Quantum polylogarithms

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To Maxim Kontsevich for his 60th birthday

Contents

Abstract

Multiple polylogarithms are periods of variations of mixed Tate motives. Conjecturally, they deliver all such periods.

We introduce deformations of multiple polylogarithms depending on a parameter $\hbar \in \mathbb{C}$. We call them *quantum polylogarithms*. Their asymptotic expansion as $\hbar \to 0$ recovers multiple polylogarithms. The quantum dilogarithm was studied by Barnes in the XIX century. Its exponent appears in many areas of Mathematics and Physics.

Quantum polylogarithms satisfy a holonomic systems of modular difference equations with coefficients in variations of mixed Hodge-Tate structures of motivic origin.

If $\hbar \in \mathbb{Q}$, the quantum polylogarithms can be expressed via multiple polylogarithms.

However if $\hbar \neq \mathbb{Q}$, quantum polylogarithms are not periods of variations of mixed motives, i.e. they can not be expressed by integrals of rational differential forms on algebraic varieties. Instead, quantum polylogarithms are integrals of differential forms built from both rational functions and exponentials of rational functions. We call them *rational exponential integrals.*

We suggest that quantum polylogarithms reflect a very general phenomenon:

Periods of variations of mixed motives should have quantum deformations.

1 Introduction

1.1 The dilogarithm and the quantum dilogarithm

The dilogarithm function is defined by the following power series:

$$\mathrm{Li}_2(z) := \sum_{k>0} \frac{z^k}{k^2}.$$

It can be continued analytically to a function on a cover of $\mathbb{CP}^1 - \{0, 1, \infty\}$ via the integral

$$\mathrm{Li}_2(z) := -\int_0^z \log(1-t)\frac{dt}{t}.$$

One of the key features of the dilogarithm is that it satisfies Abel's five term relation.

The dilogarithm power series admit a $q-$deformation:

$$\mathrm{Li}_{1,1}(z;q) := \sum_{k=1}^{\infty} \frac{z^k}{(q^k - q^{-k})k}.$$

Its exponent is the inverse of the Pochhammer symbol. Precisely, set

$$\Psi_q(z) := \frac{1}{(1+qz)(1+q^3z)(1+q^5z)\cdot \ \ldots}.$$

Then one has

$$\log \Psi_q(z) = -\mathrm{Li}_{1,1}(-z;q). \tag{1}$$

Setting $q = e^{\pi i \hbar}$, and letting $\hbar \to 0$, we get the asymptotic expansion

$$\log \Psi_q(z) \sim_{\hbar\to 0} \ -\frac{\mathrm{Li}_2(-z)}{2\pi i \hbar}. \tag{2}$$

It satisfies the quantum pentagon relation, discovered by Faddeev and Kashaev [FK]. Namely, consider variables X, Y satisfying the relation $XY = q^2YX$. Then we have the identity of $q-$commutative power series in X, Y:

$$\Psi_q(X)\Psi_q(Y) = \Psi_q(Y)\Psi_q(X+Y)\Psi_q(X),$$

Its quasiclassical limit recovers Abel's five term relation for the dilogarithm.

The power series $\Psi_q(z)$ converge only if $|q| < 1$. Remarkably, the quotient

$$\Phi_\hbar(\omega) = \frac{\Psi_q(e^\omega)}{\Psi_{q^*}(e^{\omega/\hbar})}, \qquad q = e^{i\pi\hbar}, \quad q^* = e^{-i\pi/\hbar}, \quad \mathrm{Im}\hbar > 0 \tag{3}$$

has excellent analytic properties. In particular, it is a meromorphic function in ω, depending on a complex parameter $\hbar$. To see this, recall the integral introduced and studied by Barnes [Ba]:

$$\mathcal{F}^\hbar(\omega) := \int_{\mathbb{R}+i0} \frac{e^{-ip\omega}}{\mathfrak{sh}(\pi p)\mathfrak{sh}(\pi\hbar p)}\frac{dp}{p}, \qquad \mathfrak{sh}(p) = e^p - e^{-p}. \tag{4}$$

The integration contour $\mathbb{R}+i0$ is the limit of the contour $\mathbb{R}+i\varepsilon$, $\varepsilon > 0$ when $\varepsilon \to 0$. The integral is well defined for any complex values of $\hbar$, convergent for $\mathrm{Im}(\omega) < \pi(1+|\mathrm{Re}(\hbar)|)$, and satisfies difference relations under the shift of ω by $2\pi i$ and $2\pi i\hbar$

$$\begin{aligned} \mathcal{F}^\hbar(\omega + 2\pi i) &= \mathcal{F}^\hbar(\omega) - \log(1 + e^{i\pi\hbar}e^\omega) \\ \mathcal{F}^\hbar(\omega + 2\pi i\hbar) &= \mathcal{F}^\hbar(\omega) - \log(1 + e^{i\pi/\hbar}e^{\omega/\hbar}) \end{aligned} \tag{5}$$

which allow to extend it to a multivalued analytic function in $\omega \in \mathbb{C}$. Finally, one has

$$\Phi_\hbar(\omega) = \exp(-\mathcal{F}^\hbar(\omega)). \tag{6}$$

We call the function $\mathcal{F}^\hbar(\omega)$ the *quantum dilogarithm*, although this name is often used for its exponent. The quantum dilogarithm and its relatives appear in Statistical Physics [Bax], Liouville theory[DO], [ZZ], quantum groups [F], quantum higher Teichmuller theory [K], [CF], [FKV], [FG1] - [FG2], [GS], quantization of cluster varieties [FG3], and many other areas [V]. The quantum dilogarithm satisfies the quantum pentagon relation, which plays an important role in its applications. Namely, the function $\Phi_\hbar(\omega)$ is well defined and unitary on the real line. Consider the unitary operator K in $L_2(\mathbb{R})$ given by the multiplication by $\Phi_\hbar(w)$, followed by the Fourier transform:

$$K : f(w) \longrightarrow \int_{\mathbb{R}} f(w)\Phi_\hbar(w)e^{-\frac{i\omega\xi}{2\pi\hbar}}dt.$$

Then $K^5 = c \cdot \mathrm{Id}$, where c is a constant, $|c| = 1$. Its quasiclassical limit delivers the five term relation for the dilogarithm. The remarkable analytical properties of the function $\mathcal{F}^\hbar(\omega)$ together with the quantum deformation of the five term relation convince that it is the natural quantum deformation of the dilogarithm. Let us now look at generalizations of the dilogarithm function.

1.2 Multiple polylogarithms

Recall the classical polylogarithm series:

$$\mathrm{Li}_n(z) := \sum_{k>0} \frac{z^k}{k^n}.$$

They make sense for any integer n. For $n \leq 0$ they are rational functions in z. For $n > 0$ they are convergent for $|z| < 1$, and admit an analytic continuation to a cover of $\mathbb{CP}^1 - \{0, 1, \infty\}$.

Multiple polylogarithms are defined by the power series expansion [G94]:

$$\mathrm{Li}_{n_1,\ldots,n_m}(z_1,\ldots,z_m) := \sum_{0<k_1<k_2<\ldots<k_m} \frac{z_1^{k_1} z_2^{k_2} \ldots z_m^{k_m}}{k_1^{n_1} k_2^{n_2} \ldots k_m^{n_m}}, \tag{7}$$

which are convergent if $|z_i| < 1$. Here m is the *depth*, and $|\mathbf{n}| := n_1 + \ldots + n_m$ is the *weight*.

Multiple polylogarithms admit an iterated integral presentation, which allows to continue them analytically. Namely, given meromorphic $1-$forms $\omega_i(t)$ on $\mathbb{C}$, and a path $\gamma : [0,1] \longrightarrow \mathbb{C}$ not intersecting their poles, recall the iterated integrals on the line:

$$\int_\gamma \omega_1(t) \circ \ldots \circ \omega_m(t) := \int_{0 \le t_1 \le \ldots \le t_m \le 1} \gamma^*\omega_1(t_1) \wedge \ldots \wedge \gamma^*\omega_m(t_m).$$

To present multiple polylogarithms by iterated integrals on the line, set

$$\mathrm{I}_{n_1,\ldots,n_m}(0; z_1, \ldots, z_m; z_{m+1}) := \int_0^{z_{m+1}} \underbrace{\frac{dt}{z_1 - t} \circ \frac{dt}{t} \circ \ldots \frac{dt}{t}}_{n_1 \text{ differentials}} \circ \ldots \circ \underbrace{\frac{dt}{z_m - t} \circ \frac{dt}{t} \circ \ldots \circ \frac{dt}{t}}_{n_m \text{ differentials}}. \tag{8}$$

Here the iterated integral is over a path from 0 to z_{m+1}. It is a multivalued analytic function on the space of collections of distinct points $(0, z_1, ..., z_{m+1})$ in $\mathbb{C}$. Then by [G01, Theorem 2.2]:[1]

$$\mathrm{I}_{n_1,\ldots,n_m}(0; z_1, \ldots, z_m; z_{m+1}) \ = \ \mathrm{Li}_{n_1,\ldots,n_m}\Big(\frac{z_2}{z_1}, \frac{z_3}{z_2}, \ldots, \frac{z_{m+1}}{z_m}\Big). \tag{10}$$

In Section 2 we establish a new integral presentation for multiple polylogarithms:

$$\begin{gathered}\mathrm{Li}_{n_1,\ldots,n_m}(e^{\omega_1-\omega_2}, e^{\omega_2-\omega_3}, \ldots, -e^{\omega_m}) = \\ i^{|\mathbf{n}|-m} \int_{(\mathbb{R}+i0)^m} \frac{e^{-ip_1\omega_1}}{\mathfrak{sh}(\pi p_1)} \frac{dp_1}{p_1^{n_1}} \wedge \ldots \wedge \frac{e^{-ip_m\omega_m}}{\mathfrak{sh}(\pi p_m)} \frac{dp_m}{(p_1 + \ldots + p_m)^{n_m}}.\end{gathered} \tag{11}$$

So we have three different presentations of multiple polylogarithms: as power series (8), via iterated integrals (10), and using integral presentation (11). We note that the two integrals (8) and (11) are related via power series (7) rather than directly.

Iterated integral presentation (10) shows that multiple polylogarithms are periods of mixed Tate motives. Conjecturally, they provide all such periods [G94, Conjecture 17].

We introduce a deformation of multiple polylogarithms, called *quantum polylogarithms*, depending on a parameter $\hbar \in \mathbb{C}$. Their asymptotic expansion at $\hbar \to 0$ recovers multiple polylogarithms.

Quantum polylogarithms provide quantum deformation of all periods of mixed Tate motives.

I suggest that periods of any variations of mixed motives admit quantum deformation.

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[1] The *positive locus* $\mathcal{M}^+_{0,n+3}$ of the moduli space $\mathcal{M}_{0,n+3}$ the set of ordered configurations of points on $\mathbb{RP}^1$ modulo the diagonal $\mathrm{PGL}_2(\mathbb{R})$-action, whose order is compatible with one of the circle orientations. We have

$$(\infty, z_1, ..., z_m, 1, 0) \in \mathcal{M}^+_{0,n+3} \quad \text{if and only if} \quad z_1 > \ldots > z_m > 1. \tag{9}$$

Reversing the order preserves the positive locus. Iterated integral (8) has a natural branch on the positive locus, provided by the path $\gamma = [0,1]$. The positivity just means that arguments of multiple polylogarithm series (10) are positive numbers smaller than 1, so the series are convergent.

1.3 Quantum polylogarithms.

Let us introduce first the *kernel function*.

Definition 1.1. *Let a, b be a pair of non-negative integers. The kernel function is defined by*

$$K^{\hbar}_{a,b}(p;\omega) := \frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\hbar p)}. \tag{12}$$

The *depth m* quantum polylogarithms are functions $\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,\ldots,\omega_m)$ in m complex variables ω_i which depend on a triple of m-tuples of integers

$$\mathbf{a} = (a_1,...,a_m), \quad \mathbf{b} = (b_1,...,b_m), \quad \mathbf{n} = (n_1,...,n_m), \qquad a_i, b_i \in \mathbb{Z}_{\geq 0},\ n_i \in \mathbb{Z}.$$

Definition 1.2. *The depth m quantum polylogarithms are the integrals*

$$\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,...,\omega_m) := i^{|\mathbf{n}|-m}\int_{(\mathbb{R}+i0)^m}\bigwedge_{k=1}^{m} K^{\hbar}_{a_k,b_k}(p_k;\omega_k)\frac{dp_k}{(p_1+\ldots+p_k)^{n_k}}. \tag{13}$$

We define the weight of π and ω_j to be 1, and the weight of the quantum polylogarithm to be

$$|\mathbf{n}| := n_1 + ... + n_m. \tag{14}$$

The integral converges if $|\mathrm{Im}\ \omega_i| < \pi(a_i + b_i|\mathrm{Re}(\hbar)|)$. It extends to a multivalued analytic function in $(\omega_1,...,\omega_m) \in \mathbb{C}^m$ using the difference relations (35) for quantum polylogarithms.

As an analytic function of $\hbar$, the $\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,...,\omega_m)$ extends to the complex plane with the negative real axis $\hbar < 0$ removed. The integral converges for any $n_i \in \mathbb{R}$ if one of the integers a_i, b_i is positive. Here are two examples.

1. The depth one quantum polylogarithms are the following integrals

$$\mathcal{F}^{\hbar}_{a,b,n}(\omega) := i^{n-1}\cdot\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\hbar p)}\cdot\frac{dp}{p^n}, \qquad a,b\geq 0.$$

2. The depth two quantum polylogarithms are given by

$$\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,\omega_2) := \\ i^{|\mathbf{n}|-2}\cdot\int_{(\mathbb{R}+i0)^2}\frac{e^{-ip_1\omega_1}}{\mathfrak{sh}^{a_1}(\pi p_1)\mathfrak{sh}^{b_1}(\pi\hbar p_1)}\frac{e^{-ip_2\omega_2}}{\mathfrak{sh}^{a_2}(\pi p_2)\mathfrak{sh}^{b_2}(\pi\hbar p_2)}\cdot\frac{dp_1}{p_1^{n_1}}\wedge\frac{dp_2}{(p_1+p_2)^{n_2}}. \tag{15}$$

Theorem 1.3. *Quantum polylogarithms at the rational $\hbar \in \mathbb{Q}$ are periods of variations of mixed Tate motives of the same weight.*

See the precise statement in Theorem 1.7. In sharp contrast with this, quantum polylogarithms for irrational $\hbar \notin \mathbb{Q}$ are **not** periods of variations of mixed motives.

By Theorem 3.7, quantum polylogarithms satisfy shuffle product formulas, providing an algebra structure.

Specialising $\omega_1 = ... = \omega_m = 0$ and $\mathbf{a} = \mathbf{b} = (1, ..., 1)$, we get an $\hbar-$deformation of the depth m multiple ζ-function. For example, for the depth 2 we have

$$\zeta_\hbar(s_1, s_2) := \int_{(\mathbb{R}+i0)^2} \frac{dp_1}{\mathfrak{sh}(\pi p_1)\mathfrak{sh}(\pi\hbar p_1)p_1^{s-1}} \frac{dp_2}{\mathfrak{sh}(\pi p_2)\mathfrak{sh}(\pi\hbar p_2)(p_1+p_2)^{s_2-1}}. \tag{16}$$

The analytic continuation of these functions is obtained the same way as in [G01, Theorem 2.25]. When $s_i = n_i$ are positive integers, we get an $\hbar-$deformation of Euler's multiple ζ-values.

Interesting $q-$deformations of the multiple $\zeta-$values were considered by Okounkov [O].

1.4 $q-$deformations of multiple polylogarithms

Convention. We denote by latin letters z_i coordinates of the points on the projective line, and by the greek letters ω_i the relevant logarithmic coordinates: $z_i = e^{\omega_i}$.

Multiple polylogarithm power series (7) have a $q-$deformation:

Definition 1.4. *Let $\mathbf{a}, \mathbf{n} \in \mathbb{Z}^m_{\geq 0}$. Multiple $q-$polylogarithms are power series in $z_1, ..., z_m$:*

$$\mathrm{Li}_{\mathbf{a},\mathbf{n}}(z_1, ..., z_m; q) := \sum_{k_1,...,k_m>0}^{\infty} \frac{z_1^{k_1} z_2^{k_2} \dots z_m^{k_m}}{[k_1]_q^{a_1}[k_2]_q^{a_2}\dots[k_m]_q^{a_m} \cdot k_1^{n_1}(k_1+k_2)^{n_2}\dots(k_1+...+k_m)^{n_m}}, \quad [n]_q := q^n - q^{-n}. \tag{17}$$

The weight of the multiple $q-$polylogarithm series (17) is defined to be $\mathbf{n}$, just as in (14).

Example. The weights of the $q-$dilogarithm $\mathrm{Li}_{1,1}(z; q)$ and the quantum dilogarithm $\mathcal{F}^\hbar(\omega)$ are 1. Since the weight of $2\pi\hbar$ is 1, the weight of the function in (2) is 1, consistently with (1).

Note that unlike in series (7), the summation in (17) is over the octant $k_1, ..., k_m > 0$. Therefore there are no shuffle product formulas for the series $\mathrm{L}_{\mathbf{a},\mathbf{n}}(z_1, ..., z_m; q)$.

Power series (17) are not defined when q is a root of unity. In Section 4.2 we complement them by *companion series.* Their appropriate sums coincide with the quantum polylogarithm integrals, generalising the logarithm of relation (3), thus converging for any $\hbar$.

1.5 Connections between quantum and multiple polylogarithms

The weight is compatible with major operations with quantum polylogarithms:

- The partial derivatives decrease the weight by 1, see (51).
- The difference relations preserve the weight, see (35).

Passing from quantum polylogarithms to their exponents destroys the weights. In particular, this is why we call the function $\mathcal{F}^\hbar(\omega)$, rather than its exponent $\Phi_\hbar(\omega)$, the quantum dilogarithm.

Quantum polylogarithms are related to multiple polylogarithms in several ways:

1. *Via the asymptotic expansion as $\hbar \to 0$.* The weight $|\mathbf{n}|$ quantum polylogarithms have an asymptotic expansion as $\hbar \to 0$ of the following shape, see Theorem 3.1:

$$\sum_k (2\pi i\hbar)^{-k} \times \quad \text{sums of multiple polylogarithms of the weight } |\mathbf{n}| + k.$$

All terms of the asymptotic expansion have the same weight $|\mathbf{n}|$.

2. *Via the $\hbar = 1$ specialization.* We prove in Theorem 3.3:

Theorem 1.5. *The $\mathcal{F}^1_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1, ..., \omega_m)$ is the product of the function $\mathbf{P}_{\mathbf{a},\mathbf{b}}\Big(\frac{\omega_1}{2\pi}, ..., \frac{\omega_m}{2\pi}\Big)$, where $\mathbf{P}_{\mathbf{a},\mathbf{b}}$ is a polynomial with coefficients in $\mathbb{Q}$, and the multiple polylogarithm $\mathrm{Li}_{n_1,...,n_m}$:*

$$\mathcal{F}^1_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1, ..., \omega_m) = \mathbf{P}_{\mathbf{a},\mathbf{b}}\Big(\frac{\omega_1}{2\pi}, ..., \frac{\omega_m}{2\pi}\Big) \cdot \mathrm{Li}_{n_1,...,n_m}\Big((-1)^{a_1+b_1}e^{\omega_1}, ..., (-1)^{a_m+b_m}e^{\omega_m}\Big). \tag{18}$$

Since the weight of $\frac{\omega}{2\pi}$ is zero, both parts of the equality have the same weight $|\mathbf{n}|$.

The right hand side of (18) is a period of a variation of mixed Tate motives on $(\mathbb{C}^\times)^m$:

$$(18) = \mathbf{P}_{\mathbf{a},\mathbf{b}}\Big(\frac{\log z_1}{2\pi}, ..., \frac{\log z_m}{2\pi}\Big) \cdot \mathrm{Li}_{n_1,...,n_m}\Big((-1)^{a_1+b_1}z_1, ..., (-1)^{a_m+b_m}z_m\Big).$$

3. *Via distribution relations.* Given a pair of coprime integers r, s, Theorem 3.2 relates quantum polylogarithms at $\frac{r}{s}\hbar$ to a sum of similar quantum polylogarithms at $r\hbar$. Precisely:

Theorem 1.6. *One has distribution relations:*

$$r^{|\mathbf{n}|-m}\mathcal{F}^{\frac{r}{s}\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(r\omega_1, ..., r\omega_m) = \sum_{\alpha_j=\frac{1-r}{2}}^{\frac{r-1}{2}} \sum_{\beta_j=\frac{1-s}{2}}^{\frac{s-1}{2}} \mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\ldots, \omega_k + \frac{2\pi i}{r}\sum_{i=1}^{a_k}\alpha_j + \frac{2\pi i\hbar}{s}\sum_{j=1}^{b_k}\beta_j, \ldots). \tag{19}$$

Here the sum is over half-integers α_j, β_j if the summation limits are half-integers.

Combining Theorems 1.5 and 1.6, we express quantum polylogarithms at the rational $\hbar \in \mathbb{Q}$ via sums of the multiple polylogarithms of the same weight:

Theorem 1.7. *Quantum polylogarithm functions at $\hbar \in \mathbb{Q}$ are periods of variations of mixed Tate motives, provided by multiple polylogarithms. Precisely, set*

$$\omega'_k = \omega_k + \frac{2\pi i}{r}\sum_{i=1}^{a_k}\alpha_j + \frac{2\pi i\hbar}{s}\sum_{j=1}^{b_k}\beta_j.$$

Then, using the notation of Theorem 1.5, we have

$$r^{|\mathbf{n}|-m}\mathcal{F}^{\frac{r}{s}}_{\mathbf{a},\mathbf{b},\mathbf{n}}(r\omega_1, ..., r\omega_m) = \sum_{\alpha_j=\frac{1-r}{2}}^{\frac{r-1}{2}} \sum_{\beta_j=\frac{1-s}{2}}^{\frac{s-1}{2}} \mathbf{P}_{\mathbf{a},\mathbf{b}}\Big(\frac{\omega'_1}{2\pi}, ..., \frac{\omega'_m}{2\pi}\Big)\mathrm{Li}_{n_1,...,n_m}(e^{\omega'_1}, ..., e^{\omega'_m}). \tag{20}$$

4. *Via companion $q-$polylogarithm series.* Recall that we have

$$\mathcal{F}^{\hbar}(\omega) \stackrel{(3)}{=} \log\Psi_q(e^{\omega}) - \log\Psi_{q^*}(e^{\omega/\hbar}) \stackrel{(1)}{=} -\mathrm{Li}_{1,1}(-e^{\omega}; q) + \mathrm{Li}_{1,1}(-e^{\omega/\hbar}; q^*). \tag{21}$$

The depth m quantum polylogarithms are sums of 2^m *companion $q-$polylogarithm series* of the same weight, generalizing formula (21).

So the weight of quantum polylogarithms is compatible, in several different ways, with the weight of multiple polylogarithms. This is quite remarkable since the weight of multiple polylogarithms have a deep algebraic geometric origin, while quantum polylogarithms live outside of the traditional Algebraic Geometry.

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2 A new integral presentation for multiple polylogarithms

Theorem 2.1. *Assume that* $|\mathrm{Im}\ w_i| < \pi$ *and* $\mathrm{Re}\ w_i < 0$. *Then one has*

$$\begin{gathered}
\mathrm{Li}_{n_1,\ldots,n_m}(e^{w_1},\ldots,e^{w_{m-1}},-e^{w_m}) = \\
i^{|\mathbf{n}|-m}\int_{(\mathbb{R}+i0)^m}\frac{e^{-ip_1w_1}}{\mathfrak{sh}(\pi p_1)}\frac{dp_1}{p_1^{n_1}}\wedge\ldots\wedge\frac{e^{-i(p_1+\ldots+p_m)w_m}}{\mathfrak{sh}(\pi p_m)}\frac{dp_m}{(p_1+\ldots+p_m)^{n_m}}.
\end{gathered} \tag{22}$$

Proof. Consider the integral over $(\Omega_N + i\varepsilon)^m$, where $N > 0$ is an integer, Ω_N is the square in the upper half plane with the base $[-N, N]$, and $(\Omega_N + i\varepsilon)$ its shift by a small $\varepsilon > 0$.

We claim that this integral is convergent, and the integrals over any but the bottom side decay exponentially as $N \to \infty$. Indeed, we have

$$\begin{aligned}
&|e^{-ipw}| = e^{\mathrm{Im}(p)\mathrm{Re}(w)+\mathrm{Re}(p)\mathrm{Im}(w)}, \\
&|e^{-ipw}/\mathfrak{sh}(\pi p)| \sim_{p\to\pm\infty} e^{\mathrm{Im}(p)\mathrm{Re}(w)+\mathrm{Re}(p)(\mathrm{Im}(w)\mp\pi)}.
\end{aligned} \tag{23}$$

So $e^{-ipw}/\mathfrak{sh}(\pi p)$ decays exponentially on the left and right sides as $N \to \infty$ since $\mathrm{Re}\ w < 0$. The integral over the top side of $(\Omega_N + i\varepsilon)^m$ decays exponentially since $|\mathrm{Im}\ w| < \pi$, and therefore

$$\mathrm{Re}(p)(\mathrm{Im}(w) \mp \pi) \longrightarrow -\infty \quad \text{if} \quad p \longrightarrow \pm\infty.$$

Indeed, we have either $\mathrm{Re}(p) \to \infty$ & $\mathrm{Im}(w) - \pi < 0$, or $\mathrm{Re}(p) \to -\infty$ & $\mathrm{Im}(w) + \pi > 0$. Finally, on our contour $|p| > \varepsilon$, so the integral over the bottom side converges.

Therefore we can calculate the integral using the residue theorem. The residues are at the points $(p_1, ..., p_m) = (ik_1, ..., ik_m)$, where $k_1, ..., k_m > 0$ are integers.[2] For example, in the depth 2 case the contribution of the residue at the point $(p_1, p_2) = (ik_1, ik_2)$ is equal to[3]

$$(2\pi i)^2 \cdot \frac{i^{-2}}{(2\pi)^2}\frac{(-1)^{k_1+k_2}e^{k_1w_1}e^{w_2(k_1+k_2)}}{k_1^{n_1}(k_1+k_2)^{n_2}} = \frac{e^{k_1w_1}(-e^{w_2})^{k_1+k_2}}{k_1^{n_1}(k_1+k_2)^{n_2}}. \tag{24}$$

The $(-1)^{k_1+k_2}$ amounts to the fact that $\mathfrak{sh}(\pi p_j)$ is multiplied by $(-1)^k$ after the shift by $i\pi k$. Then the sum $\sum_{k_1>0,k_2>0}$ delivers $\mathrm{Li}_{n_1,n_2}(e^{w_1}, -e^{w_2})$. The series are convergent since $\mathrm{Re}(w_i) < 0$.

[2]See also the proof of Theorem 3.3 where we explain how the calculation of residues in the more general set up at $p_1 = ik_1, ..., p_m = ik_m$ where $k_1, ..., k_m > 0$ reduces to the calculation of the residues at $p_1 = ... = p_m = 0$.

[3]The factor $i^{|\mathbf{n}|}$ in (22) cancells with the factor $i^{|\mathbf{n}|}$ from the denominator. This is how the factor $i^{|\mathbf{n}|}$ in (22), as well as in Definition 1.2 of quantum polylogarithms, helps. Next, the factor $(2\pi i)^m$ from the Cauchy theorem cancell the factor i^{-m} in (22) and Definition 1.2.

The argument in the depth m case is the similar: the residue at $(ik_1, ..., ik_m)$ is

$$(2\pi i)^m \cdot \frac{i^{-m}}{(2\pi)^m} \frac{(-1)^{k_1+...+k_m} e^{k_1 w_1} \dots e^{(k_1+...+k_m)w_m}}{k_1^{n_1} \dots (k_1+...+k_m)^{n_m}} \\ = \frac{e^{k_1 w_1} e^{(k_1+k_2)w_2} \dots (-e^{w_m})^{k_1+...+k_m}}{k_1^{n_1}(k_1+k_2)^{n_2} \dots (k_1+...+k_m)^{n_m}}. \tag{25}$$

Theorem is proved. □

Let $\gamma = [0,1]$. Denote by $\mathrm{I}^{(\gamma)}_{n_1,\dots,n_m}(z_1,\dots,z_m)$ the iterated integral defined using this path.

Corollary 2.2. *Assume that* $|\mathrm{Im}\ \omega_i| < \pi$ *and*

$$\mathrm{Re}\ \omega_1 < \mathrm{Re}\ \omega_2 < \dots < \mathrm{Re}\ \omega_m < 0. \tag{26}$$

Then one has

$$\mathrm{I}^{(\gamma)}_{n_1,\dots,n_m}(0; e^{-\omega_1},\dots,e^{-\omega_m}; -1) \overset{(10)}{=} \\ \mathrm{Li}_{n_1,\dots,n_m}(e^{\omega_1-\omega_2}, e^{\omega_2-\omega_3},\dots,-e^{\omega_m}) = \\ i^{|\mathbf{n}|-m}\int_{(\mathbb{R}+i0)^m} \frac{e^{-ip_1\omega_1}}{\mathfrak{sh}(\pi p_1)}\frac{dp_1}{p_1^{n_1}} \wedge \dots \wedge \frac{e^{-ip_m\omega_m}}{\mathfrak{sh}(\pi p_m)}\frac{dp_m}{(p_1+\dots+p_m)^{n_m}}. \tag{27}$$

Proof. The first equality is the basic equality (10). The second gives the new integral presentation (11) for multiple polylogarithms, and follows immediately from (22). Indeed, the integrands in (22) and (27) differ only by the exponentials, which match thanks to the identity

$$e^{-ip_1\omega_1}e^{-ip_2\omega_2}\dots e^{-ip_m\omega_m} = e^{-ip_1(\omega_1-\omega_2)}e^{-i(p_1+p_2)(\omega_2-\omega_3)}\dots e^{-i(p_1+...+p_m)\omega_m}. \tag{28}$$

Note that the latter is equivalent to the identity

$$p_1\omega_1 + p_2\omega_2 + \dots + p_m\omega_m = p_1(\omega_1-\omega_2) + (p_1+p_2)(\omega_2-\omega_3) + ... + (p_1+...+p_m)\omega_m. \tag{29}$$

Note that the condition on the ω_i in Corollary 2.2 is equivalent to the one in Theorem 2.1. □

So we get new integral presentations for both multiple polylogarithms and iterated integrals. For example, for the depth $m = 2$ they look as follows:

$$\mathrm{Li}_{n_1,n_2}(e^{w_1}, -e^{w_2}) = i^{|\mathbf{n}|-2}\int_{(\mathbb{R}+i0)^2} \frac{e^{-ip_1w_1}}{\mathfrak{sh}(\pi p_1)}\frac{e^{-i(p_1+p_2)w_2}}{\mathfrak{sh}(\pi p_2)}\frac{dp_1}{p_1^{n_1}} \wedge \frac{dp_2}{(p_1+p_2)^{n_2}}. \\ \mathrm{I}_{n_1,n_2}(0; e^{-\omega_1}, e^{-\omega_2}; -1) = i^{|\mathbf{n}|-2}\int_{(\mathbb{R}+i0)^2} \frac{e^{-ip_1\omega_1}}{\mathfrak{sh}(\pi p_1)}\frac{e^{-ip_2\omega_2}}{\mathfrak{sh}(\pi p_2)}\frac{dp_1}{p_1^{n_1}} \wedge \frac{dp_2}{(p_1+p_2)^{n_2}}. \tag{30}$$

Remark. Condition (26) for real ω_i just means that $e^{-\omega_1} > \dots > e^{-\omega_m} > 1$. So the arguments of the iterated integral in (27) form a positive configuration of $m+3$ points.

3 Properties of quantum polylogarithms

3.0.1 Difference relations

Recall the kernel function

$$K^{\hbar}_{a,b}(p;\omega) := \frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\hbar p)}. \tag{31}$$

Let $\Delta_a^{(\omega)}$ be difference operators in the variable ω:

$$\Delta_a^{(\omega)} f(\omega) := f(\omega+a) - f(\omega-a), \tag{32}$$

The kernel function satisfies two difference equations in ω:

$$\begin{aligned} \Delta_{i\pi}^{(\omega)} K^{\hbar}_{a,b}(p;\omega) &= K^{\hbar}_{a-1,b}(p;\omega), \\ \Delta_{i\pi\hbar}^{(\omega)} K^{\hbar}_{a,b}(p;\omega) &= K^{\hbar}_{a,b-1}(p;\omega). \end{aligned} \tag{33}$$

Indeed, one has

$$\begin{aligned} \Delta_{i\pi\hbar}^{(\omega)} e^{-ip\omega} &= \mathfrak{sh}(\pi\hbar p)\cdot e^{-ip\omega}, \\ \Delta_{i\pi}^{(\omega)} e^{-ip\omega} &= \mathfrak{sh}(\pi p)\cdot e^{-ip\omega}. \end{aligned} \tag{34}$$

Set $\mathbf{1}_k := (0,...,0,1,0,...,0)$, where 1 is on the k-th place. Then difference relations (33) for the kernel function imply difference relations for quantum polylogarithms:[4]

$$\begin{aligned} \Delta_{i\pi}^{(\omega_k)} \mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,...,\omega_m) = \mathcal{F}^{\hbar}_{\mathbf{a}-\mathbf{1}_k,\mathbf{b},\mathbf{n}}(\omega_1,...,\omega_m), \\ \Delta_{i\pi\hbar}^{(\omega_k)} \mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,...,\omega_m) = \mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b}-\mathbf{1}_k,\mathbf{n}}(\omega_1,...,\omega_m). \end{aligned} \tag{35}$$

3.0.2 The asymptotic expansion when $\hbar \to 0$

Theorem 3.1. *When $\hbar \to 0$, the function $\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,...,\omega_m)$ has an asymptotic Laurent series expansion in $2\pi\hbar$, whose coefficients are sums of quantum polylogarithms:*

$$\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,...,\omega_m) \sim_{\hbar\to 0} (2\pi\hbar)^{-|\mathbf{b}|}\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{0},\mathbf{b}+\mathbf{n}}(\omega_1,...,\omega_m) + \dots \tag{36}$$

All terms of the asymptotic expansion have the same weight.

Proof. The leading term of the $\hbar \to 0$ asymptotic expansion of the kernel function (31) is

$$\frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\hbar p)} \sim_{\hbar\to 0} \frac{1}{(2\pi\hbar)^b}\cdot\frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)}\frac{1}{p^b}. \tag{37}$$

This implies the claim about the leading term of the asymptotic expansion. For example, in the depth two case the leading term of the expansion is

$$\frac{i^{|\mathbf{n}|-2}}{(2\pi\hbar)^{|\mathbf{b}|}}\int_{(\mathbb{R}+i0)^2}\frac{e^{-ip_1\omega_1}}{\mathfrak{sh}^{a_1}(\pi p_1)}\frac{e^{-ip_2\omega_2}}{\mathfrak{sh}^{a_2}(\pi p_2)}\frac{dp_1}{p_1^{b_1+n_1}}\frac{dp_2}{p_2^{b_2}(p_1+p_2)^{n_2}}.$$

[4] We assume that $\mathbf{a}-\mathbf{1}_k$ and $\mathbf{b}-\mathbf{1}_k$ are still non-negative integers, to avoid convergence issues.

To get all terms of the asymptotic expansion, we write the Laurent series expanding $\mathfrak{sh}(\pi\hbar p_k)$ in $\pi\hbar p_k$. Since the weight of $\pi\hbar p_k$ is zero, the weight conservation is clear.

To get the rest of the terms of the asymptotic expansion, we use identity

$$\frac{1}{p_2(p_1+p_2)} = \frac{1}{p_1}\Big(\frac{1}{p_2} - \frac{1}{p_1+p_2}\Big). \tag{38}$$

and proceed by the induction on $b_2 + n_2$, till either $b_2 = 0$ or $n_2 = 0$. If $b_2 = 0$, we get the double polylogarithm. If $n_2 = 0$, we get a product of two depth one polylogarithms.

The case $m > 2$ is similar. □

3.0.3 Distribution relations

Theorem 3.2. *One has distribution relations:*

$$\begin{aligned} & r^{|\mathbf{n}|-m}\mathcal{F}^{\frac{r}{s}\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(r\omega_1, ..., r\omega_m) = \\ & \sum_{\alpha_j=\frac{1-r}{2}}^{\frac{r-1}{2}} \sum_{\beta_j=\frac{1-s}{2}}^{\frac{s-1}{2}} \mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\ldots, \omega_k + \frac{2\pi i}{r}\sum_{i=1}^{a_k}\alpha_j + \frac{2\pi i\hbar}{s}\sum_{j=1}^{b_k}\beta_j, \ldots). \end{aligned} \tag{39}$$

Equivalently, the functions $\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\zeta_1, ..., \zeta_m)$ in the bottom line have the arguments

$$\zeta_k := \omega_k + \frac{2\pi i}{r}\sum_{i=1}^{a_k}\alpha_j + \frac{2\pi i\hbar}{s}\sum_{j=1}^{b_k}\beta_j, \qquad k = 1, ..., m.$$

Here the sum is over half-integers α_j, β_j if the summation limits are half-integers.

Proof. Write the identity $\mathfrak{sh}(rx) = \mathfrak{sh}(x)(e^{(r-1)x} + e^{(r-3)x} + \ldots + e^{(1-r)x})$ as

$$\frac{1}{\mathfrak{sh}(x)} = \frac{e^{(r-1)x} + e^{(r-3)x} + \ldots + e^{(1-r)x}}{\mathfrak{sh}(rx)}. \tag{40}$$

Set $q = pr$ in the kernel function:

$$K^{r/s}_{a,b}(p; r\omega) := \frac{e^{-ipr\omega}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\hbar p)} = \frac{e^{-iq\omega}}{\mathfrak{sh}^a(\pi q/r)\mathfrak{sh}^b(\pi q/s)}.$$

Then using (40) we write this as

$$\frac{e^{-iq\omega}\Big(e^{\frac{r-1}{r}\pi q} + e^{\frac{r-3}{r}\pi q} + \ldots + e^{\frac{1-r}{r}\pi q}\Big)^a \Big(e^{\frac{s-1}{s}\pi q} + e^{\frac{s-3}{s}\pi q} + \ldots + e^{\frac{1-s}{s}\pi q}\Big)^b}{\mathfrak{sh}^a(\pi q)\mathfrak{sh}^b(\pi q)} \tag{41}$$

The claim follows immediately from this by expanding the products. □

3.0.4 The value at $\hbar = 1$

Recall the basic quantum polylogarithm functions $\mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1, ..., \omega_m)$, see (58).

Theorem 3.3. *The function $\mathcal{F}^{1}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1, ..., \omega_m)$ is the product of a polynomial $\mathbf{P}_{\mathbf{a},\mathbf{b}}\Big(\frac{\omega_1}{2\pi}...., \frac{\omega_m}{2\pi}\Big)$ with rational coefficients, and a multiple polylogarithm of the same depth & weight:*

$$\mathcal{F}^{1}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1, ..., \omega_m) = \mathbf{P}_{\mathbf{a},\mathbf{b}}\Big(\frac{\omega_1}{2\pi}...., \frac{\omega_m}{2\pi}\Big) \cdot \mathrm{Li}_{n_1,...,n_m}\Big((-1)^{a_1+b_1}e^{\omega_1}, ..., (-1)^{a_m+b_m}e^{\omega_m}\Big) \tag{42}$$

Proof. One has

$$\mathcal{F}^{1}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1, ..., \omega_m) = i^{|\mathbf{n}|-m}\int_{(\mathbb{R}+i0)^m}\prod_{j=1}^{m}\frac{e^{-ip_j\omega_j}}{\mathfrak{sh}^{a_j+b_j}(\pi p_j)}\frac{dp_j}{(p_1+\ldots+p_j)^{n_j}}. \tag{43}$$

Let us assume that $\mathrm{Re}(\omega_j) < 0$. Then if $\mathrm{Im}(p_j) \to +\infty$, the exponential $e^{-ip_j\omega_j}$ decays fast. Next, when $|\mathrm{Re}(\omega_j)| < \pi(a_j + b_j)$, the integrand decays exponentially at $|p| \to \infty$. So assuming $-\pi(a_j + b_j) < \mathrm{Re}(\omega_j) < 0$ we evaluate the integral as $(2\pi i)^m\times$ the sum over $k_1, ..., k_m > 0$ of the residues at $p_1 = ik_1, ..., p_m = ik_m$.

Lemma 3.4. *Such a residue at $p_1 = ik_1, ..., p_m = ik_m$ is equal to*

$$i^{-m}(-1)^{(a_1+b_1)k_1+...+(a_m+b_m)k_m}\ \mathrm{Res}_{p_1=...=p_m=0}\Big(\prod_{j=1}^{m}\frac{e^{-ip_j\omega_j}\,dp}{\mathfrak{sh}^{a_j+b_j}(\pi p_j)}\Big)\cdot\frac{e^{k_1\omega_1}\ldots e^{k_m\omega_m}}{k_1^{n_1}\ldots k_m^{n_m}}. \tag{44}$$

Proof. Calculating the residue at $p_1 = ik_1, ..., p_m = ik_m$ we use the following:

1. The shift by $p_j \to p_j + i\pi$ for $j = 1, ..., m$ results in the multiplication of the denominator $\prod_{j=1}^{m}\mathfrak{sh}^{a_j+b_j}(\pi p_j)$ by $(-1)^{|\mathbf{a}|+|\mathbf{b}|}$.
2. We expand the exponential at $p_j = ik_j + p'_j$ as
$$\prod_{j=1}^{m}e^{-ip_j\omega_j} = \prod_{j=1}^{m}e^{-ip'_j\omega_j}\cdot e^{k_1\omega_1}\ldots e^{k_m\omega_m}.$$
3. As $p_j \to ik_j$, we have
$$p_1^{-n_1}\cdot\ldots\cdot(p_1+...+p_m)^{-n_m} \longrightarrow i^{-|\mathbf{n}|}k_1^{-n_1}\cdot\ldots\cdot(k_1+...+k_m)^{-n_m}.$$

In particular we observe that the factor $i^{|\mathbf{n}|}$ in (43) is cancelled with the one $i^{-|\mathbf{n}|}$ from (3). $\square$

The left factor in (44) does not depend on k_j. So taking the sum over all $k_1, \ldots, k_m$ we get

$$(2\pi)^m\mathrm{Res}_{p_1=...=p_m=0}\Big(\prod_{j=1}^{m}\frac{e^{-ip_j\omega_j}}{\mathfrak{sh}^{a_j+b_j}(\pi p_j)}\Big)\cdot\mathrm{Li}_{n_1,...,n_m}\Big((-1)^{a_1+b_1}e^{\omega_1}, \ldots, (-1)^{a_k+b_k}e^{\omega_m}\Big). \tag{45}$$

The residue on the left factorises into the product of the one variable residues:

$$2\pi\ \mathrm{Res}_{p_j=0}\Big(\frac{e^{-ip_j\omega_j}\,dp_j}{\mathfrak{sh}^{a_j+b_j}(\pi p_j)}\Big) = 2\pi\ \mathrm{Res}_{p_j=0}\Big(\frac{1+(-ip_j\omega_j)+(-ip_j\omega_j)^2/2!+...}{(\pi p_j)^{a_j+b_j}}(1+...)\Big).$$

The latter is equal to a sum of rational constants times

$$2\pi\ \mathrm{Res}_{p_j=0}\Big(\frac{(-ip_j\omega_j)^m(\pi p_j)^n dp_k}{(\pi p_j)^{a_j+b_j}}\Big), \qquad m+n=a_j+b_j-1.$$

Its weight is equal to zero since the weights of π and ω_j are equal to 1.

Note also that $z_j = e^{\omega_j}$ has zero weight: indeed, we have $w_j = \log(z_j)$, so in the final answer we have polynomials in $z_j, \log(z_j)$.

□

3.0.5 The $\mathcal{I}-$variant of quantum polylogarithms.

We will need the $\mathcal{I}-$variant of quantum polylogarithms:

$$\mathcal{I}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(w_1,...,w_m) := i^{|\mathbf{n}|-m}\int_{(\mathbb{R}+i0)^m}\prod_{k=1}^{m}\frac{e^{-i(p_1+...+p_k)w_k}}{\mathfrak{sh}^{a_k}(\pi p_2)\cdot\mathfrak{sh}^{b_k}(\pi\hbar p_2)}\frac{dp_k}{(p_1+\ldots+p_k)^{n_k}}. \tag{46}$$

For example, in the depth two we get

$$i^{|\mathbf{n}|-2}\int_{(\mathbb{R}+i0)^2}\frac{e^{-ip_1w_1}}{\mathfrak{sh}^{a_1}(\pi p_1)\cdot\mathfrak{sh}^{b_1}(\pi\hbar p_1)}\frac{e^{-i(p_1+p_2)w_2}}{\mathfrak{sh}^{a_2}(\pi p_2)\cdot\mathfrak{sh}^{b_2}(\pi\hbar p_2)}\frac{dp_1}{p_1^{n_1}}\frac{dp_2}{(p_1+p_2)^{n_2}}.$$

The functions $\mathcal{F}$ and $\mathcal{I}$ are related by

$$\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,\omega_2,\ldots,\omega_m) = \mathcal{I}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1-\omega_2,\omega_2-\omega_3,\ldots,\omega_m). \tag{47}$$

Indeed, their integrands differ only by the exponentials, related by (28). Equivalently, we have

$$\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\zeta_1,\ldots,\zeta_m) = \mathcal{I}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\zeta_1+\ldots\zeta_m,\zeta_2+\ldots+\zeta_m,\ldots,\zeta_m). \tag{48}$$

3.0.6 Differential equations

Proposition 3.5. *The quantum polylogarithms satisfy the differential equation*

$$d\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,...,\omega_m) = \sum_{k=1}^{m}\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}-1_k}(\omega_1,...,\omega_m)d(\omega_k-\omega_{k+1}). \tag{49}$$

Proof. The introduced in Section 3.0.5 function

$$\mathcal{I}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\xi_1,...,\xi_m) = \\ i^{|\mathbf{n}|-m}\int_{(\mathbb{R}+i0)^m}K^{\hbar}_{\mathbf{a},\mathbf{b}}(p_1,...,p_m)\frac{e^{-ip_1\xi_1}e^{-i(p_1+p_2)\xi_2}\ldots e^{-i(p_1+...+p_m)\xi_m}dp_1dp_2\ldots dp_m}{p_1^{n_1}(p_1+p_2)^{n_2}\ldots(p_1+...+p_m)^{n_m}}. \tag{50}$$

evidently[5] satisfies the differential equations:

$$d\mathcal{I}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{c}}(\xi_1,...,\xi_m) = \sum_{k=1}^{m}\mathcal{I}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}-1_k}(\xi_1,...,\xi_m)d\xi_k. \tag{51}$$

Therefore the claim follows from (47). □

[5]since the variable ξ_k appears only in the exponential $e^{-ip_k\xi_k}$

3.0.7 Complex conjugation

We claim that one has

$$\overline{\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\omega_1,\ldots,\omega_m)} = (-1)^{|\mathbf{a}|+|\mathbf{b}|-m}\mathcal{F}^{\overline{\hbar}}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\overline{\omega}_1,\ldots,\overline{\omega}_m).$$

Done by a change of variables $q = -\overline{p}$, altering the orientation of $\mathbb{R}+i0$. Here is how it works in the depth one case.

$$\begin{aligned}\overline{\mathcal{F}^{\hbar}_{a,b,n}(\omega)} = (-i)^{n-1}\int_{\overline{\mathbb{R}+i0}} \frac{e^{i\overline{p}\overline{\omega}}}{\mathfrak{sh}^a(\pi\overline{p})\mathfrak{sh}^b(\pi\overline{\hbar}\overline{p})}\frac{d\overline{p}}{\overline{p}^n} \overset{q=-\overline{p}}{=} \\ -(-1)^{a+b}i^{n-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\overline{\omega}}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\overline{\hbar}p)}\frac{dp}{p^n} = (-1)^{a+b+1}\mathcal{F}^{\overline{\hbar}}_{a,b,n}(\overline{\omega}).\end{aligned} \tag{52}$$

3.0.8 Shuffle relations

Quantum polylogarithms satisfy shuffle relations, similar to the ones for the iterated integrals representing the multiple polylogarithms.[6] Namely, let us set

$$\mathbf{w} = (\omega_1,...,\omega_m), \quad \mathbf{u} = (u_1,...,u_m), \quad \mathbf{t} = (t_1,...,t_m). \tag{53}$$

Consider the generating series in $\mathbf{u}$ whose coefficients are quantum polylogs with indices $\mathbf{n}$:

$$\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b}}(\mathbf{w}|\mathbf{u}) := \sum_{n_1,\ldots,n_m\geq 1}\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\mathbf{w}){u_1}^{n_1-1}\ldots{u_m}^{n_m-1}. \tag{54}$$

For example,

$$\mathcal{F}^{\hbar}_{a,b}[w|u] = i^{n-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\hbar p)}\ \frac{dp}{p-iu}.$$

Then make a substitution $u_k := t_1 + ... + t_k$ for $k = 1,...,m$:

$$\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b}}[\mathbf{w}|\mathbf{t}^*] := \mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b}}(\mathbf{w}|t_1, t_1+t_2,\ldots,t_1+\ldots+t_m).$$

Lemma 3.6. *There is an integral presentation:*

$$\mathcal{F}^{\hbar}_{\mathbf{a};\mathbf{b}}[\mathbf{w}|\mathbf{t}^*] = i^{|\mathbf{n}|-m}\cdot\int_{(\mathbb{R}+i0)^m}\prod_{k=1}^m K^{\hbar}_{a_k,b_k}(p_k;\omega_k)\frac{dp_k}{(p_1+\ldots+p_k)-i(t_1+\ldots+t_k)}.$$

Proof. Follows by applying the substitution $u_k := t_1 + ... + t_k$ to the following identity:

$$\sum_{n=1}^{\infty}\frac{dp_k}{(p_1+\ldots+p_k)^n}(iu_k)^{n-1} = \frac{dp_k}{(p_1+\ldots+p_k)-iu_k}. \tag{55}$$

□

[6]It is interesting that although quantum polylogarithms do not seem to have an iterated integral presentation, they do satisfy the same shuffle relations as the iterated integrals.

Note that integrals (13) converge for any integers n_i, while the generating series use $n_i \geq 1$.

Given $\mathbf{a} := (a_1, ..., a_k)$ and $\mathbf{a}' := (a_{k+1}, ..., a_{k+l})$ and a permutation σ of the set $\{1, ..., k+l\}$, set $\sigma(\mathbf{aa}') := (a_{\sigma(1)}, ..., a_{\sigma(k+l)})$.

Theorem 3.7. *One has*

$$\mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b}}[\mathbf{w}|\mathbf{t}^*] \cdot \mathcal{F}^{\hbar}_{\mathbf{a}',\mathbf{b}'}[\mathbf{w}'|\mathbf{t}'^*] = \sum_{\sigma \in \Sigma_{k,l}} \mathcal{F}^{\hbar}_{\sigma(\mathbf{aa}'),\sigma(\mathbf{bb}')}[\sigma(\mathbf{ww}')|\sigma(\mathbf{tt}')^*]. \tag{56}$$

The sum is over the set of all permutations shuffling $\{1, ..., k\}$ and $\{k+1, ..., k+l\}$.

Proof. Follows immediately from the following identity [G01, Lemma 2.12]:

$$\begin{gathered}\frac{1}{p_1(p_1+p_2)...(p_1+...+p_k)} \cdot \frac{1}{p_{k+1}(p_{k+1}+p_{k+2})...(p_{k+1}+...+p_{k+l})} = \\ \sum_{\sigma \in \Sigma_{k,l}} \frac{1}{p_{\sigma(1)}(p_{\sigma(1)}+p_{\sigma(2)})...(p_{\sigma(1)}+...+p_{\sigma(k+l)})}.\end{gathered} \tag{57}$$

□

For example, using the identity

$$\frac{1}{p_1p_2} = \frac{1}{p_1(p_1+p_2)} + \frac{1}{p_2(p_1+p_2)}$$

we have

$$\mathcal{F}^{\hbar}_{a_1,b_1,1}(\omega_1) \cdot \mathcal{F}^{\hbar}_{a_2,b_2,1}(\omega_2) = \mathcal{F}^{\hbar}_{(a_1,a_2),(b_1,b_2),(1,1)}(\omega_1, \omega_2) + \mathcal{F}^{\hbar}_{(a_2,a_1),(b_2,b_1),(1,1)}(\omega_2, \omega_1).$$

Shuffle relations for the generating functions. In addition to (53), let us set

$$\mathbf{r} = (r_1, ..., r_m), \quad \mathbf{s} = (s_1, ..., s_m).$$

Generalizing (54), we introduce the quantum polylogarithm generating series:

$$\mathcal{F}^{\hbar}(\mathbf{w}|\mathbf{r}, \mathbf{s}, \mathbf{u}) := \sum_{a_i, b_i, n_i > 0} \mathcal{F}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\mathbf{w}) \prod_{k=1}^{m} r_k^{a_k-1} {s_k}^{b_k-1} {u_k}^{n_k-1}.$$

One can rewrite this using the kernel generating series.

Lemma 3.8. *The kernel generating function is given by*

$$\mathcal{K}^{\hbar}(p; z|r, s) := \sum_{a,b=1}^{\infty} K^{\hbar}_{a,b} r^{a-1} s^{b-1} = \frac{e^{-ipz}}{(\mathfrak{sh}(\pi p) - r)(\mathfrak{sh}(\pi \hbar p) - s)}.$$

Lemma 3.9. *The quantum polylogarithm generating series are given by the integrals*

$$\mathcal{F}^{\hbar}(\mathbf{w}|\mathbf{r}, \mathbf{s}, \mathbf{u}) := \int_{(\mathbb{R}+i0)^m} \prod_{k=1}^{m} \mathcal{K}^{\hbar}(p_k; \omega_k|r_k, s_k) \frac{dp_k}{(p_1 + \ldots + p_k) - iu_k}.$$

Proof. Follows immediately using (55). □

For example, the depth one quantum polylogarithm generating series are

$$\mathcal{F}^{\hbar}(\omega|r,s,u) := \int_{\mathbb{R}+i0} \frac{e^{-ip\omega}}{(\mathfrak{sh}(\pi p)-r)(\mathfrak{sh}(\pi\hbar p)-s)}\frac{dp}{p-iu}.$$

Theorem 3.7 immediately implies the following

Theorem 3.10. *One has*

$$\mathcal{F}^{\hbar}[\mathbf{w}|\mathbf{r},\mathbf{s},\mathbf{t}^*]\cdot\mathcal{F}^{\hbar}[\mathbf{w}'|\mathbf{r}',\mathbf{s}',\mathbf{t}'^*] = \sum_{\sigma\in\Sigma_{k,l}} \mathcal{F}^{\hbar}[\sigma(\mathbf{w}\mathbf{w}')|\sigma(\mathbf{r}\mathbf{r}'),\sigma(\mathbf{s}\mathbf{s}'),\sigma(\mathbf{t}\mathbf{t}')^*].$$

3.0.9 Analytic continuation.

Quantum polylogarithms have an analytic continuation to a cover of $\mathcal{M}_{0,m+3}(\mathbb{C})$ given by

$$(w_1,...,w_m) \longrightarrow (\infty,-1,0,e^{w_1},...,e^{w_m}) \in \mathcal{M}_{0,m+3}(\mathbb{C}).$$

The integral representation (27) is convergent at the strip

$$|\text{Im }\omega_i| < \pi a_i + \pi\hbar b_i.$$

We use difference relations (35) to extend it from that strip to $\mathbb{C}^m$, and argue by the induction on $|\mathbf{a}|+|\mathbf{b}|$. First, one checks formally by induction that

$$\Delta^{(\omega_l)}_{i\pi\hbar}\Delta^{(\omega_k)}_{i\pi}\mathcal{I}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\mathbf{w}) = \Delta^{(\omega_k)}_{i\pi}\Delta^{(\omega_l)}_{i\pi\hbar}\mathcal{I}^{\hbar}_{\mathbf{a},\mathbf{b},\mathbf{n}}(\mathbf{w}).$$

where each of the sides is defined by applying twice difference relations (35).

3.0.10 An example: basic quantum polylogarithms

We define the basic quantum polylogarithms by setting $a_j = b_j = 1$. So in the depth m case

$$\begin{aligned}&\mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1,...,\omega_m) := \\ &i^{|\mathbf{n}|-m}\int_{(\mathbb{R}+i0)^m} \frac{e^{-ip_1\omega_1}}{\mathfrak{sh}(\pi p_1)\mathfrak{sh}(\pi\hbar p_m)}\frac{dp_1}{p_1^{n_1}}\wedge\ldots\wedge\frac{e^{-ip_1\omega_1}}{\mathfrak{sh}(\pi p_m)\mathfrak{sh}(\pi\hbar p_m)}\frac{dp_m}{(p_1+\ldots+p_m)^{n_m}}.\end{aligned} \tag{58}$$

Theorem 3.11. *The function $\mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1,...,n_m)$ enjoys the following properties:*

1. *Asymptotic expansion as $\hbar\to 0$:*

$$\mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1,...,\omega_m) \sim \frac{1}{(2\pi i\hbar)^m}\cdot \text{Li}_{n_1+1,...,n_m+1}(e^{\omega_1-\omega_2},e^{\omega_2-\omega_3},\ldots,-e^{\omega_m})+\ldots$$

2. *Difference relations, connecting them with multiple polylogarithms:*

$$\begin{aligned}\Delta^{(\omega_1)}_{i\pi\hbar}\ldots\Delta^{(\omega_m)}_{i\pi\hbar}\mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1,...,\omega_m) &= \text{Li}_{n_1,...,n_m}(e^{\omega_1-\omega_2},e^{\omega_2-\omega_3},\ldots,-e^{\omega_m}).\\ \Delta^{(\omega_1)}_{i\pi}\ldots\Delta^{(\omega_m)}_{i\pi}\mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1,...,\omega_m) &= \hbar^{|\mathbf{n}|-m}\text{Li}_{n_1,...,n_m}(e^{\frac{\omega_1-\omega_2}{\hbar}},e^{\frac{\omega_2-\omega_3}{\hbar}},\ldots,-e^{\frac{\omega_m}{\hbar}}).\end{aligned} \tag{59}$$

3. The value at $\hbar = 1$:

$$\mathcal{F}^{1}_{n_1,...,n_m}(\omega_1, ..., \omega_m) = \mathbf{P_{1,1}}\Big(\frac{\omega_1}{2\pi}...., \frac{\omega_m}{2\pi}\Big) \cdot \mathrm{Li}_{n_1,...,n_m}\Big(e^{\omega_1}, ..., e^{\omega_m}\Big).$$

4. Distribution relations:

$$\begin{aligned} & r^{|\mathbf{n}|-m}\mathcal{F}^{\frac{r}{s}\hbar}_{n_1,...,n_m}(r\omega_1, ..., r\omega_m) = \\ & \sum_{\alpha_k=\frac{1-r}{2}}^{\frac{r-1}{2}} \sum_{\beta_k=\frac{1-s}{2}}^{\frac{s-1}{2}} \mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1 + \frac{2\pi i}{r}\alpha_1 + \frac{2\pi i\hbar}{s}\beta_1, \ldots, \omega_k + \frac{2\pi i}{r}\alpha_k + \frac{2\pi i\hbar}{s}\beta_k). \end{aligned} \tag{60}$$

5. The differential:

$$d\mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1, ..., \omega_m) = \sum_{j=1}^{m} \mathcal{F}^{\hbar}_{n_1,...,n_k-1,...,n_m}(\omega_1, ..., \omega_m)d(\omega_k - \omega_{k+1}).$$

6. $\hbar \longleftrightarrow 1/\hbar$ symmetry:

$$\mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1, ..., \omega_m) = \hbar^{|\mathbf{n}|-m}\mathcal{F}^{\frac{1}{\hbar}}_{n_1,...,n_m}\Big(\frac{\omega_1}{\hbar}, ..., \frac{\omega_m}{\hbar}\Big).$$

Proof. 1) Using (36) and (22), we get the leading term of the $\hbar \to 0$ asymptotic expansion:

$$\begin{aligned} & \mathcal{F}^{\hbar}_{n_1,...,n_m}(\omega_1, ..., \omega_m) \ \sim_{\hbar\to 0} \\ & \frac{i^{|\mathbf{n}|}}{(2\pi i\hbar)^m}\int_{(\mathbb{R}+i0)^m} \frac{e^{-ip_1\omega_1}}{\mathfrak{sh}(\pi p_1)}\frac{dp_1}{p_1 p_1^{n_1}} \wedge \ldots \wedge \frac{e^{-ip_m\omega_m}}{\mathfrak{sh}(\pi p_m)}\frac{dp_m}{p_m(p_1 + \ldots + p_m)^{n_m}} \\ & \overset{(22)}{\sim}_{\hbar\to 0} \ \frac{1}{(2\pi i\hbar)^m}\mathrm{Li}_{n_1,...,n_m}(e^{\omega_1-\omega_2}, e^{\omega_2-\omega_3}, \ldots, -e^{\omega_m}) + \ldots. \end{aligned} \tag{61}$$

2) For the second identity, set $q_i = p_i\hbar$, and use integral (11). The rest is straightforward. □

4 Quantum polylogarithms and multiple $q-$polylogarithms

Recall Definition 1.4 of the multiple $q-$polylogarithms:

$$\begin{aligned} & \mathrm{Li}_{\mathbf{a},\mathbf{n}}(x_1, ..., x_m; q) := \\ & \sum_{k_1,...,k_m>0}^{\infty} \frac{x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}}{[k_1]_q^{a_1}[k_2]_q^{a_2}\ldots[k_m]_q^{a_m} \cdot k_1^{n_1}(k_1+k_2)^{n_2}\ldots(k_1+...+k_m)^{n_m}}. \end{aligned} \tag{62}$$

If $\mathbf{a} = 0$, we get the multiple polylogarithms $\mathrm{Li}_{\mathbf{n}}(z_1, \ldots, z_m) = \mathrm{Li}_{n_1,...,n_m}(z_1, \ldots, z_m)$.

Multiple $q-$polylogarithms satisfy both the differential and difference equations:

The differential. Given an $\mathbf{x} = (x_1, \ldots, x_m)$, we set $\mathbf{x}^* := (x_1...x_m, x_2...x_m, \ldots x_m)$. Then

$$d\mathrm{Li}_{\mathbf{a},\mathbf{n}}(\mathbf{x}^*; q) = \sum_{k=1}^{m} \mathrm{Li}_{\mathbf{a},\mathbf{n}-\mathbf{1}_k}(\mathbf{x}^*; q)d\log x_k.$$

The difference relation. The $q-$difference operator defined by setting

$$\mathbf{\Delta}_{x,q}f(x) := f(qx) - f(q^{-1}x).$$

We have the difference relations

$$\mathbf{\Delta}_{x_k,q}\mathrm{L}_{\mathbf{a},\mathbf{n}}(\mathbf{x};q) = \mathrm{L}_{\mathbf{a}-\mathbf{1}_k,\mathbf{n}}(\mathbf{x};q). \tag{63}$$

4.1 Multiple $q-$polylogarithms by the $q-$integration

Definition 4.1. *Given a power series $f(x)$ and an integer $a \geq 0$, the $q-$integral $\mathbb{I}_x^a f(x)$ is:*

$$(\mathbb{I}_x^a f)(x) := (-1)^{a-1}\sum_{k\geq 0}\binom{k+a-1}{a-1} f(q^{2k+a}x).$$

The name $q-$integral is justified by the following Lemma.

Lemma 4.2. *Let $a > 0$. Then one has:*

$$\mathbf{\Delta}_{x,q} \circ \mathbb{I}_x^a f(x) = \mathbb{I}_x^{a-1} f(x).$$

Proof. Follows by a pretty standard calculation:

$$\begin{aligned}
&\mathbb{I}_x^a f(qx) - \mathbb{I}_x^a f(q^{-1}x) = \\
&(-1)^{a-1}\sum_{k\geq 0}\binom{k+a-1}{a-1}\Big(f(q^{2k+a+1}x) - f(q^{2k+a-1}x)\Big) = \\
&(-1)^{a-1}\sum_{k\geq 0}\Big(\binom{(k+1)+a-2}{a-1} f(q^{2(k+1)+a-1}x) - \binom{k+a-1}{a-1} f(q^{2k+a-1}x)\Big) = \\
&(-1)^{a-1}\sum_{k> 0}\Big(\binom{k+a-2}{a-1} - \binom{k+a-1}{a-1}\Big)(q^{2k+a-1}x) - (-1)^{a-1}f(q^{a-1}x) = \\
&(-1)^{a-2}\sum_{k\geq 0}\binom{k+a-2}{a-2} f(q^{2k+a-1}x) = \\
&\mathbb{I}_x^{a-1} f(x).
\end{aligned} \tag{64}$$

□

Proposition 4.3. *There is an equality of power series:*

$$\begin{aligned}
&\mathrm{Li}_{\mathbf{a},\mathbf{n}}(x_1, ..., x_m; q) = (\mathbb{I}_{x_1}^{a_1} \dots \mathbb{I}_{x_m}^{a_m}\mathrm{Li}_{\mathbf{n}})(x_1, \dots, x_m) = \\
&\sum_{k_1,\dots,k_m\geq 0} (-1)^{|\mathbf{a}|-m}\binom{k_1+a_1-1}{a_1-1}\dots\binom{k_m+a_m-1}{a_m-1}\mathrm{Li}_{\mathbf{n}}(q^{2k_1+a_1}x_1, \dots, q^{2k_m+a_m}x_m).
\end{aligned} \tag{65}$$

Proof. By Lemma 4.2 and (63) we have

$$\mathbf{\Delta}_{x_1,q}^{a_1} \circ \dots \circ \mathbf{\Delta}_{x_1,q}^{a_1}\Big(\mathrm{Li}_{\mathbf{a},\mathbf{n}}(x_1, ..., x_m; q) - (\mathbb{I}_{x_1}^{a_1} \dots \mathbb{I}_{x_m}^{a_m}\mathrm{Li}_{\mathbf{n}})(x_1, \dots, x_m)\Big) = 0.$$

Both series vanish at $x_1 = ... = x_m = 0$. □

Examples

1. *The q-polylogarithms* are power series in x:

$$\mathrm{Li}_{a,n}(x;q) := \sum_{k=1}^{\infty} \frac{x^k}{(q^k - q^{-k})^a \; k^n}, \qquad a, n \in \mathbb{Z}.$$

   They satisfy both the differential and difference equations:

$$\begin{aligned} d\mathrm{Li}_{a,n}(x;q) &= \mathrm{Li}_{a,n-1}(x;q) d\log x. \\ \mathbf{\Delta}\mathrm{Li}_{a,n}(qx;q) &= \mathrm{Li}_{a-1,n}(x;q). \end{aligned} \tag{66}$$

2. *Higher Pochhammer symbols* $\Psi_{a+1}(x;q)$ are the power series given by the infinite products

$$\Psi_{a+1}(x;q) := \prod_{n\geq 0} (1 + q^{2n+1}x)^{(-1)^{a+1}\binom{n+a}{a}}.$$

   For example,

$$\begin{aligned} \Psi_1(x;q) &= \frac{1}{(1+qx)(1+q^3x)(1+q^5x)(1+q^7x)\cdot \ldots}, \\ \Psi_2(x;q) &= (1+qx)(1+q^3x)^2(1+q^5x)^3(1+q^7x)^4 \cdot \ldots. \end{aligned} \tag{67}$$

   They are the unique power series in x, q which satisfies the recursion

$$\frac{\Psi_a(qx;q)}{\Psi_a(q^{-1}x;q)} = \Psi_{a-1}(x;q), \qquad \Psi_0(x;q) := 1 + x.$$

   Proposition 4.4. *One has*

$$\log \Psi_a(x;q) = -\mathrm{Li}_{a-1,1}(-x;q) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{(q^k - q^{-k})^{a-1} k}. \tag{68}$$

   Proof. Both power series satisfy the same difference equation, and equal to 0 at $x = 0$.

   Alternatively, here is a direct calculation for the classical case of $\Psi_1(x;q)$. Formula (68) in this case is the following identity:

$$\mathrm{Li}_{1,1}(-x;q) = \sum_{n\geq 0} \log(1 + q^{2n+1}x).$$

   To prove it, we use the expansion $\log(1+x) = -\sum_{k>0} \frac{(-x)^k}{k}$:

$$\begin{aligned} &\sum_{n\geq 0} \log(1+q^{2n+1}x) = -\sum_{n\geq 0}\sum_{k>0} \frac{(-q^{2n+1}x)^k}{k} = \\ &-\sum_{k>0}\sum_{n\geq 0} \frac{q^{2nk}q^k(-x)^k}{k} = -\sum_{k>0} \frac{q^k(-x)^k}{(1-q^{2k}) \cdot k} = \mathrm{Li}_{1,1}(-x;q). \end{aligned} \tag{69}$$

   □

3. Consider a slight modification of the classical polylogarithm power series:

$$\mathrm{L}_n(z) := -\mathrm{Li}_n(-z) = -\sum_{k>0} \frac{(-z)^k}{k^n}, \quad |z| < 1.$$

So $\mathrm{L}_1(z) = \log(1+z)$ and $\mathrm{L}_0(z) = \frac{z}{1+z}$. By Proposition 4.3,

$$\mathrm{Li}_{a,n}(-x;q) = (-1)^a \sum_{k\geq 0} \binom{k+a-1}{a-1} \mathrm{L}_n(q^{2k+a}x).$$

4. It is interesting to compare q-polylogarithms with the elliptic polylogarithms [BL]. The latter are obtained by the regularized weighted averaging over $\mathbb{Z}$ of the classical polylogarithms, while the former are obtained by a similar weighted averaging but over the non-negative integers. For example, starting with $\log(1+x)$, the regularized averaging over $\mathbb{Z}$ delivers the logarithm of a theta function, while averaging over $\mathbb{Z}_{\geq 0}$ we get the negative of the logarithm of the q-exponential.

4.2 Quantum polylogarithms as sums of 2^m companion $q-$polylogarithms

Recall that the quantum dilogarithm function can be written as a difference of two series:

$$\mathcal{F}_1^{\hbar}(w) = -\mathrm{Li}_{1,1}(-e^w; q) + \mathrm{Li}_{1,1}(-e^{w/\hbar}; q^\vee). \tag{70}$$

In Section 4.2 we show that quantum polylogarithms have similar presentation. We elaborate in detail the case of depth m *basic quantum polylogarithms*. We prove that they are sums of 2^m companion polylogarithm series. One of them is a quantum $q-$polylogarithm series, another one is a quantum $q^\vee-$polylogarithm series, and the other 2^m-2 companion series are given by more general series.

Recall the depth m *basic quantum polylogarithm*:

$$\mathcal{F}^{\hbar}_{n_1,\ldots,n_m}(\omega_1, ..., \omega_m) := i^{|\mathbf{n}|-m} \int_{(\mathbb{R}+i0)^m} \prod_{k=1}^{m} \frac{e^{-ip_i\omega_k}}{\mathfrak{sh}(\pi p_k)\mathfrak{sh}(\pi\hbar p_k)} \frac{dp_k}{(p_1+\ldots+p_k)^{n_k}}.$$

Recall also the $\mathcal{I}-$variant (46) of the basic quantum polylogarithms:

$$\begin{aligned}\mathcal{I}^{\hbar}_{n_1,\ldots,n_m}(w_1, ..., w_m) &:= i^{|\mathbf{n}|-m} \int_{(\mathbb{R}+i0)^m} \prod_{k=1}^{m} \frac{e^{-i(p_1+\ldots+p_k)w_k}}{\mathfrak{sh}(\pi p_k)\mathfrak{sh}(\pi\hbar p_k)} \frac{dp_k}{(p_1+\ldots+p_k)^{n_k}} \\ &= \mathcal{F}^{\hbar}_{n_1,\ldots,n_m}(w_1-w_2, w_2-w_3, ..., w_m).\end{aligned} \tag{71}$$

We calculate integral (71) as a sum over the residues. The sum splits into a sum of 2^m series over the cones given by the direct sum of m copies of the cones $\mathbb{Z}_{>0}$ or $\hbar^{-1}\mathbb{Z}_{>0}$. We call the resulting series *companion polylogarithm series*. They are paramatrised by the sequences

$$\underline{\varepsilon} := (\varepsilon_1, ..., \varepsilon_m), \quad \varepsilon_j \in \{1, \hbar^{-1}\}.$$

For each such $\underline{\varepsilon}$ we assign the *companion cone*:

$$\mathrm{C}_{\underline{\varepsilon}} := \mathbb{Z}_{>0}\varepsilon_1 \oplus \ldots \oplus \mathbb{Z}_{>0}\varepsilon_m \in \mathbb{C}^m.$$

Recall $q = e^{i\pi\hbar}$ and $q^\vee = e^{i\pi/\hbar}$. We also use a notation

$$[k]_{q_\varepsilon} := \begin{cases} q^k - q^{-k} & \varepsilon = 1 \\ (q^\vee)^k - (q^\vee)^{-k} & \varepsilon = \hbar^{-1}. \end{cases} \tag{72}$$

Definition 4.5. *The $\underline{\varepsilon}$−companion polylogarithm series are given by*

$$\begin{aligned} &\mathrm{Li}^{\hbar}_{\mathbf{a},\mathbf{n}}(\underline{\varepsilon}; w_1, ..., w_m) := \\ &\varepsilon_1 \dots \varepsilon_m \sum_{k_1,...,k_m>0} \frac{e^{k_1\omega_1\varepsilon_1} \dots e^{(k_1+...+k_m)\omega_m\varepsilon_m}}{[k_1]^{a_1}_{q_{\varepsilon_1}} \dots [k_m]^{a_m}_{q_{\varepsilon_m}} \cdot (\varepsilon_1 k_1)^{n_1} \dots (\varepsilon_1 k_1 + ... + \varepsilon_m k_m)^{n_m}}. \end{aligned} \tag{73}$$

The $\underline{\varepsilon}$−companion polylogarithm series (73) generalize q−polylogarithm series (62). Indeed:

$$\mathrm{Li}^{\hbar}_{\mathbf{a},\mathbf{n}}(\underline{\varepsilon}; \omega_1, ..., \omega_m) = \begin{cases} \mathrm{Li}_{\mathbf{a},\mathbf{n}}(e^{\omega_1}, ..., e^{\omega_m}; q) & \text{if } \underline{\varepsilon} = (1, \dots, 1), \\ \hbar^{|\mathbf{n}|-m} \mathrm{Li}_{\mathbf{a},\mathbf{n}}(e^{\omega_1/\hbar}, ..., e^{\omega_m/\hbar}; q^\vee) & \text{if } \underline{\varepsilon} = (\hbar^{-1}, \dots, \hbar^{-1}). \end{cases} \tag{74}$$

Theorem 4.6. *Assume $\hbar > 0$. Assume that $\mathrm{Re}(\omega_i) < 0$ and $|\mathrm{Im}(\omega_i)| < \pi$. Then we have*

$$\mathcal{I}^{\hbar}_{\mathbf{n}}(w_1, \dots, w_m) = \sum_{\underline{\varepsilon}} \mathrm{Li}^{\hbar}_{\mathbf{1},\mathbf{n}}(\underline{\varepsilon}; w_1, w_2, \dots, w_m). \tag{75}$$

Before we proceed with the proof, let us elaborate two examples.

1. $m = 1$. We get two companion cones: $\mathbb{Z}_{>0}$ and $\hbar^{-1}\mathbb{Z}_{>0}$. The related companion series are

$$\begin{aligned} \mathbb{Z}_{>0}: &\quad \sum_{k>0} \frac{(-1)^k e^{kw}}{[k]_q \; k^n} = \mathrm{Li}_{1,n}(-e^w; q). \\ \hbar^{-1}\mathbb{Z}_{>0}: &\quad \hbar^{n-1} \sum_{k>0} \frac{(-1)^k e^{kw/\hbar}}{[k]_{q^\vee} \; k^n} = \hbar^{n-1} \mathrm{Li}_{1,n}(-e^{w/\hbar}; q^\vee). \end{aligned} \tag{76}$$

   So $\mathcal{I}^{\hbar}_n(w)$ is a sum of the two companion series (76):

$$\mathcal{I}^{\hbar}_n(w) = \mathrm{Li}_{1,n}(-e^w; q) + \hbar^{n-1} \mathrm{Li}_{1,n}(-e^{w/\hbar}; q^\vee). \tag{77}$$

   When $n = 1$ we recover formula (70).

2. $m = 2$. We get four companion cones:

$$\mathbb{Z}_{>0} \oplus \mathbb{Z}_{>0}, \quad \hbar^{-1}\mathbb{Z}_{>0} \oplus \mathbb{Z}_{>0}, \quad \mathbb{Z}_{>0} \oplus \hbar^{-1}\mathbb{Z}_{>0}, \quad \hbar^{-1}\mathbb{Z}_{>0} \oplus \hbar^{-1}\mathbb{Z}_{>0}.$$

   The related companion series are:

$$\begin{aligned} \mathbb{Z}_{>0} \oplus \mathbb{Z}_{>0}: &\quad \sum_{k_1,k_2>0} \frac{e^{-k_1 w_1} e^{-k_2 w_2}}{[k_1]_q [k_2]_q} \frac{1}{k_1^{n_1}(k_1+k_2)^{n_2}}. \\ \hbar^{-1}\mathbb{Z}_{>0} \oplus \mathbb{Z}_{>0}: &\quad \hbar^{-1} \sum_{k_1,k_2>0} \frac{e^{-k_1 w_1/\hbar} e^{-k_2 w_2}}{[k_1]_{q^\vee} [k_2]_q} \frac{1}{(\hbar^{-1}k_1)^{n_1}(\hbar^{-1}k_1+k_2)^{n_2}}. \\ \mathbb{Z}_{>0} \oplus \hbar^{-1}\mathbb{Z}_{>0}: &\quad \hbar^{-1} \sum_{k_1,k_2>0} \frac{e^{-k_1 w_1} e^{-k_2 w_2/\hbar}}{[k_1]_q [k_2]_{q^\vee}} \frac{1}{k_1^{n_1}(k_1+\hbar^{-1}k_2)^{n_2}}. \\ \hbar^{-1}\mathbb{Z}_{>0} \oplus \hbar^{-1}\mathbb{Z}_{>0}: &\quad \hbar^{-2} \sum_{k_1,k_2>0} \frac{e^{-k_1 w_1/\hbar} e^{-k_2 w_2/\hbar}}{[k_1]_{q^\vee} [k_2]_{q^\vee}} \frac{1}{(\hbar^{-1}k_1)^{n_1}(\hbar^{-1}k_1+\hbar^{-1}k_2)^{n_2}}. \end{aligned} \tag{78}$$

   So $\mathcal{I}^{\hbar}_{n_1,n_2}(w_1, w_2)$ is a sum of the four companion polylogarithm series (78).

Proof. We elaborate the case of the companion cone $\hbar^{-1}\mathbb{Z}_{>0} \oplus \hbar^{-1}\mathbb{Z}_{>0}$. Let $\mathbf{n} := (n_1, n_2)$. Then

$$\mathcal{I}^{\hbar}_{\mathbf{n}}(w_1, w_2) := \int_{(\mathbb{R}+i0)^2} \frac{e^{-ip_1w_1}}{\mathfrak{sh}(\pi p_1)\mathfrak{sh}(\pi\hbar p_1)} \frac{e^{-i(p_1+p_2)w_2}}{\mathfrak{sh}(\pi p_2)\mathfrak{sh}(\pi\hbar p_2)} \frac{dp_1}{p_1^{n_1}} \frac{dp_2}{(p_1+p_2)^{n_2}}.$$

The contribution of the residues at $p_1 = ik_1/\hbar, p_2 = ik_2/\hbar$ where $k_1, k_2 > 0$ gives

$$\begin{aligned}
&\hbar^{-2} \sum_{k_1,k_2>0} \frac{(e^{w_1/\hbar})^{k_1}}{[k_1]_{q^\vee}} \frac{(-e^{w_2/\hbar})^{k_1+k_2}}{[k_2]_{q^\vee}} \frac{1}{(\hbar^{-1}k_1)^{n_1}\ (\hbar^{-1}(k_1+k_2))^{n_2}} \\
&= \hbar^{|\mathbf{n}|-2}\mathrm{Li}_{\mathbf{1},\mathbf{n}}(e^{w_1/\hbar}, -e^{w_2/\hbar}; q^\vee).
\end{aligned} \tag{79}$$

□

The analog of Theorem 4.6 for arbitrary quantum polylogarithms is obtained by a similar residue calculation. Since the function $\frac{1}{\mathfrak{sh}^{a_s}(\pi p_s)\mathfrak{sh}^{b_s}(\pi\hbar p_s)}$ has zeros of higher order at $p_s = ik_s, ik_s/\hbar$ where $k_s > 0$, we get sums of companion series multiplied by powers of ω_s. Calculating the residues at $p_s = ik_s/\hbar$ we encounter the following derivatives, evaluated then at $p_s = ik_s/\hbar$:

$$\left(\frac{d}{dp_s}\right)^{b_s-1} \frac{1}{\mathfrak{sh}^{a_s}(\pi p_s)} \prod_{j=s}^{m} \frac{e^{-i(p_1+\ldots+p_j)z_j}}{(p_1+\ldots+p_j)^{n_j}}. \tag{80}$$

For the residues at $p_s = ik_s$ we get similar derivatives, with a_s switched with b_s, at $p_s = ik_s$:

5 Depth one examples

We consider integrals the depth one integrals for different countours α:

$$i^{n-1} \cdot \int_{\alpha} \frac{e^{-ipz}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\hbar p)} \frac{dp}{p^n}.$$

If $\alpha := \alpha_0$ is a small *counterclockwise* oriented loop around zero, we get polynomials in z, generalizing Bernoulli polynomials. If $\alpha := \mathbb{R}+i0$, we get the depth one quantum polylogarithms.

5.1 Quantum Bernoulli polynomials

Recall the Bernoulli polynomials $B_n(x)$:

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

Definition 5.1. *Quantum Bernoulli polynomials are polynomials in ω and $\hbar^{\pm 1}$ given by*

$$B^{\hbar}_{a,b,n}(\omega) := i^{n-1} \int_{\alpha_0} \frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\hbar p)} \frac{dp}{p^n}.$$

To state the properties of the polynomials $B^{\hbar}_{a,b,n}(\omega)$ we need a polynomial

$$Q_m(\omega) = \frac{\Big(\omega - \pi i(m-1)\Big)\Big(\omega - \pi i(m-3)\Big) \cdot \ldots \cdot \Big(\omega - \pi i(1-m)\Big)}{(2\pi i)^m m!}.$$

It is the unique degree m polynomial with the following two properties:

- It satisfies difference relations

$$\Delta_{i\pi} Q_m(\omega) = Q_{m-1}(\omega), \quad Q_0(\omega) = 1. \tag{81}$$

- The roots of $Q_m(\omega)$ form an arithmetic progression with the step $2\pi i$, centered at 0.

The weight of $Q_m(\omega)$ is 0. For example

$$Q_0(\omega) = 1, \quad Q_1(\omega) = \frac{\omega}{2\pi i}, \quad Q_2(\omega) = \frac{(\omega - i\pi)(\omega + i\pi)}{2! \cdot (2\pi i)^2}. \tag{82}$$

Theorem 5.2. *The quantum Bernoulli polynomials $B^{\hbar}_{a,b,n}(\omega)$ have the following properties.*

1. *$B^{\hbar}_{a,b,n}(\omega)$ is a polynomial in ω of the degree $a + b + n - 1$.*
2. *Differential and difference equations:*

$$\begin{aligned} dB^{\hbar}_{a,b,n}(\omega) &= B^{\hbar}_{a,b,n-1}(z)d\omega. \\ \Delta_{i\pi\hbar} B^{\hbar}_{a,b,n}(\omega) &= B^{\hbar}_{a,b-1,n}(\omega), \\ \Delta_{i\pi} B^{\hbar}_{a,b,n}(\omega) &= B^{\hbar}_{a-1,b,n}(\omega). \end{aligned} \tag{83}$$

3. *Asymptotic expansion when $\hbar \to 0$:*

$$B^{\hbar}_{a,b,n}(\omega) \ \sim_{\hbar\to 0} \ \frac{1}{(2\pi i\hbar)^b} B_{a,0,b+n}(\omega) + \dots.$$

4. *The value at $\hbar = 1$:*

$$B^{1}_{a,b,n}(\omega) = B_{a+b,0,n}(\omega).$$

5. *Relation with Bernoulli polynomials $B_n(\omega)$ and polynomials $Q_n(\omega)$:*

$$\begin{aligned} B^{\hbar}_{1,0,n}(\omega) &= \frac{(2\pi i)^{n-1}}{(n-1)!} B_{n-1}\left(\frac{\omega}{2\pi i} + \frac{1}{2}\right), \\ B^{\hbar}_{a,0,1}(\omega) &= Q_{a-1}(\omega). \end{aligned} \tag{84}$$

6. *Modular property, or $\hbar \leftrightarrow 1/\hbar$ symmetry: for any $\hbar \in \mathbb{C}^{\times}$ one has:*

$$B^{\hbar}_{a,b,n}(\omega) = \hbar^{n-1} B^{1/\hbar}_{b,a,n}(\omega/\hbar).$$

7. *Complex conjugation:*

$$\overline{B^{\hbar}_{a,b,n}(\omega)} = (-1)^{a+b+1} B^{\overline{\hbar}}_{a,b,n}(\overline{\omega}).$$

8. *The $\omega \longleftrightarrow -\omega$ symmetry:*

$$B^{\hbar}_{a,b,n}(\omega) = (-1)^{a+b+n} B^{\hbar}_{a,b,n}(-\omega).$$

9. *The generating function*

$$B^{\hbar}(\omega|r,s,u) := \int_{\alpha_0} \frac{e^{-ip\omega}}{(\mathfrak{sh}(\pi p) - r)(\mathfrak{sh}(\pi\hbar p) - s)} \frac{dp}{p - iu}.$$

Proof. We present an argument only if it is not totally straightforward.

1) A residue calculation.

5) The first claim is an easy calculation. The second is proved in the following Lemma.

Lemma 5.3. *One has for* $m \geq 0$[7]

$$-\,i\int_{\alpha_0}\frac{e^{-ip\omega}}{\mathfrak{sh}^m(\pi p)}dp = Q_{m-1}(\omega). \tag{85}$$

For example,

$$-i\int_{\alpha_0}\frac{e^{-ip\omega}}{\mathfrak{sh}(\pi p)}dp = 1, \qquad -i\int_{\alpha_0}\frac{e^{-ip\omega}}{\mathfrak{sh}^2(\pi p)}dp = \frac{\omega}{2\pi i}.$$

Proof. Integral (85) satisfies recursion (81), which determines each next one uniquely up to a constant. Furthermore, $Q_m(\omega) = (-1)^m Q_m(-\omega)$, which tells that $Q_{2k+1}(\omega)$ is divisible by ω. This, however, does not complete the proof, so we give a proof based on a different idea. Set

$$\mathrm{I}_m(\omega) := -i\int_{\alpha_0}\frac{e^{-ip\omega}}{\mathfrak{sh}^m(\pi p)}dp.$$

Then one has a recursion

$$\mathrm{I}_{m+1}(\omega) = \frac{\omega - i\pi(m-1)}{2\pi i m}\mathrm{I}_m(\omega + i\pi). \tag{86}$$

Indeed, integrating by parts we get

$$\int_{\alpha_0} e^{-ip\omega}\frac{d}{dp}\mathfrak{sh}^{-m}(\pi p)dp = \omega \mathrm{I}_m(z).$$

Since

$$-\frac{d}{dp}\mathfrak{sh}^{-m}(\pi p) = \frac{\pi m\cdot(e^{\pi p}+e^{-\pi p})}{\mathfrak{sh}^{m+1}(\pi p)} = \frac{\pi m}{\mathfrak{sh}^m(\pi p)} + \frac{2\pi m\cdot e^{-\pi p}}{\mathfrak{sh}^{m+1}(\pi p)},$$

we get $(\omega - i\pi m)\cdot \mathrm{I}_m(\omega) = 2\pi i m\cdot \mathrm{I}_{m+1}(\omega - i\pi)$. This is equivalent to (86). Therefore

$$\mathrm{I}_{m+1}(\omega) = Q_m(z)\mathrm{I}_1(\omega + i\pi m) = Q_m(\omega).$$

□

6) Done by a change of variables $q = p/\hbar$, preserving the isotopy class of the contour α_0.

7) Done by a change of variables $q = -\overline{p}$, altering the orientation of α_0:

$$\begin{aligned}\overline{B^{\hbar}_{a,b,n}(\omega)} = (-i)^{n-1}\int_{\overline{\alpha_0}}\frac{e^{i\overline{p}\overline{\omega}}}{\mathfrak{sh}^a(\pi\overline{p})\mathfrak{sh}^b(\pi\overline{\hbar}\overline{p})}\frac{d\overline{p}}{\overline{p}^n} \overset{q=-\overline{p}}{=} \\ (-1)^{a+b+1}i^{n-1}\int_{\alpha_0}\frac{e^{-ip\overline{\omega}}}{\mathfrak{sh}^a(\pi p)\mathfrak{sh}^b(\pi\overline{\hbar}p)}\frac{dp}{p^n} = (-1)^{a+b+1}B^{\overline{\hbar}}_{a,b,n}(\overline{\omega}).\end{aligned} \tag{87}$$

The extra $-$ sign amounts to the change of the contour orientation.

8) It is obtained by a change of variables $q = -p$. It does not change the contour α_0. □

[7] The $-i$ factor is just the factor i^{n-1} in the case $n = 0$.

5.2 Depth one quantum polylogarithms

They are the following integrals:

$$\mathcal{F}^{\hbar}_{a,b,n}(\omega) := i^{n-1}\int_{\mathbb{R}+i0} \frac{e^{-ip\omega}}{\mathfrak{sh}(\pi p)^a \mathfrak{sh}(\pi\hbar p)^b}\frac{dp}{p^n}, \qquad a,b\in\mathbb{Z}_{\geq 0},\ n\in\mathbb{Z}.$$

The next Lemma tells that they reduce to the classical polylogarithms when $b=0$.

Lemma 5.4. *i) One has for $a>0$, $m\geq 0$:*

$$i^{-m-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)}p^m dp = \left(\frac{d}{d\omega}\right)^m \Big(Q_{a-1}(\omega)\mathrm{Li}_0(e^{\omega+\pi i a})\Big). \tag{88}$$

In particular, it is a single-valued meromorphic function in z

ii) One has for $a>0$, $n>0$:

$$i^{n-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)}\frac{dp}{p^n} \;=\; i^n\sum_{k\geq 0}\binom{n+k}{k}\left(-\frac{d}{d\omega}\right)^k Q_{a-1}(\omega)\cdot \mathrm{Li}_{n+k}\left(e^{\omega+\pi i a}\right). \tag{89}$$

Formulas (89) look simpler for the generating series $\mathrm{Li}(x;t) := \sum_{n>0}\mathrm{Li}_n(x)t^{n-1}$

$$\sum_{n>0}t^{n-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)}\frac{dp}{p^n}\cdot t^{n-1} \;=\; i^n\left(1+t^{-1}\frac{d}{dw}\right)^{-1} Q_{a-1}(\omega)\cdot\mathrm{Li}\left(e^{\omega+\pi i a};t\right). \tag{90}$$

Examples. 1. Note that $\mathrm{Li}_0(e^\omega)=\frac{d}{d\omega}\mathrm{Li}_1(e^\omega)=\frac{e^\omega}{1-e^\omega}$. Then one has

$$i^{-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)}dp \;=\; Q_{a-1}(\omega)\mathrm{L}_0(e^{\omega+\pi i a}) \;=\; Q_{a-1}(\omega)\frac{e^{\omega+\pi i a}}{1-e^{\omega+\pi i a}}. \tag{91}$$

For example,

$$\begin{aligned} i^{-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}(\pi p)}dp &= \frac{-e^\omega}{1+e^\omega},\\ i^{-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}^2(\pi p)}dp &= \frac{\omega}{2\pi i}\frac{e^\omega}{1-e^\omega},\\ i^{-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}^3(\pi p)}dp &= \frac{\omega^2+\pi^2}{2!(2\pi i)^2}\frac{-e^\omega}{1+e^\omega}. \end{aligned} \tag{92}$$

2. Formula (88) for $m=0$ is

$$i^{-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)}\frac{dp}{p} = \sum_{k\geq 0}\left(-\frac{d}{d\omega}\right)^k Q_{a-1}(\omega)\cdot\mathrm{Li}_k\left(e^{\omega+\pi i a}\right). \tag{93}$$

Proof. i) We start with the $m=0$ case. Let us calculate integral (91) using the residue theorem, assuming that $\mathrm{Re}(\omega)<0$, using the rectangular contour Ω_N a bit over the real axis. The residues are at the points ik, $k > 0$. The residue at $p = ik$ equals to the residue at $p = 0$ multiplied by $(-1)^{ka}e^{k\omega}$. Lemma 5.3 calculaties the integral around $p = 0$. So we get

$$Q_{a-1}(\omega) \cdot \sum_{k>0} (-1)^{ka} e^{k\omega} = Q_{a-1}(\omega) \cdot \sum_{k>0} e^{k(\omega+i\pi a)} = Q_{a-1}(\omega) \frac{e^{(\omega+i\pi a)}}{1 - e^{(\omega+i\pi a)}}.$$

Formula (88) follows from this by differentiating by ω.

ii) We prove (90) applying $1 + t^{-1}\frac{d}{d\omega}$ to the left hand side, using the fact that

$$i^n \frac{d}{d\omega} \int_{\mathbb{R}+i0} \frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)} \frac{dp}{p^n} = i^{n-1} \int_{\mathbb{R}+i0} \frac{e^{-ip\omega}}{\mathfrak{sh}^a(\pi p)} \frac{dp}{p^{n-1}}.$$

□

Theorem 5.5. *The depth one quantum polylogarithm $\mathcal{F}^{\hbar}_{a,b,n}(\omega)$ has the following features:*

1. *Differential and difference equations:*

$$\begin{aligned} d\mathcal{F}^{\hbar}_{a,b,n}(\omega) &= \mathcal{F}^{\hbar}_{a,b,n-1}(\omega)d\omega, \\ \Delta_{i\pi\hbar}\mathcal{F}^{\hbar}_{a,b,n}(\omega) &= \mathcal{F}^{\hbar}_{a,b-1,c}(\omega), \\ \Delta_{i\pi}\mathcal{F}^{\hbar}_{a,b,n}(\omega) &= \mathcal{F}^{\hbar}_{a-1,b,n}(\omega). \end{aligned} \tag{94}$$

2. *The limit when $\Re z \to -\infty$, taken along a line parallel to the real axis:*

$$\lim_{\Re z \to -\infty} \mathcal{F}^{\hbar}_{a,b,n}(z) = 0.$$

3. *Let $a, b \geq 0$ and $n \geq 1$. Then $\mathcal{F}^{\hbar}_{a,b,n}(\omega)$ is a multivalued analytic function with the singularities at the two integral positive cones:*

$$\Big\{ \pm\pi i \Big((2m + a) + (2n + b)\hbar \Big) \ | \ m, n \in \mathbb{Z}_{\geq 0} \Big\}.$$

4. *Asymptotic expansion when $\hbar \to 0$:*

$$\mathcal{F}^{\hbar}_{a,b,n}(\omega) \ \sim_{\hbar\to 0} \ \frac{i^n}{(2\pi\hbar)^b} \sum_{k\geq 0} \binom{n+k}{k} \left(-\frac{d}{d\omega}\right)^k Q_{a-1}(\omega) \cdot \mathrm{Li}_{b+n+k}\left(e^{\omega+\pi i a}\right) + \dots. \tag{95}$$

5. *The value at $\hbar = 1$:*

$$\mathcal{F}^{1}_{a,b,n}(\omega) = i^{n+b} \sum_{k\geq 0} \binom{n+k}{k} \left(-\frac{d}{d\omega}\right)^k Q_{a+b-1}(\omega) \mathrm{L}_{n+k+b}\left(e^{\omega+\pi i(a+b)}\right). \tag{96}$$

6. *Complex conjugation:*

$$\overline{\mathcal{F}^{\hbar}_{a,b,n}(\omega)} = (-1)^{a+b-1}\mathcal{F}^{\hbar}_{a,b,n}(\overline{\omega}).$$

7. *The $\omega \longleftrightarrow -\omega$ symmetry:*

$$\mathcal{F}^{\hbar}_{a,b,n}(\omega) + (-1)^{a+b+n-1}\mathcal{F}^{\hbar}_{a,b,n}(-\omega) = B_{a,b,n}(\omega; \hbar).$$

8. *The* $\hbar \longleftrightarrow 1/\hbar$ *symmetry:*

$$\mathcal{F}^{\hbar}_{a,b,n}(\omega) = \hbar^{n-1}\mathcal{F}^{1/\hbar}_{b,a,n}(\omega\hbar). \tag{97}$$

9. *Distribution relations:*

$$r^{n-1}\mathcal{F}^{\frac{r}{s}\hbar}(r\omega) = \prod_{\alpha=\frac{1-r}{2}}^{\frac{r-1}{2}}\prod_{\beta=\frac{1-s}{2}}^{\frac{s-1}{2}}\mathcal{F}^{\hbar}(\omega + \frac{2\pi i}{r}\alpha + \frac{2\pi i\hbar}{s}\beta).$$

Proof. We provide the arguments only when they are not evident.

3) If $\hbar \in \mathbb{R}_{>0}$ the integral converges when $|\mathrm{Im}z| < 1 + \hbar$. The claim follows from recursions (94), and uses Property 2) for the normalization. Precisely, let $n = 1$. Then by Lemma 5.4

1) The function $\mathcal{F}^{\hbar}_{a,0,1}(z)$ has simple poles at the rays $\pm\pi i(2\mathbb{Z}_{\geq 0} + a)$.

2) The function $\mathcal{F}^{\hbar}_{0,b,1}(z)$ has simple poles at the rays $\pm\pi i\hbar(2\mathbb{Z}_{\geq 0} + b)$.

Note that the roots of polynomials $Q_{a-1}(z)$ and $Q_{b-1}(z/\hbar)$ kill the poles at the centered at 0 segments of lengths $a-2$ and respectively $b-2$, with the steps $2\pi i$ and $2\pi i\hbar$.

The case $n > 1$ is obtained by the integration in z, and thus follows from the $n = 1$ case. The case when $\hbar \in \mathbb{C} - (-\infty, 0]$ follows by an analytic continuation.

4) Follows by (89) from

$$\mathcal{F}^{\hbar}_{a,b,n}(\omega) \ \sim_{\hbar\to 0} \ \frac{i^{n-1}}{(2\pi\hbar)^b}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}(\pi p)^a}\frac{dp}{p^{b+n}} + \dots \tag{98}$$

5) Follows from (89).

7) Change the variables $q = -p$. It changes the integration contour γ to $-\gamma$. Their sum is a clockwise contour around 0. □

Example: Basic depth one quantum polylogarithms. They are given by the integrals

$$\mathcal{F}^{\hbar}_{n}(\omega) := i^{n-1}\int_{\mathbb{R}+i0}\frac{e^{-ip\omega}}{\mathfrak{sh}(\pi p)\mathfrak{sh}(\pi\hbar p)}\frac{dp}{p^n}, \qquad n \in \mathbb{Z},$$

and have the following properties:

1. Asymptotic relation to the n-logarithm

$$\mathcal{F}^{\hbar}_{n}(\omega) \ \sim_{\hbar\to 0} \ \frac{\mathrm{L}_{n+1}(-e^{\omega})}{2\pi\hbar}.$$

2. The differential and difference relations:

$$\begin{aligned} d\mathcal{F}^{\hbar}_{n}(\omega) &= \mathcal{F}^{\hbar}_{n-1}(\omega)d\omega \\ \Delta_{i\pi\hbar}\mathcal{F}^{\hbar}_{n}(\omega) &= \mathrm{L}_n(-e^{\omega}), \\ \Delta_{i\pi}\mathcal{F}^{\hbar}_{n}(\omega) &= \hbar^{n-1}\mathrm{L}_n(-e^{\omega/\hbar}). \end{aligned} \tag{99}$$

3. Modular property:

$$\mathcal{F}^{\hbar}_{n}(\omega) + \hbar^{1-n}\mathcal{F}^{-1/\hbar}_{n}(\omega/\hbar) = 0.$$

Equivalently, the generating series $\mathcal{F}(\omega;\hbar;t) := \sum_{m=1}^{\infty}\mathcal{F}^{\hbar}_{m}(\omega)t^{m-1}$ satisfy

$$\mathcal{F}(\omega;\hbar;t) + \mathcal{F}(\omega/\hbar;-1/\hbar;t/\hbar) = 0.$$

4. Relation with q-polylogarithms:

$$\mathcal{F}_n^{\hbar}(\omega) \ = \ \mathrm{L}_{1,n}(e^{-\omega}; e^{i\pi\hbar}) - \hbar^{n-1}\mathrm{L}_{1,n}(e^{-\omega/\hbar}; e^{i\pi/\hbar}). \tag{100}$$

Equivalently, the generating series $\mathrm{L}(e^z; e^{i\pi\hbar}; t) := \sum_{m=1}^{\infty} \mathrm{L}_{1,m}(e^z; e^{i\pi\hbar})t^{m-1}$ satisfy

$$\mathcal{F}(\omega; \hbar; t) = \mathrm{L}(e^{\omega}; e^{i\pi\hbar}; t) - \mathrm{L}(e^{\omega/\hbar}; e^{-i\pi/\hbar}; \hbar t).$$

5. Distribution relations:

$$\sum_{\alpha=\frac{1-r}{2}}^{\frac{r-1}{2}} \sum_{\beta=\frac{1-s}{2}}^{\frac{s-1}{2}} \mathcal{F}_n^{\hbar}(\omega + \frac{2\pi i}{r}\alpha + \frac{2\pi i\hbar}{s}\beta) = r^{n-2}\mathcal{F}_n^{\frac{r}{s}\hbar}(r\omega).$$

6 Concluding remarks

Scattering amplitudes in the $\mathcal{N} = 4$ SUYM theory can be expressed via polylogarithms and their generalizations. For example, the MHV n particles L$-$loop scattering amplitudes are weight 2L functions on the configuration space $\mathrm{Conf}_n(\mathbb{CP}^3)$ of collections of n points in $\mathbb{CP}^3$, considered modulo the diagonal action of the group PGL_4. It is invariant under the cyclic shift of the points. The space $\mathrm{Conf}_n(\mathbb{P}^3)$ carries canonical cluster Poisson structure, invariant under the cyclic shift, and the related space $\mathrm{Conf}_n(\mathbb{C}^4)$ carries a cluster K_2-variety structure.

What is role the cluster Poisson structure on $\mathrm{Conf}_n(\mathbb{P}^3)$ for the scattering amplitudes?

Any cluster Poisson variety $\mathscr{X}$ admits cluster quantization [FG3], where the quantised algebra of functions $\mathcal{O}_q(\mathscr{X})$ acts by unbounded operators in a cluster Hilbert space $\mathcal{H}_{\mathscr{X}}$. The Hilbert space $\mathcal{H}_{\mathscr{X}} \otimes \overline{\mathcal{H}}_{\mathscr{X}}$ for a given cluster coordinate system $\mathbf{c}$ is realised as the Hilbert space $\mathrm{L}_2(\mathscr{A}(\mathbb{R}_{>0}))$ of functions on the space of real positive points of the cluster K_2-variety $\mathscr{A}(\mathbb{R}_{>0})$. These Hilbert spaces for different cluster coordinate systems are related by the quantum dilogarithm intertwiners.

Suppose that the asymptotic expansion as $\hbar \to 0$ of a vector $\varphi_{\hbar}^{\mathbf{c}}$ in the Hilbert space for a single cluster coordinate system $\mathbf{c}$ is written as

$$\varphi_{\hbar}^{\mathbf{c}} \sim e^{F_{\mathbf{c}}(a,\hbar)/\hbar}, \quad \text{where } F_{\mathbf{c}}(a,\hbar) \text{ is a function on } \mathscr{A}(\mathbb{R}_{>0}) \text{ depending on } \hbar. \tag{101}$$

The vectors $\varphi_{\hbar}^{\mathbf{c}}$ and $\varphi_{\hbar}^{\mathbf{c}'}$ for two different clusters $\mathbf{c}$ and $\mathbf{c}'$ are related by the quantum dilogarithm intertwiners. Therefore the stationary phase method shows that the $\hbar \to 0$ asymptotics of the vectors $\varphi_{\hbar}^{\mathbf{c}}$ in any cluster coordinate system $\mathbf{c}$ can be written in the form (101). Moreover the functions $F_{\mathbf{c}}(a,\hbar)_{|\hbar=0}$ and $F_{\mathbf{c}'}(a,\hbar)_{|\hbar=0}$ for any two clusters $\mathbf{c}$ and $\mathbf{c}'$ are the same functions on $\mathscr{A}(\mathbb{R}_{>0})$, expressed in the cluster coordinates for the clusters $\mathbf{c}$ and $\mathbf{c}'$.

I suggest that the scattering amplitudes should have an $\hbar-$deformation, becoming vectors $\mathcal{A}_{n,\mathrm{L}}^{\hbar}$ in the space of cluster distributions. These vectors should be expressed via quantum polylogarithms and their generalizations. In a given cluster coordinate system $\mathbf{c}$, the asymptotic expansion of the vectors $\mathcal{A}_{n,\mathrm{L}}^{\hbar}$ should have the form

$$\mathcal{A}_{n,\mathrm{L}}^{\hbar} \sim e^{\alpha_{\mathbf{c},\mathrm{L}}(a,\hbar)/\hbar}, \quad \text{where } \alpha_{\mathbf{c},\mathrm{L}}(a,\hbar) \text{ is a function on } \mathrm{Conf}_n(\mathbb{R}_{>0}^4). \tag{102}$$

One should have

$$\alpha_{n,\mathrm{L}}(a,\hbar)_{|\hbar=0} = \textit{the } \mathrm{L}-\textit{loop scatterring amplitude.}$$

Similar conjectural cluster description of the correlation functions in the Liouville and Toda theories is discussed in [GS, Section 6].

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