Resurgence and perverse sheaves

Mikhail Kapranov, Yan Soibelman

December 27, 2025

To our friend Maxim Kontsevich on occasion of his sixtieth birthday

Abstract

We propose a point of view on resurgence theory based on the study of perverse sheaves on the complex line carrying an algebraic structure with respect to additive convolution. In particular, we lift the concept of alien derivatives introduced originally by J. Écalle, to the framework of perverse sheaves and study its behavior under sheaf-theoretic convolution. The full fledged resurgence theory needs a (yet undeveloped) generalization of the concept of perverse sheaves allowing infinite, possibly dense, sets of singularities. We discuss possible approaches to defining such objects and some potential examples of them coming from Cohomological Hall Algebras, wall-crossing structures and Chern-Simons theory.

Contents

Le sommeil est plein de miracles!
Par un caprice singulier,
J'avais banni de ces spectacles
Le végétal irrégulier.

Ch. Baudelaire, *Rêve parisien*

0 Introduction

The theory of resurgent functions pioneered by J. Écalle [9] studies analytic functions "given by divergent series" in terms of singularities of their Borel (formal Fourier-Laplace) transform (see §1.1 below for discussion). As with any kind of Fourier transform, this procedure takes multiplication into additive convolution. Resurgent analysis then proceeds by using equations that involve convolution as well as the monodromy data (alien derivatives) for the Borel-transformed functions.

The goal of this paper is to propose a more conceptual point of view on resurgence theory by using the notion of perverse sheaf. It is known that operations such as convolution or Fourier transform can be defined at the sheaf-theoretical level and match, to some extent, the corresponding analytic operations for functions when such functions are realized as sections of the sheaves in question. So, it is a natural idea to import the perverse sheaves language into resurgent analysis, with the hope to achieve greater conceptual clarity.

In a few words, our proposal is to consider perverse sheaves on $\mathbb{C}$ carrying the structure of an algebra with respect to convolution (see §3.1 for more details). Using this structure one can write resurgent equations whose unknowns will be sections of such "resurgent perverse sheaves", with classical examples appearing when sections are realized as actual analytic functions. But one can also, in principle, construct resurgent perverse sheaves in a more abstract fashion, not unlike the way one constructs field extensions not necessarily embedded in $\mathbb{C}$ by adjoining roots of algebraic equations.

Already in 1985 B. Malgrange [26] gave an interpretation of the resurgent formalism in terms of M. Sato's theory of microfunctions. His main observation was that the concept of a "singularity" (or "singular part"), ubiquitous in this formalism, is but a synonym for a microfunction. Now, the space of vanishing cycles of a perverse sheaf on $\mathbb{C}$ is the same as the space of microfunction solutions of the corresponding holonomic regular D-module [12].

Therefore, working with perverse sheaves and their vanishing cycles is a natural conceptual framework for resurgence theory.

However, for true applications to resurgence the theory of perverse sheaves must be extended to match the kind of multivaluedness that resurgent functions typically possess. These functions typically have infinite or even dense sets of singular points. That is, on any "branch" the singular points are of course discrete, but going around each one leads to a new branch with new singularities etc. Such behavior is referred to as "'analytic continuation without end".

In this paper we do not attempt to generalize the theory of perverse sheaves in this direction. Instead, we develop a resurgence-like formalism involving the standard concept of perverse sheaves on $\mathbb{C}$ (i.e. with finitely many singularities). Already this allows us to highlight many of the familiar features in the sheaf-theoretic context, for example the interpretation of Stokes data via Picard-Lefschetz type formulas in the Borel plane. Further, some examples of "resurgent perverse sheaves" may be already given in this restricted context, such as the version of Cohomological Hall algebra in §3.2.

A special role in our considerations is played by the category $\overline{\text{Perv}}(\mathbb{C})$ obtained by localizing $\text{Perv}(\mathbb{C})$, the abelian category of all perverse sheaves on $\mathbb{C}$, by the Serre subcategory of (shifted) constant sheaves [13]. Objects of $\overline{\text{Perv}}(\mathbb{C})$ have well-defined "tunnelling data" consisting of the spaces of vanishing cycles $\Phi_a, a \in \mathbb{C}$ and the transport maps $m_{ab}(\gamma) : \Phi_a \to \Phi_b$ for various paths γ joining a and b.

It is known [20, 10] that $\overline{\text{Perv}}(\mathbb{C})$ can be embedded back into $\text{Perv}(\mathbb{C})$ as the subcategory $\text{Perv}^0(\mathbb{C})$ formed by perverse sheaves $\mathcal{F}$ with $H^\bullet(\mathbb{C}, \mathcal{F}) = 0$. The operation of additive convolution $\mathcal{F} * \mathcal{G}$ is most easily defined using this realization [10]. Our "toy resurgent formalism" can be seen as further study of the Tannakian Galois group of the tensor category $(\overline{\text{Perv}}(\mathbb{C}), *)$, as defined and already studied in [10]. So, our larger point is that the full fledged resurgent formalism is just a similar study but for a more general concept of perverse sheaves, still to be defined rigorously (see §3.1 below). We plan to discuss this further in a future work.

The paper consists of three chapters. In Chapter 1 we recall the elementary theory of perverse sheaves on $\mathbb{C}$ (with finitely many singularities) with emphasis on features that we need. The motivational §1.1 explains the general framework of Borel summation and the resulting "doctrine of two planes": the original one carrying functions given by divergent series and the Borel one where things become more topological. In §1.2 we recall the basic definitions and emphasize the Picard-Lefschetz formula (Proposition 1.2.8) in the context of perverse sheaves. In §1.3 we present the Gelfand-MacPherson-Vilonen classification of perverse sheaves and of objects of the localized category $\overline{\text{Perv}}(\mathbb{C}, A)$, where constant sheaves are factored out but the vanishing cycles and transport maps remain. The Fourier transform for perverse sheaves is explained in §1.4. There, we mostly follow [17]. In §1.5 we discuss a particular class of examples of perverse sheaves on $\mathbb{C}$ associated to a regular function $S : X \to \mathbb{C}$ on a complex algebraic variety. We call them *Lefschetz perverse sheaves* $\mathcal{L}_S$. The Fourier transform of $\mathcal{L}_S$ can be seen as a categorification of the exponential integral associated to S.

In Chapter 2 we build up features of perverse sheaves on $\mathbb{C}$ which are most remindful of resurgence formalism. In this, we extend the analysis of rectilinear transports given in [17], in the case when the set A of singularities is in linearly general position, to the arbitrary case, when an interval $[a, b], a, b \in A$ can contain intermediate points. In §2.1 we discuss additive convolution of perverse sheaves on $\mathbb{C}$. Motivated by the classical resurgent formalism, we consider, in §2.2, various ways of modifying the rectilinear transport so as to avoid the intermediate points. The formulas for alien derivatives appear naturally in this context as some linear combinations of such modified transports. In §2.3 we study alien derivatives more systematically; we also explain their relation with the Stokes automorphisms for the Fourier transform.

In the final Chapter 3 we discuss how our approach can be applied to actual resurgence problems. This chapter is more speculative. We start by sketching in §3.1 the general program of studying perverse sheaves which are algebras with respect to the convolution, highlighting the difficulties that are present in the general case. Then, we discuss several classes of potential examples: the example with the Cohomological Hall algebra (a.k.a COHA) of a quiver in §3.2, that of "cluster perverse sheaves" associated to wall-crossing structures in §3.3, and that of Lefschetz perverse sheaves associated to the complex Chern-Simons functional in §3.4.

Acknowledgments. The research of M.K. was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan and by the JSPS KAKENHI grant 20H01794. The work of Y.S. was partially supported by Simons Foundation grant MP-TSM-00001658, NSF grant 2401518, IHES Sabbatical Professorship and ERC grant "Renew Quantum".

We would like to express our gratitude to A. Soibelman for correcting the English in the Introduction.

1 Perverse sheaves and their Fourier transform

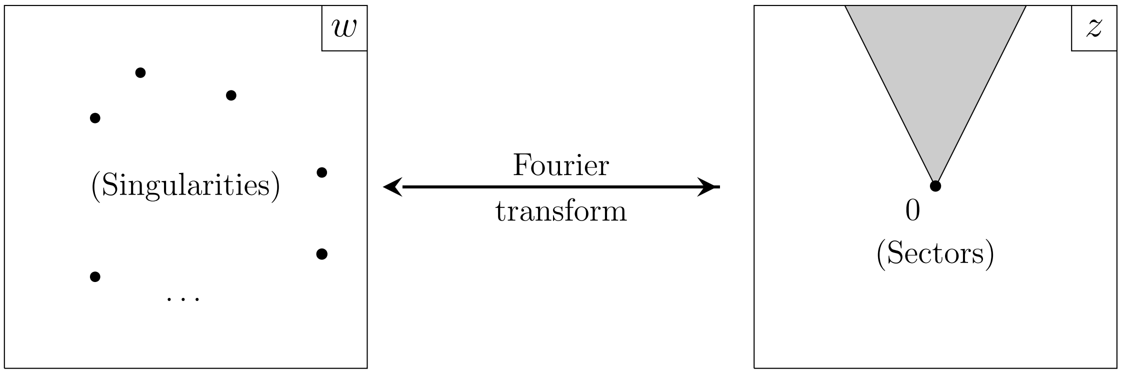
1.1 Motivation: Borel summation and the Borel plane

The famous Borel summation process for divergent series can be seen as an application of the Fourier transform in the complex domain. It connects two copies of the complex plane $\mathbb{C}$ which are loosely related "by the Fourier transform":

- The original ("irregular") plane $\mathbb{C}_z$ with coordinate ("large parameter") z in which we study possibly divergent formal power series near ∞:

(1.1.1) $$\widehat{f}(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}$$

  Such series typically satisfy linear differential equations irregular at ∞, are divergent everywhere but serve as asymptotic expansions of interesting analytic solutions in various



sectors. Quantum mechanical asymptotic series in powers of the Planck constant $\hbar$ are realized here by putting $z = 1/\hbar$.

- The dual (Borel or "regular") plane $\mathbb{C}_w$ with coordinate w where we study (or obtain) solutions of differential equations with regular singularities, given by convergent series in w, which represent multivalued functions and so can be analyzed topologically. The modern way of doing so is by using the language of perverse sheaves.

On the formal level, Borel summation of the series (1.1.1) (when it is possible) consists, first, of taking the termwise Fourier transform of the series using the identity (particular case of the general formula for the Fourier transform of z^α)

$$\mathrm{FT}\left(\frac{1}{z^{n+1}}\right) = \frac{w^n}{n!}.$$

This gives the Borel-transformed series

$$\widehat{f}^B(w) = \sum_{n=0}^{\infty} \frac{a_n}{n!} w^n$$

which has more chances to converge. In good cases it has nonzero radius of convergence and extends to an analytic function $f^B(w)$ "on the entire $\mathbb{C}_w$" but possibly multivalued, with singularities etc. Then the prescription for the sum $f(z)$ of the original series $\widehat{f}(z)$ is obtained by taking the inverse[1] Fourier (-Laplace) transform of $\widehat{f}^B(w)$:

$$f(z) := \int_0^\infty f^B(w) e^{-zw} dw.$$

Because of singularities of $f^B(w)$ there can be several inequivalent allowable choices of the integration contour leading to ambiguity of the Borel sum, which is not surprising if the series is divergent.

[1]the kind that sends $w^n/n!$ back into $1/z^{n+1}$

On the conceptual level, the Borel summation approach can be said to consist in representing irregular data[2] as Fourier transforms of regular ones. The success of this approach comes from the fact that such representation is possible in many cases of practical interest. Differential equations which are Fourier transforms of regular ones form a rather special class.

[2]Here we understand the word "irregular" in the wider sense, including but not restricted to differential equations and functions satisfying them.

1.2 Perverse sheaves on Riemann surfaces

A Generalities. We fix a base field $\mathbf{k}$. Let X be a complex analytic manifold. We denote by $\mathrm{LS}(X)$ the category of local systems of (not necessarily finite-dimensional) $\mathbf{k}$-vector spaces on X.

Let $\mathcal{S}$ a locally finite complex Whitney stratification of X. Thus each stratum $S \in \mathcal{S}$ is a complex submanifold; we denote $i_S : S \to X$ the embedding. The closure $\overline{S}$ is, in general, a singular complex space. We denote by $D^b(X, \mathcal{S})$ the triangulated category of bounded $\mathcal{S}$-constructible complexes of sheaves $\mathcal{F}$ on X. By definition, $\mathcal{S}$-constructibility of $\mathcal{F}$ means that each cohomology sheaf $\underline{H}^i(\mathcal{F})$ is $\mathcal{S}$-constructible. That is, each $i_S^* \underline{H}^q(\mathcal{F}) = \underline{H}^q(i_S^* \mathcal{F})$ is an object of $\mathrm{LS}(S)$.

We denote $\mathrm{Perv}(X, \mathcal{S}) \subset D^b(X, \mathcal{S})$ the abelian category of perverse sheaves of $\mathbf{k}$-vector spaces on X smooth with respect to $\mathcal{S}$. Explicitly, $\mathrm{Perv}(X, \mathcal{S})$ consists of complexes $\mathcal{F} \in D^b(X, \mathcal{S})$ with the following properties:

(1) For each stratum $S \in \mathcal{S}_A$ we have $\underline{H}^q(i_S^* \mathcal{F}) = 0$ for $q > -\dim_{\mathbb{C}} S$.

(2) For each stratum $S \in \mathcal{S}_A$ we have $\underline{H}^q(i_S^! \mathcal{F}) = 0$ for $q < -\dim_{\mathbb{C}} S$.

For example, if $\mathcal{L} \in \mathrm{LS}(X)$ is a local system, then $\mathcal{L}[\dim X]$, i.e., $\mathcal{L}$ put in degree $(-\dim X)$, is perverse and lies in $\mathrm{Perv}(X, \varnothing)$.

Remark 1.2.1. In [17] we used a different normalization of the perversity conditions for which a local system in degree 0 is considered to be perverse. In references to [17] later in this paper, this difference, being easy to account for, is not further highlighted.

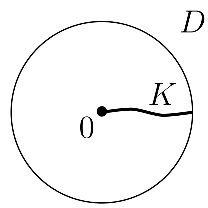
B Nearby and vanishing cycles. From now on we assume $\dim_{\mathbb{C}} X = 1$, so X is a complex curve (Riemann surface, possibly non-compact). Then a stratification $\mathcal{S}$ of X is given by a discrete subset $A \subset X$ so the strata are elements of A and the complement $X \backslash A$. In this case we use the notation $\mathrm{Perv}(X, A)$ for $\mathrm{Perv}(X, \mathcal{S})$. The definition implies that for any $\mathcal{F} \in \mathrm{Perv}(X, A)$ the restriction $\mathcal{F}|_{C \backslash A}$ is quasi-isomorphic to a local system (not necessarily of finite rank) in degree -1 and that $\underline{H}^q(\mathcal{F}) = 0$ for $q \neq -1, 0$.

Example 1.2.2. Let $X = D = \{|z| < 1\}$ be the unit disk in $\mathbb{C}$ and $A = \{0\}$. In this case it is classical [12] that $\mathrm{Perv}(D, 0)$ is equivalent to the category of diagrams

$$\Phi \underset{v}{\overset{u}{\rightleftarrows}} \Psi$$

of $\mathbf{k}$-vector spaces are linear maps such that $T_\Psi := \mathrm{Id}_\Psi - uv$ is an isomorphism (or, what is equivalent, such that $T_\Phi = \mathrm{Id}_\Phi - vu$ is an isomorphism).

More precisely, see [12] [17, Prop. 1.1.6], the equivalence above depends on a choice of a *radial cut* $K \subset D$, a simple curve starting at 0 and ending on the boundary ∂D. Given such K, the sheaves of K-supported hypercohomology $\underline{H}^q_K(\mathcal{F})$ are 0 for $q \neq 0$, the sheaf $\underline{H}^0_K(\mathcal{F})$ on



K is constructible with respect to the stratification of K into $\{0\}$ and $K - \{0\}$, the spaces Φ and Ψ associated to $\mathcal{F}$ are found as its stalks:

$$\Phi = (\underline{H}^0_K(\mathcal{F}))_0, \quad \Psi = (\underline{H}^0_K(\mathcal{F}))_\varepsilon, \ \forall \varepsilon \in K \backslash \{0\},$$

and the map $u : \Phi \to \Psi$ is the generalization map of $\underline{H}^0_K(\mathcal{F})$. See [12] and the discussion after [17, Prop. 1.1.6] for the definition of $v : \Psi \to \Phi$. The spaces Φ and Ψ are called the spaces of *vanishing cyclies* and *nearby cycles* of $\mathcal{F}$ at 0 (in the direction of K). Note also that Ψ is identified with the stalk of the local system $(\mathcal{F}[-1])|_{D\backslash\{0\}}$ at any $\varepsilon \in K - \{0\}$ (hence the name "nearby cycles").

More generally, for a Riemann surface X and a point $a \in X$, we denote by $S^1_a = S^1_a(X)$ the circle of directions at a. If K is a smooth simple curve ending at a, we denote by $\mathrm{dir}_a(K) \in S^1_a$ the direction of K at a. If $A \subset X$ is discrete and $\mathcal{F} \in \mathrm{Perv}(X, A)$, then in a small disk near any $a \in A$ we have the situation of Example 1.2.2. In particular, the vector spaces of vanishing and nearby cycles of $\mathcal{F}$ at a, being dependent on the direction of a cut, are naturally local systems on S^1_w which we denote $\mathbf{\Phi}_w(\mathcal{F})$ and $\mathbf{\Psi}_w(\mathcal{F})$. We thus have functors

$$\mathbf{\Phi}_a, \mathbf{\Psi}_a : \mathrm{Perv}(X, A) \longrightarrow \mathrm{LS}(S^1_a). \tag{1.2.3}$$

C Transport maps. We now recall the construction of *curvilinear transport maps* from [17, §1.1C] Let $\dim_{\mathbb{C}}(X) = 1$ and $A \subset X$ be discrete. Let α be a simple, piecewise smooth arc in X joining two distinct points $a, b \in A$ and not passing through any other elements of A see Fig. 1. Let us equip α with the orientation going from a to b.

Considering α as a closed subset in X, we have the sheaf $\underline{H}^0_\alpha(\mathcal{F})$ on α which is constant on the open arc $\alpha - \{a, b\}$.

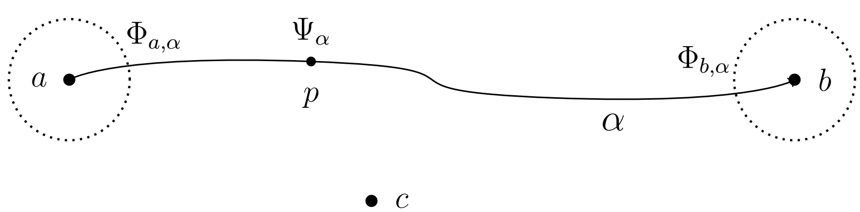


Figure 1: The transport map.

Its stalks at a and b are the vanishing cycle spaces $\Phi_{a,\alpha}$, $\Phi_{b,\alpha}$ for $\mathcal{F}$ at a or b in the direction of α, while its restriction to $\alpha\backslash\{a,b\}$ is a local system on the open interval; in particular, all the stalks of this local system are canonically identified; let us denote their common value by Ψ_α. So we have the maps

$$\Phi_{a,\alpha} \underset{v_{a,\alpha}}{\overset{u_{a,\alpha}}{\rightleftarrows}} \Psi_\alpha \underset{v_{b,\alpha}}{\overset{u_{b,\alpha}}{\leftrightarrows}} \Phi_{b,\alpha}, \tag{1.2.4}$$

obtained from the description of $\mathcal{F}$ on small disks near a and b and using γ as the choice for a cut K. We define the *transport map* along α as

$$m_{ab}(\alpha) = m_{ab}^{\mathcal{F}}(\alpha) \;:=\; v_{b,\alpha} \circ u_{a,\alpha} : \Phi_{a,\alpha} \longrightarrow \Phi_{b,\alpha}. \tag{1.2.5}$$

D Picard-Lefschetz identities for transports. The construction of the maps $m_{ab}(\alpha)$ being purely topological, it is unchanged under isotopic deformations of α which do not pass through other elements of A. More precisely [17, §1.1C], let $(\alpha_t)_{t\in[0,1]}$ be an *admissible isotopy* of paths from a to b, i.e., a continuous 1-parameter family of simple arcs $(\alpha_t)_{t\in[0,1]}$, each α_t joining a with b and not passing through any other $c \in A$. Then we have a commutative diagram

$$\begin{array}{ccc} \Phi_{a,\alpha_0} & \xrightarrow{m_{ab}(\alpha_0)} & \Phi_{b,\alpha_0} \\ {\scriptstyle t_a}\downarrow & & \downarrow{\scriptstyle t_b} \\ \Phi_{a,\alpha_1} & \xrightarrow{m_{ab}(\alpha_1)} & \Phi_{b,\alpha_1}, \end{array} \tag{1.2.6}$$

where t_a is the monodromy of the local system $\mathbf{\Phi}_a$ on $S^1_{w_i}$ from $\mathrm{dir}_a(\alpha_0)$ to $\mathrm{dir}_a(\alpha_1)$, and similarly for t_b.

We now recall what happens when a path crosses a single point of A. That is, we consider a situation as in Fig. 2, where a path γ' from a to c approaches the composite path formed by β from a to b and α from b to c. After crossing b, the path γ' is changed to γ.

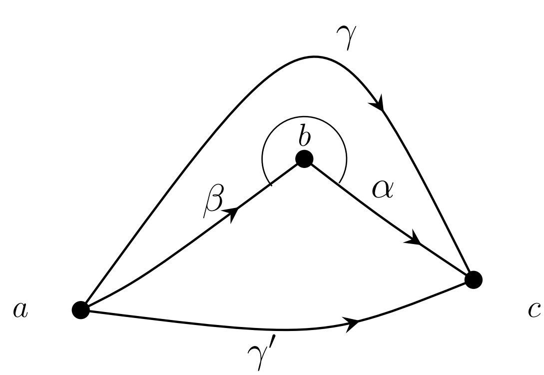


Figure 2: The Picard-Lefschetz situation.

In this case we have identifications

$$\Phi_{a,\gamma} \to \Phi_{a,\beta} \to \Phi_{a,\gamma'}, \quad \Phi_{c,\gamma'} \to \Phi_{c,\alpha} \to \Phi_{c,\gamma}, \quad \Phi_{b,\beta} \to \Phi_{b,\alpha}, \tag{1.2.7}$$

given by *clockwise* monodromies of the local systems $\mathbf{\Phi}$ around the corresponding arcs in the circles of directions. So after these identications we can assume that we deal with single vector spaces denoted by Φ_1, Φ_3 and Φ_2 respectively. Then we have [15, Prop.1.8] [17, Prop.1.1.12]:

Proposition 1.2.8 (Abstract Picard-Lefschetz identity)**.** *We have the equality of linear operators $\Phi_1 \to \Phi_2$:*

$$m_{ac}(\gamma') = m_{ac}(\gamma) - m_{bc}(\alpha) m_{ab}(\beta).$$

□

1.3 (Localized) perverse sheaves on $\mathbb{C}$

A The category $\overline{\mathrm{Perv}}(X, A)$. We start with the case of an arbitrary Riemann surface, i.e. let X, A be as before. Let $\mathrm{LS}(X)$ be the category of local systems of $\mathbf{k}$-vector spaces on X. For each $\mathcal{L} \in \mathrm{LS}(X)$ the shifted sheaf $\mathcal{L}[1]$ is an object of $\mathrm{Perv}(X, A)$. It is straightforward that this defines an embedding of the shifted category $\mathrm{LS}(X)[1]$ (identified with $\mathrm{LS}(X)$) as a Serre subcategory on $\mathrm{Perv}(X, A)$ and so we have the quotient abelian category

$$\overline{\mathrm{Perv}}(X, A) = \mathrm{Perv}(X, A)/(\mathrm{LS}(X)[1]).$$

Explicitly, $\mathrm{Ob}\,\overline{\mathrm{Perv}}(X, A) = \mathrm{Ob}\,\mathrm{Perv}(X, A)$ while

$$\mathrm{Hom}_{\overline{\mathrm{Perv}}(X,A)}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{Perv}(X,A)}(\mathcal{F}, \mathcal{G})/I_{\mathcal{F},\mathcal{G}},$$

where $I_{\mathcal{F},\mathcal{G}}$ is the subset (actually a $\mathbf{k}$-vector subspace) formed by morphisms factoring as $\mathcal{F} \to \mathcal{L}[1] \to \mathcal{G}$ for some $\mathcal{L} \in \mathrm{LS}(X)$. Thus we have a functor $\mathrm{Perv}(X, A) \to \overline{\mathrm{Perv}}(X, A)$ bijective on objects and surjective on morphisms.

Note that the first (but not the second) functor in (1.2.3) vanishes on $\mathrm{LS}(X)[1]$ and so descends to a functor which we denote by the same symbol:

$$\mathbf{\Phi}_a : \overline{\mathrm{Perv}}(X, A) \longrightarrow \mathrm{LS}(S^1_a). \tag{1.3.1}$$

Further, let γ be a simple path joining $a, b \in A$ as in §**C**. The transport map $m_{ab}(\alpha)$ from (1.2.5) descends to a natural transformations (also denoted $m_{ab}(\alpha)$) between functors (1.3.1) evaluated on $\mathrm{dir}_a(\alpha)$ and $\mathrm{dir}_b(\alpha)$. These transformation satisfy deformation invariance (1.2.6) and the Picard-Lefschetz identities (Proposition 1.2.8).

B The Gelfand-MacPherson-Vilonen description of $\mathrm{Perv}(\mathbb{C}, A)$ and $\overline{\mathrm{Perv}}(\mathbb{C}, A)$ for finite A. From now on we assume that our Riemann surface X is the complex line $\mathbb{C}$ with coordinate w. We further assume that the set A is finite.

Recall the descriptions given in [13]. From the topological point of view, we can replace $\mathbb{C}$ by an open disk D which we view as the interior of a closed disk $\overline{D}$, i.e., $D = \overline{D} \backslash \partial \overline{D}$, so $A = \{a_1. \cdots, a_N\} \subset D$.

Fix a point $\mathbf{v} \in \partial D$. Call a $\mathbf{v}$-*spider* for (D, A) a system $K = \{\gamma_1, \cdots, \gamma_N\}$ of simple closed piecewise smoorth arcs in D so that γ_i joins $\mathbf{v}$ with a_i and different γ_i do not meet except at $\mathbf{v}$, see Fig. 3.

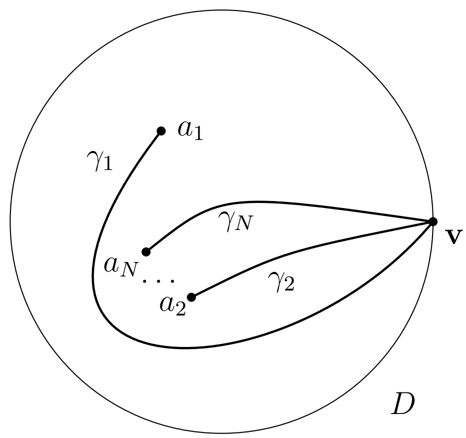
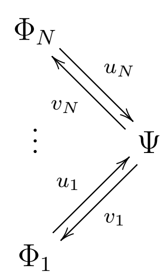


Figure 3: A spider defining the GMV-equivalence.

A $\mathbf{v}$-spider for (D, A) defines a total order on A, by clockwise ordering of the slopes of the γ_i at $\mathbf{v}$ (assumed distinct) and we choose the numbering $A = \{a_1, \cdots, a_N\}$ in this order, as in Fig. 3.

Denote by $\mathcal{Q}_N$ the category of diagrams of finite-dimensional $\mathbf{k}$-vector spaces



such that $T_{\Phi_i} := \mathrm{Id}_{\Phi_i} - v_i u_i$ is invertible for each i. This implies that each $T_{i,\Psi} = \mathrm{Id}_\Psi - u_i v_i$ is invertible as well.

Given a $\mathbf{v}$-spider K for (D, A) and $\mathcal{F} \in \mathrm{Perv}(D, A)$, we consider, for each $i = 1, \cdots, N$, the space $\Phi_{i,K}(\mathcal{F}) = \Phi_{a_i,\gamma_i}(\mathcal{F})$, i.e., the stalk of the local system $\mathbf{\Phi}_{a_i}(\mathcal{F})$ at the point in $S^1_{a_i}$ represented by the direction of γ_i. Let us also identify the stalk $\Psi_{\gamma_i}(\mathcal{F})$ with the stalk $\mathcal{F}_{\mathbf{v}}$, using the monodromy along γ_i. This gives a diagram

$$\Theta_K(\mathcal{F}) = (\Psi(\mathcal{F}), \Phi_{i,K}(\mathcal{F}), u_{i,\gamma_i}, v_{i,\gamma_i}) \in \mathcal{Q}_N.$$

Here u_{a_i,γ_i} and v_{a_i,γ_i} are the canonical maps along γ_i as in (1.2.5).

Proposition 1.3.2 ([13])**.** *Let K be a $\mathbf{v}$-spider for (D, A). The functor $\Theta_K : \mathrm{Perv}(D, A) \to \mathcal{Q}_N$ is an equivalence.* □

Further, a spider K defines, for each $i \neq j$, a path α_{ij}^K joining a_i and a_j by first going from a_i to $\mathbf{v}$ and then from $\mathbf{v}$ to a_j.

Let $\mathcal{M}_N$ be the category whose objects are diagrams consisting of:

(0) Vector spaces Φ_i, $i = 1, \cdots, N$.

(1) Linear operators $m_{ij} : \Phi_i \to \Phi_j$ given for all i, j (including $i = j$) such that $\mathrm{Id}_{\Phi_i} - m_{ii}$ is invertible.

Given an object $\mathcal{F} \in \overline{\mathrm{Perv}}(D, A)$ and a spider K as above, we construct a diagram $\Xi_K(\mathcal{F}) = (\Phi_i, m_{ij}) \in \mathcal{M}_N$ by putting $\Phi_i = \Phi_{i,\gamma_i}(\mathcal{F})$ as before and

$$m_{ij} = \begin{cases} m_{ij}(\alpha_{ij}^K), & \text{if } i \neq j; \\ \mathrm{Id} - T_i(\mathcal{F}), & \text{if } i = j, \end{cases}$$

Proposition 1.3.3 ([13])**.** *The functor $\Xi_K : \overline{\mathrm{Perv}}(D, A) \to \mathcal{M}_N$ is an equivalence.* □

Remark 1.3.4. We can think of the point $\mathbf{v} \in \partial D$ as being far away at the infinity ("Vladivostok"). This becomes even more natural if we use D as a model for the compelx plane $\mathbb{C}$. For this reason we will sometimes refer to the description of $\overline{\mathrm{Perv}}(D, A)$ given by Proposition 1.3.3 as the *Vladivostok description* and call the path m_{ij}^K the *Vladivostok path* joining a_i and a_j. From the naive "physical" point of view this is not the most natural way to connect a_i and a_j by a path.

C $\overline{\mathrm{Perv}}(\mathbb{C}, A)$ **inside** $\mathrm{Perv}(\mathbb{C}, A)$**.** For $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A)$ the hypercohomology $H^i(\mathbb{C}, \mathcal{F})$ of $\mathbb{C}$ with coefficients in $\mathcal{F}$ vanish for $i \neq -1, 0$. Let $\mathrm{Perv}^0(\mathbb{C}, A) \subset \mathrm{Perv}(\mathbb{C}, A)$ be the full subcategory formed by $\mathcal{F}$ such that $H^i(\mathbb{C}, \mathcal{F}) = 0$ for all i. The following statement is a reformulation of the results of [13, 20].

Proposition 1.3.5. *(a) The localization functor* $\mathrm{Perv}(\mathbb{C}, A) \to \overline{\mathrm{Perv}}(\mathbb{C}, A)$ *restricts to an equivalence of categories* $\mathrm{Perv}^0(\mathbb{C}, A) \to \overline{\mathrm{Perv}}(\mathbb{C}, A)$.

(b) Each object of $\mathrm{Perv}^0(\mathbb{C}, A)$ *reduces to a single sheaf in degree* (-1).

Proof: (a) In the proof of [13, Prop. 2.3] the authors construct a full embedding $\lambda : \mathcal{M}_N \to \mathcal{Q}_N$, where $\mathcal{M}_N$ is the category of diagrams describing $\overline{\mathrm{Perv}}(\mathbb{C}, A)$ by Proposition 1.3.3 and $\mathcal{Q}_N$ is the category of diagrams describing $\mathrm{Perv}(\mathbb{C}, A)$ by Proposition 1.3.2. The image $\mathrm{Im}(\lambda)$ is a subcategory in $\mathcal{Q}_N$ which maps equivalently to $\mathcal{M}_N$ under the localization functor $\mathcal{Q}_N \to \mathcal{M}_N$. It is then verified directly from the definition of λ , see the end of the proof of [20, Thm.2.29] that $\mathrm{Im}(\lambda) = \mathrm{Perv}^0(\mathbb{C}, A)$.

(b) This is shown in the first part of the proof of [20, Thm.2.29] . □

Remark 1.3.6. Se we can write the lifting functor λ in a geometric way, as

$$\lambda : \overline{\mathrm{Perv}}(\mathbb{C}, A) \xrightarrow{\sim} \mathrm{Perv}^0(\mathbb{C}, A) \subset \mathrm{Perv}(\mathbb{C}, A).$$

By Proposition 1.3.3, an object of $\overline{\mathrm{Perv}}(\mathbb{C}, A)$ is determined by its vanishing cycles Φ_a and transport maps between them, but does not have well defined stalks at points outside A, which for an actual perverse sheaf form a (shifted) local system on $\mathbb{C}\backslash A$. The functor λ provides a preferred way to supply such local system. Interpreting the construction of λ from [13] in a geometric fashion, we see that the typical stalk of this system is identified with $\bigoplus_{a\in A} \Phi_a$.

Example 1.3.7. let $a \in A$ and $\mathcal{F} = \mathbf{k}_a \in \mathrm{Perv}(\mathbb{C}, A)$ be the skyscraper sheaf at a. Denoting $\overline{\mathcal{F}} \in \overline{\mathrm{Perv}}(\mathbb{C}, A)$ the image of $\mathcal{F}$, the lift $\lambda(\overline{\mathcal{F}}) \in \mathrm{Perv}^0(\mathbb{C}, A)$ is the sheaf $j_!(\underline{\mathbf{k}}_{\mathbb{C}\backslash\{a\}})[1]$, where $j : \mathbb{C}\backslash\{a\} \to \mathbb{C}$ is the embedding. It becomes isomorphic to $\mathcal{F} = \mathbf{k}_a$ in $\overline{\mathrm{Perv}}(\mathbb{C}, A)$ because of the exact sequence of sheaves

$$0 \to j_!(\underline{\mathbf{k}}_{\mathbb{C}\backslash\{a\}}) \to \underline{\mathbf{k}}_{\mathbb{C}} \to \mathbf{k}_a \to 0$$

gives an exact sequence in $\mathrm{Perv}(\mathbb{C}, A)$

$$0 \to \mathbf{k}_a \to j_!(\underline{\mathbf{k}}_{\mathbb{C}\backslash\{a\}})[1] \to \underline{\mathbf{k}}_{\mathbb{C}}[1] \to 0$$

with third term in $\mathrm{LS}(\mathbb{C})[1]$.

1.4 Fourier transform of perverse sheaves and their Stokes data

In this section we assume $\mathbf{k} = \mathbb{C}$.

A The formal Fourier transform. Let $D_w = \mathbb{C}\langle w, \partial_w\rangle$ be the Weyl algebra of polynomial differential operators on $\mathbb{C}$ and $D_w\text{-}\mathrm{Mod}^{\mathrm{h}} \supset D_w\text{-}\mathrm{Mod}^{hrs}$ be the categories of holonomic and holonomic regular singular D_w-modules. It is well known that the *solution functor* (on all holonomic modules, regular singular or not)

$$\mathrm{Sol} : D_w\text{-}\mathrm{Mod}^{\mathrm{h}} \longrightarrow \mathrm{Perv}(\mathbb{C}), \quad M \mapsto \mathrm{Sol}(M) = \underline{\mathrm{RHom}}_{D_w}(M, \mathcal{O}_{\mathbb{C}})[1]$$

takes values in $\mathrm{Perv}(\mathbb{C})$. Further, its restriction to $D_w\text{-}\mathrm{Mod}^{hrs}$ is an equivalence (Riemann-Hilbert correspondence). See, e.g., [27]. So we can realize any $\mathcal{F} \in \mathrm{Perv}(\mathbb{C})$ as $\mathrm{Sol}(M)$ for a unique $M = M_{\mathcal{F}} \in D_w\text{-}\mathrm{Mod}^{\mathrm{hr}}$.

The *formal Fourier transform* is the isomorphism

$$D_w = \mathbb{C}\langle w, \partial_w\rangle \longrightarrow D_z = \mathbb{C}\langle z, \partial_z\rangle, \quad w \mapsto -\partial_z,\ \partial_w \mapsto z,$$

matching the analytic Fourier transform of solutions. Given $M \in D_w\text{-}\mathrm{Mod}^{\mathrm{h}}$, its *Fourier transform* $\widehat{M}$ is the same M but considered as a D_z-module using the above isomorphism. We refer to [27] for background on this construction. In particular, it is known that $\widehat{M}$ is again holonomic, so we have the perverse sheaf $\mathrm{Sol}(\widehat{M}) = \underline{\mathrm{RHom}}_{D_z}(M, \mathcal{O}_{\mathbb{C}})[1]$.

If M is regular, then $\widehat{M}$ is typically not regular. In this case it is also known that the perverse sheaf $\widehat{\mathcal{F}} = \mathrm{Sol}(\widehat{M})$ has 0 as the only possible singularity, so we get the functor (also called the *Fourier transform*)

$$\text{(1.4.1)} \qquad \mathrm{FT} : \mathrm{Perv}(\mathbb{C}) \longrightarrow \mathrm{Perv}(\mathbb{C}, 0), \quad \mathcal{F} \mapsto \widehat{\mathcal{F}} := \mathrm{Sol}(\widehat{M}_{\mathcal{F}}).$$

The restriction of $\mathrm{FT}(\mathcal{F})$ to $\mathbb{C}\backslash\{0\}$ has thus the form $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})[1]$ for a local system $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})$ on $\mathbb{C}^*$, so we have the functor

$$\mathrm{FT}_{\mathrm{gen}} : \mathrm{Perv}(\mathbb{C}) \longrightarrow \mathrm{LS}(\mathbb{C}^*).$$

As $\mathbb{C}^*$ is homotopy equivalent to the unit circle $S^1 \subset \mathbb{C}$, for any local system $\mathcal{L}$ on $\mathbb{C}^*$ we can speak about the stalk $\mathcal{L}_\zeta$ at ant $\zeta \in S^1$. Further, for any $a \in \mathbb{C}$, the circle S^1_a of directions at a is identified with this fixed S^1, so we can view any $\mathbf{\Phi}_a(\mathcal{F})$ as a local system on this S^1. With this understanding, the following is true [27, Ch.XII] [5, Prop.6.1.4].

Proposition 1.4.2. *Let $\mathcal{F} \in \mathrm{Perv}(\mathbb{C})$. We have a natural isomorphism*

$$\mathrm{FT}_{\mathrm{gen}}(\mathcal{F}) \simeq \bigoplus_{a\in\mathbb{C}} \mathbf{\Phi}_a(\mathcal{F})$$

of local systems on S^1. □

In particular, the functor $\mathrm{FT}_{\mathrm{gen}}$ factors through $\overline{\mathrm{Perv}}(\mathbb{C})$, which is one of the reasons to consider this localized category.

Proof: For future reference, we give a sketch of the construction of the identification of the proposition for generic ζ. Let $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A)$. A general result [6, Th.3.1.1] expresses the stalk of $\mathrm{FT}(\mathcal{F})$ at $\zeta \in S^1$ (considered as the unit circle in $\mathbb{C}$) as the cohomology with support,namely

$$\text{(1.4.3)} \qquad \mathrm{FT}(\mathcal{F})_\zeta = H^0_{\{\mathrm{Re}(\zeta\overline{w}) \geqslant -R\}}(\mathbb{C}, \mathcal{F}), \quad R \gg 0,$$

with R large enough so that the shifted half-plane contains A. For $a \in A$ let $K_a(\zeta) = a + \zeta \cdot \mathbb{R}_+$ be the half-line in the direction ζ issuing from a, see Fig. 4.

The embedding of the complements

$$\mathbb{C}\backslash \, \{\mathrm{Re}(\zeta\overline{w}) \geqslant -R\} \;\hookrightarrow\; \mathbb{C}\backslash \bigcup_{a\in A} K_a(\zeta)$$

is a homotopy equivalence and $\mathcal{F}$ is locally constant on both of them. So we can replace the support in (1.4.3) by the union $\bigcup_{a\in A} K_a(\zeta)$. Further, if ζ is generic enough, then the union is disjoint and so we have

$$\mathrm{FT}(\mathcal{F})_\zeta \simeq H^0_{\bigsqcup_{a\in A} K_a(\zeta)}(\mathbb{C}, \mathcal{F}) \;=\; \bigoplus_{a\in A} H^0_{K_a(\zeta)}(\mathbb{C}, \mathcal{F}) \;=\; \bigoplus_{a\in A} \Phi_a(\mathcal{F}). \quad \square$$

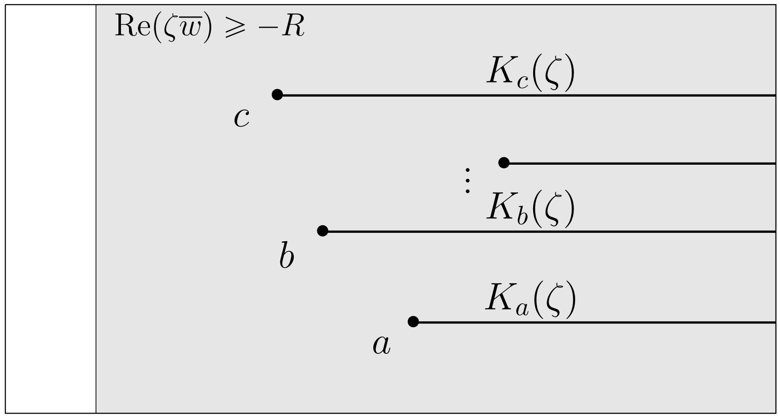


Figure 4: The half-lines $K_a(\zeta)$. Here $\zeta = 1$.

B The Stokes filtration of $\widehat{\mathcal{F}}$. As $\widehat{M}$ is typically irregular, its solutions (i.e., sections of $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})$) can grow exponentially in sectors near ∞. So the local system $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})$ carries the additional *Stokes structure* given by the data of such exponential growth in various sectors. In our case, the growth is at most of the form $e^{\lambda z}$ where λ is a constant.

Thus each stalk $\mathrm{FT}(\mathcal{F})_\zeta)$ carries the *Stokes filtration* $(\Sigma_\lambda)_{\lambda\in\mathbb{C}}$ labelled by the set $\mathbb{C}$ with partial order $\leqslant_\zeta$ given by $\lambda \leqslant_\zeta \mu$ if $\mathrm{Re}(\zeta\lambda) \leqslant \mathrm{Re}(\zeta\mu)$, i.e., $e^{\lambda z}$ is dominated by $e^{\mu z}$, as $z \to \infty$ on the ray $\zeta \cdot \mathbb{R}_+$. The subspace $\Sigma_\lambda \mathrm{FT}(\mathcal{F})_\zeta \subset \mathrm{FT}(\mathcal{F})_\zeta$ consists of solution of $\widehat{M}$ which grow in the direction ζ at most as $e^{\lambda z}$.

Proposition 1.4.4. *For a generic ζ we have*

$$\Sigma_\lambda \mathrm{FT}(\mathcal{F})_\zeta = \bigoplus_{\mathrm{Re}(\zeta a)\geqslant\ -\lambda} \Phi_a(\mathcal{F}).$$

Proof: This follows from representing solutions of $\widehat{M}$ as actual Fourier integrals $g_{i,\zeta}(\varphi)(z)$ corresponding to $\varphi \in \Phi_{a_i}(\mathcal{F})$ over the half-lines $K_i(\zeta)$ as in Fig. 4. See [27, Ch. XII] for more details. The growth near $\zeta\infty$ of $g_{i,\zeta}(\varphi)(z)$ is of the rate $e^{-a_i z}$. □

1.5 The Lefschetz perverse sheaf and its Fourier transform

In this section we take $\mathbf{k}$ to be an arbitrary field.

A The exponential integral as Fourier transform. Let X be a smooth complex algebraic variety of dimension n and $S : X \to \mathbb{C}$ be a regular function. Given a regular volume form dx on X, we can consider exponential integrals

(1.5.1) $$I(z) = I(\hbar) = \int_\Gamma e^{\frac{i}{\hbar}S(x)}dx, \quad z = 1/\hbar,$$

where Γ is a locally finite n-cycle in X such that the integral converges. We can consider $I(z)$ as a multivalued function whose determinations are labelled by choices of Γ.

It is classical that one can split the integration in $I(z)$ into two stages. First, we have the relative volume form dx/dS, a rational section of $\Omega^{n-1}_{X/\mathbb{C}}$ and form the multivalued function L_S on $\mathbb{C}$

$$L_S(w) = \int_{\Gamma_w \subset S^{-1}(w)} \frac{dx}{dS}$$

where Γ_w is an $(n-1)$-cycle in the fiber varying with w via the Gauss-Manin connection. Then formally (assuming that Γ is formed out of the $\Gamma_w, w \in \gamma$ for a cycle γ in $\mathbb{C}$)

$$I(z) = \int_\gamma L_S(w) e^{izw} dw$$

is (a determination of) the Fourier transform of L_S. Note that L_S is a Nielsen type function, satisfying a differential equation with regular singularities, so all transcendental nontriviality of $I(\hbar)$ comes from the 1-dimensional Fourier transform.

B The Lefschetz perverse sheaf. A sheaf-theoretic analog of the function L_S is given by the collection of perverse sheaves

$$\mathcal{L}_S^i = \underline{H}^i_{\text{perv}}(RS_*(\underline{\mathbf{k}}_X[\dim(X)]), \quad i \in \mathbb{Z}$$

on $\mathbb{C}$. Here $\underline{H}^i_{\text{perv}}$ is the degree i perverse cohomology taken with respect to the perverse t-structure. We will be particularly interested in the case $i = 0$ and write simply $\mathcal{L}_S = \mathcal{L}_S^0$ while using the notation $\mathcal{L}_S^\bullet$ for the graded perverse shead $\bigoplus_i \mathcal{L}_S^i$.

Proposition 1.5.2. *Assume that the function $S : X \to \mathbb{C}$ has only isolated singularities and is proper as a morphism of algebraic varieties (so each $S^{-1}(a)$ is compact). Then:*

(a) If $a \in \mathbb{C}$ is a non-critical value for S, then $\mathcal{L}_S[-1]$ is a local system near w and its stalk at a is identified with $H^n(S^{-1}(a), \mathbf{k})$.

(b) If a is a critical value then

$$\Phi_a(\mathcal{L}_S) = \bigoplus_{x \in S^{-1}(a)} \Phi_S(\underline{\mathbf{k}}_X[\dim(X)])_x$$

is the direct sum of the classical (Lefschetz) spaces of vanishing cycles for S at the critical points over a.

Thus, for $\mathbf{k} = \mathbb{C}$ sections of the local system $\mathcal{L}_S[-1]$ over the open set of non-critical values of S give determinations of L_S.

Proof: (a) follows since the perverse t-structure is centered around the middle dimension. Part (b) follows from the definition of vanishing cycles. □

Remark 1.5.3. One can say that the first step towards categorification of the exponential integral (1.5.1) is given by $\mathrm{FT}(\mathcal{L}_S)$, the Fourier transform of the perverse sheaf $\mathcal{L}_S$. [3]As we saw in §1.4, the structure of $\mathrm{FT}(\mathcal{L}_S)$ near ∞ is entirely given by the image of $\mathcal{L}_S$ in $\overline{\mathrm{Perv}}(\mathbb{C})$, i.e., by the vanishing cycles of $\mathcal{L}_S$ and the transport maps $m_{ab}(\gamma)$ between then. To find these data, we do not need to compactify S to a proper morphism $X \to \mathbb{C}$. For example, if S is a Morse function, then it suffices to have the part of X containing the critical points and the Lefschetz thimbles emanating from critical points towards other critical points.

2 Alien derivatives for perverse sheaves: elementary theory

2.1 Additive convolution of localized perverse sheaves and Fourier transform

Convolution of étale perverse sheaves on commutative algebraic groups was studied by N. Katz [19]. We will need a simplified version for analytic perverse sheaves on the additive group $\mathbb{C}$, see e.g. [10].

A Additive convolution. Let $\mathbf{k}$ be an aribtrary field. Let $A, B \subset \mathbb{C}$ be finite subsets and let $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A)$, $\mathcal{G} \in \mathrm{Perv}(\mathbb{C}, B)$. Then $\mathcal{F} \boxtimes \mathcal{G}$ is a constructible complex (in fact, a perverse sheaf) on $\mathbb{C} \times \mathbb{C}$. We have the addition map

$$+ : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}, \quad (w', w'') \mapsto w' + w''.$$

Let $A + B = +(A \times B)$ be the set of sums $a + b$, $a \in A$, $b \in B$. The map $+$ gives the *additive convolution*

$$\mathcal{F} * \mathcal{G} := R(+)_*(\mathcal{F} \boxtimes \mathcal{G})$$

Note that $\mathcal{F} * \mathcal{G}$ is a priori a constructible complex on $\mathbb{C}$ with singularities (points of local non-constancy of cohomology) contained in $A + B$.

Proposition 2.1.1. *[10, Prop.2.4.3] If* $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A)$ *and* $\mathcal{G} \in \mathrm{Perv}^0(\mathbb{C}, B)$*, then* $\mathcal{F} * \mathcal{G} \in \mathrm{Perv}^0(\mathbb{C}, A + B)$. □

Let $\mathrm{Perv}(\mathbb{C}) = \bigcup_A \mathrm{Perv}(\mathbb{C}, A)$ be the category of all perverse sheaves on $\mathbb{C}$ with finitely many singularities. In a similar way we define the quotient category $\overline{\mathrm{Perv}}(\mathbb{C})$ and its lift $\mathrm{Perv}^0(\mathbb{C}) \subset \mathrm{Perv}(\mathbb{C})$. By the above, $\overline{\mathrm{Perv}}(\mathbb{C})$ and $\mathrm{Perv}^0(\mathbb{C})$ are equivalent.

Corollary 2.1.2. *[10, Th.2.4.11] The operation* $*$ *makes* $\overline{\mathrm{Perv}}(\mathbb{C}) \simeq \mathrm{Perv}^0(\mathbb{C})$ *into a symmetric monoidal category with unit object* $\mathbf{1}$ *being the class of* $\underline{\mathbf{k}}_0$ *in* $\overline{\mathrm{Perv}}(\mathbb{C})$ *or, equivalently, its lift* $j_! \underline{\mathbf{k}}_{\mathbb{C}\setminus\{0\}} \in \mathrm{Perv}^0(\mathbb{C})$*, see Example 1.3.7.*

[3]For the next step one should replace cohomology groups by appropriate categories, cf. [17].

B Comparison with the Hurwitz convolution for analytic functions. The operation $*$ for perverse sheaves is a categorical analog of the additive (or Hurwitz) convolution of holomorphic functions $f, g \in \mathbb{C}\{\{w\}\}$ defined near 0:

$$(f * g)(w) = \int_0^w f(u)g(w-u)du. \tag{2.1.3}$$

If

$$f(w) = \sum_{n=0}^{\infty} a_n \frac{w^n}{n!}, \quad g(w) = \sum_{n=0}^{\infty} b_n \frac{w^n}{n!},$$

then

$$(f * g)(w) = \sum_{n=0}^{\infty} \Big(\sum_{i+j=n} a_i b_j \Big) \frac{w^{n+1}}{(n+1)!}$$

(series without constant term, in fact we have $1 * 1 = w$). The theorem of Hurwitz (additive version of the Hadamard theorem of multiplication of singularities) says that if f, g extend to possibly multivalued analytic functions in $\mathbb{C}$ with singularities in possible infinite sets A, B respectively, then $f * g$ similarly extends to a possibly multivalued analytic function in $\mathbb{C}$ with singularities in $A \cup B \cup (A + B)$. See [14] and later treatments in [29] and [28] §6.4. The additional possible singularities at $A = A + \{0\}$ and $B = \{0\} + B$ in Hurwitz's theorem as compared to Proposition 2.1.1 come from the fact that the integration path in (2.1.3), starting from 0, is a chain but not a cycle with coefficients in the local system of determinations of the integrand.

C Additive convolution and Fourier transform. Let $\mathbf{k} = \mathbb{C}$. The classical principle that "Fourier transform takes convolution into product" has in our case the following form.

Proposition 2.1.4. *For $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}^0(\mathbb{C})$ we have a natural isomorphism of local systems on S^1*

$$\mathrm{FT}_{\mathrm{gen}}(\mathcal{F} * \mathcal{G}) \simeq \mathrm{FT}_{\mathrm{gen}}(\mathcal{F}) \otimes \mathrm{FT}_{\mathrm{gen}}(\mathcal{G}).$$

Proof: By the Riemann-Hilbert correspondence, the direct image $R+_*$ in the definition of $\mathcal{F} * \mathcal{G}$ can be calculated at the level of D-modules. That is, let $M, N \in D_w\text{-}\mathrm{Mod}^{hrs}$. We define the D-module additive convolution to be the complex of D_w-modules

$$M *^D N = R +_*^D (M \boxtimes N),$$

where:

(1) $M \boxtimes N = M \otimes_{\mathbb{C}} N$ considered as a module over $D_w \otimes_{\mathbb{C}} D_w = \mathbb{C}\langle w', w'', \partial_{w'}, \partial_{w''} \rangle$ (polynomial differential operators on $\mathbb{C} \times \mathbb{C}$);

(2) $R+_*^D$ is the derived D-module direct image, i.e., the de Rham complex along the fibers of $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

So defined, $M *^D N$ is a complex with holonomic regular singular cohomology and $\mathrm{Sol}(M *^D N) \simeq \mathrm{Sol}(M) * \mathrm{Sol}(N)$ in the derived category of constructible complexes.

Further, the tensor product of local systems in the claim of the proposition corresponds to the tensor product of D_z-modules over $\mathbb{C}[z]$. So our claim is a consequence of the following one.

Lemma 2.1.5. *Let $M, N \in D_w\text{-}\mathrm{Mod}^{hrs}$. Then we have a natural isomorphism in derived category*

$$\widehat{M *^D N} \simeq \widehat{M} \otimes^L_{\mathbb{C}[z]} \widehat{N}$$

Proof of lemma: The fibers of the map $+$ are 1-dimensional with the relative tangent bundle generated by the vector field $\partial_{w'} - \partial_{w''}$. So $M *^D N$ is the complex

$$M \otimes_{\mathbb{C}} N \xrightarrow{\partial_{w'} - \partial_{w''}} M \otimes_{\mathbb{C}} N$$

with the differential $\partial_{w'} - \partial_{w''} = \partial_w \otimes 1 - 1 \otimes \partial_w$. Now, $\widehat{M} \otimes^L_{\mathbb{C}[z]} \widehat{N}$ is the complex

$$\widehat{M} \otimes_{\mathbb{C}} \widehat{N} \xrightarrow{z \otimes 1 - 1 \otimes z} \widehat{M} \otimes_{\mathbb{C}} \widehat{N}.$$

But $\widehat{M}$ is M in which z acts as ∂_w and ∂_z as $-w$, and similarly for $\widehat{N}$, so the second complex is identified with the first after the Fourier transform. This proves the lemma and Proposition 2.1.4. □

D The Thom-Sebastiani theorem. Propositions 1.4.2 and 2.1.4 lead to a multiplicativity property (a version of the Thom-Sebastiani theorem) which does not involve Fourier transform and can be proved directly at the level of perverse sheaves. We take $\mathbf{k}$ to be an arbitrary field.

Let $\mathrm{LS}(S^1)^{\mathbb{C}}$ be the category of $\mathbb{C}$-*graded local systems* on S^1, i.e. of collections $\mathcal{L} = (\mathcal{L}_c)_{c \in \mathbb{C}}$ of local systems on S^1 such that $\mathcal{L}_a = 0$ for almost all a. This category has a symmetric monoidal structure given by

$$(\mathcal{L} \otimes \mathcal{M})_c = \bigoplus_{a+b=c} \mathcal{L}_a \otimes \mathcal{M}_b.$$

We have the *total vanishing cycle functor*

$$\mathbf{\Phi} : \mathrm{Perv}(\mathbb{C}) \longrightarrow \mathrm{LS}^{\mathbb{C}}(S^1), \quad \mathcal{F} \mapsto \mathbf{\Phi}(\mathcal{F}) := (\mathbf{\Phi}_c(\mathcal{F}))_{c \in \mathbb{C}}.$$

Theorem 2.1.6. *[10, Th.2.8.3] The functor $\mathbf{\Phi}$ is symmetric monoidal. In other words, for any $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}^0(\mathbb{C})$ and any $c \in \mathbb{C}$ we have a natural isomorphism of local systems on S^1*

$$\mathbf{\Phi}_c(\mathcal{F} * \mathcal{G}) \simeq \bigoplus_{a+b=c} \mathbf{\Phi}_a(\mathcal{F}) \otimes \mathbf{\Phi}_b(\mathcal{G}).$$

For convenience of the reader and future reference we give a direct proof. It is enough to consider the case $c = 0$, the case of arbitrary c is similar. We further identify the stalks of the local systems of vanishing cycles at the point $1 \in S^1$, i.e., in the direction of $\mathbb{R}_+$, the identification of monodromy following by the same arguments as below. We will use the letter Φ to mean such stalks.

For a perverse sheaf $\mathcal{E} \in \mathrm{Perv}(\mathbb{C})$ we have a canonical identification

$$\Phi_0(\mathcal{E}) \;=\; R\Gamma_{\{\mathrm{Re}(w)\geqslant 0\}}(\{|w|<r\},\mathcal{E}),$$

where $r>0$ is small enough. Indeed, the discussion in §1.2 **B** gives $\Phi_0(\mathcal{E})$ as $R\Gamma_{\mathbb{R}_+}(\{|w|<r\},\mathcal{E})$ but the half-plane $\mathrm{Re}(w)\geqslant 0$ (the part of it lying in the disk $|w|<r$) is stratified homotopy equivalent (w.r.t. to the stratification of $|w|<r$ by 0 and everything else) to the real half-line. Therefore, using w', w'' as coordinates in $\mathbb{C}\times\mathbb{C}$, we have

$$\Phi_0(\mathcal{F}*\mathcal{G}) \;=\; R\Gamma_Z(\{|w'+w''|<r\},\mathcal{F}\boxtimes\mathcal{G}), \quad Z:=\{\mathrm{Re}(w'+w'')\geqslant 0\}, \tag{2.1.7}$$

see Fig. 5. Now consider the subset

$$W \;:=\; \bigcup_{a,b\in A, a+b=0} \big\{\mathrm{Re}(w'-a)\geqslant 0,\, \mathrm{Re}(w''-b)\geqslant 0\big\} \;\subset\; Z.$$

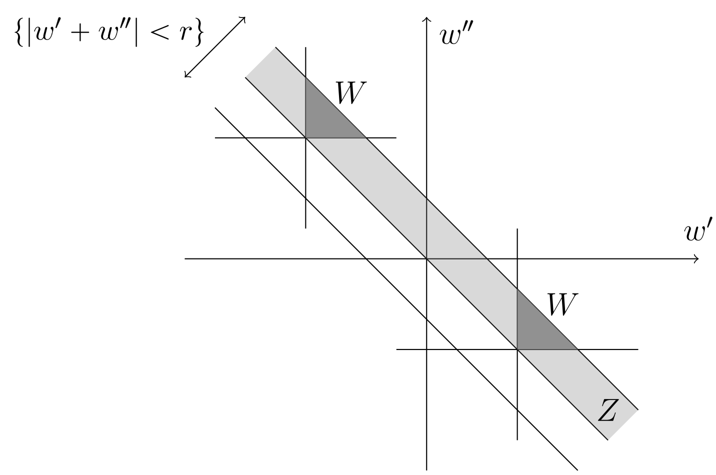


Figure 5: The two sets of supports in the Thom-Sebastiani theorem.

If r is small enough, then the union in the definition of W is disjoint and

$$R\Gamma_W(\{|w'+w''|<r\},\mathcal{F}\boxtimes\mathcal{G}) \;=\; \bigoplus_{a,b\in A, a+b=0} \Phi_a(\mathcal{F})\otimes\Phi_b(\mathcal{G}) \tag{2.1.8}$$

by the Künneth theorem. It remains to notice that the complements of the two supports, i.e., the open subsets

$$U \;=\; \{|w'+w''|<r\}\setminus Z \quad \text{and} \quad V \;=\; \{w'+w''|<r\}\setminus W$$

are stratified homotopy equivalent (with respect to the stratification given by the singularities of $\mathcal{F}\boxtimes\mathcal{G}$). More precisely, U can be obtained as a deformation retract of V with deformations affecting only the area where $\mathcal{F}\boxtimes\mathcal{G}$ is locally constant, so $R\Gamma_Z = R\Gamma_W$ and therefore (2.1.7) and (2.1.8) give the same answer. □

2.2 Rectilinear transports with avoidances and alien transports

A Rectilinear transports. We start by taking $\mathbf{k}$ to be an arbitrary field. Let $a, b \in A$ be two distinct points. The most natural path joining a and b is the straight line interval $[a, b]$. We denote by

$$\zeta_{ab} = \frac{b-a}{|b-a|} \in S^1 := \{\zeta \in \mathbb{C} : |\zeta| = 1\} \tag{2.2.1}$$

the slope of $[a, b]$ in the direction from a to b.

If $[a, b]$ does not contain any other elements of A, then for any $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A)$ we can use $[a, b]$ to define the *rectilinear transport*

$$m_{ab} = m_{ab}^{\mathcal{F}} = m_{ab}^{\mathcal{F}}([a, b]) : \mathbf{\Phi}_{a,[a,b]}(\mathcal{F}) \longrightarrow \mathbf{\Phi}_{b,[a,b]}(\mathcal{F}). \tag{2.2.2}$$

We say that A is *in linearly general position including* ∞, if no three points of A lie on a real line in $\mathbb{C}$ and no ζ_{ab} belongs to $\mathbb{R}$.

Suppose A is in linearly general position including ∞. Then all rectilinear transports $m_{ab}([a, b])$, $a, b \in A$, are defined. Let $\mathcal{M}_N$ be the category of diagrams (Φ_i, m_{ij}) as in Proposition 1.3.3. Let us number $A = \{a_1, \cdots, a_N\}$ in an arbitrary way, and denote $\zeta_{ij} = \zeta_{a_i,a_j}$, $i \neq j$ and $\mathbf{\Phi}_i(\mathcal{F}) = \mathbf{\Phi}_{a_i}(\mathcal{F})$ (a local system on S^1). In this notation, we define a functor

$$\Xi_{\mathrm{rect}} : \overline{\mathrm{Perv}}(\mathbb{C}, A) \longrightarrow \mathcal{M}_N, \quad \mathcal{F} \mapsto (\Phi_i, m_{ij}),$$

as follows. We put

$$\Phi_i = \mathbf{\Phi}_{a_i, a_i+\mathbb{R}}(\mathcal{F}) = \mathbf{\Phi}_i(\mathcal{F})_1$$

(the stalk of the local system $\mathbf{\Phi}_i(\mathcal{F})$ at $1 \in S^1$). Further, m_{ii} is defined as $\mathrm{Id} - T_i(\mathcal{F})$ where $T_i(\mathcal{F}) : \mathbf{\Phi}_i(\mathcal{F})_1 \to \mathbf{\Phi}_i(\mathcal{F})_1$ is the counterclockwise monodromy. For $i \neq j$ we define m_{ij} as the composition

$$\mathbf{\Phi}_i(\mathcal{F})_1 \xrightarrow{T_1^{\zeta_{ij}}} \mathbf{\Phi}_i(\mathcal{F})_{\zeta_{ij}} = \mathbf{\Phi}_{a_i,[a_i,a_j]}(\mathcal{F}) \xrightarrow{m^{\mathcal{F}}_{a_i,a_j}} \mathbf{\Phi}_{a_j,[a_j,a_i]}(\mathcal{F}) = \mathbf{\Phi}_j(\mathcal{F})_{\zeta_{ji}} \xrightarrow{T^1_{\zeta_{ji}}} \mathbf{\Phi}_j(\mathcal{F})_1, \tag{2.2.3}$$

where $T_1^{\zeta_{ij}}$ (resp. $T^1_{\zeta_{ij}}$) is the monodromy of the local system $\mathbf{\Phi}_i(\mathcal{F})$ from 1 to ζ_{ij} (resp. from ζ_{ji} to 1) taken in the counterclockwise direction, if $\mathrm{Im}(w_i) < \mathrm{Im}(w_j)$ and in the clockwise direction, if $\mathrm{Im}(w_i) > \mathrm{Im}(w_j)$.

Proposition 2.2.4. *If A is in linearly general position including ∞, then the functor Ξ_{rect} is an equivalence of categories.*

Proof: This statement, which is [17, Prop.2.1.7], is deduced from Proposition 1.3.3 by deforming the set A to the convex position. □

B Rectilinear transport with avoidances. Let us now allow $[a, b]$ to contain other elements of A, say $[a, b] \cap A = \{a_0 = a, a_1, \cdots, a_r, a_{r+1} = b\}$, numbered in the direction from a to b , with $r \geqslant 0$. Let $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_r)$, $\varepsilon_i \in \{+, -\}$ be a sequence formed by $+$ and $-$ signs. We define the *rectilinear transport with avoidances* given by ε to be the map

$$m^{\varepsilon}_{ab} = m^{\varepsilon,\mathcal{F}}_{ab} : \Phi_{a,[a,b]}(\mathcal{F}) \longrightarrow \Phi_{b,[a,b]}(\mathcal{F}), \quad m^{\varepsilon}_{ab} := m_{ab}(\gamma_\varepsilon), \tag{2.2.5}$$

where γ_ε is the perturbation of the path $[a, b]$ obtaining by avoiding a_i on the left, if $\varepsilon_i = -$, and on the right, if $\varepsilon_i = +$, see Fig. 6.

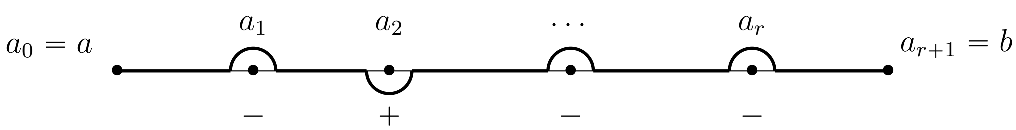


Figure 6: Transport with avoidances.

We define $m^{+}_{ab} := m^{+,\cdots,+}_{ab}$, resp. $m^{-}_{ab} := m^{-,\cdots,-}_{ab}$ to be the transport with all avoidances on the right, resp. on the left (both understood as $m_{ab}[a, b]$ for $r = 0$). Let us note the following consequences of the Picard-Lefschetz identities (Proposition 1.2.8). Here and later in the paper the identification of the stalks of the local Φ-systems at the intermediate points is done by clockwise rotation as in the Picard-Lefschetz formula, see (1.2.7).

Proposition 2.2.6. *(a) We have*

$$m^{-}_{ab} = \sum_{s=0}^{r} \sum_{1 \leqslant i_1 < \cdots < i_s \leqslant r} m^{+}_{a_{i_s},b} m^{+}_{a_{i_{s-1}},a_{s_r}} \cdots m^{+}_{a,a_{i_1}}.$$

(b) Equivalently, we have

$$m^{-}_{ab} = m^{+}_{ab} + \sum_{i=1}^{r} m^{+}_{a_i,b} m^{-}_{a,a_i}.$$

(c) For the composition of rectilinear transports we have the identity

$$m_{a_r,b} m_{a_{r-1},a_r} \cdots m_{a,a_1} = \sum_{\varepsilon \in \{+,-\}^r} (-1)^{|+(\varepsilon)|} m^{\varepsilon}_{ab},$$

where $| + (\varepsilon)|$ is the number of $+$ signs in ε.

Proof: All three statements follow easily by iterated application of Proposition 1.2.8. Part (b) is reduced to (a) by expanding each m^{-}_{a,a_i} according to (a). Part (c) is an instance of [17, Cor.1.1.19] for the case of the composition of paths which are rectilinear. □

C The alien derivative transport. Assume now that $\mathrm{char}(\mathbf{k}) = 0$. Adapting the approach of Écalle, we give the following

Definition 2.2.7. The *alien derivative transport* from a to b for $\mathcal{F} \in \mathrm{Perv}(\mathbb{C})$ is the map

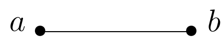
$$m_{ab}^{\Delta} = m_{ab}^{\Delta,\mathcal{F}} = \sum_{s=0}^{r} \frac{(-1)^{s+1}}{s+1} \sum_{1 \leqslant i_1 < \cdots < i_s \leqslant r} m_{a_{i_s},b}^{+} m_{a_{i_{s-1}},a_{s_r}}^{+} \cdots m_{a,a_{i_1}}^{+} : \Phi_{a,[a,b]}(\mathcal{F}) \longrightarrow \Phi_{b,[a,b]}(\mathcal{F}).$$

Here the superscript Δ is just a symbol chosen to invoke the standard notation for alien derivatives [9, 28]. Further, the formulas of Écalle extend to the context of perverse sheaves in the form:

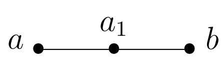
Proposition 2.2.8. *We have*

$$m_{ab}^{\Delta} = \sum_{\varepsilon \in \{+,-\}^r} \frac{(|+(\varepsilon)|!) \cdot (|-(\varepsilon)|!)}{(r+1)!} m_{ab}^{\varepsilon}.$$

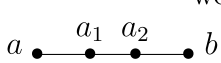
where $|+(\varepsilon)|$ *and* $|-(\varepsilon)|$ *are the numbers of* $+$ *and* $-$ *signs in* ε

Example 2.2.9. For $r = 0$: 

we have $m_{ab}^{\Delta} = m_{ab}[a,b]$.

For $r = 1$: 

we have $m_{ab}^{\Delta} = \frac{1}{2} m_{ab}^{+} + \frac{1}{2} m_{ab}^{-}$.

For $r = 2$: 

we have

$$m_{ab}^{\Delta} = \frac{1}{3} m_{ab}^{++} + \frac{1}{6} m_{ab}^{+-} + \frac{1}{6} m_{ab}^{-+} + \frac{1}{3} m_{ab}^{--}.$$

Remarks 2.2.10. (a) Proposition 2.2.8 represents m_{ab}^{Δ} as a linear combination of the m_{ab}^{ε} with positive coefficients summing to 1.

(b) It also shows that m_{ab}^{Δ} is stable under introducing dummy singularities and depends only on $[a,b]$ and not on A. That is, if $\mathcal{F}$ does not really have a singularity at some a_i, i.e., $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A \setminus \{a_i\}) \subset \mathrm{Perv}(\mathbb{C}, A)$, then calculating m_{ab}^{Δ} while taking a_i into account and while not taking it into account gives the same answer.

Proof of Proposition 2.2.8: This is a formal consequence of the Picard-Lefschetz identities (Proposition 1.2.8). To organize the calculations, let us extend the notation m_{ab}^{ε} to the case when $a = a_0, a_1, \cdots, a_r, a_{r+1} = b$ lie, in this order, on a possibly curvilinear simple path γ from a to b which contains no other elements of A. That is, we define the path γ_ε as in (2.2.5) but as the perturbation of γ, not $[a,b]$, according to ε. To indicate the dependence on γ, we write $m_{ab}^{\varepsilon}(\gamma)$. A curvilinear version of Proposition 2.2.6(c), i.e., [17, Cor.1.1.19] gives:

Proposition 2.2.11. *Let* γ_i *denote the part of* γ *between* a_i *and* a_{i+1}. *Then*

$$m_{a_{r+1},a_r}(\gamma_r) m_{a_r,a_{r-1}}(\gamma_{r-1}) \cdots m_{a_0,a_1}(\gamma_0) = \sum_{\delta \in \{+,-\}^r} (-1)^{|+(\delta)|} m_{ab}^{\delta}(\gamma). \quad \square$$

Now, to prove Proposition 2.2.8, we apply Proposition 2.2.11 to each composition in the RHS of Definition 2.2.7. That is, we take as γ the path which goes from $a_0 = a$ to a_{i_1} avoiding the intermediate a_i on the left, then from a_{i_1} to a_{i_2} with similar avoidances and continues like this, ending in the curved segment from a_{i_s} to $a_{i_{s+1}} = b$ with similar avoidances. The intermediate points are $a_{i_1}, \cdots, a_{i_s}$. Then the LHS of the formula of Proposition 2.2.11 for such γ and such choice of the intermetiate points is the composition in Proposition 2.2.8. So we get

$$m^+_{a_{i_s}, a_{i_{s+1}}} m^+_{a_{i_{s-1}}, a_{sr}} \cdots m^+_{a_{i_0}, a_{i_1}} = \sum_{\delta \in \{+,-\}^s} (-1)^{|+(\delta)|} m_{ab}^{(+^{i_1 - 1}, \delta_1, +^{i_2 - i_1 - 1}, \delta_2, \cdots, \delta_s, +^{r - i_s})},$$

where $+^m$ stands for the sequence of m plus signs.

Now we need to find the coefficient at each m^ε_{ab}, $\varepsilon \in \{+,-\}^r$ after we sum these expansions over all s and all $1 \leqslant i_1 < \cdots i_s \leqslant r$ with coefficients $(-1)^{s+1}/(s+1)$. For this, let us encode ε by the subset $I = +(\varepsilon) = \{i | \varepsilon_i = +\} \subset \{1, \cdots, r\}$. The coefficient is then

$$\sum_{J \supset I} \frac{(-1)^{|J \setminus I|}}{|J| + 1} = \sum_{k=0}^{r - |I|} \frac{(-1)^k}{|I| + k + 1} \binom{r - |I|}{k}.$$

So Proposition 2.2.8 reduces to the following.

Lemma 2.2.12. *For any integer $a, m > 0$ we have*

$$\sum_{k=0}^{m} \frac{(-1)^k}{a + k + 1} \binom{m}{k} = \frac{m! a!}{(m + a + 1)!}.$$

Proof of Lemma: For any function $f = f(a)$ of an integer variable a let Δf be its difference derivative: $(\Delta f)(a) = f(a) - f(a+1)$. The mth iteration of Δ has the form

$$(\Delta^m f)(a) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(a+k).$$

Let $f_m(a) = m! a!/(m + a + 1)!$, $m \geqslant 0$. The lemma means that $f_m = \Delta^m f_0$. To see this, it is enough to show that $\Delta f_m = f_{m+1}$, which is straightforward:

$$(\Delta f_m)(a) = \frac{m! a!}{(m+a+1)!} - \frac{m!(a+1)!}{(m+a+2)!} = m! \frac{a!(m+a+2)! - (a+1)!}{(m+a+2)!} =$$
$$= m! \frac{a!(m+1)}{(m+a+2)!} = \frac{a!(m+1)!}{(m+a+2)!} = f_{m+1}(a). \quad \square$$

D Description of $\overline{\mathrm{Perv}}(\mathbb{C}, A)$ via alien transports. We now generalize Proposition 2.2.4 to the case when $A = \{a_1, \cdots, a_N\} \subset \mathbb{C}$ is an arbitrary finite subset. We keep the notation of §**A** above and assume only that no $[a_i, a_j]$ is horizontal, i.e, all $\zeta_{ij} \notin \mathbb{R}$, $i \neq j$.

Define the functor

$$\Xi_\Delta : \overline{\mathrm{Perv}}(\mathbb{C}, A) \longrightarrow \mathcal{M}_N, \quad \mathcal{F} \mapsto (\Phi_i, m_{ij}),$$

where, as before,

$$\Phi_i = \mathbf{\Phi}_{a_i, a_i+\mathbb{R}}(\mathcal{F}) = \mathbf{\Phi}_i(\mathcal{F})_1, \quad m_{ii} = \mathrm{Id} - T_i(\mathcal{F})$$

and for $i \neq j$ the map m_{ij} is the composition

$$\mathbf{\Phi}_i(\mathcal{F})_1 \xrightarrow{T_1^{\zeta_{ij}}} \mathbf{\Phi}_i(\mathcal{F})_{\zeta_{ij}} = \mathbf{\Phi}_{a_i,[a_i,a_j]}(\mathcal{F}) \xrightarrow{m^{\Delta,\mathcal{F}}_{a_i,a_j}} \mathbf{\Phi}_{a_j,[a_j,a_i]}(\mathcal{F}) = \mathbf{\Phi}_j(\mathcal{F})_{\zeta_{ji}} \xrightarrow{T^1_{\zeta_{ji}}} \mathbf{\Phi}_j(\mathcal{F})_1.$$

That is, we replace the rectilinear transform in (2.2.3) (which may no longer make sense because of the presence of intermediate points) by $m^{\Delta,\mathcal{F}}_{a_i,a_j}$, the alien transform from a_i to a_j for $\mathcal{F}$.

Proposition 2.2.13. *The functor Ξ_Δ is an equivalence of categories.*

Proof: Let us make a small deformation of the set A, replacing it with $A' = \{a'_1, \cdots, a'_N\}$ with $|a_i - a'_i| \ll 1$ such that A' is now in linearly general position including ∞. Since perverse sheaves are topological objects, the continuous deformation $a_i(t) = (1-t)a_i + ta'_i$, $t \in [0,1]$ of the sets of singularities gives rise to an equivalence $u : \overline{\mathrm{Perv}}(\mathbb{C}, A) \to \overline{\mathrm{Perv}}(\mathbb{C}, A')$ ("isomonodromic deformation of perverse sheaves", see, e.g., [17]). For any $\mathcal{F} \in \overline{\mathrm{Perv}}(\mathbb{C}, A)$ we denote $\mathcal{F}'$ the corresponding object of $\overline{\mathrm{Perv}}(\mathbb{C}, A')$, so that $\mathbf{\Phi}_i(\mathcal{F})$ is identified with $\mathbf{\Phi}_{a'_i}(\mathcal{F})$ as a local system on S^1.

Applying Proposition 2.2.4 to A', we see that any $\mathcal{F} \in \overline{\mathrm{Perv}}(\mathbb{C}, A)$ is uniquely determined by the data of:

(1) The monodromies of $\mathbf{\Phi}_{a'_i}(\mathcal{F}')$ which are identified with the monodromies of $\mathbf{\Phi}_i(\mathcal{F})$.

(2) The rectilinear transports for $\mathcal{F}'$.

Now, each rectilinear transport m_{ij} for $\mathcal{F}'$ corresponds, under the equivalence u, to the rectilinear transport with avoidances $m_{ij}^{\varepsilon(i,j)}$ for some $\varepsilon(i,j)$ describing to which side of $[a'_i, a'_j]$ the (formerly) intermediate points a'_k now lie. Let $r_{ij} = |A \cap (a_i, a_j)|$ be the number of the intermediate points on $[a_i, a_j]$, so $\varepsilon(i,j)$ is a sequence of length r_{ij}.

The alien derivative transport $m^\Delta_{a_i,a_j}$ is a linear combination of all $2^{r_{ij}}$ transports $m^\varepsilon_{a_i,a_j}$ with strictly positive (in particular, nonzero) coefficients. Knowing any one $m^\varepsilon_{a_i,a_j}$, any other $m^{\varepsilon'}_{a_i,a_j}$ is expressed, in virtue of the Picard-Lefschetz formulas, by adding or subtracting compositions of transports with avoidances for smaller subintervals of $[a_i, a_j]$. Hence the data of all $\{m^\varepsilon_{a_i,a_j}\}$ for all distinct $1 \leqslant i, j \leqslant N$ and all $\varepsilon \in \{+,-\}^{r_{ij}}$, is uniquely recovered (by triangular-type formulas) from the data of $\{m^{\varepsilon(i,j)}_{a_i,a_j}\}$, $1 \leqslant i, j \leqslant N$, where we choose one representative $\varepsilon(i,j)$ for each ordered pair (i,j). Therefore the data of such $\{m^{\varepsilon(i,j)}_{a_i,a_j}\}$ are in bijection with the data of $\{m^\Delta_{a_i,a_j}\}$, and the proposition is proved. □

2.3 Alien derivatives and Stokes automorphisms for perverse sheaves

A Multiplicative properties of one-sided avoidances. Again, we start by taking $\mathbf{k}$ to be an arbitrary field. For $a, b \in \mathbb{C}$ and a perverse sheaf $\mathcal{F} \in \operatorname{Perv}(\mathbb{C})$ we use notation $m_{ab}^{\pm} = m_{ab}^{\pm,\mathcal{F}}$ to mean either $m_{ab}^{+,\mathcal{F}} = m_{ab}^{+,\cdots,+,\mathcal{F}}$ or $m_{ab}^{-,\mathcal{F}} = m_{ab}^{-,\cdots,-,\mathcal{F}}$, this meaning to be used consistently in any formula.

Let $\mathcal{F} \in \operatorname{Perv}^0(\mathbb{C}, A')$, $\mathcal{G} \in \operatorname{Perv}^0(\mathbb{C}, A'')$ and $A = A' + A''$. By Theorem 2.1.6, for any direction $\theta \in S^1$ and any $a, b \in A$ we have

(2.3.1)

$$\mathbf{\Phi}_a(\mathcal{F} * \mathcal{G})_\theta \simeq \bigoplus_{\substack{a' \in A', a'' \in A'' \\ a'+a''=a}} \mathbf{\Phi}_{a'}(\mathcal{F})_\theta \otimes \mathbf{\Phi}_{a''}(\mathcal{G})_\theta, \quad \mathbf{\Phi}_b(\mathcal{F} * \mathcal{G})_{-\theta} \simeq \bigoplus_{\substack{b' \in A', b'' \in A'' \\ b'+b''=b}} \mathbf{\Phi}_{b'}(\mathcal{F})_{-\theta} \otimes \mathbf{\Phi}_{b''}(\mathcal{G})_{-\theta}.$$

Let $a = a' + a'', b = b' + b'' \in A$ be distinct, with $a', b' \in A'$ and $a'', b'' \in A''$. Note that it is possible that $a' = a''$ or $b' = b''$ (but not both). Take $\theta = \zeta_{ab}$ to be the direction from a to b. Let us view the rectilinear transport with one-sided avoidances as a linear map

$$m_{ab}^{\pm,\mathcal{F}*\mathcal{G}} : \mathbf{\Phi}_a(\mathcal{F} * \mathcal{G})_\theta \longrightarrow \mathbf{\Phi}_b(\mathcal{F} * \mathcal{G})_{-\theta}.$$

With respect to the decompositions (2.3.1), we then have the matrix element

$$\big(m_{ab}^{\pm,\mathcal{F}*\mathcal{G}}\big)_{a',a''}^{b',b''} : \mathbf{\Phi}_{a'}(\mathcal{F})_\theta \otimes \mathbf{\Phi}_{a''}(\mathcal{G})_\theta \longrightarrow \mathbf{\Phi}_{b'}(\mathcal{F})_{-\theta} \otimes \mathbf{\Phi}_{b''}(\mathcal{G})_{-\theta}.$$

Theorem 2.3.2. *(a) Unless the intervals $[a', b'], [a'', b'']$ and $[a, b]$ are parallel with the same direction, $\big(m_{ab}^{\pm,\mathcal{F}*\mathcal{G}}\big)_{a',a''}^{b',b''} = 0$. Here a degenerate interval $[a', a']$ or $[b', b']$ is considered parallel (with the same direction) to any other interval.*

(b) If the intervals $[a', b'], [a'', b'']$ and $[a, b]$ are parallel with the same direction, then

$$\big(m_{ab}^{\pm,\mathcal{F}*\mathcal{G}}\big)_{a',a''}^{b',b''} = m_{a',b'}^{\pm,\mathcal{F}} \otimes m_{a'',b''}^{\pm,\mathcal{G}}.$$

Here we understand $m_{a',a'}^{\pm,\mathcal{F}}$ or $m_{a'',a''}^{\pm,\mathcal{G}}$ as the identity map.

Let us express the above condition of three intervals being parallel with the same direction by $[a', b'] \parallel [a'', b''] \parallel [a, b]$. Then we can reformulate Theorem 2.3.2 as follows:

Reformulation 2.3.3. *In the above notation, we have*

$$m_{ab}^{\pm,\mathcal{F}*\mathcal{G}} = \sum_{\substack{a'+a''=a,\, b'+b''=b \\ [a',b'] \parallel [a'',b''] \parallel [a,b]]}} m_{a',b'}^{\pm,\mathcal{F}} \otimes m_{a'',b''}^{\pm,\mathcal{G}}. \quad \square$$

Theorem 2.3.2 and Reformulation 2.3.3 as well as the proof below are inspired by Theorem 6.83 of [28] and its purely analytic proof.

B Proof of Theorem 2.3.2. Let us treat the case of m^- (avoidances on the left), the case of m^+ being similar. Fix $a' \in A'$, $a'' \in A''$. Let $a = a' + a'' \in A$ and

$$K' = a' + \mathbb{R}_+\theta = \{a' + t'\theta |\ t' \geqslant 0\}, \quad K'' = a'' + \mathbb{R}_+\theta = \{a'' + t''\theta |\ t'' \geqslant 0\}, \\ K = a + R_+\theta = \{a + t\,\theta |\ t \geqslant 0\} \tag{2.3.4}$$

be the straight half-lines issuing from a', a'' and a in the direction θ, see Fig. 7(b). We use t', t'' and t as coordinates on these half-lines.

By Example 1.2.2,

$$\mathbf{\Phi}_{a'}(\mathcal{F})_\theta \;=\; \underline{H}^0_{K'}(\mathcal{F})_{a'}, \quad \mathbf{\Phi}_{a''}(\mathcal{G})_\theta \;=\; \underline{H}^0_{K''}(\mathcal{F})_{a''} \quad \text{and therefore} \\ \mathbf{\Phi}_{a'}(\mathcal{F})_\theta \otimes \mathbf{\Phi}_{a''}(\mathcal{G})_\theta \;=\; \mathcal{H}_{(a',a'')}, \quad \text{where} \quad \mathcal{H} := \underline{H}^0_{K'\times K''}(\mathcal{F} \boxtimes \mathcal{G}). \tag{2.3.5}$$

The stalk $\mathcal{H}_{(a',a'')}$ can be seen as the 0th cohomology of $\mathcal{F} \boxtimes \mathcal{G}$ with support in the dark shaded area near the left of (a', a'') in Fig. 7(a). Denote

$$\widetilde{A}' = A' \cap K' \;=\; \{a' = b'_0, b'_1, b'_2, \cdots\}, \quad \widetilde{A}'' = A'' \cap K'' \;= \{a'' = b''_0, b''_1, b''_2, \cdots\}$$

in order given by the direction of K', K'', see Fig. 7(b).

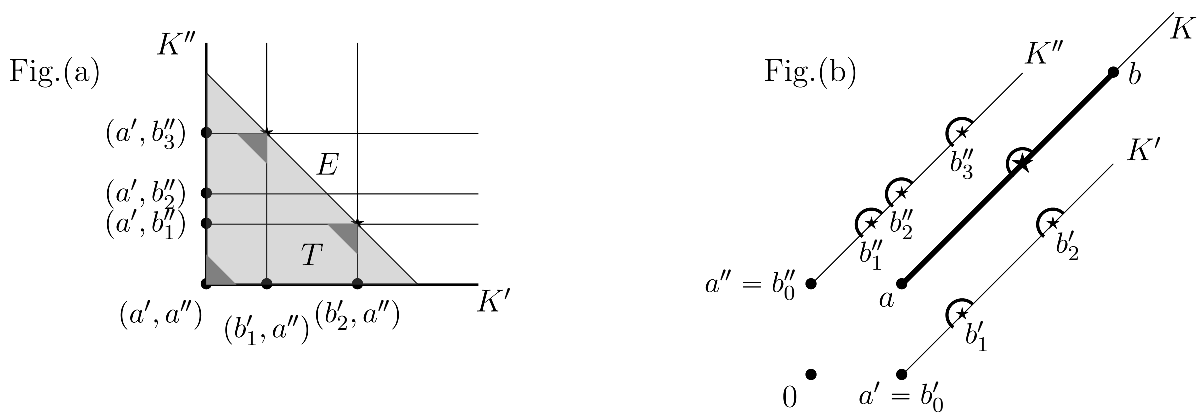


Figure 7: The area $T \subset K' \times K''$ and the transport with avoidances for $\mathcal{F} * \mathcal{G}$.

The sheaf $\mathcal{H}$ on $K' \times K''$ is constructible with respect to the stratification cut out by $\widetilde{A}' \times K''$, $K' \times \widetilde{A}''$ and their intersection $\widetilde{A}' \times \widetilde{A}''$, see Fig. 7(a). This is because $\mathcal{F} \boxtimes \mathcal{G}$ is constructible w.r.t. a similar stratification of $\mathbb{C} \times \mathbb{C}$. In particular, $\mathcal{H}$ is locally constant on the interior of $K' \times K''$ near (a', a'').

Let $\widetilde{K}'$, $\widetilde{K}''$ be small perturbations of K', K'' obtained by avoiding the $b'_i, b''_i, i > 0$, on the left. Then $\widetilde{K}' \times \widetilde{K}''$ coincides with $K' \times K''$ near (a', a''). By construction,

$$\widetilde{\mathcal{H}} \;=\; \underline{H}^0_{\widetilde{K}'\times\widetilde{K}''}(\mathcal{F} \boxtimes \mathcal{G})$$

is locally constant on the entire interior of $\widetilde{K}' \times \widetilde{K}''$ and coincides with $\mathcal{H}$ near (a', a''). Let also $\widetilde{K}$ be a similar perturbation of $[a, b] \subset K$ avoiding all the elements of A other than a and b on the left.

Now look at the composite map

$$(2.3.6)\quad \mathbf{\Phi}_{a'}(\mathcal{F})_\theta \otimes \mathbf{\Phi}_{a''}(\mathcal{G})_\theta \xrightarrow{\varepsilon_{a',a''}} \mathbf{\Phi}_a(\mathcal{F} * \mathcal{G})_\theta \xrightarrow{m_{ab}^{-,\mathcal{F}*\mathcal{G}}} \mathbf{\Phi}_b(\mathcal{F} * \mathcal{G})_{-\theta} = \bigoplus_{b'+b''=b} \mathbf{\Phi}_{b'}(\mathcal{F})_{-\theta} \otimes \mathbf{\Phi}_{b''}(\mathcal{G})_{-\theta}.$$

Here $\varepsilon_{a',a''}$ is an embedding given by the Thom-Sebastiani theorem 2.1.6 which is also indicated in the equality on the right. The transport $m_{ab}^{-,\mathcal{F}*\mathcal{G}}$ is defined using $\widetilde{K}$. As explained in §1.2 **C** (applied to $\alpha = \widetilde{K}$) it is composed of three maps:

(1) The generalization map $u_{a,\widetilde{K}} = u_{a,\widetilde{K}}^{\mathcal{F}*\mathcal{G}}$ (in the notation of (1.2.4)) from $\mathbf{\Phi}_a(\mathcal{F} * \mathcal{G})_\theta = \underline{H}^0_{\widetilde{K}}(\mathcal{F} * \mathcal{G})_a$ to the stalk at a nearby point $c \in \widetilde{K}$ which is $\underline{H}^0_{\widetilde{K}}(\mathcal{F} * \mathcal{G})_c = (\mathcal{F} * \mathcal{G})[-1]_c$, the same as the stalk at c of the local system $\mathcal{F} * \mathcal{G}[-1]$.

(2) The parallel transport of the result of (1) along $\widetilde{K}$ in the local system $\mathcal{F} * \mathcal{G}[-1]$ until we almost reach b.

(3) After approaching close to b using (2), applying the variation map $v_{b,\widetilde{K}} = v_{b,\widetilde{K}}^{\mathcal{F}*\mathcal{G}}$ (in the notation of (1.2.4)) at b which is the dual of the generalization map $u_{b,\widetilde{K}}^{(\mathcal{F}*\mathcal{G})^\vee}$ for the Verdier dual perverse sheaf $(\mathcal{F} * \mathcal{G})^\vee \simeq \mathcal{F}^\vee * \mathcal{G}^\vee$.

Let $\tau \in K' \times K''$ be an interior point close to the point (a', a''). It also lies in $\widetilde{K}' \times \widetilde{K}''$ and

$$\gamma_{a',a''} : \mathcal{H}_{(a',a'')} = \widetilde{\mathcal{H}}_{(a',a'')} \longrightarrow \mathcal{H}_\tau = \widetilde{\mathcal{H}}_\tau$$

be the generalization map of the constructible sheaves $\mathcal{H}, \widetilde{\mathcal{H}}$ which coincide in the area containing (a', a'') and τ.

Let $\varphi \in \mathbf{\Phi}_{a'}(\mathcal{F})_\theta \otimes \mathbf{\Phi}_{a''}(\mathcal{G})_\theta = \mathcal{H}_{(a',a'')}$. As follows from the construction of the identification in the Thom-Sebastiani theorem (proof of Theorem 2.1.6), the composition $u_{a,\widetilde{K}}\varepsilon_{a',a''}$ can be seen as the composition of $\gamma_{a',a''}$ followed by the "forgetting of support" morphism

$$R(+)_* \, \underline{H}^0_{K' \times K''}(\mathcal{F} \boxtimes \mathcal{G}) \longrightarrow R(+)_*(\mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F} * \mathcal{G}$$

evaluated in the stalks over c. Therefore we can replace parallel transport of $u_{a,\widetilde{K}}\varepsilon_{a',a''}(\varphi)$ along $\widetilde{K}$ by parallel transport of $\gamma_{a',a''}(\varphi)$ in the local system given by $\widetilde{\mathcal{H}}$ on the interior of $\widetilde{K}' \times \widetilde{K}''$. As this interior is contractible, we have a well defined section $\widetilde{\varphi}$ of $\widetilde{\mathcal{H}}$ on it, extending $\gamma_{a',a''}(\varphi)$.

The map $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ restricts to $+_K : K' \times K'' \to K$ which in coordinates t', t'' from (2.3.4) has the form $t = t' + t''$. Let $T = +_K^{-1}([a, b)) \subset K' \times K''$ be the preimage of the half-open interval $[a, b)$, depicted as the large shaded area on Fig. 7(a), and $\widetilde{T} \subset \widetilde{K}' \times \widetilde{K}''$ be corresponding perturbation of T. On the edge $E = \{t' + t'' = |b - a|\}$ of T we have the points (b'_i, b''_j) with $b'_i + b''_j = b$, i.e. precisely the points (b', b'') such that $[a', b']$, $[a', b'']$ and $[a, b]$ are parallel in the same direction, as in the statement of Theorem 2.3.2.

Since the variation map $v^{\mathcal{F}*\mathcal{G}}_{b,\widetilde{K}}$ in (3) above is dual to $u^{\mathcal{F}^\vee * \mathcal{G}^\vee}_{b,\widetilde{K}}$ which has been just described above, we see that only (b'_i, b''_j) on E will receive a component of $m^{-,\mathcal{F}}_{ab}(\varepsilon_{a,a'}(\varphi))$: the other (b'_i, b''_j) will map far from b. This is precisely part (a) of the theorem.

Let us now prove (b). Let $b' = b'_i, b'' = b''_j$ be such that $b' + b'' = b$ and $\psi \in \mathbf{\Phi}_{b'}(\mathcal{F}^\vee)_{-\theta} \otimes \mathbf{\Phi}_{b''}(\mathcal{G}^\vee)_{-\theta}$. As before, we use the duality between $v^{\mathcal{F}*\mathcal{G}}_{b,\widetilde{K}}$ and $u^{\mathcal{F}^\vee * \mathcal{G}^\vee}_{b,\widetilde{K}}$ and interpret the latter in terms of the generalization map $\gamma_{(b',b'')} : \mathcal{H}_{(b',b'')} \to \mathcal{H}_\sigma$ where σ is a point of T in the area near (b', b''), depicted as the darker shaded area near the edge E on Fig. 7(b). So we have the equality of the pairings

$$\big(m^{-,\mathcal{F}*\mathcal{G}}_{ab}(\varepsilon_{a',a''}(\varphi)), \psi\big) = \big(\widetilde{\varphi}(\sigma), \gamma_{(b',b'')}(\psi)\big).$$

where on the right the sections of the two dual local systems are evaluated at a nearby points so the pairing is well defined. But since $\widetilde{\varphi}$ is a section of the local system $\mathcal{F} \boxtimes \mathcal{G}[-2]$, pairing on the right is precisely $\big((m^{-,\mathcal{F}}_{a',b'} \otimes m^{-,\mathcal{G}}_{a'',b''})(\varphi), \psi\big)$. This proves the theorem.

C Example: an elementary parallelogram. As an illustration of Theorem 2.3.2 consider the following particular case. Let $A' = \{a', b''\}$ and $A'' = \{a'', b''\}$ each consist of two elements such that the intervals $[a', b']$ and $[a'', b'']$ are not parallel, so

$$A = A' + A'' = \big\{a := a' + a'',\ b' + a'',\ a' + b'',\ b' + b'' =: b\big\}$$

is the set of vertices of a nondegenerate parallelogram, see Fig. 8. Then by Theorem 2.1.6 the vanishing cycle spaces of $\mathcal{F} * \mathcal{G}$ at these vertices are the tensor products, as indicated in Fig. 8. As there are no intermediate points, the rectilinear transports for $\mathcal{F}*\mathcal{G}$ between these vertices do not need avoidances: $m^+ = m^- = m$. In this situation, Theorem 2.3.2 says that the transports along the faces of the parallelogram are tensor products of $m^{\mathcal{F}}_{a',b'}$ or $m^{\mathcal{G}}_{a'',b''}$ with Id, so these maps look (up to isomorphisms of the stalks of the local systems $\mathbf{\Phi}$) as forming a commutative square.

But the diagonal transport $m^{\mathcal{F}*\mathcal{G}}_{ab}$ is equal to 0. This last statement can be seen directly by noticing that the rays K', K'' coming from a' and a'' in the direction $\theta = \zeta_{ab}$ will contain no other elements of A' or A''. So $\widetilde{K}' = K'$, $\widetilde{K}'' = K''$ and the sheaf $\mathcal{H} = \widetilde{\mathcal{H}}$ will be locally constant everywhere inside $K' \times K''$. This means that for $\varphi \in \Phi_{a'}(\mathcal{F}) \otimes \Phi_{a''}(\mathcal{G})$ the image $u_{a,K}(\varphi)$ of φ under the generalization map, can be continued along K all the way through b and so its variation at b (image under $v_{b,K}$) is zero.

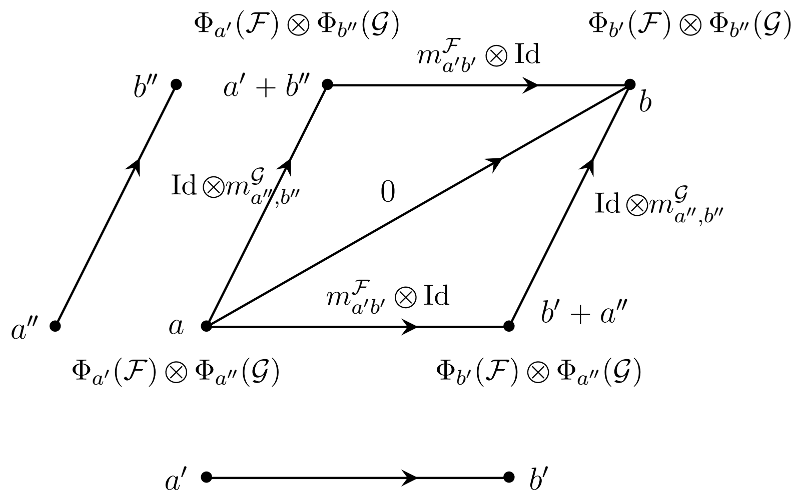


Figure 8: An elementary parallelogram: the diagonal transport is 0.

D Matrix formulation. The Stokes operator. As before, let $S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ be the circle of directions. Let $\omega \in \mathbb{C}$ be a nonzero number. For any $\mathcal{F} \in \mathrm{Perv}(\mathbb{C})$ we write $\mathbf{\Phi}_a(\mathcal{F})_\omega$ for the stalk of $\mathbf{\Phi}_a(\mathcal{F})$ at $\omega/|\omega| \in S^1$ and put $\mathbf{\Phi}(\mathcal{F})_\omega = \bigoplus_{a\in\mathbb{C}} \mathbf{\Phi}_a(\mathcal{F})_\omega$.

Define the operator $C^{\pm,\mathcal{F}}_\omega : \mathbf{\Phi}(\mathcal{F})_\omega \to \mathbf{\Phi}(\mathcal{F})_\omega$ by defining its matrix elements $(C^{\pm,\mathcal{F}}_\omega)^b_a : \mathbf{\Phi}_a(\mathcal{F})_\omega \to \Phi_b(\mathcal{F})_\omega$ as follows:

$$(C^{\pm,\mathcal{F}}_\omega)^b_a = \begin{cases} \mathrm{Id}, & \text{if } a = b; \\ T^\omega_{-\omega} \circ m^{\pm,\mathcal{F}}_{ab}, & \text{if } b = a + \omega; \\ 0, & b \text{ otherwise.} \end{cases}$$

Here $T^\omega_{-\omega}$ is the clockwise half-monodromy of $\mathbf{\Phi}_b(\mathcal{F})$ from the direction $-\omega/|\omega|$ to $+\omega/|\omega|$ (same identification as used in composing rectilinear transports).

Reformulation 2.3.3 can be further reformulated as follows.

Corollary 2.3.7. *For $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}^0(\mathbb{C})$ we have*

$$C^{\pm,\mathcal{F}*\mathcal{G}}_\omega = \sum_{\substack{\omega'+\omega''=\omega \\ \omega',\omega''\in[0,\omega]}} C^{\pm,\mathcal{F}}_{\omega'} \otimes C^{\pm,\mathcal{G}}_{\omega''}.$$

Here for $\omega' = 0$ or $\omega'' = 0$ (only one case can occur, as $\omega \neq 0$) we understand $C^\pm_0$ to be Id. □

Definition 2.3.8. Let $\zeta \in S^1$ and $\mathcal{F} \in \mathrm{Perv}(\mathbb{C})$. We define the *Stokes operator* associated to ζ and $\mathcal{F}$ as

$$\mathrm{St}_\zeta = \mathrm{St}^{\mathcal{F}}_\zeta = \mathrm{Id} + \sum_{\omega\in\mathbb{R}_{>0}\zeta} C^-_\omega : \mathbf{\Phi}(\mathcal{F})_\zeta \longrightarrow \mathbf{\Phi}(\mathcal{F})_\zeta.$$

The operator St_ζ is invertible because it is represented by a block-upper triangular matrix with respect to the order $\leqslant_\zeta$ on A. It has Id on the diagonals since it gives identity on the associated graded space. Proposition 2.3.7 can be reformulated even more concisely.

Proposition 2.3.9. *Let $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}^0(\mathbb{C})$. For any $\zeta \in S^1$.*

$$\mathrm{St}_\zeta^{\mathcal{F}*\mathcal{G}} \;=\; \mathrm{St}_\zeta^{\mathcal{F}} \otimes \mathrm{St}_\zeta^{\mathcal{G}}. \quad \square$$

In other words, St_ζ is an automorphism of the tensor functor $\mathbf{\Phi}(-)_\zeta$.

E Alien derivatives via matrix elements. Assume now that $\mathrm{char}(\mathbf{k}) = 0$. For $\mathcal{F} \in \mathrm{Perv}(\mathbb{C})$ and a nonzero $\omega \in \mathbb{C}$ we call the *alien derivative for $\mathcal{F}$ in the direction* ω the operator $\Delta_\omega = \Delta_\omega^{\mathcal{F}} : \mathbf{\Phi}(\mathcal{F})_\omega \to \mathbf{\Phi}(\mathcal{F})_\omega$ whose matrix elements $(\Delta_\omega)_a^b : \mathbf{\Phi}_a(\mathcal{F})_\omega \to \mathbf{\Phi}_b(\mathcal{F})_\omega$ are defined as follows:

$$(\Delta_\omega)_a^b = \begin{cases} T_{-\omega}^{\omega} \circ m_{ab}^{\Delta}, & \text{if } b = a + \omega; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\Delta_\omega = 0$ for almost all ω.

F Alien derivatives as functor derivations. As $\mathrm{St}_\zeta^{\mathcal{F}}$ is given by a block-upper triangular matrix with Id on the diagonal, its logarithm is a well defined operator.

Theorem 2.3.10. *(a) We have*

$$\log \mathrm{St}_\zeta^{\mathcal{F}} \;=\; \sum_{\omega \in \mathbb{R}_{>0}\zeta} \Delta_\omega^{\mathcal{F}}.$$

(b) Let $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}^0(\mathbb{C})$. With respect to the identification $\mathbf{\Phi}(\mathcal{F} * \mathcal{G})_\omega \simeq \Phi(\mathcal{F})_\omega \otimes \Phi(\mathcal{G})_\omega$ we have the Leibniz rule

$$\Delta_\omega^{\mathcal{F}*\mathcal{G}} \;=\; \Delta_\omega^{\mathcal{F}} \otimes \mathrm{Id} \;+\; \mathrm{Id} \otimes \Delta_\omega^{\mathcal{G}}.$$

In other words, the alien derivative is a derivation of the tensor functor $\mathbf{\Phi}(-)_\omega$.

Proof: (a) follows by comparison of Definition 2.2.7 of the m_{ab}^{Δ} with the logarithmic series $\log(1+x) = \sum_{s=0}^{\infty} (-1)^s x^{s+1}/(s+1)$.

(b) Since St_ζ is an automorphism of the tensor functor $\mathbf{\Phi}(-)_\zeta$, its logarithm $\Delta_{\mathbb{R}_{>0}\zeta} = \log \mathrm{St}_\zeta$ is a derivation by formal reasons. Now, $\mathbf{\Phi}(-)_\zeta$ takes values in the tensor category of $\mathbb{C}$-graded vector spaces (with the graded tensor product). Any endomorphism D of this functor can be split into homogeneous components $D = \sum_{\omega \in \mathbb{C}} D_\omega$, where D_ω raises the degree by ω. Clearly, D is a derivation if and only if each D_ω is a derivation. It remains to notice that the Δ_ω, $\omega \in \mathbb{R}_{>0}\zeta$ are precisely the homogeneous components of $\Delta_{\mathbb{R}_{>0}\zeta}$, in virtue of (a). $\square$

G Stokes automorphisms in terms of the Fourier transform. Let $\mathbf{k} = \mathbb{C}$ and $\mathcal{F} \in Perv(\mathbb{C}, A)$. The directions ζ_{ab} for all distinct $a, b \in A$, see (2.2.1), will be called *Stokes directions* for A.

As in §1.4, Fourier transform gives a local system $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})$ on $\mathbb{C}^*$ or, equivalently, on S^1. In the proof of Proposition 1.4.2 we constructed an identification of $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})$ with $\mathbf{\Phi}(\mathcal{F}) = \bigoplus_{a\in A} \mathbf{\Phi}_a(\mathcal{F})$ outsides of the Stokes directions. Indeed, $\zeta \in S^1$ is non-Stokes if and only if all the half-rays $K_a(\zeta) = a + \zeta\mathbb{R}_+$ are disjoint.
Let us now complete that construction by describing how these identifications glue together at a given Stokes direction ζ. Let ζ^+ and ζ^- be nearby non-Stokes directions clockwise and anti-clockwise from ζ. The gluing along ζ for the local system $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})$). with respect to our prior identifications is given by the map S_ζ defined as the composition

$$\mathbf{\Phi}(\mathcal{F})_\zeta \xrightarrow{T_\zeta^{\zeta^+}(\mathbf{\Phi})} \mathbf{\Phi}(\mathcal{F})_{\zeta^+} \xrightarrow{(1.4.2)} \mathrm{FT}_{\mathrm{gen}}(\mathcal{F})_{\zeta^+} \xrightarrow{T_{\zeta^+}^{\zeta^-}(\mathrm{FT})} \mathrm{FT}_{\mathrm{gen}}(\mathcal{F})_{\zeta^-} \xrightarrow{(1.4.2)} \mathbf{\Phi}(\mathcal{F})_{\zeta^-} \xrightarrow{T_{\zeta^-}^{\zeta}(\mathbf{\Phi})} \mathbf{\Phi}(\mathcal{F})_\zeta.$$

Here, say, $T_\zeta^{\zeta^+}(\mathbf{\Phi})$ is the monodromy from ζ to ζ^+ (along the shortest path) for the local system $\mathbf{\Phi}(\mathcal{F})$, and $T_{\zeta^+}^{\zeta^-}(\mathrm{FT})$ is the monodromy from ζ^+ to ζ^- for $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})$. The following result up to notation coincides with [5, Th.5.2.2].

Theorem 2.3.11. *S_ζ coincides with the Stokes automorphism* St_ζ. □

Remark 2.3.12. Because of the triangular nature of St_ζ, it preserves the Stokes filtration which was described in Proposition 1.4.4 for generic (non-Stokes) directions θ: the terms that are added after crossing ζ, have lower rate of exponential growth. So we obtain a well defined filtration on the local system $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})$ on S^1 labelled by the sheaf of posets $(\underline{A}, \leqslant_\theta)_{\theta\in S^1}$ on S^1. This means that the sheaf of sets $\underline{A}$ is constant, but the order $\leqslant_\theta$ varies with θ, see [8, 7] and [17, §2.4].

H Meaning and reformulation of Theorem 2.3.11. For the convenience of the reader let us discuss the meaning of Theorem 2.3.11 in more detail. Let $V = \mathrm{FT}_{\mathrm{gen}}(\mathcal{F})_\zeta$, identified with $\mathrm{FT}_{\mathrm{gen}}(\mathcal{F})_{\zeta^\pm}$ by monodromy (along the shortest path). Choose $R \gg 0$ and let $\mathfrak{H}$ be the half-plane $\{\mathrm{Re}(\zeta\overline{w}) \geqslant -R\}$ as in the proof of Proposition 1.4.2, so that

$$V = H^0_{\mathfrak{H}}(\mathbb{C}, \mathcal{F}).$$

For any closed subset $Z \subset \mathfrak{H}$ let $\tau_Z : H^0_Z(\mathbb{C}, \mathcal{F}) \to V$ be the morphism induced by the inclusion of supports.

As before, for any $a \in \mathbb{C}$ and $\theta \in S^1$ denote $K_a(\theta) = a + \mathbb{R}_+\theta$ the ray in the direction θ issuing from a. We denote

$$K_a^\pm = K_a(\zeta^\pm), \quad K_a = K_a(\zeta), \quad K^\pm = \bigsqcup_{a\in A} K_a^\pm, \quad K = \bigcup_{a\in A} K_a,$$

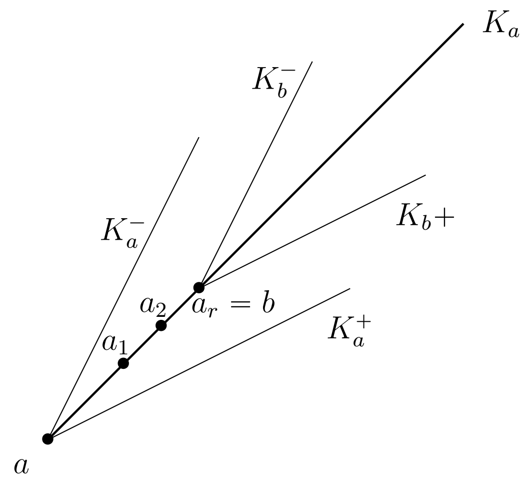


Figure 9: Crossing a Stokes direction $\zeta = \zeta_{ab}$.

see Fig. 9. As $\zeta^\pm$ is non-Stokes, the rays $K_a^\pm$, $a \in A$, are all distinct, but not so for the K_a. Let us write Φ_a for the stalk $\mathbf{\Phi}_a(\mathcal{F})_\theta$ for any $\theta \in [\zeta^+, \zeta^-]$, these stalks being identified by monodromy and put $\Phi = \bigoplus_{a\in A} \Phi_a$. Then we have identifications

$$\Phi \simeq \bigoplus_a H^0_{K_a^\pm}(\mathbb{C}, \mathcal{F}) \ = \ H^0_{K^\pm}(\mathbb{C}, \mathcal{F}).$$

Note that the morphisms

$$\tau_{K^\pm} : \Phi = H^0_{K^\pm}(\mathbb{C}, \mathcal{F}) \to V \text{ as well as } \tau_K : H^0_K(\mathbb{C}, \mathcal{F}) \to V$$

are isomorphisms, since the complements $\mathbb{C}\backslash K^\pm$ and $\mathbb{C}\backslash K$ are homotopy equivalent to $\mathbb{C}\backslash\mathfrak{H}$ and $\mathcal{F}$ is locally constant on these complements. It follows from the identifications constructed in Proposition 1.4.2, that $S_\zeta = \tau_{K^-}^{-1}\tau_{K^+}$, i.e., it is equal to the composition

$$\Phi = H^0_{K^+}(\mathbb{C}, \mathcal{F}) \xrightarrow{\tau_{K^+}} H^0_{\mathfrak{H}}(\mathbb{C}, \mathcal{F}) \xrightarrow{\tau_{K^-}^{-1}} H^0_{K^-}(\mathbb{C}, \mathcal{F}) = \Phi$$

Denote for short the inclusion of support maps for individual rays by

$$\tau_a^\pm = \tau_{K^\pm(a)} : H^0_{K_a^\pm}(\mathbb{C}, \mathcal{F}) \to V, \quad \tau_a = \tau_{K(a)} : H^0_{K(a)}(\mathbb{C}, \mathcal{F}) \to V,$$

so that

$$\tau_{K^\pm} = \sum_{a\in A} \tau_a^\pm : \ \Phi = \bigoplus_{a\in A} \Phi_a \longrightarrow V.$$

Fix now $a \in A$ and let $a_0 = a, a_1, \cdots, a_r = b$ be all elements of A on K_a, see Fig. 9. Recalling Definition 2.3.8 of St_ζ, we can reformulate Theorem 2.3.11 as follows.

Reformulation 2.3.13. *For any choice of a as above and any $\varphi \in \Phi_a$ we have*

$$\tau_a^-(\varphi) \ = \ \tau_a^+(\varphi) + \sum_{i=1}^r \tau_{a_i}^+(m_{a,a_i}^-(\varphi)). \quad \square$$

Remarks 2.3.14. (a) Note the similarity of the above formula with Proposition 2.2.6(b). Here, instead of the final transport $m^+_{a_i,b}$ to Φ_b, we have the map $\tau^+_{a_i}$ to V which corresponds, informally, to putting b at ∞.

(b) Reformulation 2.3.13 matches rather directly the identities among exponential integrals of multivalued functions along various paths, traditionally used in resurgence theory.

3 Resurgence theory: convolution algebras in $\overline{\mathrm{Perv}}(\mathbb{C})$

3.1 The general program

We now outline an approach to resurgence as a program of extending and applying the above elementary theory to a more general concept of perverse sheaves.

A Resurgent perverse sheaves: algebras in the convolution category. We propose to consider perverse sheaves on $\mathbb{C}_w$ (the Borel plane) carrying some algebraic structures with respect to the convolution operation $*$. For example, associative (commutative or not) algebras, i.e., perverse sheaves $\mathcal{A}$ with an operation (i.e., morphism) $\mathcal{A}*\mathcal{A} \to \mathcal{A}$ satisfying the associativity and possibly commutativity condition. Or, given such algebra $\mathcal{A}$, we can consider $\mathcal{A}$-modules, i.e., perverse sheaves $\mathcal{M}$ with an operation $\mathcal{A} * \mathcal{M} \to \mathcal{M}$. Other algebraic structures can be considered (e.g., Lie algebras).

An algebraic structure with respect to convolution defined on a perverse sheaf $\mathcal{A}$ would give a formal convolution operation on its sections over various domains or on the spaces of vanishing cycles (whose intuitive meaning is to describe singularity data of sections). So various formulas of resurgent analysis involving convolutions, alien derivatives and such could be written intrinsically inside the data associated to $\mathcal{A}$. Therefore we propose to call perverse sheaves equipped with such algebra structures *resurgent perverse sheaves.*[4] In various concrete examples sections of resurgent perverse sheaves will be represented by actual resurgent functions in the classical sense.

As Fourier transform takes convolution to fiberwise multiplication, applying it to a resurgent perverse sheaf $\mathcal{A}$ would give a local system on $\mathbb{C}^*_z$ the punctured z-plane with an algebra structure (of the corresponding type) in the fibers and with a Stokes structure such that the Stokes matrices are isomorphisms of algebras. For example, if $\mathcal{A}$ is a commutative algebra, the Stokes matrices, being isomorphisms of commutative algebras, can be thought of as coordinate changes. This would fit, e.g., into the interpretation of cluster transformations as Stokes data for appropriate differential (or rather integral) equations, see [11, §7] and §3.3 below.

B Generalized perverse sheaves and their convolution. In order to realize the above program, we need to generalize the concept of perverse sheaves.

[4] To be more precise, we suggest to use this term for a generalization of the notion of perverse sheaf discussed in the next subsection.

First of all, we need perverse sheaves on $\mathbb{C}$ whose set of singularities A is an arbitrary countable (for example, everywhere dense) subset in $\mathbb{C}$. A typical example of such A is a (free) abelian subgroup in $\mathbb{C}$ of finite rank r; it cannot be discrete, if $r \geqslant 3$. Examples like this are inevitable since in the classical case (Proposition 2.1.1) the singularities of $\mathcal{F} * \mathcal{G}$ are typically all the sums of a singularity of $\mathcal{F}$ and a singularity of $\mathcal{G}$. So to have an interesting map $\mathcal{A} * \mathcal{A} \to \mathcal{A}$, the set of singularities of $\mathcal{A}$ needs to be closed under addition.

The fundamental object of study should be the localized category $\overline{\mathrm{Perv}}(\mathbb{C}, A)$. Its objects should have well-defined spaces (more precisey, local systems on S^1) of vanishing cycles Φ_a, $a \in A$ and the transport maps $m_{ab}(\gamma) : \Phi_a \to \Phi_b$ for some class of paths γ joining a and b.

Note that case of a discrete $A \subset \mathbb{C}$ may seen to be covered by the theory of $\mathcal{D}$-modules and perverse sheaves in the analytic context. However, already the lifting of $\overline{\mathrm{Perv}}(\mathbb{C}, A)$ back into $\mathrm{Perv}(\mathbb{C}, A)$ in this context is not obvious, since in the classical case $|A| < \infty$ the generic stalk of the lifted sheaf is the direct sum of all the Φ_a (see [13, 20, 10]) which can be infinite-dimensional for $|A| = \infty$ and so falls outside of the theory of analytic $\mathcal{D}$-modules.

For this and other reasons we need perverse sheaves with possibly infinite-dimensional stalks or, more generally, perverse sheaves with values in a more or less arbitrary abelian category $\mathcal{C}$. For example, when $\mathcal{C}$ is the category of pro-finite-dimensional (= locally linearly compact linearly topological) vector spaces, the dual to the category of all vector spaces, this approach would give (perverse) cosheaves of [25]. Also, one needs to consider various analytic completions (e.g. of the infinite direct sum of the vanishing cycles above), intermediate between direct sums and direct products and involving convergence conditions.

C Lefschetz perverse sheaves in infinite dimensions. It is a very appealing idea to generalize the construction of the Lefschetz perverse sheaves $\mathcal{L}_S^i$ from §1.5 to the case when X is some complex function space of "fields" and S is the classical action functional corresponding to some physical theory.

Indeed, $\mathrm{Crit}(S)$, the critical locus of S, is the space of solutions of the classical equations of motion; if the problem is set up appropriately, connected components of $\mathrm{Crit}(S)$ are finite-dimensional. The behavior of S in the directions "transverse" to $\mathrm{Crit}(S)$ typically has the form

$$S(x) = f(x_1, \cdots, x_m) + \sum_{i=m+1}^{\infty} x_i^2,$$

the direct sum of a function of finitely many variables and an infinite sum of independent squares. This means that perverse sheaf $\Phi =$ "$\Phi_S(\underline{\mathbf{k}}_X[\dim X])$" on $\mathrm{Crit}(S)$, or at least, on some patches of $\mathrm{Crit}(S)$, can be defined[5] "by hand" starting from the Φ_f. As adding an extra independent square transforms vanishing cycles in a known way (Knörrer periodicity), we are lead to a natural gerbe (of orientation data) whose trivialization defines Φ completely. This by

[5]Strictly speaking, on a compoment $C \subset \mathrm{Crit}(X)$ with $S(C) = a \in \mathbb{C}$ we should define Φ_{S-a}, not Φ_S.

now well known procedure is axiomatized using the framework of (-1)-shifted symplectic structures [2]. The hypercohomology groups of connected components $C \subset \mathrm{Crit}(S)$ with coefficients in Φ appear in the framework of motivic Donaldson-Thomas (DT for short) theory.

However, such a procedure treats different components independently, as the actual values of S on the components are ignored or lost. A natural refinement of the above data would be perverse sheaves[6] $\mathcal{L}_S^i$ on $\mathbb{C}$ whose stalks at $a \in \mathbb{C}$ would be $H^i(C_a, \Phi)$, where C_a is the union of components $C \subset \mathrm{Crit}(S)$ with $S(C) = a$. This additional structure would also provide the transport maps $m_{ab}(\gamma)$ between motivic DT-invariants for a certain class of paths γ as long as stability structures are incorporated in our framework (e.g. in the case when X is the stack of objects of a 3-dimensional Calabi-Yau category of "geometric origin").

D Unlimited analytic continuation: the analytic pro-étale site. Multivalued analytic functions $f(w)$ on the Borel plane appearing in resurgence theory, have the remarkable property of unlimited analytic continuation which has been made precise using slightly different concepts of continuation "without cut" (sans coupure) in [9] or "without end" (sans fin) in [4]. Intuitively, such a formalization needs to accomodate two features of the functions in question:

(1) They possess no natural boundaries, beyond which analytic continuation is not possible (such as the unit circle being the natural boundary for the function $\sum_n w^{n^2}$).

(2) But they can have isolated singularities including ramification points that can accumulate on further and further sheets of the Riemann surface.

To explain (2), any "branch" of $f(w)$ is defined over a "sheet" obtained by removing from $\mathbb{C}$ a discrete set of cuts emanating from a discrete set of ramification points "visible on this sheet". But after crossing a cut we arrive on a new sheet where $f(w)$ has a new, still discrete but possibly larger set of ramification points etc. At the end one can have a seemingly paradoxical outcome that f has a non-discrete, e.g., everywhere dense set of singularities (understood as points in $\mathbb{C}$).

The features (1) and (2) make one think about the Bhatt-Scholze theory of the pro-étale site [1]. Indeed, (1) suggests some étale property while going to further and further sheets in (2) resembles some projective limit procedure. So let us sketch a version of this theory in the analytic situation. We plan to discuss it in detail in the future.

Let X be a complex manifold of dimension d. We can consider on X the *analytic Zariski* (or *ana-Zariski* for short) topology, in which the closed sets are analytic subsets $S \subset X$. Then the open sets are complements $X \backslash S$ of such S. For example, if $d = 1$, then an analytic subset in X is just a discrete subset, possibly infinite. Thus an ana-Zariski open subset is a complement of a discrete subset.

Let now X and Y be complex manifolds of the same dimension d.

[6] or objects of the localized category $\overline{\overline{\mathrm{Perv}}}$.

Definition 3.1.1. A holomorphic map $f : Y \to X$ is called an *analytic étale* (*ana-étale* for short), if:

- f is a local biholomorphism, i.e., the differential df is invertible everywhere.
- There is an ana-Zariski open sets $Y' \subset Y$, $X' \subset X$ such that $f(Y') = X'$ and moreover, $f : Y' \to X'$ is an unramified covering (with possibly infinite fibers).

An example is given by the exponential map $\exp : \mathbb{C} \to \mathbb{C}$, with $S = \{0\} \subset X = \mathbb{C}$.

Definition 3.1.2. A holomorphic map $f : Y \to X$ is called a *pro-ana-étale*, if there exist:

- A projective system
$$Y_0 = X \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots$$
with each arrow being ana-étale;
- An injective morphism $\widetilde{f} : Y \to \varprojlim Y_i$ whose composition with the projection $\varprojlim Y_i \to Y_0 = X$ coincides with f.

Thus Definition 3.1.1 accounts for the desired feature (1), while Definition 3.1.2 accounts for (2).

Examples 3.1.3. (a) The embedding map $Y = \{|w| < 1\} \hookrightarrow X = \mathbb{C}_w$ is a local biholomorphism but not pro-ana-étale. Such Y can be seen as the Riemann surface of a function with natural boundary.

(b) It seems plausible that the classical example of the Riemann surface of the inverse of the hyperelliptic integral discussed in [4, §3] can be included into the framework of pro-ana-étale theory outlined above. We plan to discuss this as well as more general examples in the future.

To define constructible and then perverse sheaves with possible non-discrete sets of singularities, one can follow one of the two paths.

First, each sheaf $\mathcal{F}$ on X in the analytic topology has the *étale space* $\widetilde{X}_{\mathcal{F}} \to X$ obtained by topologizing the union of all the stalks $\mathcal{F}_x, x \in X$. One can consider sheaves whose étale spaces have maximal Hausdorff parts of their connected components satisfying the property of being pro-ana-étale in the above sense, with some constructibility conditions imposed on the sheaves.

Alternatively, one can consider directly some version of pro-ana-étale Grothendieck site on X and work with sheaves on this site.

.

3.2 A finitistic example: COHA of a quiver with potential

Let $Q = (I, E)$ be a finite quiver, with I being the set of vertices and $E \to I \times I$ being the set of oriented edges. For any dimension vector $d = (d_i)_{i \in I}$, $d_i \in \mathbb{Z}_+$ we denote $\mathrm{Rep}_d(Q)$ the stack of complex d-dimensional representations V of Q. By definition, such a representation associates to each $i \in I$ a d_i-dimensional $\mathbb{C}$-vector space V_i and to each arrow $e \in E$ between i and j a linear operator $\rho_e : V_i \to V_j$ with no further relations.

We denote by $\mathbb{C}\langle Q\rangle$ the path algebra of Q, so representations of Q are the same as left $\mathbb{C}\langle Q\rangle$-modules. Fix a *potential*, or a *cyclic word* in $\mathbb{C}\langle Q\rangle$, i.e., an element $s \in \mathbb{C}\langle Q\rangle / \big[\mathbb{C}\langle Q\rangle, \mathbb{C}\langle Q\rangle\big]$, the quotient by the commutator subspace (not the ideal generated by commutators). More explicitly, s is represented as a linear combination of closed edge paths in Q. A choice of s gives for any d a regular function

$$S_d : \mathrm{Rep}_d(Q) \longrightarrow \mathbb{C}, \quad V \mapsto \mathrm{tr}_V(s)$$

given by taking the trace. Such functions are additive in the following sense: in the *induction diagram* of stacks

$$\begin{array}{ccc} & \{0 \to E' \to E \to E'' \to 0\} & \\ \swarrow & & \searrow \\ \mathrm{Rep}_{d'} \times \mathrm{Rep}_{d''} & & \mathrm{Rep}_d \end{array} \qquad \begin{array}{c} \dim(E') = d', \dim(E'') = d'', \\ \dim(E) = d = d' + d'' \end{array}$$

the pullback of $S_{d'} + S_{d''}$ from the left is equal to the pullback of S_d from the right.

The Cohomological Hall Algebra (COHA) associated to (Q, s) is [22]

$$A = \bigoplus_{d \in \mathbb{Z}_+^I} A_d, \quad A_d = H^\bullet\big(\mathrm{Rep}_d(Q), \Phi_{S_d}(\underline{\mathbf{k}}[\dim \mathrm{Rep}_d(Q)])\big).$$

Here $\dim \mathrm{Rep}_d(Q)$ is the dimension in the sense of stacks. The multiplication $A_{d'} \otimes A_{d''} \to A_{d'+d''}$ is given by the pullback and pushforward in the induction diagram above.

Now consider the (bi)graded perverse sheaf

$$\mathcal{A} = \bigoplus_d \mathcal{A}_d, \quad \mathcal{A}_d = \mathcal{L}^\bullet_{S_d}$$

where $\mathcal{L}^\bullet_{S_d}$ is the graded Lefschetz perverse sheaf (see §1.5 **B**) on $\mathbb{C}$ associated to S_d. Each $\mathcal{A}_d$ is a graded perverse sheaf with finitely many singularities only. We will consider $\mathcal{A}_d$ as an object of the localized category $\overline{\mathrm{Perv}}(\mathbb{C})$. It seems plausible that the induction diagram defines morphisms of graded perverse sheaves $\mathcal{A}_{d'} * \mathcal{A}_{d''} \to \mathcal{A}_{d''}$ in $\overline{\mathrm{Perv}}(\mathbb{C})$, i.e., makes $\mathcal{A}$ into an associative convolution algebra refining A, cf. [22, §4].

Remark 3.2.1. The above definition of COHA depends only on the quiver with potential. One can make an additional choice consisting of the central charge $Z : \mathbb{Z}^I \to \mathbb{C}$. This choice gives rise to the wall-crossing structure in the sense of [23, 24]. If the wall-crossing structure is analytic in the sense of [24] then it was conjectured in the loc.cit. that germs of sections of the associated non-linear fiber bundle over $\mathbb{C}$ are resurgent. Assuming the conjecture we obtain a perverse sheaf on the Borel plane with singularities belonging to the image $Z(\mathbb{Z}^I)$. This perverse sheaf seems quite different from the $\mathcal{L}^\bullet_{S_d}$.

3.3 Cluster perverse sheaves and wall-crossing structures

A Two types of examples: Lefschetz type and cluster type. Two main classes of perverse sheaves on $\mathbb{C}$ of infinite rank with possibly infinite set of singularities appear naturally in resurgence theory:

- Those which comes from holomorphic functions on infinite-dimensional manifolds (natural generalization of Lefschetz sheaves)
- Those which come from wall-crossing structures [24]. In this section we consider this class. They can be called *cluster perverse sheaves"* or *wall-crossing perverse sheaves.* We will use the former term, since cluster transformations play the key role in the story.

These classes have nonempty intersection.

B Reminder on stability data on graded Lie algebras. Let us recall the relevant structures following [21, 23, 24].

Let Γ be a free abelian group of finite rank n endowed with a skew-symmetric integer-valued bilinear form $\langle -, - \rangle : \Gamma \times \Gamma \to \mathbb{Z}$. Consider the vector space $\mathfrak{g} = \mathfrak{g}_\Gamma = \oplus_{\gamma \in \Gamma} \mathbb{Q} \cdot e_\gamma$. This space is made into a Poisson algebra with the commutative (associative) product and the Poisson bracket given by

$$\begin{aligned} e_{\gamma_1} e_{\gamma_2} &= (-1)^{\langle \gamma_1, \gamma_2 \rangle} e_{\gamma_1 + \gamma_2} \\ \{e_{\gamma_1}, e_{\gamma_2}\} &= (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2} \ . \end{aligned} \tag{3.3.1}$$

Let $\mathbb{T} = \mathbb{T}_\Gamma := \mathrm{Spec}(\mathfrak{g})$ be the algebraic Poisson manifold obtained as the spectrum of the commutative algebra $\mathfrak{g}$. It is a torsor over the algebraic torus $\mathrm{Hom}(\Gamma, \mathbb{G}_m)$ and the Poisson structure on $\mathbb{T}$ is invariant with respect to the torus action.

Let $\Gamma_\mathbb{R} = \Gamma \otimes \mathbb{R}$. For any strictly convex cone $C \subset \Gamma_\mathbb{R}$ we denote

$$\widehat{\mathfrak{g}}_C = \prod_{\gamma \in \Gamma \cap C - \{0\}} \mathbb{Q} \cdot e_\gamma$$

the completion of $\mathfrak{g}$ associated to C. It inherits the Poisson algebra structure.

Recall [21] that *stability data* on $\mathfrak{g}$ consist of a pair (Z, a), where $Z : \Gamma \to \mathbb{C}$ is a homomorphism of abelian groups ("central charge") and a is a collection of elements $a_\gamma \in \mathbb{Q} \cdot e_\gamma \subset \mathfrak{g}$ given for each $\gamma \in \Gamma \backslash \{0\}$. These data satisfy the so-called support condition which means that there exists a non-zero quadratic form on $\Gamma \otimes \mathbb{R}$ which is non-negative on those $\gamma \in \Gamma$ for which $a_\gamma \neq 0$ and which is negative on $\mathrm{Ker}(Z \otimes \mathbb{R})$, see [21, §2] for details.

It is convenient to transform the collection $a = (a_\gamma)$ into a collection of numbers $\Omega = (\Omega(\gamma) \in \mathbb{Q})_{\gamma \in \Gamma \backslash \{0\}}$ defined uniquely by the identities

$$a(\gamma) = - \sum_{n \geqslant 1, \frac{1}{n}\gamma \in \Gamma \backslash \{0\}} \frac{\Omega(\gamma/n)}{n^2} e_\gamma \ .$$

Then for any cone C as above containing γ we have a formal identity in $\widehat{\mathfrak{g}}_C$:

$$\exp\left(\sum_{n\geqslant 1} a(n\gamma)\right) = \exp\left(-\sum_{n\geqslant 1}\Omega(n\gamma)\sum_{k\geqslant 1}\frac{e_{kn\gamma}}{k^2}\right) := \exp\left(-\sum_{n\geqslant 1}\Omega(n\gamma)\mathrm{Li}_2(e_{n\gamma})\right),$$

where $\mathrm{Li}_2(t) = \sum_{k\geqslant 1} t^k/k^2$ is the dilogarithm series.

As for any Poisson manifold, the Lie algebra $\mathfrak{g} = (\mathcal{O}(\mathbb{T}), \{-,-\})$ acts on $\mathbb{T}$ by Hamiltonian vector fields (derivations of the coordinate ring $(\mathfrak{g} = \mathcal{O}(\mathbb{T}), \bullet)$) which have the form $\{f, -\}$ for $f \in \mathfrak{g}$. For any $\gamma \in \Gamma\backslash\{0\}$ denote by S_γ the formal Poisson automorphism defined by

$$S_\gamma = \exp(\{-\mathrm{Li}_2(e_\gamma), -\}), \quad S_\gamma(e_\mu) = (1-e_\gamma)^{\langle\gamma,\mu\rangle} e_\mu$$

Here the second equality exhibits S_γ as a *birational automorphism* of $\mathbb{T}$, i.e., an automorphism of the field of fractions of $\mathfrak{g}$. The first equality shows its formal series expansion in $\widehat{\mathfrak{g}}_C$ for any $C \ni \gamma$ as above. It is explained in [21] how the $\Omega(\gamma)$ are related to enumerative Donaldson-Thomas invariants of 3-dimensional Calabi-Yau categories.

Further, if the above data satisfy a certain *analyticity assumption*, then they give rise to an analytic fiber bundle E over $\mathbb{C}$ with fibers isomorphic to $\mathbb{T}$. The gluing functions of E come from transformations

$$S_l = \prod_{Z(\gamma)\in l} S_\gamma^{\Omega(\gamma)} \quad \text{(product in the order given by } l\text{)} \tag{3.3.2}$$

associated to various rays $l \subset \mathbb{C}$. In this case, the *resurgence conjecture* of [24] says that with any analytic section of E and each $\gamma \in \Gamma$ one can naturally associate a resurgent series in the standard coordinate z on $\mathbb{C}$.

The notions of wall-crossing structure and analytic wall-crossing structure (see [24]) generalizes the notion of stability data on a graded Lie algebra roughly by considering sheaves of stability data and analytic stability data.

C A perverse sheaf interpretation. From the point of view of the present paper, it is natural to think of the target of the central charge map $Z : \Gamma \to \mathbb{C}$ as the Borel plane. Let us assume for simplicity that Z is a set-theoreical embedding. The transformations S_l are suggestive of Stokes multiplies and are in fact interpreted as such (in the more general context of integral equations) in [11], cf. also [3].

So it is natural to look for a perverse sheaf[7] $\mathcal{F}$ on $\mathbb{C}$ in some generalized sense as above (in fact, an object of an apprioriate localized category $\overline{\mathrm{Perv}}$) with the properties:

(0) $\mathcal{F}$ is a Poisson algebra with respect to convolution.

(1) The set A of singularities of $\mathcal{F}$ is a subgroup of $Z(\Gamma)$. So it can be discrete (though still infinite), if $\mathrm{rk}(\Gamma) \leqslant 2$.

[7] with $\mathbf{k} = \mathbb{Q}$.

(2) For $a = Z(\gamma)$ the space $\Phi_a(\mathcal{F})$ is 1-dimensional, identified with $\mathbb{Q} \cdot e_\gamma$.

(3) For $\zeta \in S^1$ the Stokes operator St_ζ for $\mathcal{F}$ is given by $S_{\mathbb{R}_+\zeta}$ from (3.3.2).

Then $\mathfrak{g}$ would be realized as $\bigoplus_{a\in A} \Phi_a(\mathcal{F})$ with the operations coming from those on $\mathcal{F}$. The Fourier transform $\widehat{\mathcal{F}}$ would then be a generalized perverse sheaf of infinite rank (the latter needs a rigorous definition), and the stalks of the local system $\mathrm{FT}(\mathcal{F})_{\mathrm{gen}}$ on $\mathbb{C}^*$ would involve some analytic completions of $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathbb{Q} \cdot e_\gamma$. For a ray $l \subset \mathbb{C}$ the transformation S_l need not preserve $\mathfrak{g}$ which is an algebraic direct sum: it maps $\mathfrak{g}$ into the direct product. But as such it has a well defined matrix element between any two summands which we can write as $(S_l)_{ab} : \Phi_a(\mathcal{F}) \to \Phi_b(\mathcal{F})$. Note that $(S_l)_{ab} = 0$ unless $(b - a) \in l$. Therefore it is indeed meaningful to interpret each S_l as the Stokes operator of some would-be generalized perverse sheaf $\mathcal{F}$. If we do so, the finite case analysis of §2.3 provides an answer for what should be the (say, left-)avoiding transport $m_{ab}^{-,\mathcal{F}}$ or the alien transport $m_{ab}^{\Delta,\mathcal{F}}$ for any $a, b \in A = Z(\Gamma)$. Proposition 2.2.13 suggests that his should be sufficient information to recover $\mathcal{F}$ as an object of the localized category.

This picture is expecially compelling when $\mathrm{rk}(\Gamma) = 2$ and A is a discrete lattice in $\mathbb{C}$. Then one can look for $\mathcal{F}$ as a perverse sheaf in the classical sense but with possibly infinite-dimensional stalks.

3.4 Perverse sheaves in Chern-Simons theory

An example of a holomorphic function on an infinite-dimensional complex manifold (or stack) is provided by the complexified Chern-Simons functional (CS functional for short) associated to a compact oriented 3-manifold M and a complex semisimple Lie group G. Let e.g. $G = SL_n(\mathbb{C})$. We take $\mathcal{X}$ to be the moduli stack of C^∞-connections on the trivial SL_n-bundle on M, so $\mathcal{X}$ is the quotient of the vector space $\Omega^1(M) \otimes \mathfrak{sl}_n$ by the gauge group $C^\infty(M, SL_n(\mathbb{C}))$, and the CS functional is

$$\text{(3.4.1)} \qquad \mathrm{CS} : \mathcal{X} \longrightarrow \mathbb{C}/4\pi^2\mathbb{Z}, \quad \mathrm{CS}(A) = \int_{M^3} \mathrm{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A\right) \bmod 4\pi^2\mathbb{Z}.$$

In fact, quotienting by the subgroup of gauge transformations equal to 1 over a fixed point $m_0 \in M$, gives an infinite-dimensional *manifold* $\mathcal{X}_{\mathrm{fr}}$ (framed connections) with $\mathcal{X} = \mathcal{X}_{\mathrm{fr}}/SL_n(\mathbb{C})$ and we can consider CS on this manifold.

In order to get rid of the multivaluedness we pass to the maximal abelian covering $\widetilde{\mathcal{X}}_{\mathrm{fr}} \xrightarrow{\mathbb{Z}} \mathcal{X}_{\mathrm{fr}}$, so we get a well defined holomorphic function

$$\text{(3.4.2)} \qquad \widetilde{\mathrm{CS}} : \widetilde{\mathcal{X}}_{\mathrm{fr}} \longrightarrow \mathbb{C}.$$

One can then attempt to define and study the corresponding Lefschetz perverse sheaf $\mathcal{L}_{\mathrm{CS}}$ along the lines discussed in §3**C**. Such a study was in fact initiated in [25] although very little is known about $\mathcal{L}_{\widetilde{\mathrm{CS}}}$. Some known facts and some expectations can be summarized as follows.

(1) The functional CS in (3.4.1) has finitely many critical values, which are known to be the regulators of some elements in algebraic K-group $K_3(\mathbb{C})$. Accordingly, the critical values of $\widetilde{\mathrm{CS}}$ fall into finitely many arithmetic progressions with step $4\pi^2\mathbb{Z}$.

(2) The generic stalks of the expected perverse sheaf $\mathcal{L}_{\widetilde{\mathrm{CS}}}$ (i.e., intuitively, the middle-dimensional cohomology of the generic fiber of $\widetilde{\mathrm{CS}}$) are infinite-dimensional. In fact, it is easier to first define the Verdier dual cosheaf, as done in [25]. But the spaces of vanishing cycles are finite-dimensional.

(3) $\mathcal{L}_{\widetilde{\mathrm{CS}}}$ can be defined as the perverse extension of the local system formed by the middle cohomology of the fibers on the complement to the set of critical values, see [25, §8.3].

(4) In addition, $R\Gamma(\mathbb{C}, \mathcal{L}_{\widetilde{\mathrm{CS}}}) = 0$, so $\mathcal{L}_{\widetilde{\mathrm{CS}}}$ is, formally, an object of the category $\mathrm{Perv}^0(\mathbb{C})$ (although with infinitely many singularities and with infinite-dimensional stalks).

Some further conjectures can be found in [25, §8.3]. It was also explained in *loc.cit.* how the wall-crossing structure and resurgence properties of the perturbative expansions of the Chern-Simons functional integral are related to the perverse sheaf $\mathcal{L}_{\widetilde{\mathrm{CS}}}$.

References

[1] B. Bhatt, P. Scholze. The pro-étale topology for schemes. arXiv:1309.1198.

[2] C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendröi. Symmetries and stabilization for sheaves of vanishing cycles. *J. of Singularities* **11** (2015), 85-151.

[3] T. Bridgeland, B. Toledano Laredo. Stability conditions and Stokes factors. *Invent. math.* **187** (2012) 61-98 .

[4] B. Candelperger, J. C. Nosmas, F. Pham. Approche de résurgence. Hermann, Paris, 1993.

[5] A. D'Agnolo, M. Hien, G. Morando, C. Sabbah. Topological computation of some Stokes phenomena on the affine line. arXiv: 1705.07610.

[6] L. Daia. La transformation de Fourier pour les $\mathcal{D}$-modules. *Ann. Inst. Fourier* **50** (2000) 1891-1944.

[7] P. Deligne, B. Malgrange, J.-P. Ramis (Eds.) Singularités Irrégulières: Correspondance et Documents. Soc. Math. France, 2007.

[8] P. Deligne. Letter to V. S. Varadarajan of Jan. 4, 1986, in: [7] p. 47-51.

[9] J. Écalle. Fonctions Résurgentes, Vol. 1. Algèbres des fonctions résurgentes. Publ. Math. Orsay 81.05 (1981), available at `https://www.imo.universite-paris-saclay.fr/~jean.ecalle/fichiersweb/E_ECALLE_81_05.pdf`

[10] J. Fresán, P. Jossen. Exponential motives. Book manuscript, available at `http://javier.fresan.perso.math.cnrs.fr/expmot.pdf`

[11] D. Gaiotto, G. W. Moore, A. Neitzke. Four-dimensional wall-crossing via three-dimensional field theory. arXiv:0807.4723.

[12] A. Galligo, M. Granger, P. Maisonobe. $\mathcal{D}$-modules et faisceaux pervers dont le support singulier est un croisement normal. *Ann.Inst. Fourier* **35** (1985) 1-48.

[13] S. Gelfand, R. D. MacPherson, K. Vilonen. Perverse sheaves and quivers. *Duke Math. J.* **33** (1996) 621-643.

[14] A. Hurwitz. Sur un théorème de M. Hadamard. *C. R. Acad. Sci. Paris* **128** (1899) 350-353.

[15] M. Kapranov, V. Schechtman. Perverse schobers. arXiv:1411.2772.

[16] M. Kapranov, V. Schechtman. Perverse sheaves and graphs on surfaces. arXiv:1601.10789.

[17] M. Kapranov, Y. Soibelman, L. Soukhanov. Perverse schobers and the algebra of the infrared. arXiv:2011.00845.

[18] M. Kapranov, Y. Soibelman, Algebra of the Infrared, secondary polytopes and perverse schobers. arXiv:2509.13716.

[19] N. Katz. Rigid Local Systems. Princeton Univ. Press, 1996.

[20] L. Katzarkov, M. Kontsevich, T. Pantev. Hodge-theoretic aspects of mirror symmetry, in: *Proc. Symp. Pure Math.* **78** (2008) "From Hodge theory to integrability and TQFT: tt^*-geometry" (R. Donagi, K. Wendland Eds.) p. 87-174.

[21] M. Kontsevich, Y. Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. arXiv:0811.2435.

[22] M. Kontsevich, Y. Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants. arXiv:1006.2706.

[23] M. Kontsevich, Y. Soibelman. Wall-crossing structures in Donaldson-Thomas invariants, integrable systems and Mirror Symmetry. arXiv:1303.3253.

[24] M. Kontsevich, Y. Soibelman. Analyticity and resurgence in wall-crossing formulas. arXix:2005.10651.

[25] M. Kontsevich, Y. Soibelman. Holomorphic Floer theory I: exponential integrals in finite and infinite dimensions. arXiv:2402.07343.

[26] B. Malgrange. Introduction aux travaux de J. Écalle. *L' Enseignement Math.* **31** (1985) 261-282.

[27] B. Malgrange. Équations Différentielles à Coefficients Polynomiaux. Birhkäuser, Boston, 1991.

[28] C. Mitschi, D. Sauzin. Divergent Series, Summability and Resurgence I. *Lect. Notes in Math.* **2153**, Springer-Verlag, 2016.

[29] S. Schottlaender. Der Hadamardsche Multiplikationssatz und weitere Kompositionssätze der Funktiontheorie. *Math. Nachr.* **11** (1954) 239-294.

M. K.: Kavli IPMU, 5-1-5 Kashiwanoha, Kashiwa, Chiba, 277-8583 Japan. mikhail.kapranov@protonmail.com

Y.S.: Department of Mathematics, Kansas State University, Manhattan KS 66508 USA. soibel@math.ksu.edu