \cdot \bar{\mathbf{u}} F(\bar{\mathbf{u}}) \ldots F^{n-1}(\bar{\mathbf{u}})$. To show that this equals $F^m V^n$, use (2.5) if $p > 2$ and (2.6) if $m \geq 1$. □

The next two lemmas describe $I_{n,m}/I'_{n,m}$ if $m \geq \delta_p$. Let

$$\hat{I}_{n,m} := \{(x,y) \in \hat{W}^2 \mid F^{m+n}x = p^n y\},$$
$$\hat{I}'_{n,m} := I'_{n,m} \cap \hat{I}_{n,m} = \{(x,y) \in \hat{W}^2 \mid x \in V^n(\hat{W}),\ y = F^m V^{-n} x\}.$$

Lemma 8.3.3. *(i) If $m \geq \delta_p$ then the canonical map $\hat{I}_{n,m}/\hat{I}'_{n,m} \to I_{n,m}/I'_{n,m}$ is an isomorphism.*

(ii) For any $m \geq 0$ the canonical map $\hat{I}_{n,m,\mathbb{F}_p}/\hat{I}'_{n,m,\mathbb{F}_p} \to I_{n,m,\mathbb{F}_p}/I'_{n,m,\mathbb{F}_p}$ is an isomorphism.

Proof. We will only prove (i); the proof of (ii) is similar. Let

$$Y_{n,m} := \{(x,y) \in W^2 \mid F^{m+n}x = p^n y,\ x_i = 0 \text{ for } i > n\}, \tag{8.6}$$

where x_i's are the components of the Witt vector x. Let

$$\hat{Y}_{n,m} := Y_{n,m} \cap \hat{W}^2 = \{(x,y) \in \hat{W}^2 \mid F^{m+n}x = p^n y,\ x_i = 0 \text{ for } i > n\}. \tag{8.7}$$

If $m \geq \delta_p$ then the map $Y_{n,m} \cap I_{n,m} \to I_{n,m}/I'_{n,m}$ is an isomorphism. So to prove (i), it suffices to show that the inclusion $\hat{Y}_{n,m} \subset Y_{n,m} \cap I_{n,m}$ is an equality.

Indeed, let $(x,y) \in Y_{n,m}(R) \cap I_{n,m}(R)$, where R is a p-nilpotent ring. Since $F^{m+n}x = p^n y$, we see that x_i is nilpotent for each $i < n$; on the other hand, $x_i = 0$ for $i \geq n$. So $x \in \hat{W}(R)$. Since $F^m x - \tilde{V}^n y \in \hat{W}(R)$, we see that $\tilde{V}^n y \in \hat{W}(R)$. So $y \in \hat{W}(R)$. Thus $(x,y) \in \hat{Y}_{n,m}(R)$. □

Lemma 8.3.4. *(i) The map*

$$\hat{W}^{(F^{m+n})} \to \hat{I}_{n,m} \subset \hat{W}^2, \quad x \mapsto (x, 0) \tag{8.8}$$

induces an isomorphism

$$\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)}) \xrightarrow{\sim} \hat{I}_{n,m}/\hat{I}'_{n,m}. \tag{8.9}$$

(ii) $W^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$ *is an ind-finite ind-scheme over* $\operatorname{Spf}\mathbb{Z}_p$ *whose group of* $\bar{\mathbb{F}}_p$*-points is zero. This ind-scheme is isomorphic to the ind-scheme* $\hat{Y}_{n,m}$ *from formula* (8.7).

Additional information about the group ind-scheme $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$ can be found in §8.4.

Proof. Statement (i) follows from surjectivity of $F:\hat{W}\to\hat{W}$. Statement (ii) follows from (i) and the fact that the map $\hat{Y}_{n,m}\to\hat{I}_{n,m}/\hat{I}'_{n,m}$ is an isomorphism, where $\hat{Y}_{n,m}$ is defined by (8.7). ∎

8.3.5. *Remark.* The composition $\hat{Y}_{n,m}\xrightarrow{\sim}\hat{I}_{n,m}/\hat{I}'_{n,m}\xrightarrow{\sim}\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$ takes (x,y) to $x-V^nF^{-m}(y)$, where $F^{-m}(y)$ is a point of $\hat{W}/\hat{W}^{(F^m)}$; note that $F^{m+n}(x-V^nF^{-m}(y))=F^{m+n}x-p^ny=0$.

8.3.6. *The economic models.* Let $m\geq\delta_p$, where δ_p is as in (8.1). Then $I'_{n,m}\subset I_{n,m}$ by Lemma 8.3.2(i). The map $d:I_{n,m}\to W$ from §8.2 induces an isomorphism

$$I'_{n,m}\xrightarrow{\sim}V^n(W)=\operatorname{Ker}(W\twoheadrightarrow W_n).$$

So for each $m\geq\delta_p$ we have ${}^s\mathscr{R}_n=\operatorname{Cone}(I_{n,m}/I'_{n,m}\xrightarrow{d}W_n)$. By Lemmas 8.3.3(i) and 8.3.4, this can be rewritten as

$$ {}^s\mathscr{R}_n=\operatorname{Cone}(\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})\to W_n), \tag{8.10}$$

where the map from $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$ to W_n is the tautological one and the action of W_n on $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$ comes from the obvious action of W on $\hat{W}^{(F^{m+n})}$. Similarly, using Lemma 8.3.2(ii) and Lemma 8.3.3(ii), we see that for *all* m one has

$$ {}^s\mathscr{R}_{n,\mathbb{F}_p}=\operatorname{Cone}(\hat{W}_{\mathbb{F}_p}^{(F^{m+n})}/V^n(\hat{W}_{\mathbb{F}_p}^{(F^m)})\to W_{n,\mathbb{F}_p}). \tag{8.11}$$

8.3.7. *Examples.* (i) Let $p>2$. Then one can set $m=0$ in (8.10). So

$$ {}^s\mathscr{R}_n=\operatorname{Cone}(\hat{W}^{(F^n)}\xrightarrow{d}W_n), \tag{8.12}$$

where d is the tautological map. Note that in the case $p=2$ the r.h.s. of (8.12) is *not a* $\mathbb{Z}/2^n\mathbb{Z}$*-algebra* (unlike ${}^s\mathscr{R}_n$); this follows from Lemma 8.6.1 below.

(ii) Setting $m=0$ in (8.11), we see that for *all* p (including $p=2$) one has

$$ {}^s\mathscr{R}_{n,\mathbb{F}_p}=\operatorname{Cone}(\hat{W}_{\mathbb{F}_p}^{(F^n)}\xrightarrow{d}W_{n,\mathbb{F}_p}). \tag{8.13}$$

(iii) Let $n=1$. As noted in [Vo], $\hat{W}^{(F^{m+1})}/V(\hat{W}^{(F^m)})$ has a description in the spirit of P. Berthelot [Ber]: namely, the quasi-ideal $\hat{W}^{(F^{m+1})}/V(\hat{W}^{(F^m)})$ in $W_1=\mathbb{G}_a$ identifies with the nilpotent PD neighborhood of $Z\subset\mathbb{G}_a$, where Z is the kernel of Fr^m acting on $\mathbb{G}_a\otimes\mathbb{F}_p$. In the case $m=0$ this is well known. In the case $m\geq 1$ (or more generally, $m\geq\delta_p$) this can be deduced from the isomorphism $\hat{W}^{(F^{m+1})}/V(\hat{W}^{(F^m)})\xrightarrow{\sim}\hat{Y}_{1,m}$ from the proof of Lemma 8.3.4. Note that $\hat{Y}_{1,m}=\{(x_0,y)\in\mathbb{G}_a\times\hat{W}\mid[x_0^{p^m}]-Vy\in\hat{W}^{(F)}\}$, and the map $\hat{Y}_{1,m}\to\mathbb{G}_a$ is given by x_0, so we have a pullback square of ind-schemes

$$\begin{array}{ccc} \hat{Y}_{1,m} & \twoheadrightarrow & \mathbb{G}_a \\ \downarrow & & \downarrow{\scriptstyle z\mapsto z^{p^m}} \\ \hat{W}^{(F)} & \twoheadrightarrow & \mathbb{G}_a \end{array}$$

8.4. **More on $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$.** By Lemma 8.3.4, $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$ is an ind-scheme.

Proposition 8.4.1. *(i) The ind-scheme $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$ can be represented as an inductive limit of a diagram*

$$\operatorname{Spf} C_1 \hookrightarrow \operatorname{Spf} C_2 \hookrightarrow \dots, \tag{8.14}$$

where each C_i is a finite flat $\mathbb{Z}_p$-algebra.

(ii) The canonical nondegenerate pairing[10] *$\hat{W} \times W \to \mathbb{G}_m$ induces isomorphisms*

$$G_{n,m} \xrightarrow{\sim} \underline{\operatorname{Hom}}(\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)}), \mathbb{G}_m), \tag{8.15}$$

$$\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)}) \xrightarrow{\sim} \underline{\operatorname{Hom}}(G_{n,m}, \mathbb{G}_m), \tag{8.16}$$

where $G_{n,m} := \operatorname{Ker}(W_{m+n} \overset{F^n}{\twoheadrightarrow} W_m)$.

Note that $G_{n,m}$ is a flat affine group scheme of finite type over $\operatorname{Spf} \mathbb{Z}_p$; it is smooth if and only if $m = 0$.

Proof. We will use the ind-scheme $\hat{Y}_{n,m}$ defined by (8.7). (If $m = 0$ this is not necessary.)

(i) As explained in the proof of Lemma 8.3.4, there is an isomorphism of ind-schemes $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)}) \xrightarrow{\sim} \hat{Y}_{n,m}$. We also have the affine scheme $Y_{n,m}$ over $\operatorname{Spf} \mathbb{Z}_p$ defined by (8.6). The group $\mathbb{G}_m$ acts on $Y_{n,m}$ and $\hat{Y}_{n,m}$: namely, $\lambda \in \mathbb{G}_m$ acts by

$$(x, y) \mapsto ([\lambda^{p^{m+n}}]x, [\lambda]y).$$

The relation between $Y_{n,m}$ and $\hat{Y}_{n,m}$ is as follows. Let A be the coordinate ring of $Y_{n,m}$. The group $\mathbb{G}_m$ acts on $Y_{n,m}$: namely, $\lambda \in \mathbb{G}_m$ acts by $(x, y) \mapsto ([\lambda^{p^{m+n}}]x, [\lambda]y))$. This action defines a $\mathbb{Z}$-grading[11] on A. This grading is non-negative, and each graded component A_i is a finitely generated $\mathbb{Z}_p$-module. Then $\hat{Y}_{n,m}$ is the inductive limit of the closed subschemes $Y^{<r}_{n,m} \subset Y_{n,m}$, where $Y^{<r}_{n,m}$ corresponds to the ideal $\bigoplus\limits_{i\geq r} A_i \subset A$ (here $\bigoplus$ stands for the p-completed direct sum).

So it remains to show that each A_i is flat over $\mathbb{Z}_p$. Indeed, $Y_{n,m}$ is the kernel of the homomorphism $W^2 \to W$ given by $(x, y) \mapsto F^{m+n}x - p^n y$. This homomorphism is flat because it becomes flat after base change to $\operatorname{Spec} \mathbb{F}_p$.

(ii) The isomorphism (8.15) is straightforward. Statement (i) ensures that the map from $\hat{W}^{(F^{m+n})}/V^n(\hat{W}^{(F^m)})$ to its double Cartier dual[12] is an isomorphism. So (8.16) follows from (8.15). □

[10] E.g., see [Dr25a, Appendix A] and references therein.

[11] A is p-complete, and the word "grading" is understood in the p-complete sense.

[12] Cartier duality between commutative affine group schemes and commutative ind-affine group ind-schemes over arbitrary bases is discussed in the lecture [M3] at 46:00. The material from [M3] is closely related to the Appendices of [Bou].

8.5. **An economic model for ${}^s\mathscr{R}^{\oplus}_{n,\mathbb{F}_p}$.** As before, we use the subscript $\mathbb{F}_p$ to denote base change to $\operatorname{Spec}\mathbb{F}_p$.

8.5.1. *The model.* By §8.3.7(ii), the DG ring ind-scheme

$$C_{n,\mathbb{F}_p} := \operatorname{cone}(\hat{W}^{(F^n)}_{\mathbb{F}_p} \xrightarrow{d} W_{n,\mathbb{F}_p}) \tag{8.17}$$

is a model for ${}^s\mathscr{R}_{n,\mathbb{F}_p}$. The DG ring (8.17) is equipped with operators F, V satisfying the usual identities; they come from the maps $F, V : W_{\mathbb{F}_p} \to W_{\mathbb{F}_p}$. So we can apply to (8.17) the Lau equivalence $\mathfrak{L}$ (see Proposition 4.1.4) and get a $\mathbb{Z}$-graded DG ring ind-scheme $C^{\oplus}_{n,\mathbb{F}_p}$, where

$$C^{\oplus}_{n,\mathbb{F}_p} := \operatorname{cone}((\hat{W}^{(F^n)}_{\mathbb{F}_p})^{\oplus} \xrightarrow{d} W^{\oplus}_{n,\mathbb{F}_p}). \tag{8.18}$$

We claim that $C^{\oplus}_{n,\mathbb{F}_p}$ is a model for ${}^s\mathscr{R}^{\oplus}_{n,\mathbb{F}_p}$. To justify this claim, we will construct[13] a quasi-isomorphism

$$B^{\oplus}_{n,\mathbb{F}_p} \twoheadrightarrow C^{\oplus}_{n,\mathbb{F}_p}, \tag{8.19}$$

where $B^{\oplus}_{n,\mathbb{F}_p}$ is the model from §7.3.

8.5.2. *Constructing* (8.19). $C_{n,\mathbb{F}_p}$ is the quotient of the DG ring $B_{n,\mathbb{F}_p}$ from §7.2.3 by the acyclic ideal $\operatorname{cone}(I'_{n,\mathbb{F}_p} \to V^n(W_{\mathbb{F}_p}))$, where

$$I'_{n,\mathbb{F}_p} = \{(x,y) \in W^2_{\mathbb{F}_p} \mid x = V^n y\};$$

note that $I'_{n,\mathbb{F}_p} \subset I_{n,\mathbb{F}_p}$, where I_n is given by (7.2). The map $B_{n,\mathbb{F}_p} \twoheadrightarrow C_{n,\mathbb{F}_p}$ commutes with F and V. So we get (8.19)

8.6. **The ring stack** $\operatorname{Cone}(\hat{W}^{(F^n)} \longrightarrow W_n)$ **for arbitrary** p.

Lemma 8.6.1. *For any prime p and any $n \in \mathbb{N}$, the ring stack* $\operatorname{Cone}(\hat{W}^{(F^n)} \longrightarrow W_n)$ *is canonically isomorphic to* $\operatorname{Cone}({}^sW \xrightarrow{p^n\mathbf{u}} {}^sW)$*, where $\mathbf{u} \in W(\mathbb{Z}_p)^{\times}$ is as in §3.2.1.*

Note that $p^n\mathbf{u} \in {}^sW(\mathbb{Z}_p)$ because $p\mathbf{u} = \mathbf{p} \in {}^sW(\mathbb{Z}_p)$.

Proof. Using Lemma 3.2.5 and mimicking the proof of [Dr24, Prop. 3.5.1], one gets a canonical isomorphism $\operatorname{Cone}(\hat{W}^{(F^n)} \to W_n) \xrightarrow{\sim} \operatorname{Cone}({}^sW \xrightarrow{F^n\tilde{V}^n} {}^sW)$. It remains to use §3.2.6. □

9. A class of models for ${}^s\mathscr{R}^{\oplus}_n$

9.1. **Ind-affineness of certain morphisms.** A morphism of stacks $\mathscr{X} \to \mathscr{Y}$ is said to be ind-affine if for any p-nilpotent affine scheme S and any morphism $S \to \mathscr{Y}$ the stack $\mathscr{X} \times_{\mathscr{Y}} S$ is an ind-affine ind-scheme (i.e., it can be represented as an inductive limit of a directed family of affine schemes with respect to closed immersions).

Proposition 9.1.1. *The diagonal morphism ${}^s\mathscr{R}_n \to {}^s\mathscr{R}_n \times {}^s\mathscr{R}_n$ is ind-affine.*

The proof will be given in §9.1.5.

Corollary 9.1.2. *The morphism of stacks ${}^s\mathscr{R}^{\oplus}_n \to {}^s\mathscr{R}_n[u,u^{-1}] \times {}^s\mathscr{R}_n[t,t^{-1}]$ from §5.2 is ind-affine.*

[13]The construction is the $m = 0$ case of §8.3.6.

Proof. Combine Propositions 5.2.1 and 9.1.1. □

To prove Proposition 9.1.1, we need the following lemma.

Lemma 9.1.3. *Let R be a ring. Let H be a commutative affine group R-scheme such that the R-module $H^0(H, \mathcal{O}_H)$ is projective. Let $\mathcal{F}$ be an H^*-torsor*[14]*, where H^* is the Cartier dual. Then $\mathcal{F}$ is representable by an ind-affine ind-scheme over R.*

Remark 9.1.4. H-torsors are representable by affine schemes: this is a well known consequence of the theory of flat descent.

Proof of Lemma 9.1.3. Interpreting H^* as the automorphism group of the trivial extension

$$0 \to \mathbb{G}_m \to H \oplus \mathbb{G}_m \to H \to 0$$

and using the theory of flat descent, we see that $\mathcal{F}$ defines an extension

$$0 \to \mathbb{G}_m \to \tilde{H} \to H \to 0, \tag{9.1}$$

where $\tilde{H}$ is an affine R-scheme. Then $\mathcal{F}$ is the sheaf of splittings of (9.1).

Let $M_+ := H^0(H, L)$, $M_- := H^0(H, L^{-1})$, where L is the line bundle on H corresponding to the $\mathbb{G}_m$-torsor $\tilde{H} \to H$. A splitting of (9.1) can be viewed as a pair

$$(s_+, s_-), \quad s_+ \in M_+,\ s_- \in M_-$$

satisfying certain equations (one of them is $s_+ s_- = 1$). So it remains to show that the functor

$$\{R\text{-algebras}\} \to \{\text{Sets}\}, \quad \tilde{R} \mapsto \tilde{R} \otimes_R (M_+ \oplus M_-)$$

is an ind-affine ind-scheme over R. To see this, represent $M_+ \oplus M_-$ as a direct summand of a free R-module; this is possible because the R-module $H^0(H, \mathcal{O}_H)$ is assumed to be projective. □

9.1.5. *Proof of Proposition 9.1.1.* Let S be a p-nilpotent affine scheme equipped with a morphism $f : S \to {}^s\mathscr{R}_n \times {}^s\mathscr{R}_n$. Let $X := S \times_{{}^s\mathscr{R}_n \times {}^s\mathscr{R}_n} {}^s\mathscr{R}_n$, where the map ${}^s\mathscr{R}_n \to {}^s\mathscr{R}_n \times {}^s\mathscr{R}_n$ is the diagonal. Then X can be described as follows.

Let $g := f_1 - f_2$, where $f_1, f_2 : S \to {}^s\mathscr{R}_n$ are the components of f. By §6,

$${}^s\mathscr{R}_n = \text{Cone}(\tilde{A}_n^{-1} \to \tilde{A}_n^0),$$

where A_n^{-1} and A_n^0 are additively isomorphic to $W \oplus \hat{W}$. So $g : S \to {}^s\mathscr{R}_n$ is given given by a $\tilde{A}_n^{-1}$-torsor $T \to S$ and a $\tilde{A}_n^{-1}$-equivariant map $h : T \to B_n$. In these terms, $X = h^{-1}(0)$.

A priori, T and X are fpqc sheaves. We have to prove that X is an ind-affine ind-scheme. It suffices to show that T is an ind-affine ind-scheme. This follows from Lemma 9.1.3 and Remark 9.1.4 because the group ind-scheme $\tilde{A}_n^{-1}$ is isomorphic to $W \oplus \hat{W}$, and $\hat{W}$ is Cartier dual to W. □

9.2. A class of models for ${}^s\mathscr{R}_n^{\oplus}$ and ${}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus}$.

[14] In other words, $\mathcal{F}$ is an fpqc sheaf on the category of R-schemes which is a torsor over the sheaf of sections of H^*.

9.2.1. Suppose we have a quasi-ideal pair

$$I \xrightarrow{d} C \tag{9.2}$$

with $\mathrm{Cone}(I \xrightarrow{d} C) = {}^s\mathscr{R}_n$ such that C and I are ind-schemes over $\mathrm{Spf}\,\mathbb{Z}_p$. (E.g., formula (8.10) provides such a pair by Lemma 8.3.3; this pair depends on the choice of a number $m \geq \delta_p$.) Using (9.2) as an input datum, we are going to construct a model for ${}^s\mathscr{R}_n^{\oplus}$ in a rather tautological way.

Let $\tilde{C}$ be the fpqc sheaf of graded rings defined by the pullback diagram

$$\begin{array}{ccc} \tilde{C} & \twoheadrightarrow & {}^s\mathscr{R}_n^{\oplus} \\ \downarrow & & \downarrow \\ C[u,u^{-1}] \times C[t,t^{-1}] & \twoheadrightarrow & {}^s\mathscr{R}_n[u,u^{-1}] \times {}^s\mathscr{R}_n[t,t^{-1}] \end{array} \tag{9.3}$$

in which the lower horizontal arrow comes from the epimorphism $C \twoheadrightarrow {}^s\mathscr{R}_n$ and the right vertical arrow was constructed in §5.2. By Corollary 9.1.2, $\tilde{C}$ is an ind-scheme. The two horizontal arrows of (9.3) have the same fibers over 0, namely $I[u,u^{-1}] \times I[t,t^{-1}]$. Thus we have constructed a quasi-ideal pair

$$I[u,u^{-1}] \times I[t,t^{-1}] \xrightarrow{d} \tilde{C} \tag{9.4}$$

whose Cone equals ${}^s\mathscr{R}_n^{\oplus}$. Moreover, $\tilde{C}$ is a $\mathbb{Z}$-graded ring ind-scheme, and the map d from (9.4) is compatible with the gradings (assuming that $\deg t = -1$, $\deg u = 1$).

By construction, we get a homomorphism of quasi-ideal pairs

$$\begin{array}{ccc} I[u,u^{-1}] \times I[t,t^{-1}] & \longrightarrow & \tilde{C} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ I[u,u^{-1}] \times I[t,t^{-1}] & \longrightarrow & C[u,u^{-1}] \times C[t,t^{-1}] \end{array} \tag{9.5}$$

such that the corresponding morphism of Cones is the right vertical arrow of (9.3).

From diagram (9.5) one gets the following two complexes of $\mathbb{Z}$-graded commutative group ind-schemes

$$0 \to I[t,t^{-1}] \to \tilde{C} \to C[u,u^{-1}] \to 0, \tag{9.6}$$

$$0 \to I[u,u^{-1}] \to \tilde{C} \to C[t,t^{-1}] \to 0. \tag{9.7}$$

Proposition 9.2.2. *The complex* (9.6) *is exact in positive degrees, and* (9.7) *is exact in non-positive degrees. Thus we have exact sequences*

$$0 \to I[t^{-1}] \to \tilde{C}_{>0} \to uC[u] \to 0, \tag{9.8}$$

$$0 \to I[u^{-1}] \to \tilde{C}_{\leq 0} \to C[t] \to 0. \tag{9.9}$$

where $\tilde{C}_{>0}$ (resp. $\tilde{C}_{\leq 0}$) is the positively (resp. non-positively) graded part of $\tilde{C}$.

Proof. Consider the diagram

$$(9.10)\qquad \begin{array}{ccc} I[u,u^{-1}]\times I[t,t^{-1}] & \longrightarrow & \tilde{C} \\ \downarrow & & \downarrow \\ I[u,u^{-1}] & \longrightarrow & C[u,u^{-1}] \end{array}$$

coming from (9.5). The Cone of the upper row of (9.10) is ${}^s\mathscr{R}_n^{\oplus}$, and the Cone of the lower row is ${}^s\mathscr{R}_n[u,u^{-1}]$. By Proposition 5.2.1, the map ${}^s\mathscr{R}_n^{\oplus}\to {}^s\mathscr{R}_n[u,u^{-1}]$ is an isomorphism in positive degrees. So the complex corresponding to the bicomplex (9.10) is acyclic in positive degrees. This means that (9.6) is acyclic in positive degrees. The statement about (9.7) is proved similarly. □

9.2.3. *A model for* ${}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus}$. Of course, the statements from §9.2.1 and Proposition 9.2.2 remain valid if one works over $\operatorname{Spec}\mathbb{F}_p$ (instead of $\operatorname{Spf}\mathbb{Z}_p$), i.e., if the input datum is a model for ${}^s\mathscr{R}_{n,\mathbb{F}_p}$; then the output is a model for ${}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus}$.

Suppose that the input datum is the quasi-ideal pair

$$\hat{W}_{\mathbb{F}_p}^{(F^n)}\to W_{n,\mathbb{F}_p}$$

from §8.3.7(ii). Define $\tilde{C}$ by the pullback diagram

$$(9.11)\qquad \begin{array}{ccc} \tilde{C} & \twoheadrightarrow & {}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus} \\ \downarrow & & \downarrow \\ W_{n,\mathbb{F}_p}[u,u^{-1}]\times W_{n,\mathbb{F}_p}[t,t^{-1}] & \twoheadrightarrow & {}^s\mathscr{R}_{n,\mathbb{F}_p}[u,u^{-1}]\times {}^s\mathscr{R}_{n,\mathbb{F}_p}[t,t^{-1}] \end{array}$$

similar to (9.3). Our next goal is to give an explicit description of $\tilde{C}$ (see §9.2.4 below).

Note that F,V act on $\operatorname{cone}(\hat{W}_{\mathbb{F}_p}^{(F^n)}\to W_{n,\mathbb{F}_p})$ *on the nose.* So we get the model

$$\operatorname{cone}((\hat{W}_{\mathbb{F}_p}^{(F^n)})^{\oplus}\to W_{n,\mathbb{F}_p}^{\oplus})$$

for ${}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus}$. Moreover, we have a commutative diagram

$$(9.12)\qquad \begin{array}{ccccc} (\hat{W}_{\mathbb{F}_p}^{(F^n)})^{\oplus} & \longrightarrow & W_{n,\mathbb{F}_p}^{\oplus} & \twoheadrightarrow & {}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus} \\ \downarrow & & \downarrow & & \downarrow \\ \hat{W}_{\mathbb{F}_p}^{(F^n)}[u^{\pm1}]\times \hat{W}_{\mathbb{F}_p}^{(F^n)}[t^{\pm1}] & \longrightarrow & W_{n,\mathbb{F}_p}[u^{\pm1}]\times W_{n,\mathbb{F}_p}[t^{\pm1}] & \twoheadrightarrow & {}^s\mathscr{R}_{n,\mathbb{F}_p}[u^{\pm1}]\times {}^s\mathscr{R}_{n,\mathbb{F}_p}[t^{\pm1}] \end{array}$$

whose rows are fiber sequences. Using the right square of (9.12) and the pullback diagram (9.11), we get a homomorphism $W_{n,\mathbb{F}_p}^{\oplus}\to\tilde{C}$. Thus we get a commutative diagram

$$(9.13)\qquad \begin{array}{ccccc} (\hat{W}_{\mathbb{F}_p}^{(F^n)})^{\oplus} & \longrightarrow & W_{n,\mathbb{F}_p}^{\oplus} & \twoheadrightarrow & {}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \hat{W}_{\mathbb{F}_p}^{(F^n)}[u,u^{-1}]\times \hat{W}_{\mathbb{F}_p}^{(F^n)}[t,t^{-1}] & \longrightarrow & \tilde{C} & \twoheadrightarrow & {}^s\mathscr{R}_{n,\mathbb{F}_p}^{\oplus} \end{array}$$

whose rows are fiber sequences. From this diagram we get the following description of $\tilde{C}$.

9.2.4. *Description of $\tilde{C}$.* The left square of (9.13) is a pushout diagram. The middle vertical arrow of (9.13) is injective (because the left vertical arrow is). The ring structure on $\tilde{C}$ is such that the action of $W^{\oplus}_{n,\mathbb{F}_p}$ on $\hat{W}^{(F^n)}_{\mathbb{F}_p}[u,u^{-1}] \times \hat{W}^{(F^n)}_{\mathbb{F}_p}[t,t^{-1}]$ via the homomorphism

$$W^{\oplus}_{n,\mathbb{F}_p} \hookrightarrow \tilde{C}$$

is equal to the action via the homomorphism $W^{\oplus}_{n,\mathbb{F}_p} \to W_{n,\mathbb{F}_p}[u,u^{-1}] \times W_{n,\mathbb{F}_p}[t,t^{-1}]$.

9.2.5. *Remark.* The lower row of (9.13) provides a model

$$\operatorname{cone}(\hat{W}^{(F^n)}_{\mathbb{F}_p}[u,u^{-1}] \times \hat{W}^{(F^n)}_{\mathbb{F}_p}[t,t^{-1}] \to \tilde{C})$$

for ${}^s\mathscr{R}^{\oplus}_{n,\mathbb{F}_p}$; this model is rather non-economic, but it has the following advantage: *in the case $p>2$ it lifts to a model for ${}^s\mathscr{R}^{\oplus}_n$.* This follows from §8.3.7(i) combined with §9.2.1.

Appendix A. A description of ${}^sW(R)$ for a class of p-nilpotent rings R

Recall that ${}^sW := W \times_Q Q^{\mathrm{perf}}$, where $Q := W/\hat{W}$. If $R \in$ p-Nilp is such that R_{red} is perfect then $Q(R) = W(R)/\hat{W}(R)$ by §2.2.1, so ${}^sW(R)$ is rather explicit. We will give an even more explicit description of ${}^sW(R)$ if $R \in$ p-Nilp is *weakly semiperfect* in the sense of §A.1 below (this condition is stronger than perfectness of R_{red}); see Corollary A.3.2 and §A.3.3.

A.1. **Weakly semiperfect $\mathbb{F}_p$-algebras.**

Recall that an $\mathbb{F}_p$-algebra A is said to be *semiperfect* if the Frobenius homomorphism $\mathrm{Fr}_A : A \to A$ is surjective. We say that an $\mathbb{F}_p$-algebra is *weakly semiperfect* if it has the equivalent properties from the following lemma.

Lemma A.1.1. *The following properties of an $\mathbb{F}_p$-algebra A are equivalent:*
(i) there exists $n \in \mathbb{N}$ such that $\operatorname{Im} \mathrm{Fr}_A^n = \operatorname{Im} \mathrm{Fr}_A^{n+1}$;
(ii) there exists $n \in \mathbb{N}$ such that $A/(\operatorname{Ker} \mathrm{Fr}_A^n)$ is semiperfect;
(iii) there exists an ideal $I \subset A$ such that A/I is semiperfect and $I \subset \operatorname{Ker} \mathrm{Fr}_A^n$ for some n.

Proof. Both (i) and (ii) are equivalent to the following property:

$$\exists n\, \forall a\, \exists a' \text{ such that } \mathrm{Fr}_A^n(a - \mathrm{Fr}_A(a')) = 0.$$

It is clear that (ii)⇔(iii). □

A.2. **Some lemmas.**

Lemma A.2.1. *Let $R \in$ p-Nilp. Let $I \subset R$ be an ideal whose image in R/pR is killed by a power of Frobenius. Then $W(I)$ is killed by a power of F.*

Proof. The lemma clearly holds if R is an $\mathbb{F}_p$-algebra. So it remains to prove the lemma if $I \subset pR$ and $pI = 0$. In this case $W(I)$ is killed by F (e.g., because I is an $\mathbb{F}_p$-algebra with zero Frobenius endomorphism). □

Corollary A.2.2. *Let R and I be as in Lemma A.2.1. Then the pro-objects corresponding to the projective systems*

$$\dots \xrightarrow{F} W(R) \xrightarrow{F} W(R),$$

$$\dots \xrightarrow{F} \hat{W}(R) \xrightarrow{F} \hat{W}(R)$$

do not change (up to canonical isomorphism) if R is replaced by R/I. □

Applying Corollary A.2.2 for $I = pR$, one gets the following statement.

Corollary A.2.3. *For any $R \in$ p-Nilp, one has canonical isomorphisms*

$$\hat{W}^{\mathrm{perf}}(R) \xrightarrow{\sim} \hat{W}^{\mathrm{perf}}(R/pR), \quad W^{\mathrm{perf}}(R) \xrightarrow{\sim} W^{\mathrm{perf}}(R/pR) = W(R^\flat),$$

where $R^\flat := (R/pR)^{\mathrm{perf}}$. □

A.3. **The case where R/pR is weakly semiperfect.** By §2.2.1, for any $R \in$ p-Nilp such that R_{red} is perfect, one has an exact sequence

$$0 \to \hat{W}(R) \to W(R) \to Q(R) \to 0. \tag{A.1}$$

Lemma A.3.1. *If R/pR is weakly semiperfect then* (A.1) *induces an exact sequence*

$$0 \to \hat{W}^{\mathrm{perf}}(R) \to W^{\mathrm{perf}}(R) \to Q^{\mathrm{perf}}(R) \to 0. \tag{A.2}$$

Proof. The problem is to check surjectivity of the map $W^{\mathrm{perf}}(R) \to Q^{\mathrm{perf}}(R)$. It suffices to show that the projective system

$$\dots \xrightarrow{F} \hat{W}(R) \xrightarrow{F} \hat{W}(R)$$

satisfies the Mittag-Leffler condition. Weak semiperfectness of $A := R/pR$ means that $\operatorname{Im} \mathrm{Fr}_A^n = \operatorname{Im} \mathrm{Fr}_A^{n+1}$ for some n, so the projective system

$$\dots \xrightarrow{F} \hat{W}(A) \xrightarrow{F} \hat{W}(A)$$

satisfies the Mittag-Leffler condition. It remains to use Corollary A.2.2. □

Corollary A.3.2. *If R/pR is weakly semiperfect then the map*

$$(W \times_Q W^{\mathrm{perf}})(R) \to (W \times_Q Q^{\mathrm{perf}})(R) = {}^sW(R)$$

is surjective. So the sequence

$$0 \to \hat{W}^{\mathrm{perf}}(R) \to (W \times_Q W^{\mathrm{perf}})(R) \to {}^sW(R) \to 0. \tag{A.3}$$

induced by (3.8) *is exact.* □

A.3.3. *Remarks.* (i) As explained at the end of §3.1.4, $W \times_Q W^{\mathrm{perf}}$ canonically identifies with the semidirect product $W^{\mathrm{perf}} \ltimes \hat{W}$. By Corollary A.2.3, $W^{\mathrm{perf}}(R) \simeq W(R^\flat)$. So the description of ${}^sW(R)$ by the exact sequence (A.3) is quite explicit.

(ii) Recall that the map $W \times_Q W^{\mathrm{perf}} \to {}^sW$ commutes with F and $\tilde{V}$ if the operators F and $\tilde{V}$ on $W \times_Q W^{\mathrm{perf}}$ are defined as in §3.3.2. So the description of $F, \tilde{V} : {}^sW \to {}^sW$ in terms of (A.3) is also quite explicit.

A.3.4. In §A.4-A.5 we apply Corollary A.3.2 to two classes of p-nilpotent rings.

A.4. ${}^sW(R)$ **if R is admissible in the sense of** [L14]**.**

A.4.1. *Admissible rings.* Let $R \in$ p-Nilp be admissible in the sense of [L14]; by definition, this means that R_{red} is perfect and the nilpotent radical of R/pR is killed by a power of Frobenius. Let I be the nilradical of R. Applying Corollary A.2.2 to I, we get an isomorphism $W^{\mathrm{perf}}(R) \xrightarrow{\sim} W^{\mathrm{perf}}(R_{\mathrm{red}}) = W((R_{\mathrm{red}})^{\mathrm{perf}}) = W(R_{\mathrm{red}})$. Let $s : W(R_{\mathrm{red}}) \to W(R)$ be the composition $W(R_{\mathrm{red}}) \xrightarrow{\sim} W^{\mathrm{perf}}(R) \to W(R)$; then s is a splitting[15] for the epimorphism $W(R) \twoheadrightarrow W(R_{\mathrm{red}})$. So one has $W(R) = s(W(R_{\mathrm{red}})) \oplus W(I)$.

We have $\hat{W}(R) = \hat{W}(I) \subset W(I)$. Following [L14], set $\mathbb{W}(R) := s(W(R_{\mathrm{red}})) \oplus W(I)$; this is a subring of $W(R)$.

Proposition A.4.2. *If $R \in$ p-Nilp is admissible then the canonical map ${}^sW(R) \to W(R)$ is injective, and its image equals $\mathbb{W}(R)$.*

Proof. Admissibility implies that R/pR is weakly semiperfect, so we can apply Corollary A.3.2. Since $W^{\mathrm{perf}}(R) = W(R_{\mathrm{red}})$, we have $(W \times_Q W^{\mathrm{perf}})(R) = \mathbb{W}(R)$. By Corollary A.2.2, $\hat{W}^{\mathrm{perf}}(R) = \hat{W}^{\mathrm{perf}}(R_{\mathrm{red}}) = 0$. □

A.5. ${}^sW(R)$ if R is a semiperfect $\mathbb{F}_p$-algebra.

A.5.1. *Notation.* Let R be a semiperfect $\mathbb{F}_p$-algebra. Then $W(R) = W(R^{\mathrm{perf}})/W(J)$, where $J := \mathrm{Ker}(R^{\mathrm{perf}} \twoheadrightarrow R)$. The ideal $W(J) \subset W(R^{\mathrm{perf}})$ is the projective limit of the diagram

$$\ldots \xrightarrow{F} W^{(F^2)}(R) \xrightarrow{F} W^{(F)}(R).$$

Define $\hat{W}_{\mathrm{top}}(J) \subset W(J)$ to be the projective limit of the diagram

$$\ldots \xrightarrow{F} \hat{W}^{(F^2)}(R) \xrightarrow{F} \hat{W}^{(F)}(R). \tag{A.4}$$

Then $\hat{W}_{\mathrm{top}}(J)$ is an ideal in $W(R^{\mathrm{perf}})$.

One can also describe $\hat{W}_{\mathrm{top}}(J)$ as the set of those Witt vectors over J whose components converge to 0 with respect to the natural topology of R^{perf} (i.e., the projective limit topology).

A.5.2. *The formula for ${}^sW(R)$.* The exact sequence (A.3) implies that

$${}^sW(R) = W(R^{\mathrm{perf}})/\hat{W}_{\mathrm{top}}(J). \tag{A.5}$$

One can also deduce (A.5) directly from the formulas

$${}^sW := W \times_Q Q^{\mathrm{perf}} \quad \text{and} \quad Q(R) = W(R)/\hat{W}(R).$$

In [BMVZ] one can find more material about ${}^sW(R)$, where R is a semiperfect $\mathbb{F}_p$-algebra; e.g., there is a very explicit description if $R = \mathbb{F}_p[x^{\frac{1}{p^\infty}}]/(x)$.

A.5.3. *Derived p-completeness of ${}^sW(R)$.* By Lemma 3.5.1 and Remark 3.5.2, ${}^sW(R)$ is derived p-complete for *all* $R \in$ p-Nilp. If R is a semiperfect $\mathbb{F}_p$-algebra then derived p-completeness of ${}^sW(R)$ can also be proved as follows. $W(R^{\mathrm{perf}})$ is clearly p-complete (in the sense of §2.1.5). Each term of (A.4) is killed by a power of p and therefore p-complete. So $\hat{W}_{\mathrm{top}}(J)$ is p-complete. Since $W(R^{\mathrm{perf}})$ and $\hat{W}_{\mathrm{top}}(J)$ are p-complete, the r.h.s. of (A.5) is derived p-complete.

[15] In fact, s is the unique splitting, see [L14, §1B].

A.5.4. *Remark (E. Lau).* If R is semiperfect but not perfect then ${}^sW(R)$ is *not p-complete.* Indeed, (A.5) implies that ${}^sW(R)/(p^n) = W_n(R)$, so the projective limit of the rings ${}^sW(R)/(p^n)$ equals $W(R) = W(R)^{\mathrm{perf}}/W(J)$. But $\hat{W}_{\mathrm{top}}(J) \neq W(J)$ unless $J = 0$, i.e., unless R is perfect.

Appendix B. The notion of derived p-completeness

In §B.1 we recall the notion of derived p-completeness for $\mathbb{Z}$-modules. In §B.2 we recall the notion of derived p-completeness for *sheaves* of $\mathbb{Z}$-modules.

B.1. Derived p-completeness for $\mathbb{Z}$-modules.

B.1.1. Given a ring R and an ideal $I \subset R$, there is a notion of derived completeness (with respect to I) for R-modules, see [Sta, Tag 091S]. If $R = \mathbb{Z}$ and $I = p\mathbb{Z}$ one gets the notion of *derived p-complete* $\mathbb{Z}$*-module.* The class of derived p-complete $\mathbb{Z}$-modules is larger and "better" than the class of p-complete ones in the sense of §2.1.5. E.g., if $f : A \to A'$ is a homomorphism of $\mathbb{Z}$-modules and A, A' are derived p-complete then so are $\operatorname{Ker} f$ and $\operatorname{Coker} f$ (see [Sta, Tag 091U]); on the other hand, if A, A' are p-complete then $\operatorname{Coker} f$ is p-complete if and only if $f(A)$ is closed for the p-adic topology of A'. For more details, see [Sta] (starting with [Sta, Tag 091N]) and references therein.

Recall that derived p-completeness of a $\mathbb{Z}$-module A is equivalent to each of the following conditions:

(i) the map $A \to \varprojlim\limits_n (A \overset{L}{\otimes} (\mathbb{Z}/p^n\mathbb{Z}))$ is an isomorphism;

(ii) the derived projective limit of the diagram

$$\ldots \xrightarrow{p} A \xrightarrow{p} A.$$

is zero.

The above derived projective limit equals $R\operatorname{Hom}(\mathbb{Z}[p^{-1}], A)$. So (ii) can be rewritten as

$$R\operatorname{Hom}(\mathbb{Z}[p^{-1}], A) = 0. \qquad \text{(B.1)}$$

Once we choose an exact sequence

$$0 \to \mathbb{Z}^{(\mathbb{N})} \xrightarrow{f} \mathbb{Z}^{(\mathbb{N})} \to \mathbb{Z}[p^{-1}] \to 0,$$

where $\mathbb{Z}^{(\mathbb{N})} := \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$, we can rewrite condition (B.1) as follows:

(iii) the map $A^{\mathbb{N}} \xrightarrow{f^*} A^{\mathbb{N}}$ induced by f is an isomorphism (here $A^{\mathbb{N}} := A \times A \times \ldots$).

B.1.2. *Remarks.* (i) A $\mathbb{Z}$-module killed by a power of p is derived p-complete: indeed, derived p-completeness is equivalent to (B.1).

(ii) A projective limit of derived p-complete $\mathbb{Z}$-modules is derived p-complete: this is clear from §B.1.1(iii).

(iii) A is p-complete in the sense of §2.1.5 if and only if the map from A to the projective limit of A/p^nA is an isomorphism. So p-completeness implies derived p-completeness by the above Remarks (i)-(ii).

B.2. **Derived p-completeness for sheaves of $\mathbb{Z}$-modules.** Now let A be a *sheaf* of $\mathbb{Z}$-modules on a site $\mathcal{C}$.

B.2.1. According to [Sta, Tag 0995], A is said to be derived p-complete if it satisfies the condition from §B.1.1(i), which is equivalent to the one from §B.1.1(ii). The latter can be rewritten as

$$R\,\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}[p^{-1}], A) = 0, \tag{B.2}$$

where $\underline{\mathbb{Z}}[p^{-1}]$ is the constant sheaf with fiber $\mathbb{Z}[p^{-1}]$ and $\underline{\mathrm{Hom}}$ denotes the sheaf of homomorphisms.

It is clear that if two terms of an exact sequence of sheaves of $\mathbb{Z}$-modules are derived p-complete then so is the third one.

Lemma B.2.2. *Assume that the site $\mathcal{C}$ satisfies the following condition: a countable product of exact sequences of sheaves of $\mathbb{Z}$-modules on $\mathcal{C}$ is exact. Then*

(a) derived p-completeness of A is equivalent to the condition from §B.1.1(iii);

(b) A is derived p-complete if and only if the $\mathbb{Z}$-module $A(c) = H^0(c, A)$ is derived p-complete for each $c \in \mathcal{C}$.

Proof. By assumption, $\mathrm{Ext}^i(\underline{\mathbb{Z}}^{(\mathbb{N})}, A) = 0$ for $i > 0$. So (B.2) is equivalent to the condition from §B.1.1(iii). This proves statement (a). It implies (b) because a Cartesian product of sheaves is just their product in the sense of presheaves. $\square$

B.2.3. *Remark.* By [BS, Prop. 3.1.9]), the condition from Lemma B.2.2 holds if $\mathcal{C}$ is replete in the sense of [BS, Def. 3.1.1].

Lemma B.2.4. *Let A be a sheaf of $\mathbb{Z}$-modules. Let S_n denote $\mathrm{Cone}(A \xrightarrow{p^n} A)$ viewed as a Picard stack. If A is derived p-complete then the map $A \to \varprojlim_n S_n$ is an isomorphism.*

Proof. S_n is the stack of extensions of $p^{-n}\underline{\mathbb{Z}}/\underline{\mathbb{Z}}$ by A. So $\varprojlim_n S_n$ is the stack of extensions of $\underline{\mathbb{Z}}[p^{-1}]/\underline{\mathbb{Z}}$ by A. It remains to use (B.2). $\square$

References

[Ber] P. Berthelot, *$\mathcal{D}$-modules arithmétiques. I. Opérateurs différentiels de niveau fini,* Ann. Sci. École Norm. Sup. **29** (1996), 185–272.

[Bh] B. Bhatt, *Prismatic F-gauges.* Available at https://www.math.ias.edu/~bhatt/teaching.html.

[BMVZ] B. Bhatt, A. Mathew, V. Vologodsky, and M. Zhang, *Sheared Witt vectors*, work in progress.

[BKMVZ] B. Bhatt, A. Kanaev, A. Mathew, V. Vologodsky, and M. Zhang, *Sheared prismatization*, work in progress.

[BS] B. Bhatt and P. Scholze, *The pro-étale topology for schemes*, Astérisque **369** (2015), 99–201.

[Bou] A. Bouthier, *Support singulier et homologie des fibres de Springer affines*, arXiv:2202.12017, version 3.

[D21] P. Daniels, *A Tannakian framework for G-displays and Rapoport-Zink spaces*, Int. Math. Res. Not. 2021, no. 22, 16963–17024.

[Dr21] V. Drinfeld, *On a notion of ring groupoid*, arXiv:2104.07090.

[Dr24] V. Drinfeld, *Prismatization*, Selecta Mathematica New Series **30** (2024), article no. 49, https://doi.org/10.1007/s00029-024-00937-3.

[Dr23] V. Drinfeld, *On Shimurian generalizations of the stack* $\mathrm{BT}_1 \otimes \mathbf{F}_p$, Journal of Mathematical Physics, Analysis, Geometry, **21** (2025), No. 3, 276–301. Also available as arXiv:2304.11709.

[Dr25a] V. Drinfeld, *On the Lau group scheme*, arXiv:2307.06194.

[Dr25b] *The ring space ${}^sW$ and Barsotti-Tate groups*, a talk whose recording and slides are available at https://archive.mpim-bonn.mpg.de/id/eprint/5179

[GM] Z. Gardner and K. Madapusi, *An algebraicity conjecture of Drinfeld and the moduli of p-divisible groups*, arXiv:2201.06124, version 3.

[L14] E. Lau, *Relations between Dieudonné displays and crystalline Dieudonné theory*, Algebra & Number Theory **8** (2014), no. 9, 2201–2262.

[L21] E. Lau, *Higher frames and G-displays*, Algebra Number Theory **15** (2021), no. 9, 2315–2355.

[L25] E. Lau, *The Shimurian stack is a gerbe over truncated displays*, in preparation.

[M1] A. Mathew, recording of the first talk on sheared prismatization, available at https://www.youtube.com/watch?v=o_bg1Ym9qZM

[M2] A. Mathew, recording of the second talk on sheared prismatization, available at https://www.youtube.com/watch?v=broATZdMQ_0

[M3] A. Mathew, Cartier duality and flatness for ind-schemes, available at https://www.youtube.com/watch?v=vsFSDAoeKBs

[Sta] *The Stacks Project*, http://stacks.math.columbia.edu.

[Sw] R. G. Swan, *On seminormality*, J. Algebra **67** (1980), no. 1, 210–229.

[Vo] V. Vologodsky, recording of a talk on sheared prismatization, available at https://www.youtube.com/watch?v=STvzF5pS-PY

[Zi01] Th. Zink, *A Dieudonné theory for p-divisible groups.* In: Class field theory – its centenary and prospect (Tokyo, 1998), Adv. Stud. Pure Math. **30**, 139–160, Mathematical Society of Japan, Tokyo, 2001.

University of Chicago, Department of Mathematics, Chicago, IL 60637