ON LAGRANGIANITY OF p-SUPPORTS OF HOLONOMIC D-MODULES AND q-D-MODULES

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To Maxim Kontsevich on his 60th birthday with admiration

Abstract. M. Kontsevich conjectured and T. Bitoun proved that if M is a nonzero holonomic D-module then the p-support of a generic reduction of M to characteristic $p > 0$ is Lagrangian. We provide a new elementary proof of this theorem and also generalize it to q-D-modules. The proofs are based on Bernstein's theorem that any holonomic D-module can be transformed by an element of the symplectic group into a vector bundle with a flat connection, and a q-analog of this theorem. We also discuss potential applications to quantizations of symplectic singularities and to quantum cluster algebras.

Contents

1. Introduction

One of the most important basic results in noncommutative algebra is Gabber's theorem, which in particular implies that the singular support of a nonzero holonomic D-module on a smooth variety X over a field $\mathbf{k}$ of characteristic zero is Lagrangian. M. Kontsevich conjectured and T. Bitoun proved the following refinement of this theorem: if M is a nonzero holonomic D-module over $\mathbf{k}$ then the p-support of a generic reduction of M to characteristic $p>0$ is Lagrangian. Later Bitoun's proof was simplifed by M. van den Bergh.

We give a new self-contained and elementary proof of this result based on Bernstein's theorem that a holonomic D-module on $\mathbb{A}^r$ can be transformed by an element of $\mathrm{Sp}(2r,\mathbf{k})$ into a vector bundle with a flat connection ∇ (Section 2). This allows us to reduce the problem to checking a Lagrangian property of ∇, which can be expressed in terms of its p-curvature and then verified directly. At the end of the section, we discuss potential applications to quantizations of conical symplectic singularities.

Then we generalize this story to q-D-modules (Section 3). We first generalize Bernstein's theorem to the q-case, and then develop a theory of p-supports for q-D-modules and use the method of Section 2 to prove a q-version of Bitoun's theorem.[1] At the end of the section we discuss potential applications to quantum cluster algebras.

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2. Lagrangianity of p-supports of a holonomic D-module

2.1. **Flat connections and p-curvature.** Let $\mathbf{k}$ be a commutative unital ring and ∇ be a connection on the trivial vector bundle over $\mathbb{A}^r$ of rank n defined over $\mathbf{k}$.[2] Thus $\nabla=(\nabla_1,...,\nabla_r)$, where $\nabla_j=\partial_j+a_j(x_1,...,x_r)$, with $a_j\in \mathrm{Mat}_n(\mathbf{k}[x_1,...,x_r])$. One says that ∇ is **flat** if $[\nabla_i,\nabla_j]=0$, i.e., $\partial_i a_j-\partial_j a_i+[a_i,a_j]=0$.

Let $D_r(\mathbf{k})$ be the algebra of polynomial differential operators on $\mathbb{A}^r$ defined over $\mathbf{k}$. Then any flat connection ∇ on $\mathbb{A}^r$ defines a $D_r(\mathbf{k})$-module M_∇ generated by $f_1,...,f_r$ with defining relations $\nabla\mathbf{f}=0$, where $\mathbf{f}=(f_1,...,f_r)^T$.

Now suppose $\mathbf{k}$ is a field of characteristic $p>0$. Then the p-**curvature** $C(\nabla)$ of a flat connection ∇ is the collection of operators $C_j=\nabla_j^p$, $1\le j\le r$. Since $[\nabla_j,x_i]=\delta_{ij}$, we have $[x_i,C_j]=0$, hence $C_j\in \mathrm{Mat}_n(\mathbf{k}[x_1,...,x_r])$. Also we have $[\nabla_i,C_j]=0$, so $[C_i,C_j]=0$ and the coefficients of the characteristic polynomial $\chi_{\mathbf{t}}$ of $\sum_{j=1}^r t_jC_j$ are annihilated by all ∂_i. Hence they belong to $\mathbf{k}[t_1,...,t_r,X_1,...,X_r]$, where $X_i:=x_i^p$. For example, if $\nabla=\partial+a$ is a rank 1 connection on $\mathbb{A}^1$ then $C(\nabla)=\partial^{p-1}a+a^p$.

2.2. **Lagrangian flat connections in characteristic p.** Let us say that ∇ is **separable** if the eigenvalues of C_j lie in the separable closure $\overline{\mathbf{k}(X_1,...,X_r)}_s$. E.g., every flat connection of rank $n<p$ is automatically separable. Let ∇ be separable and suppose v is a common eigenvector of C_j, i.e. $C_jv=\Lambda_jv$, $\Lambda_j\in\overline{\mathbf{k}(X_1,...,X_r)}_s$. Recall that the derivations $\frac{\partial}{\partial X_j}$ of $\mathbf{k}(X_1,...,X_r)$ extend in a unique way to $\overline{\mathbf{k}(X_1,...,X_r)}_s$. Let us say that ∇ is **Lagrangian** if for each such v the 1-form $\Lambda:=\sum_{j=1}^r\Lambda_jdX_j$ is closed, i.e., $\frac{\partial\Lambda_i}{\partial X_j}=\frac{\partial\Lambda_j}{\partial X_i}$.

The motivation for this terminology is as follows. Recall that the center of $D_r(\mathbf{k})$ is $Z:=\mathbf{k}[X_1,...,X_r,P_1,...,P_r]$, where $P_j:=\partial_j^p$. Let $\mathrm{supp}\nabla$ be the support of M_∇ as a Z-module, a closed subvariety in $\mathrm{Spec}(Z)=T^*(\mathbb{A}^r)^{(1)}$. It is clear that $\mathrm{supp}\nabla$ is given in the

[1] Note that since for q-D-modules with fixed q there is no notion of singular support (as there is no filtration on the algebra of q-difference operators with commutative associated graded algebra), considering the p-supports are the only available way to treat such modules semiclassically.

[2] Note that by the Quillen-Suslin theorem (formerly Serre's conjecture, see [La]), if $\mathbf{k}$ is a field then every vector bundle on $\mathbb{A}^r$ defined over $\mathbf{k}$ is trivial.

coordinates X_i, P_i by the equations $\chi_{\mathbf{t}}(\sum_{j=1}^r t_j P_j) = 0$ $\forall \mathbf{t} = (t_1, ..., t_r) \in \mathbf{k}^r$, i.e., it consists of points $(x_1^p, ..., x_r^p, \Lambda_1, ..., \Lambda_r)$ such that $\sum_{j=1}^r t_j \Lambda_j$ is an eigenvalue of $\sum_{j=1}^r t_j C_j(x_1, ..., x_r)$ for all $\mathbf{t}$. Thus, recalling that the graph in T^*Y of a 1-form ω on a smooth variety Y is Lagrangian if and only if ω is closed, we obtain

Proposition 2.1. *∇ is Lagrangian if and only if* $\mathrm{supp}\nabla$ *is a Lagrangian subvariety of* $T^*(\mathbb{A}^r)^{(1)}$.

Analogous definitions and statements apply to connections on the formal polydisk D^r, with the polynomial algebra $\mathbf{k}[x_1, ..., x_r]$ replaced by the formal series algebra $\mathbf{k}[[x_1, ..., x_r]]$. They also apply to connections on a smooth affine variety U equipped with coordinates $x_1, ..., x_r$, with the polynomial algebra $\mathbf{k}[x_1, ..., x_r]$ replaced by $\mathbf{k}[U]$. By gluing such charts U, the story extends to connections on arbitrary vector bundles on any smooth variety.

2.3. **Liftable connections.** In general, a separable flat connection, even one of rank 1, need not be Lagrangian.

Example 2.2. Let ∇ be the flat connection of rank 1 on $\mathbb{A}^2$ over $\mathbb{F}_p$ with

$$\nabla_1 = \partial_1, \ \nabla_2 = \partial_2 + x_1^p x_2^{p-1}.$$

Then the p-curvature of ∇ is $C_1 = 0, C_2 = -X_1 + X_1^p X_2^{p-1}$, so ∇ is not Lagrangian.

However, the Lagrangian property holds under some assumptions. Namely, say that a flat connection ∇ defined over $\mathbf{k}$ is **liftable** if it is the reduction to $\mathbf{k}$ of a flat connection defined over some local Artinian ring S with residue field $\mathbf{k}$ in which $p \neq 0$.

Lemma 2.3. *If a flat connection of rank* 1 *on a formal polydisk D^r is liftable then it is Lagrangian.*

Proof. It suffices to prove the statement for $r = 2$. In this case we have $\nabla = (\nabla_x, \nabla_y)$ with $\nabla_x = \partial_x + a(x, y)$ and $\nabla_y = \partial_y + b(x, y)$ and lifting $\widetilde{\nabla} = (\widetilde{\nabla}_x, \widetilde{\nabla}_y)$ with $\widetilde{\nabla}_x = \partial_x + \widetilde{a}(x, y)$ and $\widetilde{\nabla}_y = \partial_y + \widetilde{b}(x, y)$. Let $a_{ij}, b_{ij}, \widetilde{a}_{ij}, \widetilde{b}_{ij}$ be the coefficients of $a, b, \widetilde{a}, \widetilde{b}$. The zero curvature equation $\widetilde{a}_y = \widetilde{b}_x$ yields

$$j\widetilde{a}_{i-1,j} = i\widetilde{b}_{i,j-1}, \ i, j \geq 1. \tag{2.1}$$

On the other hand, the p-curvature of ∇ is $C(\nabla) = (A, B)$, where

$$A(X, Y) = \partial_x^{p-1} a(x, y) + a^{(1)}(X, Y), \ B(X, Y) = \partial_y^{p-1} b(x, y) + b^{(1)}(X, Y),$$

where $X = x^p, Y = y^p$, and $a^{(1)}, b^{(1)}$ are the Frobenius twists of a, b (all coefficients raised to power p). Now, reducing equation (2.1) modulo p and then raising it to p-th power, we get $ja_{i-1,j}^p = ib_{i,j-1}^p, \ i, j \geq 1$, hence

$$\partial_Y a^{(1)}(X, Y) = \partial_X b^{(1)}(X, Y).$$

So it remains to show that

$$\partial_Y \partial_x^{p-1} a(x, y) = \partial_X \partial_y^{p-1} b(x, y),$$

which is equivalent to the equations

$$ja_{pi-1,pj} = ib_{pi,pj-1}, \ i, j \geq 1.$$

But this follows from (2.1) by replacing i with pi and j with pj, as $p \neq 0$ in S, so pS surjects onto $\mathbf{k}$ as an S-module. □

On the other hand, a liftable separable flat connection of rank $n \geq p$ need not be Lagrangian.

Example 2.4. Let $\sigma = (12...p)$ be a cyclic permutation matrix, $L = \mathrm{diag}(0, 1, ..., p-1)$, so $\sigma^{-1}L\sigma = L + 1 - pE_{pp}$. Let $\widetilde{\nabla}$ be the connection of rank p on $\mathbb{A}^2$ over $\mathbb{Z}[\sqrt{p}]/p^2$ with

$$\widetilde{\nabla}_1 = \partial_1 - \sqrt{p}x_1^{p-1}\sigma^{-1}L,\ \widetilde{\nabla}_2 = \partial_2 + x_1^p x_2^{p-1}(1 - pE_{pp}) + \sqrt{p}x_2^{p-1}\sigma;$$

it is easy to check that it is flat. But the reduction ∇ of $\widetilde{\nabla}$ to $\mathbf{k}$ is the direct sum of p copies of the connection of Example 2.2, which is not Lagrangian (even though liftable).

However, there are no such examples for $n < p$.

Lemma 2.5. *Let ∇ be a liftable flat connection of rank $n < p$ on D^r. Assume that the joint eigenvalues of the p-curvature of ∇ at 0 and at the generic point of D^r have the same multiplicities. Then ∇ is Lagrangian.*

Proof. We may assume that $p^2 = 0$ in S. Let $\widetilde{\nabla}$ be a lift of ∇ over S. Let $\Lambda^{(k)} = \sum_{j=1}^r \Lambda_j^{(k)} dX_j$ be the distinct eigenvalue 1-forms of $C(\nabla)$ and m_k be their multiplicities. The constant multiplicity condition implies that we have a decomposition $\widetilde{\nabla} \cong \oplus_k \widetilde{\nabla}^{(k)}$, where $\widetilde{\nabla}^{(k)}$ is a flat connection of rank m_k whose reduction $\nabla^{(k)}$ modulo p has a single p-curvature eigenvalue $\Lambda^{(k)}$. Namely, let $\{\mathbf{e}_k\}$ be the complete system of $C_i = \nabla_i^p$-invariant orthogonal idempotents projecting to its generalized eigen-bundles with eigenvalues $\Lambda^{(k)}$, and let $\{\widetilde{\mathbf{e}}_k\}$ be its lift over S to a complete system of orthogonal idempotents invariant under $\nabla_i^{p^2}$; then we $\widetilde{\nabla}^{(k)} = \mathrm{Im}\widetilde{\mathbf{e}}_k$ and $\nabla^{(k)} = \mathrm{Im}\mathbf{e}_k$.

Since $\mathrm{supp}(\oplus_k \nabla^{(k)}) = \cup_k \mathrm{supp} \nabla^{(k)}$, we may thus assume without loss of generality that $C(\nabla)$ has a single eigenvalue 1-form Λ. Then

$$C_i(\wedge^n \nabla) = \sum_{j=1}^n 1^{\otimes j-1} \otimes C_i(\nabla) \otimes 1^{\otimes n-j-1}|_{\wedge^n \mathbf{k}^n} = \mathrm{Tr}(C_i(\nabla)) = n\Lambda_i.$$

Since the connection $\wedge^n \nabla$ is liftable and has rank 1 and $n < p$, Lemma 2.3 implies that $d\Lambda = 0$, as desired. □

Corollary 2.6. *If a flat connection ∇ of rank $n < p$ on a smooth variety X defined over $\mathbf{k}$ is liftable, then ∇ is Lagrangian.*

Proof. It suffices to prove the corollary in the formal neighborhood of a generic point of X. But this follows from Lemma 2.5, since the constant multiplicity assumption of this lemma holds at a generic point. □

2.4. **Bernstein's theorem.** Let $\mathbf{k}$ be an algebraically closed field and V be a symplectic vector space over $\mathbf{k}$ of dimension $2r$. Let $Y \subset V$ be an affine cone of dimension r.

Lemma 2.7. *(J. Bernstein;* [AGM]*, Lemma 3.15) (i) There exists a Lagrangian subspace $L \subset V$ such that $L \cap Y = \{0\}$.*

(ii) If $V = E \oplus E^*$ with the standard symplectic form, then there exists $g \in \mathrm{Sp}(V)$ such that the natural projection $gY \to V/E^* \cong E$ is finite.

Proof. (i) Let $\mathrm{LGr}(V)$ be the Lagrangian Grassmannian of V, then

$$\mathrm{LGr}(V) \cong \mathrm{LGr}_r := \mathrm{Sp}(2r, \mathbf{k})/(GL(r, \mathbf{k}) \ltimes S^2\mathbf{k}^r),$$

so $\dim \mathrm{LGr}(V) = \dim \mathrm{LGr}_r = \frac{r(r+1)}{2}$. Consider the closed subvariety $X \subset \mathbb{P}Y \times \mathrm{LGr}(V)$ consisting of pairs (y, L), $y \in \mathbb{P}Y$, $L \in \mathrm{LGr}(V)$ such that $y \subset L$. The fiber $p^{-1}(y)$ of the projection $p : X \to \mathbb{P}Y$ consists of all $L \in \mathrm{LGr}(V)$ containing y, so it is isomorphic to $\mathrm{LGr}(y^\perp/y) \cong \mathrm{LGr}_{r-1}$ via $L \mapsto L/y$. Thus we get

$$\dim X = \dim \mathbb{P}Y + \dim \mathrm{LGr}_{r-1} = r - 1 + \frac{r(r-1)}{2} = \frac{r(r+1)}{2} - 1.$$

So the projection $\pi : X \to \mathrm{LGr}(V)$ is not surjective for dimension reasons, i.e., there exists $L \in \mathrm{LGr}(V)$ not containing any $y \in \mathbb{P}Y$. Then $L \cap Y = \{0\}$.

(ii) By (i) there is a Lagrangian subspace $L \subset V$ such that $L \cap Y = \{0\}$. There exists $g \in \mathrm{Sp}(V)$ such that $gL = E^*$. Then $E^* \cap gY = \{0\}$, so the projection $gY \to V/E^*$ is finite. □

Now assume that $\mathrm{char}(\mathbf{k}) = 0$ and let M be a nonzero holonomic $D_r(\mathbf{k})$-module. The algebra $D_r(\mathbf{k})$ is the Weyl algebra of the symplectic space $V = \mathbf{k}^r \oplus (\mathbf{k}^r)^* = \mathbf{k}^{2r}$, hence the group $\mathrm{Sp}(2r, \mathbf{k})$ acts on $D_r(\mathbf{k})$ by automorphisms. For $g \in \mathrm{Sp}(2r, \mathbf{k})$ let gM be the twist of M by the automorphism g of $D_r(\mathbf{k})$.

Theorem 2.8. *(J. Bernstein;* [AGM], *Corollary 3.16) There exists $g \in \mathrm{Sp}(2r, \mathbf{k})$ such that gM is $\mathcal{O}$-coherent (a vector bundle on $\mathbf{k}^r$ with a flat connection).*

Proof. Let $Y := SS_a(M) \subset V$ be the arithmetic singular support of M, i.e. the support of $\mathrm{gr}_F M$, where F is a good filtration on M with respect to the Bernstein filtration on $D_r(\mathbf{k})$ ($\deg(x_j) = \deg(\partial_j) = 1$). Then Y is an affine cone of dimension r. Let g be chosen as in Lemma 2.3(ii). Then the projection $gY \to \mathbf{k}^r$ is finite, so $\mathrm{gr}_F M$ and hence M is finitely generated as an $\mathcal{O}(\mathbf{k}^r)$-module. □

2.5. **Bitoun's theorem.** Now let $\mathbf{k}$ be an algebraically closed field of characteristic 0 and M be a finitely generated $D_r(\mathbf{k})$-module. Pick a finitely generated subring $R \subset \mathbf{k}$ and a finitely generated $D_r(R)$-module M_R with an isomorphism $\mathbf{k} \otimes_R M_R \cong M$. For every prime p which is not a unit in R and a maximal ideal $\mathfrak{m} \subset R$ lying over p (i.e., the preimage of a maximal ideal $\overline{\mathfrak{m}} \subset R/pR$) with residue field $k_\mathfrak{m} = R/\mathfrak{m}$ let $M_\mathfrak{m}$ be the finitely generated $D_r(k_\mathfrak{m})$-module $k_\mathfrak{m} \otimes_R M$. Let $Z_\mathfrak{m}$ be the center of $D_r(k_\mathfrak{m})$. The support $\mathrm{supp}(M_\mathfrak{m})$ of $M_\mathfrak{m}$ in $\mathrm{Spec}(Z_\mathfrak{m})$ is called the p-**support** of M at $\mathfrak{m}$.

The following result was conjectured by M. Kontsevich ([K], Conjecture 2) and proved by T. Bitoun [B]. The proof was later simplified by M. van den Bergh [vdB].

Theorem 2.9. *If $M \neq 0$ is a holonomic module then the p-support $\mathrm{supp}(M_\mathfrak{m})$ is Lagrangian for generic $\mathfrak{m} \in \mathrm{Spec}(R)$.*

Example 2.10. If $\mathbf{k} = \overline{\mathbb{Q}}$ then there are finitely many $\mathfrak{m}$ over each prime p, so Theorem 2.9 holds for almost all p.

Proof. By Theorem 2.8, it suffices to prove the theorem when M is a vector bundle with a flat connection ∇. In this case, the result follows from Corollary 2.6. □

As explained in [B],[vdB], Theorem 2.9 implies its generalization to any smooth variety, which is the formulation conjectured in [K] and proved in [B].

Corollary 2.11. *(*[B]*) If X is any smooth variety over $\mathbf{k}$ and M a nonzero holonomic $D(X)$-module, then $\mathcal{Z}_\mathfrak{m} := \mathrm{supp}(M_\mathfrak{m}) \subset T^*X^{(1)}$ is Lagrangian for generic $\mathfrak{m} \in \mathrm{Spec}(R)$.*

Proof. It suffices to treat the case of affine X. Pick a closed embedding $i : X \hookrightarrow E = \mathbb{A}^r$ defined over R and let $\widetilde{\mathcal{Z}}_{\mathfrak{m}} := \iota(\pi^{-1}(\mathcal{Z}_{\mathfrak{m}}))$, where

$$\pi : X^{(1)} \times E^{(1)*} \twoheadrightarrow T^*X^{(1)}, \quad \iota = i^{(1)} \times \mathrm{id} : X^{(1)} \times E^{(1)*} \hookrightarrow E^{(1)} \times E^{(1)*}$$

are the natural maps. Then $\mathrm{supp}((i_*M)_{\mathfrak{m}}) = \widetilde{\mathcal{Z}}_{\mathfrak{m}}$, so it is Lagrangian by Theorem 2.9. Hence so is $\mathcal{Z}_{\mathfrak{m}}$. □

Note that Corollary 2.11 implies Gabber's theorem that the singular support $SS(M)$ of a nonzero holonomic D-module M on any smooth variety is Lagrangian, since the singular support of M is obtained by conical degeneration of the p-support for large p.

Remark 2.12. Let $\mathrm{char}(\mathbf{k}) = 0$ and $\mathcal{D}_r^{\mathrm{add}}(\mathbf{k})$ be the algebra of additive difference operators, generated by $x_i, T_i^{\pm 1}$, $1 \le i \le r$, with defining relations

$$[x_i, x_j] = [T_i, T_j] = 0,\ T_i x_j = (x_j + \delta_{ij}) T_i.$$

This algebra acts naturally on $\mathbf{k}[x_1, ..., x_r]$, with x_i acting by multiplication and T_i acting by translation operators

$$(T_i f)(x_1, ..., x_r) = f(x_1, ..., x_i + 1, ..., x_r).$$

The algebra $\mathcal{D}_r^{\mathrm{add}}(\mathbf{k})$ is isomorphic to the algebra $D((\mathbf{k}^\times)^r)$ of differential operators on the torus $(\mathbf{k}^\times)^r$ with coordinates y_i, via $T_i \mapsto y_i, x_i \mapsto -y_i \frac{\partial}{\partial y_i}$. Thus Corollary 2.11 applies directly to $\mathcal{D}_r^{\mathrm{add}}(\mathbf{k})$-modules.

Remark 2.13. Besides providing a new proof of Bitoun's theorem, our approach provides a simplification of its previous proofs of [B],[vdB]. Namely, both proofs rely on Theorem 5.3.2 of [B] (Theorem 3.4 in [vdB]), which is easy for $\mathcal{O}$-coherent D-modules M (the number $e(M)$ is just the rank of the vector bundle). Thus this theorem follows immediately from Theorem 2.8.

2.6. **p-supports of holonomic modules over quantized conical symplectic singularities.** Recall that a *conical symplectic singularity* over $\mathbb{C}$ is an irreducible affine conical Poisson $\mathbb{C}$-variety X such that X is normal with symplectic smooth locus X° and its symplectic form extends to a regular 2-form on some (equivalently, any) resolution of singularities of X ([Be]). Namikawa showed that X is rigid, hence is always defined over $\overline{\mathbb{Q}}$ ([N1]), and Kaledin showed that X has finitely many symplectic leaves ([Ka]).

It is known that a conical symplectic singularity X admits a homogeneous partial resolution of singularities $\phi : \widetilde{X} \to X$ called a $\mathbb{Q}$-*factorial terminalization*, such that the smooth locus $\widetilde{X}^\circ \subset \widetilde{X}$ has codimension ≥ 4 and the symplectic form of $X^\circ \subset \widetilde{X}$ extends to one on $\widetilde{X}^\circ$ (this can be deduced from the main result of [BCHM], see [Lo3], Proposition 2.1). Moreover, the second Poisson cohomology $HP^2(X)$ is naturally isomorphic to $H^2(\widetilde{X}, \mathbb{C})$ ([N3]). Using this, Namikawa showed ([N2]) that there exists a semi-universal filtered Poisson deformation X_λ of X parametrized by $\lambda \in H^2(\widetilde{X}, \mathbb{C})$ (more precisely, there is a crystallographic reflection group W acting on $H^2(\widetilde{X}, \mathbb{Z})$ called the Namikawa Weyl group such that X_λ is induced by the universal Poisson deformation of X parametrized by the vector space $H^2(\widetilde{X}, \mathbb{C})/W$). Moreover, all X_λ have symplectic singularities (in particular, finitely many symplectic leaves).

Furthermore, Losev showed in [Lo1] that this deformation lifts to the quantum level, giving a family of filtered algebras $A_{t,\lambda}$, $t \in \mathbb{C}$, such that $A_{0,\lambda} = \mathbb{C}[X_\lambda]$ and $\mathrm{gr}(A_{t,\lambda}) = \mathbb{C}[X]$. Namely, this family is obtained as the graded part of the completed algebra $\widehat{A}_{t,\lambda} = H^0(\widetilde{X}^\circ, \mathcal{A}_{t,\lambda})$, where $\mathcal{A}_{t,\lambda}$ is a sheaf of deformation quantizations of $\widetilde{X}^\circ$ defined by Bezrukavnikov and

Kaledin ([BK1]). The family $A_{t,\lambda}$ is a semi-universal filtered quantization of X (it does not depend on the choice of $\widetilde{X}$), and $A_{t,\lambda} \cong A_{st,s\lambda}$ for $s \in \mathbb{C}^\times$. We denote $A_{1,\lambda}$ by A_λ.

Moreover, it is strongly expected and in many cases known[3] that the reduction $A_{\lambda,p}$ of A_λ to a field $\mathbf{k}$ characteristic p (defined for sufficiently large p) has a big center $Z_{\lambda,p}$ such that $\mathrm{gr} Z_{\lambda,p} = \mathbf{k}[X]^p = \mathbf{k}[X^{(1)}]$, where $X^{(1)}$ is the Frobenius twist of X (in particular, it is module-finite over the center), and $\mathrm{Spec} Z_{\lambda,p}$ is isomorphic to $X^{(1)}_{\lambda^p-\lambda}$.[4] Namely, this is satisfied for quotient singularities (by the Appendix to [BFG]), for Higgs branches, such as quiver varieties (by virtue of the hamiltonian reduction procedure), for the Springer resolution for a simple Lie algebra $\mathfrak{g}$ (by the structure of the p-center of $U(\mathfrak{g})$ characteristic p), hence for Slodowy slices, for Coulomb branches (by [L]), etc.

Thus to every finitely generated module M over A_λ, we can attach its sequence of p-supports $\mathrm{supp}_p M \subset X^{(1)}_{\lambda^p-\lambda}$ which are defined for sufficiently large p and independent for almost all p on the details of reduction.

Let Y be a Poisson variety with finitely many symplectic leaves. We say that a subvariety $Z \subset Y$ is **Lagrangian** if it is coisotropic and the intersection of Z with every symplectic leaf is isotropic. Also recall from [Lo2] that a finitely generated A_λ-module M is called *holonomic* if $\mathrm{supp} M$ is Lagrangian, i.e. if the intersection of $\mathrm{supp} M$ with every symplectic leaf of X is isotropic (note that $\mathrm{supp} M$ is automatically coisotropic by Gabber's theorem). The following conjecture gives a potential generalization of Bitoun's theorem to conical symplectic singularities.

Conjecture 2.14. If M is holonomic then $\mathrm{supp}_p M$ is Lagrangian in $X^{(1)}_{\lambda^p-\lambda}$ for sufficiently large p.

If X admits a conical symplectic resolution $\widetilde{X}$ then for all λ, X_λ also admits a (non-conical) symplectic resolution $\widetilde{X}_\lambda$ which deforms $\widetilde{X}$ ([GK, N4]). Thus by Lemma 5.1 in the appendix to [Lo2], in this case Conjecture 2.14 is equivalent to the statement that the preimage of $\mathrm{supp}_p M$ in $\widetilde{X}$ is isotropic.

We expect that if X is a Higgs branch (i.e., obtained by Hamiltonian reduction from an action of a connected reductive group G on a symplectic representation V, with flat moment map) then this conjecture can be deduced from Bitoun's theorem by Hamiltonian reduction.

3. Lagrangianity of p-supports of a holonomic q-D-module

3.1. **Flat q-connections and q-D-modules.** In this section we generalize Bitoun's theorem to q-D-modules. To this end, for $q \in \mathbf{k}^\times$, define the algebra $\mathcal{D}_{r,q}(\mathbf{k})$ generated by $x_i^{\pm 1}, T_i^{\pm 1}$, $1 \le i \le r$, with defining relations

$$[x_i, x_j] = [T_i, T_j] = 0, \ T_i x_j = q^{\delta_{ij}} x_j T_i.$$

This algebra acts naturally on $\mathbf{k}[x_1^{\pm 1}, ..., x_r^{\pm 1}]$, with x_i acting by multiplication and T_i acting by translation operators

$$(T_i f)(x_1, ..., x_r) = f(x_1, ..., q x_i, ..., x_r).$$

Thus $\mathcal{D}_{r,q}(\mathbf{k})$ may be viewed as the algebra of q-difference operators (at least when q is not a root of 1). Unlike the case of additive difference operators (Remark 2.12), this algebra is

[3]Possibly this follows in general from the known results, but I was unable to find a suitable reference.

[4]If $\{c_j\}$ is a basis of $H^2(\widetilde{X}, \mathbb{Z})$ and $\lambda = \sum_i \lambda_i c_i$, $\lambda_i \in \mathbf{k}$, then $\lambda^p := \sum_i \lambda_i^p c_i$.

not isomorphic to the algebra of differential operators on any variety, so the results of the previous sections don't apply and we have to start from scratch.

The basic theory of $\mathcal{D}_{r,q}(\mathbf{k})$-modules (a.k.a. q-**D-modules**) for q not a root of 1 parallel to the Bernstein-Kashiwara theory of usual D-modules was developed in the first two sections of [S]. Note that in these sections it is assumed that $\mathrm{char}(\mathbf{k}) = 0$, but this assumption is not used, so we will not make it.

Let us summarize the basic properties of $\mathcal{D}_{r,q}(\mathbf{k})$ and its modules, mostly following [S]. If q is not a root of unity, then $\mathcal{D}_{r,q}(\mathbf{k})$ is simple ([S], Proposition 1.2.1) and has trivial center. On the other hand, if q is a root of unity of order N then the center of $\mathcal{D}_{r,q}(\mathbf{k})$ is

$$Z = \mathbf{k}[X_1^{\pm 1}, ..., X_r^{\pm 1}, P_1^{\pm 1}, ..., P_r^{\pm 1}],$$

where $X_i := x_i^N, P_i := T_i^N$, and it carries a natural symplectic structure induced by varying q, with Poisson bracket given by

$$\{X_i, X_j\} = \{P_i, P_j\} = 0, \ \{P_i, X_j\} = \delta_{ij} P_i X_j.$$

In this case, if M is a finitely generated $\mathcal{D}_{r,q}(\mathbf{k})$-module then we can define its **support** $\mathrm{supp}(M)$ as a Z-module, which is a closed subvariety of the $2r$-dimensional symplectic torus $\mathbb{G}_m^{2r} = \mathrm{Spec}(Z)$.

The algebra $\mathcal{D}_{r,q}(\mathbf{k})$ has Gelfand-Kirillov dimension $2r$, and if q is not a root of unity, it satisfies **Bernstein's inequality**: every nonzero finitely generated module has Gelfand-Kirillov dimension $\geq r$ ([S], Corollary 2.1.2). In this case, finitely generated modules of Gelfand-Kirillov dimension $\leq r$ (i.e., of dimension exactly r and the zero module) are called **holonomic** ([S], Section 2).

Examples of holonomic modules are obtained from q-connections with rational coefficients. Namely, a (rational) q-**connection** ∇ on $(\mathbf{k}^\times)^r$ is a collection of elements

$$\nabla_i = T_i^{-1} a_i, \ a_i \in GL_n(\mathbf{k}(x_1, ..., x_r)).$$

A q-connection ∇ is said to be **flat** if $[\nabla_i, \nabla_j] = 0$, i.e.,

$$a_i(x_1, ..., qx_j, ..., x_r) a_j(x_1, ..., x_r) = a_j(x_1, ..., qx_i, ..., x_r) a_i(x_1, ..., x_r).$$

It is clear that a flat q-connection is the same thing as a $\mathcal{D}_{r,q}(\mathbf{k})$-module structure on $\mathbf{k}(x_1, ..., x_r)^n$ extending the obvious $\mathbf{k}[x_1^{\pm 1}, ..., x_r^{\pm 1}]$-module structure. It is shown in [S], Corollary 2.5.2 that if M is a holonomic $\mathcal{D}_{r,q}(\mathbf{k})$-module then the vector space

$$M_{\mathrm{loc}} := \mathbf{k}(x_1, ..., x_r) \otimes_{\mathbf{k}[x_1^{\pm 1}, ..., x_r^{\pm 1}]} M$$

over $\mathbf{k}(x_1, ..., x_r)$ is finite dimensional, so M_{loc} gives rise to a flat q-connection ∇_M defined uniquely up to conjugation by $GL_n(\mathbf{k}(x_1, ..., x_r))$. (It may, of course, happen that $M \neq 0$ but $M_{\mathrm{loc}} = 0$). Conversely, for any flat q-connection ∇ there exists a holonomic $\mathcal{D}_{r,q}(\mathbf{k})$-module M such that $\nabla_M = \nabla$ ([S], Theorem 2.7.1).

3.2. **Freeness of $\mathcal{O}$-coherent q-D-modules.** Let $f \in \mathbf{k}[x_1^{\pm 1}, ..., x_r^{\pm 1}]$. For every vector $\mathbf{m} := (m_1, ..., m_r) \in \mathbb{Z}^r$, let $f_{\mathbf{m}} := T_1^{m_1} ... T_r^{m_r} f$.

Lemma 3.1. *If $f \neq 0$ then there exist $h_{\mathbf{m}} \in \mathbf{k}[x_1^{\pm 1}, ..., x_r^{\pm 1}]$, almost all zero, such that $\sum_{\mathbf{m}} h_{\mathbf{m}} f_{\mathbf{m}} = 1$.*

Proof. Since q is not a root of unity, every orbit of q-translations of coordinates is Zariski dense in $(\mathbf{k}^\times)^r$. Thus the system of equations $f_{\mathbf{m}} = 0$ for all $\mathbf{m} \in \mathbb{Z}^r$ has no solutions. Hence the result follows from the Nulstellensatz. □

Proposition 3.2. *Let M be a holonomic $\mathcal{D}_{r,q}(\mathbf{k})$-module finite over $\mathbf{k}[x_1^{\pm 1},...,x_r^{\pm 1}]$. Then M is a free $\mathbf{k}[x_1^{\pm 1},...,x_r^{\pm 1}]$-module.*

Proof. Let $0 \neq f \in \mathbf{k}[x_1^{\pm 1},...,x_r^{\pm 1}]$ be such that $M[1/f]$ is a free $\mathbf{k}[x_1^{\pm 1},...,x_r^{\pm 1}][1/f]$-module. Then for all $\mathbf{m}$, $M[1/f_{\mathbf{m}}]$ is a free $\mathbf{k}[x_1^{\pm 1},...,x_r^{\pm 1}][1/f_{\mathbf{m}}]$-module. By Lemma 3.1, this means that M is locally free over $\mathbf{k}[x_1^{\pm 1},...,x_r^{\pm 1}]$. But every finite locally free module over a Laurent polynomial algebra over a field is free by Swan's theorem ([La], Corollary 4.10). □

3.3. **q-analog of Bernstein's theorem.** Note that similarly to the differential case, the group $\mathrm{Sp}(2r,\mathbb{Z})$ acts by automorphisms on $\mathcal{D}_{r,q}(\mathbf{k})$ ([S], Subsection 1.3). Part (iii) of the following theorem is a q-analog of Theorem 2.8.

Theorem 3.3. *Let M be a holonomic $\mathcal{D}_{r,q}(\mathbf{k})$-module which is finite over the subalgebra $\mathcal{D}_{s,q}(\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}])$. Then*

(i) there exists $g \in \mathrm{Sp}(2s,\mathbb{Z})$ such that gM is finite over $\mathbf{k}[x_1^{\pm 1},...,x_r^{\pm 1}]$;

(ii) M is a free $\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}]$-module.

(iii) For every holonomic $\mathcal{D}_{r,q}(\mathbf{k})$-module M there is $g \in \mathrm{Sp}(2r,\mathbb{Z})$ such that gM is finite over $\mathbf{k}[x_1^{\pm 1},...,x_r^{\pm 1}]$.

The rest of this subsection is dedicated to the proof of Theorem 3.3.

Let R be a commutative domain and consider the algebra $\mathcal{D}_{s,q}(R)$ with generators $x_i^{\pm 1}, T_i^{\pm 1}$, $1 \leq i \leq s$.

Lemma 3.4. *Let M be a finite module over $\mathcal{D}_{s,q}(R)$ which is torsion-free as an R-module. If M does not contain a nonzero free submodule then there exists a nonempty Zariski open subset $U \subset \mathrm{Sp}(2s,\mathbb{C})$ such that for any $g \in \mathrm{Sp}(2s,\mathbb{Z}) \cap U$ there exists nonzero $f \in R$ for which the module $gM[1/f]$ is finite over $\mathcal{D}_{s-1,q}(R[1/f][x_s^{\pm 1}]) \subset \mathcal{D}_{s,q}(R[1/f])$.*

Proof. Let $v_1,...,v_k$ be generators of M and $M_1,...,M_k$ be the cyclic submodules of M generated by $v_1,...,v_k$. It suffices to prove the statement for each M_i, and every M_i satisfies the assumptions of the lemma. Thus we may assume without loss of generality that $M \neq 0$ is cyclic on some generator v: indeed, if U_i are the open sets corresponding to M_i then we can take $U = \cap_i U_i$, and if $g \in \mathrm{Sp}(2s,\mathbb{Z}) \cap U$ and $f_i \in R$ correspond to M_i, g then we may take $f = \prod_i f_i$ for M.

Pick nonzero $D \in D_{s,q}(R)$ such that $Dv = 0$ (it exists since M is not free). Since M is torsion-free over R, D contains at least two monomials. Let U be the open set of $g \in \mathrm{Sp}(2s,\mathbb{C})$ such that $g\mu$ for all monomials μ occurring in D have different last coordinates (where we view Laurent monomials in the generators x_i, T_i as elements of $\mathbb{Z}^{2s}$). The module gM is then generated by an element w satisfying the equation $g(D)w = 0$, which can be written as

$$(a_0\mu_0 + ... + a_{n-1}\mu_{n-1}T_s^{n-1} + a_n\mu_n T_s^n)w = 0,$$

where $n \geq 1$, $a_i \in R$, $a_0, a_n \neq 0$, and μ_i are monomials in $x_i^{\pm 1}$, $1 \leq i \leq s$ and $T_i^{\pm 1}$, $1 \leq i \leq s-1$. This equation multiplied by powers of T_s can be used to express $T_s^j w$, $j \in \mathbb{Z}$ in terms of $w, T_s w, ..., T_s^{n-1}w$ with coefficients whose denominators are powers of a_n if $j \geq n$ and powers of a_0 if $j < 0$. Thus, setting $f := a_0 a_n$, we see that for every $u \in M$ there exists N and $D_0,...,D_{n-1} \in \mathcal{D}_{r-1,q}(R[x_r^{\pm 1}])$ with

$$f^N u = \sum_{i=0}^{n-1} D_i T_s^i w.$$

This implies the statement. □

Lemma 3.5. *Let M be a $\mathcal{D}_{r,q}(\mathbf{k})$-module finite over $\mathcal{D}_{s,q}(\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}])\subset \mathcal{D}_{r,q}(\mathbf{k})$. Then M is torsion-free over $\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}]$.*

Proof. Let $M_{\rm tors}\subset M$ be the subspace of $\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}]$-torsion elements. Then $M_{\rm tors}$ is a $\mathcal{D}_{r,q}(\mathbf{k})$-submodule, and it is finite over $\mathcal{D}_{s,q}(\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}])$ since this algebra is Noetherian. Let $v_1,...,v_k$ be generators of $M_{\rm tors}$ over $\mathcal{D}_{s,q}(\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}])$. There exists $f\in \mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}]$ such that $fv_i=0$ for all i, so $fM_{\rm tors}=0$.

It follows that $f_{\mathbf{m}}M_{\rm tors}=0$ for all $\mathbf{m}$. By Lemma 3.1, there exist $h_{\mathbf{m}}$, almost all zero, such that $\sum_{\mathbf{m}}h_{\mathbf{m}}f_{\mathbf{m}}=1$. Thus $M_{\rm tors}\subset \sum_{\mathbf{m}}h_{\mathbf{m}}(f_{\mathbf{m}}M_{\rm tors})=0$. □

Corollary 3.6. *Let M be a $\mathcal{D}_{r,q}(\mathbf{k})$-module which is finite over $\mathcal{D}_{s,q}(\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}])$ for some $0<s\le r$ and does not contain a nonzero free submodule over this subalgebra. Then there exists $g\in {\rm Sp}(2s,\mathbb{Z})$ such that gM is finite over $\mathcal{D}_{s-1,q}(\mathbf{k}[x_s^{\pm 1},...,x_r^{\pm 1}])$.*

Proof. By Lemma 3.5, M is torsion-free over $\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}]$. Thus Lemma 3.4 applied to $R=\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}]$ implies that for some g,f, the module $gM[1/f]$ is finite over the algebra $\mathcal{D}_{s-1,q}(\mathbf{k}[x_s^{\pm 1},...,x_r^{\pm 1}][1/f])$

Since gM is a $\mathcal{D}_{r,q}(\mathbf{k})$-module, it follows that for all $\mathbf{m}$, the module $gM[1/f_{\mathbf{m}}]$ is finite over $\mathcal{D}_{s-1,q}(\mathbf{k}[x_s^{\pm 1},...,x_r^{\pm 1}][1/f_{\mathbf{m}}])$. Let $v_{\mathbf{m},i}$, $1\le i\le \ell_{\mathbf{m}}$ be generators of this module.

By Lemma 3.1, there exist $\mathbf{m}_1,...,\mathbf{m}_k$, such that the system $f_{\mathbf{m}_j}=0$, $1\le j\le k$ has no solutions. Let $f_j:=f_{\mathbf{m}_j}$, $w_{ji}:=v_{\mathbf{m}_j,i}$, $\ell_j:=\ell_{\mathbf{m}_j}$. We claim that w_{ji}, $1\le j\le k$, $1\le i\le \ell_j$ are generators of gM over $\mathcal{D}_{s-1,q}(\mathbf{k}[x_s^{\pm 1},...,x_r^{\pm 1}])$. Indeed, for every $u\in M$ and each j there are N_j and $D_{ji}\in \mathcal{D}_{s-1,q}(\mathbf{k}[x_s^{\pm 1},...,x_r^{\pm 1}])$ such that

$$f_j^{N_j}u=\sum_{i=1}^{\ell_j}D_{ji}w_{ji}.$$

Pick $h_1,...,h_k\in \mathbf{k}[x_s^{\pm 1},...,x_r^{\pm 1}]$ such that $\sum_{j=1}^k h_jf_j^{N_j}=1$ (they exist by the Nulstellensatz since the system $f_j=0$, $1\le j\le k$ has no solutions). Then

$$u=\sum_{j=1}^k h_jf_j^{N_j}u=\sum_{j=1}^k\sum_{i=1}^{\ell_j}h_jD_{ji}w_{ji},$$

as claimed. □

Now we are ready to prove Theorem 3.3. The proof of part (i) is by induction in s, starting from the trivial case $s=0$. Suppose the result is known for $s-1$ with $0<s\le r$ and let's prove it for s. So assume that M is a holonomic $\mathcal{D}_{r,q}(\mathbf{k})$-module which is finite over $\mathcal{D}_{s,q}(\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}])$. Since the GK dimension of M is r, it cannot contain a free submodule over $\mathcal{D}_{s,q}(\mathbf{k}[x_{s+1}^{\pm 1},...,x_r^{\pm 1}])$ (as this algebra has GK dimension $r+s>r$). Thus by Corollary 3.6 there is $g_1\in {\rm Sp}(2s,\mathbb{Z})$ for which g_1M is finite over $\mathcal{D}_{s-1,q}(\mathbf{k}[x_s^{\pm 1},...,x_r^{\pm 1}])$. By the induction assumption, there is $g_2\in {\rm Sp}(2s-2,\mathbb{Z})\subset {\rm Sp}(2s,\mathbb{Z})$ such that g_2g_1M is finite as a $\mathbf{k}[x_1^{\pm 1},...,x_r^{\pm 1}]$-module. This completes the induction step, with $g=g_1g_2$.

Part (iii) is a special case of (i) with $s=r$.

Finally, (ii) follows from (i) and Proposition 3.2.

3.4. N**-curvature.** If q is a primitive N-th root of 1, then there is a notion of N-**curvature** of a flat q-connection, which is a q-deformation of the notion of p-curvature of a usual flat connection in characteristic p (see e.g. [EV], Section 6). Namely, the N-curvature of ∇ is the collection of commuting operators $C_i := \nabla_i^N$. These operators are just $\mathbf{k}(x_1, ..., x_r)$-linear automorphisms of $\mathbf{k}(x_1, ..., x_r)^N$, namely,

$$C_i = \prod_{k=N-1}^{0} T_i^k(a_i) = T_i^{N-1}(a_i)...T_i(a_i)a_i.$$

As for usual connections, since $[\nabla_i, C_j] = 0$, we have $[C_i, C_j] = 0$ and the coefficients of the characteristic polynomial $\chi_{\mathbf{t}}$ of $\sum_{j=1}^r t_j C_j$ are preserved by all T_i, hence belong to $\mathbf{k}[t_1, ..., t_r, X_1^{\pm 1}, ..., X_r^{\pm 1}]$.

3.5. **Lagrangian flat q-connections.** Let q be a primitive N-th root of unity. Again parallel to the differential case, let us say that a flat q-connection ∇ is **separable** if the eigenvalues of its N-curvature operators C_j lie in the separable closure $\overline{\mathbf{k}(X_1, ..., X_r)}_s$. E.g., every flat connection is separable in characteristic zero, or if it has rank $n < p$ in characteristic p. Let ∇ be separable and suppose v is a common eigenvector of C_j, i.e. $C_j v = \Lambda_j v$, $\Lambda_j \in \overline{\mathbf{k}(X_1, ..., X_r)}_s$. Let us say that ∇ is **Lagrangian** if for each such v we have

$$\frac{X_j}{\Lambda_i}\frac{\partial \Lambda_i}{\partial X_j} = \frac{X_i}{\Lambda_j}\frac{\partial \Lambda_j}{\partial X_i}.$$

The motivation for this terminology is similar to the differential case. Let ∇ be a regular flat q-connection, i.e., $a_i \in GL_n(\mathbf{k}[x_1^{\pm 1}, ..., x_r^{\pm 1}])$. Consider the $\mathcal{D}_{r,q}(\mathbf{k})$-module M_∇ generated by $f_1, ..., f_n$ with defining relations $\nabla_i \mathbf{f} = \mathbf{f}$, where $\mathbf{f} = (f_1, ..., f_n)^T$. Let $\mathrm{supp}\nabla$ be the support of M_∇ as a Z-module. It is clear that $\mathrm{supp}\nabla$ is given in the coordinates X_i, P_i by the equations $\chi_{\mathbf{t}}(\sum_{j=1}^r t_j P_j) = 0$ $\forall \mathbf{t} = (t_1, ..., t_r) \in \mathbf{k}^r$, i.e., consists of $(x_1^N, ..., x_r^N, \Lambda_1, ..., \Lambda_r) \in (\mathbf{k}^\times)^{2r}$ such that $\sum_{j=1}^r t_j \Lambda_j$ is an eigenvalue of $\sum_{j=1}^r t_j C_j(x_1, ..., x_r)$ for all $\mathbf{t}$. Thus, we obtain

Proposition 3.7. *A flat q-connection ∇ is Lagrangian if and only if* $\mathrm{supp}\nabla$ *is a Lagrangian subvariety of* $\mathrm{Spec}(Z)$.

3.6. **Liftable q-connections.** As in the differential case, a separable flat q-connection, even one of rank 1, need not be Lagrangian in general.

Example 3.8. Let $\mathrm{ord}(q) = N$ and ∇ be the flat q-connection of rank 1 on $(\mathbf{k}^\times)^2$ with

$$\nabla_1 = T_1^{-1},\ \nabla_2 = T_2^{-1} x_1^N.$$

Then the N-curvature of ∇ is $C_1 = 1, C_2 = X_1^N$, so ∇ is not Lagrangian.

However, the Lagrangian property holds under some assumptions. Namely, suppose $\mathrm{ord}(q) = N$. Let us say that a flat q-connection ∇ defined over $\mathbf{k}$ is **liftable** if it is the reduction to $\mathbf{k}$ of a flat $\widetilde{q}$-connection defined over some local Artinian ring S with residue field $\mathbf{k}$ in which $\widetilde{q}^N \neq 1$.

Proposition 3.9. *Let n be a positive integer which is $< p$ if $p = \mathrm{char}(\mathbf{k}) > 0$. Let q be a primitive N-th root of unity in $\mathbf{k}$. Then a liftable q-connection of rank n over $\mathbf{k}$ is Lagrangian.*

Proof. It suffices to prove the statement for $r = 2$. In this case we have $\nabla = (\nabla_x, \nabla_y)$ with $\nabla_x = T_x^{-1} a(x, y)$ and $\nabla_y = T_y^{-1} b(x, y)$ and lifting $\widetilde{\nabla} = (\widetilde{\nabla}_x, \widetilde{\nabla}_y)$ with $\widetilde{\nabla}_x = T_x^{-1} \widetilde{a}(x, y)$ and $\widetilde{\nabla}_y = T_y^{-1} \widetilde{b}(x, y)$, where T_x, T_y shift x, y by $\widetilde{q}$. Thus

$$\widetilde{\nabla}_x^N = T_x^{-N} \widetilde{A}(x, y),\ \widetilde{\nabla}_y^N = T_y^{-N} \widetilde{B}(x, y),$$

where

$$\widetilde{A}(x,y)=\widetilde{a}(\widetilde{q}^{N-1}x,y)...\widetilde{a}(x,y),\ \ \widetilde{B}(x,y)=\widetilde{b}(x,\widetilde{q}^{N-1}y)...\widetilde{b}(x,y).$$

Note that the reductions of $\widetilde{A},\widetilde{B}$ to $\mathbf{k}$ are the components $A(x,y),B(x,y)$ of the N-curvature $C(\nabla)$. The zero curvature equation

$$\widetilde{A}(x,\widetilde{q}^N y)\widetilde{B}(x,y)=\widetilde{B}(\widetilde{q}^N x,y)\widetilde{A}(x,y)$$

implies that

$$(\widetilde{A}(x,\widetilde{q}^N y)-\widetilde{A}(x,y))\widetilde{B}(x,y)+[\widetilde{A}(x,y),\widetilde{B}(x,y)]=(\widetilde{B}(\widetilde{q}^N x,y)-\widetilde{B}(x,y))\widetilde{A}(x,y).$$

Note that for any n-by-n matrices P,Q we have

$$\sum_{j=0}^{m-1}\mathrm{Tr}(Q^j[P,Q]Q^{m-1-j}P^{\ell-1})=\sum_{j=1}^{m-1}\mathrm{Tr}([P,Q^m]P^{\ell-1})=0.$$

Thus for $\ell,m\ge 1$ we have

$$\sum_{j=0}^{m-1}\mathrm{Tr}\widetilde{B}(x,y)^j\left((\widetilde{A}(x,\widetilde{q}^N y)-\widetilde{A}(x,y))\widetilde{B}(x,y)-(\widetilde{B}(\widetilde{q}^N x,y)-\widetilde{B}(x,y))\widetilde{A}(x,y)\right)\widetilde{B}(x,y)^{m-1-j}\widetilde{A}(x,y)^{\ell-1}=0.$$

The left hand side of this equation belongs to $(\widetilde{q}^N-1)S$. Since $\widetilde{q}^N-1\ne 0$ but belongs to the maximal ideal of S, this yields

$$\sum_{j=0}^{m-1}\mathrm{Tr}B(x,y)^j\left(y\tfrac{\partial A(x,y)}{\partial y}B(x,y)-x\tfrac{\partial B(x,y)}{\partial x}A(x,y)\right)B(x,y)^{m-1-j}A(x,y)^{\ell-1}=0.$$

Since $[A,B]=0$, for $m<p$ this implies

$$\mathrm{Tr}\left(y\tfrac{\partial A(x,y)}{\partial y}B(x,y)-x\tfrac{\partial B(x,y)}{\partial x}A(x,y)\right)A(x,y)^{\ell-1}B(x,y)^{m-1}=0. \tag{3.1}$$

Near a generic point (x,y), the matrices A,B have constant multiplicities m_k of joint eigenvalues (α_k,β_k). Let $\mathbf{e}_k=\mathbf{e}_k(x,y)$ be the projectors onto the corresponding generalized joint eigenspaces which commute with A,B. These projectors are polynomials of A,B which can be taken to be of degree $<n$ in each variable, so (3.1) implies

$$\mathrm{Tr}\left(\alpha_k^{-1}y\tfrac{\partial A}{\partial y}-\beta_k^{-1}x\tfrac{\partial B}{\partial x}\right)\mathbf{e}_k=0. \tag{3.2}$$

Recall that if $e=e(t),M=M(t)$ are 1-parameter families in Mat_n such that $e^2=e$ and $[M,e]=0$ then $\mathrm{Tr}(Me')=0$: indeed, we have $ee'=e'(1-e)$, hence $ee'e=(1-e)e'(1-e)=0$, and $M=eMe+(1-e)M(1-e)$, so $\mathrm{Tr}(Me')=\mathrm{Tr}(eMee')+\mathrm{Tr}((1-e)M(1-e)e')=\mathrm{Tr}(Mee'e)+\mathrm{Tr}(M(1-e)e'(1-e))=0$. Thus $\mathrm{Tr}(A\partial_y\mathbf{e}_k)=\mathrm{Tr}(B\partial_x\mathbf{e}_k)=0$. Hence equation (3.2) can be written as

$$\alpha_k^{-1}y\frac{\partial\mathrm{Tr}A\mathbf{e}_k}{\partial y}-\beta_k^{-1}x\frac{\partial\mathrm{Tr}B\mathbf{e}_k}{\partial x}=0,$$

i.e.,

$$m_k(\alpha_k^{-1}y\tfrac{\partial\alpha_k}{\partial y}-\beta_k^{-1}x\tfrac{\partial\beta_k}{\partial x})=0.$$

Since $m_k\le n<p$ and $(N,p)=1$ if $p>0$, this is equivalent to

$$\frac{Y}{\alpha_k}\frac{\partial\alpha_k}{\partial Y}=\frac{X}{\beta_k}\frac{\partial\beta_k}{\partial X},$$

where $X=x^N$, $Y=y^N$, as claimed. □

Example 3.10. However, if $\mathrm{char}(\mathbf{k}) = p$ then a liftable separable q-connection of rank $n \geq p$ need not be Lagrangian. Namely, let $\mathbf{k}$ be a finite field of characteristic p and K be a local field (necessarily of characteristic zero) with residue field $\mathbf{k}$ such that its ring of integers $\mathcal{O}_K$ contains a primitive root of unity q of order pN. Let L, σ be as in Example 2.4, and ∇ be the q-connection of rank p on the 2-torus $\mathbb{G}_m^2$ defined over $S := \mathcal{O}_K/(q^N - 1)^2$ by

$$\nabla_x = T_x^{-1} q^{NL},\ \nabla_y = T_y^{-1} x^N \sigma.$$

It is easy to check that this q-connection is flat. However, reduction of ∇ to $\mathbf{k}$ yields the q-connection

$$\nabla_x = T_x^{-1},\ \nabla_y = T_y^{-1} x^N \sigma$$

with N-curvature $C_x = 1, C_y = X^N \sigma^N$, which is not Lagrangian.

3.7. **The q-analog of Bitoun's theorem.** Let $\mathbf{k}$ be an algebraically closed field (of any characteristic). Suppose that $q \in \mathbf{k}$ is not a root of unity; thus if $\mathrm{char}(\mathbf{k}) = p > 0$ then q is necessarily transcendental over $\mathbb{F}_p$. Let M be a finitely generated $\mathcal{D}_{r,q}(\mathbf{k})$-module. Analogously to $D_r(\mathbf{k})$-modules, M is defined over a finitely generated subring $R \subset \mathbf{k}$ and we can pick a finitely generated $\mathcal{D}_{r,q}(R)$-module M_R with an isomorphism $\mathbf{k} \otimes_R M_R \cong M$. For every maximal ideal $\mathfrak{m}$ of R with residue field $k_\mathfrak{m}$ let $M_\mathfrak{m}$ be the finitely generated $D_{r,q}(k_\mathfrak{m})$-module $k_\mathfrak{m} \otimes_R M$.

Let us say that a maximal ideal $\mathfrak{m} \subset R$ is a q-**torsion point** if there exists a positive integer N such that q is a primitive root of unity of order N in $R/\mathfrak{m}$. It is clear that q-torsion points are dense in $\mathrm{Spec}(R)$.

If $\mathfrak{m}$ is a q-torsion point then we may define the support $\mathrm{supp}(M_\mathfrak{m})$ of $M_\mathfrak{m}$ as a Z-module, which by analogy with the differential case we will call the p-**support** of M at $\mathfrak{m}$ (even though the field $k_\mathfrak{m}$ may have the same characteristic as $\mathbf{k}$, in particular characteristic zero).

Theorem 3.11. *If $M \neq 0$ is a holonomic $\mathcal{D}_{r,q}(\mathbf{k})$-module then* $\mathrm{supp}(M_\mathfrak{m})$ *is Lagrangian for a generic q-torsion point $\mathfrak{m} \in \mathrm{Spec}(R)$.*

Example 3.12. If $\mathbf{k} = \overline{\mathbb{Q}}$ then for each maximal ideal $\mathfrak{m} \in \mathrm{Spec}(R)$, $R/\mathfrak{m}$ is a finite field, so $\mathfrak{m}$ is a torsion point. Thus Theorem 3.11 holds for all $\mathfrak{m}$ lying over p for almost all primes p.

Proof. By Theorem 3.3, it suffices to consider the case when M is a finite $\mathbf{k}[x_1^{\pm 1}, ..., x_r^{\pm 1}]$-module. But then the result follows from Proposition 3.9. □

It is useful to make a slight generalization of Theorem 3.11. Let $\Lambda = (\lambda_{ij})$ be a skew-symmetric integer s-by-s matrix. Denote by $\mathcal{D}_{\Lambda,q} = \mathcal{D}_{\Lambda,q}(\mathbf{k})$ the $\mathbf{k}$-algebra with invertible generators $X_1, ..., X_s$ and defining relations

$$X_i X_j = q^{\Lambda_{ij}} X_j X_i.$$

Let $K := \mathrm{Ker}\Lambda \subset \mathbb{Z}^s$ be the subgroup of vectors ℓ such that $\Lambda \ell = 0$. This is a saturated subgroup, so it admits a complement L such that $\mathbb{Z}^s = K \oplus L$. Picking a basis of L, we see that $\mathcal{D}_{\Lambda,q}$ is isomorphic to $\mathcal{D}_{\Lambda',q}$ where $\Lambda' = \Lambda_* \oplus 0$ and Λ_* is nondegenerate. Thus we may assume that Λ is in this form to start with, i.e., $\mathcal{D}_{\Lambda,q} = \mathcal{D}_{\Lambda_*,q} \otimes \mathbf{k}[\mathbb{Z}^{s-2r}]$, where $2r$ is the size of Λ_* (i.e., the rank of Λ). Moreover, pick a symplectic $\mathbb{Q}$-basis v_i for Λ_* and set $Y_j := \prod_{i=1}^{2r} X_i^{dv_{ij}}$ where d is the common denominator of v_{ij}. Then Y_i generate a subalgebra $\mathcal{D}_{r,q^{d^2}} \subset \mathcal{D}_{\Lambda_*,q}$. Thus $\mathcal{D}_{\Lambda,q}$ is a finite extension of $\mathcal{D}_{r,q^{d^2}} \otimes \mathbf{k}\mathbb{Z}^{s-2r}$ (isogeny of quantum tori).

Similarly to $\mathcal{D}_{r,q}$, when we specialize $\mathcal{D}_{\Lambda,q}$ at a q-torsion point, it acquires a big center Z such that $\mathrm{Spec}(Z)$ is a Poisson torus $\mathbb{G}_m^s$ with the log-canonical Poisson bracket given by

$$\{X_i, X_j\} = \lambda_{ij} X_i X_j.$$

If Λ is degenerate, this bracket is also, but one can still talk about Lagrangian subvarieties of $\mathbb{G}_m^s$ - coisotropic subvarieties of dimension r. Also by analogy with the case of $\mathcal{D}_{r,q}$ one can define the notion of a holonomic module - a $\mathcal{D}_{\Lambda,q}$-module of GK dimension $\le r$. The fact that $\mathcal{D}_{\Lambda,q}$ is a finite extension of $\mathcal{D}_{r,q^{d^2}} \otimes \mathbf{k}\mathbb{Z}^{s-2r}$ implies that such modules have finite length, and Theorem 3.11 implies

Theorem 3.13. *If $M \neq 0$ is a holonomic $\mathcal{D}_{r,q}(\mathbf{k})$-module then* $\mathrm{supp}(M_{\mathfrak{m}})$ *is Lagrangian for a generic q-torsion point $\mathfrak{m} \in \mathrm{Spec}(R)$.*

3.8. **p-supports of holonomic modules over quantum cluster algebras.** Now we would like to apply Theorem 3.13 to quantum cluster algebras as defined by Berenstein and Zelevinsky ([BZ]). We first recall the basic definitions of the theory of cluster algebras and Poisson cluster algebras ([FZ1, GSV]).

Let $m \ge n$, and $B \in \mathrm{Mat}_n(\mathbb{Z})$ be skew-symmetrizable, i.e., there exist positive integers d_i, $1 \le i \le n$, such that for $D = \mathrm{diag}(d_1, ..., d_n)$ the matrix DB is skew-symmetric. Let M be an integer $n \times (m-n)$ matrix, and let $\widetilde{B} = (b_{ij}) = \binom{B}{M}$ be the corresponding $m \times n$ matrix. The matrix $\widetilde{B}$ is then called a **seed** (or **exchange**) matrix. Given a seed matrix $\widetilde{B}$ and $1 \le k \le n$, we can define the **cluster mutation** of $\widetilde{B}$ at k by

$$\mu_k(\widetilde{B}) = \widetilde{B}' = (b'_{ij}),\ \ b'_{ij} := \begin{cases} -b_{ij} \text{ if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} \text{ else} \end{cases}$$

This is accompanied by the change of generators $X_1, ..., X_m$ in the field $\mathcal{F} = \mathbb{Q}(X_1, ..., X_m)$ (the initial **cluster** in $\mathcal{F}$ corresponding to the seed $\widetilde{B}$) which changes only the generator X_k to

$$X'_k := \frac{\prod_{i: b_{ik} > 0} X_i^{b_{ik}} + \prod_{i: b_{ik} < 0} X_i^{-b_{ik}}}{X_k} \qquad (3.3)$$

(i.e., $X'_j = X_j$ if $j \neq k$). One says that $X'_1, ..., X'_m$ is the cluster in $\mathcal{F}$ corresponding to the seed $\widetilde{B}'$, and X'_i are called the **cluster variables** of this cluster. This way one can create more and more clusters and seeds, by successively performing mutations at various k (noting however that $\mu_k^2 = \mathrm{id}$). Note that the last $m - n$ variables are not changed under mutation, which is why they are called **frozen variables**.

Now one defines the **upper cluster algebra** $\mathbf{U}(\widetilde{B})$ to be the intersection of all the subalgebras $\mathbf{U}_C$ generated inside $\mathcal{F}$ by the cluster variables and their inverses for all clusters C that can be obtained by successive mutations from the initial cluster $C_0 = (X_1, ..., X_m)$.

It is nontrivial even that $\mathbf{U}(\widetilde{B})$ contains anything beyond constants. But it is in fact quite large due to the **Laurent phenomenon** [FZ1] which is what really makes the cluster algebras tick. Namely, $\mathbf{U}(\widetilde{B})$ contains the **cluster algebra** $\mathbf{A}(\widetilde{B})$ generated by the cluster variables of all clusters (but without inverses) and also inverses of the frozen variables.

Now suppose $\Lambda \in \mathrm{Mat}_m(\mathbb{Z})$ is a skew-symmetric matrix. We say that Λ is compatible to an integer $m \times n$-matrix $\widetilde{B}$, or that $(\Lambda, \widetilde{B})$ is a **compatible pair**, if

$$\sum_{k=1}^{m} b_{ki}\lambda_{kj} = \delta_{ij}d_j,$$

where d_i for $1 \le i \le n$ are positive integers, i.e, $\widetilde{B}^T\Lambda = (D|0)$ where $D = \mathrm{diag}(d_1, ..., d_n)$. In this case, $\widetilde{B}$ has full rank n, and $DB = \widetilde{B}^T\Lambda B$ is skew-symmetric, so $\widetilde{B}$ is a seed matrix.

Now let $(\Lambda, \widetilde{B})$ is a compatible pair and endow $\mathcal{F}$ with a Poisson bracket

$$\{X_i, X_j\} = \lambda_{ij}X_iX_j,$$

i.e. the log-canonical bracket in the initial cluster C_0 attached to the matrix Λ. Then we have

$$\{X_i', X_j'\} = \lambda_{ij}'X_i'X_j'$$

for an appropriate matrix $\Lambda' = (\lambda_{ij}')$ compatible to $\widetilde{B}' = \mu_k(\widetilde{B})$, and the pair $(\Lambda', \widetilde{B}')$ is called the mutation at k of $(\Lambda, \widetilde{B})$ and denoted $\mu_k(\Lambda, \widetilde{B})$. Namely,

$$\Lambda' = E_k^T\Lambda E_k$$

where $(E_k)_{ij} = \delta_{ij}$ unless $j = k$, $(E_k)_{kk} = -1$, and $(E_k)_{ik} = \max(0, b_{ik})$. This implies that the algebra $\mathbf{U}(\widetilde{B})$ is naturally a Poisson algebra. This algebra is called a **cluster Poisson algebras** and $\mathbf{U}(\Lambda, \widetilde{B})$.

A great feature of log-canonical Poisson brackets is that they are easy to quantize: the quantization is given by the quantum torus algebra $\mathcal{D}_{\Lambda,q}$. Remarkably, it turns out ([BZ]) that this process is compatible with cluster mutations (due to the **quantum Laurent phenomenon**) which gives rise to the **upper quantum cluster algebra** $\mathbf{U}_q(\Lambda, \widetilde{B})$, the intersection of the algebras $\mathcal{D}_{\Lambda_C,q}$ for various clusters C inside their common noncommutative fraction field $\mathcal{F}_q$, which is a flat deformation of $\mathbf{U}_q(\Lambda, \widetilde{B})$.

If the algebra $\mathbf{U}(\widetilde{B})$ for a given seed matrix $\widetilde{B}$ is finitely generated, then $\mathrm{Spec}\mathbf{U}(\widetilde{B})$ is an irreducible normal affine variety $Y(\widetilde{B})$ equipped with a smooth open subset $Y^\circ(\widetilde{B}) \subset Y(\widetilde{B})$ with complement of codimension ≥ 2 covered by toric charts $Y_C(\widetilde{B})$ labeled by various clusters C, so that the clutching maps between charts corresponding to clusters connected by a single mutation are given by (3.3). Note that in this case $Y^\circ(\widetilde{B})$ is covered by a finite subset of such charts, since it is quasi-compact. If in addition $Y(\widetilde{B}) = Y(\Lambda, \widetilde{B})$ is equipped with a cluster Poisson bracket defined by a skew-symmetric matrix Λ then the toric charts carry log-canonical Poisson structures of rank $2r$, so the Poisson structure of $Y(\Lambda, \widetilde{B})$ has constant rank $2r$ on $Y^\circ(\Lambda, \widetilde{B})$ (outside of this set the rank can drop). In this case, the algebra $\mathbf{U}_q(\Lambda, \widetilde{B})$ is a quantization of the Poisson variety $Y(\Lambda, \widetilde{B})$.

In this case, let $\mathcal{C}_q(\Lambda, \widetilde{B})$ be the category of finitely generated $\mathbf{U}_q(\Lambda, \widetilde{B})$-modules, and $\mathcal{C}_q^0(\Lambda, \widetilde{B})$ be its Serre subcategory of modules M for which $M_C := \mathcal{D}_{\Lambda_C,q} \otimes_{\mathbf{U}_q(\Lambda,\widetilde{B})} M$ vanishes for each cluster C. Let $\overline{\mathcal{C}}_q(\Lambda, \widetilde{B}) := \mathcal{C}_q(\Lambda, \widetilde{B})/\mathcal{C}_q^0(\Lambda, \widetilde{B})$ be the Serre quotient. For each $M \in \overline{\mathcal{C}}_q(\Lambda, \widetilde{B})$, the localization M_C is well defined. Let us say that $M \in \overline{\mathcal{C}}_q(\Lambda, \widetilde{B})$ is holonomic if M_C is holonomic for all clusters C. If q is specialized to a q-torsion point $\mathfrak{m} \subset R$, for every C consider the p-support $\mathrm{supp}(M_{C,\mathfrak{m}}) \subset Y_C^{(1)}(B)$, where $Y_C^{(1)}(B) = \mathrm{Spec}(Z_C)$, the spectrum of the center Z_C of $\mathcal{D}_{\Lambda_C,q}$. It is easy to see that these supports glue together into a closed subvariety $\mathrm{supp}(M_{\mathfrak{m}})$ of $Y^{(1)}(B)$, the cluster Poisson variety glued from the toric charts $Y_C^{(1)}(B)$. We call this subvariety the p-support of M.

Theorem 3.13 immediately implies the following theorem.

Theorem 3.14. *If $M \neq 0$ is a holonomic object of $\overline{\mathcal{C}}_q(\Lambda, \widetilde{B})$ then* $\mathrm{supp}(M_{\mathfrak{m}})$ *is Lagrangian for a generic q-torsion point* $\mathfrak{m} \in \mathrm{Spec}(R)$.

Remark 3.15. 1. Theorem 3.14 straightforwardly generalizes to the setting when some of the frozen variables are not inverted.

2. It would be interesting to strengthen Theorem 3.14 so that it applies to actual finitely generated $\mathbf{U}_q$-modules, not just objects of the Serre quotient, with p-support possibly contained in the Frobenius twist of the complement of the toric charts, $Y \setminus Y^\circ$, since taking the Serre quotient actually kills the most interesting modules. The methods of this paper are too crude to do this in general, but one can possibly treat the case of multiplicative Higgs branches (quasi-Hamiltonian reductions from q-differential operators on a vector space, such as multiplicative quiver varieties) similarly to how we proposed to treat ordinary Higgs branches in Subsection 2.6.

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