

A generalization of Barannikov-Kontsevich theorem

Takuro Mochizuki*

Abstract

We study the twisted de Rham complex associated with a holomorphic function on a Kähler manifold whose critical point set is compact. We prove the E_1 -degeneration of the Hodge-to-de Rham spectral sequence. It is a generalization of Barannikov-Kontsevich Theorem.

MSC:

Keywords: twisted de Rham complex, integrable mixed twistor D -modules, strictness, E_1 -degeneration

1 Introduction

1.1 Twisted de Rham complexes

Let X be a complex manifold. Let f be a holomorphic function on X such that the set of critical points $\text{Cr}(f)$ is compact.

Let Ω_X^k denote the sheaf of holomorphic k -forms on X . There exists the exterior derivative $d : \Omega_X^k \rightarrow \Omega_X^{k+1}$. The exterior product of df induces morphisms $df : \Omega_X^k \rightarrow \Omega_X^{k+1}$, i.e., $\tau \mapsto df \wedge \tau$.

Let λ be a variable. Let $\Omega_X^k[[\lambda]]$ denote the sheaf of formal power series with Ω_X^k -coefficients. Namely, for any open subset U of X , let $\Omega_X^k[[\lambda]](U)$ denote the space of formal power series $\sum_{j \geq 0} \tau_j \lambda^j$, where $\tau_j \in \Omega_X^k(U)$. We obtain the differential $\lambda d + df : \Omega_X^k[[\lambda]] \rightarrow \Omega_X^{k+1}[[\lambda]]$ determined by $\tau \mapsto \lambda d\tau + df \wedge \tau$. It satisfies $(\lambda d + df)^2 = 0$. The complex $\Omega_X^\bullet[[\lambda]]_f = (\Omega_X^\bullet[[\lambda]], \lambda d + df)$ is called the twisted de Rham complex, or the formal twisted de Rham complex when we emphasize to consider the formal series. We obtain the cohomology group $H^*(X, \Omega_X^\bullet[[\lambda]]_f)$. It is naturally a $\mathbb{C}[[\lambda]]$ -module, where $\mathbb{C}[[\lambda]]$ denotes the ring of formal power series with \mathbb{C} -coefficients.

There are several different versions of twisted de Rham complex. For example, if X and f are algebraic, let $\Omega_{X^{\text{alg}}}^k$ denote the sheaf of algebraic k -forms on the algebraic variety X^{alg} with Zariski topology, and it is also natural to consider the sheaf $\Omega_{X^{\text{alg}}}^k[\lambda]$ of polynomials with $\Omega_{X^{\text{alg}}}^k$ -coefficients on X^{alg} . We obtain the algebraic twisted de Rham complex $\Omega_{X^{\text{alg}}}^\bullet[\lambda]_f = (\Omega_{X^{\text{alg}}}^\bullet[\lambda], \lambda d + df)$ and the cohomology group $H^*(X^{\text{alg}}, \Omega_{X^{\text{alg}}}^\bullet[\lambda]_f)$.

1.2 A basic question

By setting $\mathcal{F}_\lambda^j(\Omega_X^\bullet[[\lambda]]_f) = \lambda^j \Omega_X^\bullet[[\lambda]]_f$ for any non-negative integer j , we obtain the filtered complex $\mathcal{F}_\lambda \Omega_X^\bullet[[\lambda]]_f$. The associated complexes $\text{Gr}_{\mathcal{F}_\lambda}^j(\Omega_X^\bullet[[\lambda]]_f) = \mathcal{F}_\lambda^j / \mathcal{F}_\lambda^{j+1}$ ($j \geq 0$) are isomorphic to the complex (Ω_X^\bullet, df) . Note that the cohomological support of (Ω_X^\bullet, df) is contained in $\text{Cr}(f)$. Because $\text{Cr}(f)$ is compact, $H^*(X, (\Omega_X^\bullet, df))$ is a finite dimensional complex vector space. There is the spectral sequence associated with the filtered complex, for which

$$E_1^{p,q} = H^{p+q}(X, \text{Gr}_{\mathcal{F}_\lambda}^p(\Omega_X^\bullet[[\lambda]]_f)) = H^{p+q}(X, (\Omega_X^\bullet, df)).$$

We study the following question.

Question 1.1 *Is the spectral sequence for $\mathcal{F}_\lambda \Omega_X^\bullet[[\lambda]]_f$ degenerates at the E_1 -level?*

We can rephrase the condition in several ways.

Lemma 1.2 *The spectral sequence degenerates at the E_1 -level if and only if the following equivalent conditions are satisfied.*

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8512, Japan, takuro@kurims.kyoto-u.ac.jp

- For any ℓ and j , the exact sequences $0 \rightarrow \mathrm{Gr}_{\mathcal{F}_\lambda}^j \rightarrow \Omega_X^\bullet[\lambda]_f / \mathcal{F}_\lambda^{j+1} \rightarrow \Omega_X^\bullet[\lambda]_f / \mathcal{F}_\lambda^j \rightarrow 0$ induce the following exact sequences:

$$0 \rightarrow H^\ell(X, \mathrm{Gr}_{\mathcal{F}_\lambda}^j(\Omega_X^\bullet[\lambda]_f)) \rightarrow H^\ell(X, \Omega_X^\bullet[\lambda]_f / \mathcal{F}_\lambda^{j+1}) \rightarrow H^\ell(X, \Omega_X^\bullet[\lambda]_f / \mathcal{F}_\lambda^j) \rightarrow 0.$$

- For any ℓ , $H^\ell(X, \Omega_X^\bullet[\lambda]_f)$ are isomorphic to $H^\ell(X, (\Omega_X^\bullet, df)) \otimes \mathbb{C}[\lambda]$ as $\mathbb{C}[\lambda]$ -modules. In particular, the formal twisted de Rham cohomology groups $H^\ell(X, \Omega_X^\bullet[\lambda]_f)$ are free $\mathbb{C}[\lambda]$ -modules of finite rank.

Remark 1.3 It is easy to see that the second condition implies the first. For the convenience of readers, we shall recall that the E_1 -degeneration is equivalent to the first condition in §3.1.3. We shall later explain that the first condition implies the second condition in a generalized context. (See Corollary 3.10.)

1.2.1 Classical results related to the question

There are two classical results related to this question. One appeared in the classical Hodge theory. As Deligne observed, if X is projective and $f = 0$, the desired E_1 -degeneration follows from the Hodge decomposition of $H^*(X, \mathbb{C})$. Deligne generalized it to more general algebraic varieties by using mixed Hodge theory. This E_1 -degeneration and its generalizations are not only deep results in the Hodge theory, but also useful for various applications including some vanishing theorems. Deligne and Illusie gave an alternative proof of the E_1 -degeneration using the reduction to the positive characteristic.

The other appeared in the singularity theory, in particular, the study of Brieskorn lattices. If f has only one critical point, the E_1 -degeneration holds because the ℓ -th cohomology group of (Ω_X^\bullet, df) is 0 unless ℓ equals $\dim X$. In this case, the E_1 -degeneration is an important starting point of the deep theory of primitive forms of Kyoji Saito. (See [19] and [20] for more backgrounds.)

1.3 Barannikov-Kontsevich Theorems and variations

The modern study of twisted de Rham complexes was opened by the celebrated theorem of Barannikov and Kontsevich.

Theorem 1.4 (Barannikov-Kontsevich) *The E_1 -degeneration for $\mathcal{F}_\lambda \Omega_X^\bullet[\lambda]_f$ holds in the case where X and f are quasi-projective.*

This is a fundamental theorem in the study of the holomorphic Landau-Ginzburg model of the mirror symmetry. For example, it is essential in the proof of smoothness of some moduli spaces associated with Landau-Ginzburg models. (See [6].)

The original proof of Barannikov and Kontsevich was given by a generalization of the method of Deligne and Illusie. Indeed, Barannikov and Kontsevich proved the following theorem for the algebraic version of the twisted de Rham complexes, which implies Theorem 1.4.

Theorem 1.5 (Barannikov-Kontsevich) *Suppose that $f : X \rightarrow \mathbb{C}$ is a projective morphism of algebraic varieties. Then,*

$$\dim H^j(X^{\mathrm{alg}}, (\Omega_{X^{\mathrm{alg}}}^\bullet, df)) = \dim H^j(X^{\mathrm{alg}}, (\Omega_{X^{\mathrm{alg}}}^\bullet, d + df))$$

holds for any j and for any complex number λ . ■

Theorem 1.5 implies that $H^j(X^{\mathrm{alg}}, \Omega_{X^{\mathrm{alg}}}^\bullet[\lambda]_f)$ are free $\mathbb{C}[\lambda]$ -modules, and that the E_1 -degeneration of the spectral sequence for the filtered complex $\mathcal{F}_\lambda \Omega_{X^{\mathrm{alg}}}^\bullet[\lambda]_f$. In the setting of Theorem 1.4, there exists a projective morphism of algebraic varieties $F : Y \rightarrow \mathbb{C}$ with an open embedding $\iota : X \rightarrow Y$ such that $f = F \circ \iota$. Under the assumption that $\mathrm{Cr}(f)$ is compact, the set $\mathrm{Cr}(F)$ of critical points of F is decomposed as $\mathrm{Cr}(F) = \mathrm{Cr}(f) \sqcup (\mathrm{Cr}(F) \cap (Y \setminus X))$. Then, we obtain the E_1 -degeneration for $\mathcal{F}_\lambda \Omega_X^\bullet[\lambda]_f$ from the E_1 -degeneration for $\mathcal{F}_\lambda \Omega_{Y^{\mathrm{alg}}}^\bullet[\lambda]_f$.

The theorem of Barannikov-Kontsevich for the algebraic twisted de Rham complexes (Theorem 1.5) has attracted many mathematicians because of its significance in the non-commutative Hodge theory (see [6]), and because the theorem and its generalization are deeply related with various fields of mathematics. Indeed, alternative proofs for Theorem 1.5 with different methods have been found by Sabbah [15] using Hodge modules and microlocalization, and by Ogus and Vologodsky [14] using their non-abelian Hodge correspondence in positive characteristic. Later, Arinkin, Căldăraru and Hablicsek [1] revisited it in their study of Deligne-Illusie method from the viewpoint of derived algebraic geometry. Sabbah also studied generalizations to the case where f is not necessarily projective but cohomologically tame [17]. See [2] and [12] for a generalization to the Kontsevich complexes.

The L^2 -analogue of the twisted de Rham complex associated with (X, f) has been also studied in [3, 7]. In particular, Li and Wen [7] studied the case where X has a complete Kähler metric with bounded curvature, and f is strongly elliptic, which is a kind of non-degeneracy condition at infinity. They established an analogue of Theorem 1.5 in this context.

1.4 Main result

In this paper, we shall study a generalization of Theorem 1.4. It is an affirmative answer to a question asked by Kontsevich to the author.

Theorem 1.6 *If X is Kähler, the E_1 -degeneration for $\mathcal{F}_\lambda \Omega_X^\bullet[[\lambda]]_f$ holds.*

Note that X can be a small neighbourhood of $\text{Cr}(f)$, and that we do not need any assumption on the behaviour of (X, f) at infinity. It is our purpose to show that a local assumption around $\text{Cr}(f)$ is enough for the E_1 -degeneration for the formal twisted de Rham complex, though global assumptions are useful to obtain stronger consequences as in Theorem 1.5.

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2 Integrable mixed twistor \mathcal{D} -modules

2.1 Mixed twistor \mathcal{D} -modules

The theory of twistor \mathcal{D} -modules has been developed in [16, 18] and [9, 10, 11] as a twistor version of Hodge modules [21, 22] inspired by the principle called Simpson's Meta theorem [23].

2.1.1 \mathcal{R}_X -triples

For any complex manifold X , we set $\mathcal{X} = \mathbb{C}_\lambda \times X$. Let $p_\lambda : \mathcal{X} \rightarrow X$ denote the projection. Let $\mathcal{D}_\mathcal{X}$ denote the sheaf of holomorphic linear differential operators on \mathcal{X} , and let Θ_X denote the tangent sheaf of X . We obtain the sheaf of subalgebras $\mathcal{R}_X \subset \mathcal{D}_\mathcal{X}$ generated by $\lambda \cdot (p_\lambda^* \Theta_X)$ over $\mathcal{O}_\mathcal{X}$. If X is an open subset in \mathbb{C}^n , we have $\mathcal{R}_X = \mathcal{O}_\mathcal{X} \langle \lambda \partial_1, \dots, \lambda \partial_n \rangle$.

We set $\mathcal{S} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Let $\sigma : \mathcal{S} \rightarrow \mathcal{S}$ be defined by $\sigma(\lambda) = -\lambda$. Let $\mathfrak{Db}_{\mathcal{S} \times X/\mathcal{S}}$ denote the sheaf of distributions on $\mathcal{S} \times X$ which are continuous with respect to \mathcal{S} . (See [16, §0.5].) It is naturally an $\mathcal{R}_{X|\mathcal{S} \times X} \otimes_{\mathcal{O}_{\mathcal{S}|\mathcal{S}}} \sigma^{-1}(\mathcal{R}_{X|\mathcal{S} \times X})$ -module by the action $(P_1 \otimes \sigma^{-1}(P_2)) \cdot \tau = P_1 \sigma^{-1}(P_2) \tau$. A sesqui-linear pairing of \mathcal{R}_X -modules \mathcal{M}' and \mathcal{M}'' is a morphism of $\mathcal{R}_{X|\mathcal{S} \times X} \otimes_{\mathcal{O}_{\mathcal{S}|\mathcal{S}}} \sigma^{-1}(\mathcal{R}_{X|\mathcal{S} \times X})$ -modules $\mathcal{M}'_{|\mathcal{S} \times X} \otimes_{\mathcal{O}_{\mathcal{S}|\mathcal{S}}} \sigma^{-1}(\mathcal{M}''_{|\mathcal{S} \times X}) \rightarrow \mathfrak{Db}_{\mathcal{S} \times X/\mathcal{S}}$. Such a tuple $(\mathcal{M}', \mathcal{M}'', C)$ is called an \mathcal{R}_X -triple.

For \mathcal{R}_X -triples $\mathcal{T}_i = (\mathcal{M}'_i, \mathcal{M}''_i, C_i)$ ($i = 1, 2$), a morphism $\mathcal{T}_1 \rightarrow \mathcal{T}_2$ is defined to be a pair of \mathcal{R}_X -homomorphisms $\varphi' : \mathcal{M}'_2 \rightarrow \mathcal{M}'_1$ and $\varphi'' : \mathcal{M}''_1 \rightarrow \mathcal{M}''_2$ such that $C_1 \circ (\varphi' \times \text{id}) = C_2 \circ (\text{id} \times \varphi'')$. The category of \mathcal{R}_X -triples is an abelian category.

For an increasing filtration W of \mathcal{T} in the category of \mathcal{R}_X -triples, we have the increasing filtrations $W(\mathcal{M}')$ and $W(\mathcal{M}'')$ such that $W_j(\mathcal{T}) = (\mathcal{M}'/W_{-j-1}(\mathcal{M}'), W_j(\mathcal{M}''), C_j)$, where C_j denote the induced sesqui-linear pairings.

2.1.2 Direct image of \mathcal{R}_X -triples

Let $F : X \rightarrow Y$ be a morphism of complex manifolds. We set $\omega_X := \lambda^{-\dim X} p_X^* \omega_X$, where ω_X denotes the canonical line bundle of X . Similarly, we set $\omega_Y := \lambda^{-\dim Y} p_Y^* \omega_Y$. We set $\mathcal{R}_{Y \leftarrow X} := \omega_X \otimes_{F^{-1}(\mathcal{O}_Y)} F^{-1}(\mathcal{R}_Y \otimes \omega_Y^{-1})$. For any \mathcal{R}_X -module \mathcal{M} , we obtain the following \mathcal{R}_Y -modules:

$$F_{\dagger}^j(\mathcal{M}) := R^j(\mathrm{id}_{\mathbb{C}} \times F)_! (\mathcal{R}_{Y \leftarrow X} \otimes_{\mathcal{R}_X}^L \mathcal{M}) \quad (j \in \mathbb{Z}).$$

For any \mathcal{R}_X -triple $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$, we obtain the \mathcal{R}_X -triples $F_{\dagger}^j(\mathcal{T}) = (F_{\dagger}^{-j}(\mathcal{M}'), F_{\dagger}^j(\mathcal{M}''), F_{\dagger} C)$ ($j \in \mathbb{Z}$) on Y . (See [16, §1.4].) When \mathcal{T} is equipped with an increasing filtration W , let $W_{j+\ell}(F_{\dagger}^j(\mathcal{T}))$ denote the image of $F_{\dagger}^j(W_{\ell} \mathcal{T}) \rightarrow F_{\dagger}^j(\mathcal{T})$ by which we obtain the filtration W on $F_{\dagger}^j(\mathcal{T})$.

2.1.3 Pure twistor \mathcal{D} -modules

A polarizable pure twistor \mathcal{D} -module of weight w on a complex manifold X is an \mathcal{R}_X -triple satisfying some conditions. (See [18, 10].) Let $\mathrm{MT}(X, w)$ denote the category of polarizable pure twistor \mathcal{D}_X -modules of weight w .

Theorem 2.1 ([16, 9, 10]) *Let $F : X \rightarrow Y$ be a projective morphism of complex manifolds. For any $\mathcal{T} \in \mathrm{MT}(X, w)$, we have $F_{\dagger}^j(\mathcal{T}) \in \mathrm{MT}(Y, w + j)$.* ■

There exists the full subcategory $\mathrm{MT}_{\mathrm{reg}}(X, w) \subset \mathrm{MT}(X, w)$ of regular polarizable pure twistor \mathcal{D}_X -modules of weight w . (See [16, 9] for the regularity condition.) We obtain the following generalization of Theorem 2.1 in the regular case.

Theorem 2.2 ([13]) *Let $F : X \rightarrow Y$ be a morphism of complex manifolds. Let $\mathcal{T} \in \mathrm{MT}_{\mathrm{reg}}(X, w)$. Suppose that X is Kähler, and that the support of \mathcal{T} is proper over Y . Then, we have $F_{\dagger}^j(\mathcal{T}) \in \mathrm{MT}_{\mathrm{reg}}(Y, w + j)$.* ■

2.1.4 Mixed twistor \mathcal{D} -modules

A mixed twistor \mathcal{D} -module on X is a filtered \mathcal{R}_X -triple (\mathcal{T}, W) and satisfying some additional conditions. (See [11, §7].) Let $\mathrm{MTM}(X)$ denote the category of mixed twistor \mathcal{D} -modules on X . The following theorem is fundamental.

Theorem 2.3 ([11]) *Let $F : X \rightarrow Y$ be a projective morphism of complex manifolds. For $(\mathcal{T}, W) \in \mathrm{MTM}(X)$, we have $(F_{\dagger}^j(\mathcal{T}), W) \in \mathrm{MTM}(Y)$ for any j .* ■

For any $(\mathcal{T}, W) \in \mathrm{MTM}(X)$, we have $\mathrm{Gr}_w^W(\mathcal{T}) \in \mathrm{MT}(X, w)$. There exists the full subcategory $\mathrm{MTM}_{\mathrm{reg}}(X) \subset \mathrm{MTM}(X)$ of mixed twistor \mathcal{D}_X -modules (\mathcal{T}, W) such that $\mathrm{Gr}_w^W(\mathcal{T}) \in \mathrm{MT}_{\mathrm{reg}}(X, w)$. Theorem 2.3 is generalized as follows in the regular case.

Theorem 2.4 ([13]) *Let $F : X \rightarrow Y$ be a morphism of complex manifolds. Let $\mathcal{T} \in \mathrm{MTM}_{\mathrm{reg}}(X)$. Suppose that X is Kähler, and that the support of \mathcal{T} is proper over Y . Then, we have $F_{\dagger}^j(\mathcal{T}) \in \mathrm{MTM}_{\mathrm{reg}}(Y)$.* ■

2.1.5 Pure and mixed twistor \mathcal{D} -modules on a point

Let pt denote the set of one point. An $\mathcal{R}_{\mathrm{pt}}$ -module is an $\mathcal{O}_{\mathbb{C}_{\lambda}^*}$ -module. Let $\sigma : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the anti-holomorphic map defined by $\sigma(\lambda) = (-\bar{\lambda})^{-1}$. Let $\sigma : \mathbb{C}_{\lambda}^* \rightarrow \mathbb{C}_{\lambda}^*$ and $\sigma : \mathbb{P}^1 \setminus \{0\} \rightarrow \mathbb{C}_{\lambda}^*$ denote the induced maps. Let $\mathcal{M}', \mathcal{M}''$ be locally free $\mathcal{O}_{\mathbb{C}_{\lambda}^*}$ -modules of finite rank with $\mathrm{rank} \mathcal{M}' = \mathrm{rank} \mathcal{M}''$. Let

$$C : \mathcal{M}'|_{\mathbb{C}_{\lambda}^*} \otimes \sigma^*(\mathcal{M}''|_{\mathbb{C}_{\lambda}^*}) \rightarrow \mathcal{O}_{\mathbb{C}_{\lambda}^*}$$

be an $\mathcal{O}_{\mathbb{C}_{\lambda}^*}$ -homomorphism which is perfect in the sense the induced morphism $\Psi_C : \sigma^*(\mathcal{M}'')|_{\mathbb{C}_{\lambda}^*} \rightarrow (\mathcal{M}')^{\vee}|_{\mathbb{C}_{\lambda}^*}$ is an isomorphism, where $(\mathcal{M}')^{\vee} = \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_{\lambda}^*}}(\mathcal{M}', \mathcal{O}_{\mathbb{C}_{\lambda}^*})$. Such a triple $(\mathcal{M}', \mathcal{M}'', C)$ is called a smooth $\mathcal{R}_{\mathrm{pt}}$ -triple. For any smooth $\mathcal{R}_{\mathrm{pt}}$ -triple $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$, we obtain the locally free $\mathcal{O}_{\mathbb{P}^1}$ -module $\Upsilon(\mathcal{T})$ by gluing $(\mathcal{M}')^{\vee}$ on \mathbb{C}_{λ} and $\sigma^*(\mathcal{M}'')$ on $\mathbb{P}^1 \setminus \{0\}$ with Ψ_C .

A polarizable pure twistor \mathcal{D} -module on pt is a smooth $\mathcal{R}_{\mathrm{pt}}$ -triple \mathcal{T} such that $\Upsilon(\mathcal{T})$ is isomorphic to a direct sum of $\mathcal{O}_{\mathbb{P}^1}(w)$. A mixed twistor \mathcal{D} -module on pt is a smooth $\mathcal{R}_{\mathrm{pt}}$ -triple \mathcal{T} with a weight filtration W such that $\mathrm{Gr}_w^W(\mathcal{T})$ are pure of weight w .

Let $a_X : X \rightarrow \text{pt}$ denote the canonical morphism. We set $\tilde{\Omega}_{X/\mathbb{C}}^k := \lambda^{-k}(p_X^* \Omega_X^k)$. We have the exterior derivative $d : \tilde{\Omega}_{X/\mathbb{C}}^k \rightarrow \tilde{\Omega}_{X/\mathbb{C}}^{k+1}$. For a coherent \mathcal{R}_X -module \mathcal{M} with compact support, we obtain the complex of sheaves $\mathcal{M} \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet$ on \mathcal{X} . We have

$$a_{X\dagger}^\ell(\mathcal{M}) = R^{\dim X + \ell}(\text{id} \times a_X)_*(\mathcal{M} \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet)$$

as the $\mathcal{O}_{\mathbb{C}_\lambda}$ -module.

Corollary 2.5 *Suppose that \mathcal{M} underlies a regular mixed twistor \mathcal{D} -module on X with compact support. We also assume that X is Kähler. Then, $a_{X\dagger}^\ell(\mathcal{M})$ are locally free $\mathcal{O}_{\mathbb{C}_\lambda}$ -modules.* \blacksquare

This is closely related with the E_1 -degeneration property. We consider the subcomplexes $\mathcal{F}_\lambda^j(\mathcal{M} \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet) = \lambda^j \mathcal{M} \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet$. Because $\lambda^j : a_{X\dagger}^\ell(\mathcal{M}) \rightarrow a_{X\dagger}^{\ell+j}(\mathcal{M})$ are monomorphisms, the following is exact for any ℓ and j :

$$0 \longrightarrow R^{\dim X + \ell}(\text{id} \times a_X)_*(\tilde{\Omega}_{X/\mathbb{C}}^\bullet \otimes \mathcal{M}) \longrightarrow R^{\dim X + \ell}(\text{id} \times a_X)_*(\tilde{\Omega}_{X/\mathbb{C}}^\bullet \otimes \mathcal{M}) \longrightarrow R^{\dim X + \ell}(\text{id} \times a_X)_*((\tilde{\Omega}_{X/\mathbb{C}}^\bullet \otimes \mathcal{M})/\mathcal{F}_\lambda^j) \longrightarrow 0. \quad (1)$$

Hence, the following is an epimorphism for any j and ℓ :

$$R^{\dim X + \ell}(\text{id} \times a_X)_*((\tilde{\Omega}_{X/\mathbb{C}}^\bullet \otimes \mathcal{M})/\mathcal{F}_\lambda^{j+1}) \longrightarrow R^{\dim X + \ell}(\text{id} \times a_X)_*((\tilde{\Omega}_{X/\mathbb{C}}^\bullet \otimes \mathcal{M})/\mathcal{F}_\lambda^j). \quad (2)$$

This means the E_1 -degeneration of the spectral sequence associated with the filtration \mathcal{F}_λ on $\tilde{\Omega}_{X/\mathbb{C}}^\bullet \otimes \mathcal{M}$.

2.2 Integrable mixed twistor \mathcal{D} -modules

We set $\tilde{\mathcal{R}}_X = \mathcal{R}_X \langle \lambda^2 \partial_\lambda \rangle \subset \mathcal{D}_X$. If X is an open subset in \mathbb{C}^n , we have $\tilde{\mathcal{R}}_X = \mathcal{O}_X \langle \lambda \partial_1, \dots, \lambda \partial_n, \lambda^2 \partial_\lambda \rangle$. By the identification $\mathbf{S} = \{e^{\sqrt{-1}\theta}\}$, we obtain the vector field ∂_θ on \mathbf{S} .

Let $\mathcal{M}', \mathcal{M}''$ be $\tilde{\mathcal{R}}_X$ -modules. Let $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ be an \mathcal{R}_X -triple. For any section m' of \mathcal{M}' for $U \subset \mathbf{S} \times X$, we set $\partial_\theta m' = \sqrt{-1} \lambda \partial_\lambda m' = (\sqrt{-1} \lambda \partial_\lambda - \sqrt{-1} \lambda \partial_{\bar{\lambda}}) m'$. Similarly, $\partial_\theta m''$ is defined for a section m'' of \mathcal{M}'' . The \mathcal{R} -triple \mathcal{T} is called integrable if

$$\partial_\theta C(m', \overline{\sigma^{-1}(m'')}) = C(\partial_\theta m', \overline{\sigma^{-1}(m'')}) + C(m', \overline{\sigma^{-1}(\partial_\theta m'')}).$$

(See [11, §2.1.5] for integrable \mathcal{R}_X -triples, which originally goes back to [16].) An integrable \mathcal{R}_X -triple is called $\tilde{\mathcal{R}}_X$ -triple. A morphism of $\tilde{\mathcal{R}}_X$ -triples $\mathcal{T}_i = (\mathcal{M}'_i, \mathcal{M}''_i, C)$ ($i = 1, 2$) is defined to be a morphism (φ', φ'') of \mathcal{R}_X -triples such that φ' and φ'' are $\tilde{\mathcal{R}}_X$ -homomorphisms. For a morphism of complex manifolds $F : X \rightarrow Y$ and for an $\tilde{\mathcal{R}}_X$ -triple \mathcal{T} , the \mathcal{R}_Y -triples $F_\dagger^j(\mathcal{T})$ are naturally $\tilde{\mathcal{R}}_Y$ -triples.

An integrable mixed twistor \mathcal{D} -module on X is a filtered $\tilde{\mathcal{R}}_X$ -triple (\mathcal{T}, W) satisfying some conditions. (See [11, §7.2.3].) Let $\text{MTM}_{\text{reg}}^{\text{int}}(X)$ denote the category of integrable mixed twistor \mathcal{D}_X -modules whose underlying mixed twistor \mathcal{D}_X -modules are regular. Let $\mathcal{C}_{\text{reg}}(X)$ denote the full subcategory of $\tilde{\mathcal{R}}_X$ -modules underlying regular integrable mixed twistor \mathcal{D}_X -modules, i.e., an $\tilde{\mathcal{R}}_X$ -module \mathcal{M}'' is an object of $\mathcal{C}_{\text{reg}}(X)$ if and only if there exists $((\mathcal{M}', \mathcal{M}'', C), W) \in \text{MTM}_{\text{reg}}^{\text{int}}(X)$.

2.3 $\tilde{\mathcal{R}}_X$ -modules induced by Hodge modules

Let \mathcal{D}_X denote the sheaf of holomorphic linear differential operators on X . Let $F_j(\mathcal{D}_X)$ denote the subsheaf of differential operators of degree at most j . We set $R^F(\mathcal{D}_X) := \sum_{j \in \mathbb{Z}} \lambda^j F_j(\mathcal{D}_X)$ and $\tilde{R}^F(\mathcal{D}_X) := R^F(\mathcal{D}_X) \langle \lambda^2 \partial_\lambda \rangle$.

Let M be a regular holonomic \mathcal{D}_X -module. Let $F(M)$ be a good filtration of M . We obtain $R^F(\mathcal{D}_X)$ -module $R^F(M) = \sum_{j \in \mathbb{Z}} \lambda^j F_j(M)$. It is naturally an $\tilde{R}^F(\mathcal{D}_X)$ -module. By the analytification, it induces an $\tilde{\mathcal{R}}_X$ -module denoted by $\mathcal{R}^F(M)$. In this way, we obtain a functor from the category of good filtered regular holonomic \mathcal{D}_X -modules to the category of $\tilde{\mathcal{R}}_X$ -modules.

Lemma 2.6 *If (M, F) is a filtered regular holonomic \mathcal{D} -module underlying a mixed Hodge module, we have $\mathcal{R}^F(M) \in \mathcal{C}_{\text{reg}}(X)$. (See [21, 22] for Hodge modules.)*

Proof There exists a natural functor from the category of mixed Hodge modules on X to $\text{MTM}_{\text{reg}}^{\text{int}}(X)$ as explained in [11, §13.5]. In the level of filtered \mathcal{D} -modules, it is given as above. \blacksquare

3 Main Theorem

3.1 Preliminary

Let X be a complex manifold. Let $\mathcal{M} \in \mathcal{C}_{\text{reg}}(X)$ be an $\tilde{\mathcal{R}}_X$ -module underlying an integrable regular mixed twistor \mathcal{D} -module induced by a mixed Hodge module. Let $\text{Ch}(\Xi_{\text{DR}}\mathcal{M}) \subset T^*X$ denote the characteristic variety of the underlying \mathcal{D}_X -module $\Xi_{\text{DR}}(\mathcal{M})$. Let $0_X : X \rightarrow T^*X$ denote the 0-section. We assume the following.

- The set $\text{Cr}(f)$ is compact.
- Any $c \neq 0$ is a regular value of f .
- $\text{Ch}(\Xi_{\text{DR}}(\mathcal{M})) \cap df(X) \subset 0_X(\text{Cr}(f))$.

The second condition implies $\text{Cr}(f) \subset f^{-1}(0)$. Because the characteristic varieties are cone, the third condition implies $\text{Ch}(\Xi_{\text{DR}}(\mathcal{M})) \cap (\alpha df)(X) \subset 0_X(\text{Cr}(f))$ for any non-zero constant α .

3.1.1 Cohomology group of the restriction to $\lambda = 0$

Let $\mathcal{L}(f)$ denote the $\tilde{\mathcal{R}}_X$ -module given by \mathcal{O}_X with the meromorphic integrable connection $d + d(\lambda^{-1}f)$. We obtain the $\tilde{\mathcal{R}}_X$ -module $\mathcal{M}_f = \mathcal{M} \otimes \mathcal{L}(f)$ on X , and the complex of sheaves $\mathcal{M}_f \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet$ on X . We recall that $\mathcal{M}_f \in \mathcal{C}_{\text{reg}}(X)$. Let $\iota_\lambda : \{0\} \times X \rightarrow \mathbb{C}_\lambda \times X$ denote the inclusion. We obtain the complex of coherent $\text{Sym } \Theta_X$ -modules $\iota_\lambda^*(\mathcal{M}_f \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet)$. It induces a complex of coherent \mathcal{O}_{T^*X} -modules denoted by $(\iota_\lambda^*(\mathcal{M}_f \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet))^\sim$.

Lemma 3.1 *The cohomological support of $(\iota_\lambda^*(\mathcal{M}_f \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet))^\sim$ is contained in $0_X(\text{Cr}(f))$. As a result,*

$$H^*(X, \iota_\lambda^*(\mathcal{M}_f \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet))$$

are finite dimensional.

Proof Let $(\iota_\lambda^*\mathcal{M})^\sim$ and $(\iota_\lambda^*\mathcal{L}(f))^\sim$ denote the coherent \mathcal{O}_{T^*X} -modules induced by $\iota_\lambda^*\mathcal{M}$ and $\iota_\lambda^*\mathcal{L}(f)$, respectively. The support of $(\iota_\lambda^*\mathcal{M})^\sim$ is the characteristic variety $\text{Ch}(\Xi_{\text{DR}}\mathcal{M})$. The support of $(\iota_\lambda^*\mathcal{L}(f))^\sim$ is the image of $df(X)$. The support of $(\iota_\lambda^*\mathcal{M}_f)^\sim$ is $df(X) + \text{Ch}(\Xi_{\text{DR}}(\mathcal{M}))$ in T^*X .

Let ω_X denote the canonical bundle of X . Let $\text{Sym } \Theta_X \otimes \bigwedge^\bullet \Theta_X \otimes \omega_X$ be the Koszul resolution of \mathcal{O}_X by $\text{Sym } \Theta_X$ -free modules. It induces an \mathcal{O}_{T^*X} -free resolution $\mathcal{O}_{T^*X} \otimes \pi^*(\bigwedge^\bullet \Theta_X \otimes \omega_X)$ of $0_{X*}(\mathcal{O}_X)$, where $\pi : T^*X \rightarrow X$ denotes the projection.

We have

$$(\iota_\lambda^*(\mathcal{M}_f \otimes \tilde{\Omega}_{X/\mathbb{C}}^\bullet))^\sim \simeq (\iota_\lambda^*\mathcal{M}_f)^\sim \otimes_{\mathcal{O}_{T^*X}} \left(\mathcal{O}_{T^*X} \otimes \pi^* \left(\bigwedge^\bullet \Theta_X \otimes \omega_X \right) \right).$$

Hence, the cohomological support is contained in the intersection of the support of $(\iota_\lambda^*\mathcal{M}_f)^\sim$ and the $0_X(X)$, which is contained in $0_X(\text{Cr}(f))$. \blacksquare

3.1.2 Cohomology group of the vanishing cycle sheaf

Let $\iota_f : X \rightarrow X \times \mathbb{C}_t$ be the graph embedding, i.e., $\iota_f(x) = (x, f(x))$. There exists the V -filtration $V(\iota_{f\uparrow}(\Xi_{\text{DR}}(\mathcal{M})))$ along t . We obtain the regular holonomic \mathcal{D}_X -module

$$\phi_f(\Xi_{\text{DR}}(\mathcal{M})) := \bigoplus_{-1 < a \leq 0} \text{Gr}_a^V(\iota_{f\uparrow}\Xi_{\text{DR}}(\mathcal{M})).$$

Lemma 3.2 *The support of $\phi_f(\Xi_{\text{DR}}(\mathcal{M}))$ is contained in the compact subset $\text{Cr}(f)$. As a result, the cohomology group $H^*(X, \phi_f(\Xi_{\text{DR}}\mathcal{M}) \otimes \Omega_X^\bullet)$ is finite dimensional.*

Proof The third condition implies that $\Xi_{\text{DR}}(\mathcal{M})$ is non-characteristic to the hypersurfaces $f^{-1}(c)$ on $X \setminus \text{Cr}(f)$. Hence, $\phi_f(\Xi_{\text{DR}}(\mathcal{M})) = 0$ on $X \setminus \text{Cr}(f)$. \blacksquare

3.1.3 E_1 -degeneration and long exact sequences

We recall that the E_1 -degeneration condition of a spectral sequence associated with a filtered complex for the convenience of readers. Let \mathcal{A} be an abelian category. Let (K^\bullet, d) be a complex with a decreasing filtration $F^\bullet(K^\bullet)$. To simplify the notation, we set $F^{p,p+r}(K^j) := F^p(K^j)/F^{p+r}(K^j)$.

Lemma 3.3 *Let $r_0 \in \mathbb{Z}_{\geq 1}$. The following conditions are equivalent.*

$A(r_0)$: $H^j(F^{p,p+r}(K^\bullet)) \rightarrow H^j(F^{p,p+1}(K^\bullet))$ are epimorphisms for any $j, p \in \mathbb{Z}$ and $0 \leq r \leq r_0$.

$B(r_0)$: $H^j(F^{p,p+r}(K^\bullet)) \rightarrow H^j(F^{p,p+r-1}(K^\bullet))$ are epimorphisms for any $j, p \in \mathbb{Z}$ and $0 \leq r \leq r_0$.

Proof It is easy to see that $B(r_0)$ implies $A(r_0)$. Suppose that $A(r_0)$ holds. We shall prove $H^j(F^{p,p+r}(K^\bullet)) \rightarrow H^j(F^{p,p+r-1}(K^\bullet))$ are epimorphisms for any $j, p \in \mathbb{Z}$ and $0 \leq r \leq r_0$, by an induction on r . We have the following commutative diagram:

$$\begin{array}{ccccccc} H^j(F^{p,p+r-1}) & \longrightarrow & H^j(F^{p,p+1}) & & & & \\ a \downarrow & & b \downarrow & & & & \\ 0 & \longrightarrow & H^{j+1}(F^{p+r-1,p+r}) & \xrightarrow{c} & H^{j+1}(F^{p+1,p+r}) & \longrightarrow & H^{j+1}(F^{p+1,p+r-1}) \longrightarrow 0. \end{array}$$

By $A(r_0)$, we have $b = 0$. By the assumption of the induction on r , c is a monomorphism. Hence, we obtain $a = 0$, and the induction can proceed. \blacksquare

Recall that the spectral sequence $E_r^{p,q}$ for the filtered complex $F^\bullet(K^\bullet)$ is given as

$$E_r^{p,q} = Z_r^{p,q} / (Z_{r-1}^{p+1} + dZ_{r-1}^{p-r+1,q+r-2})$$

by setting $Z_r^{p,q} = \text{Ker}(d : F^p K^{p+q} \rightarrow F^{p,p+r}(K^{p+q+1}))$. (See [4] for more details.) There exist the natural morphisms $d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$ such that $d_r \circ d_r = 0$, induced by $d : F^p K^{p+q} \rightarrow F^p K^{p+q+1}$. There exist natural isomorphisms

$$E_{r+1}^{p,q} \simeq \text{Ker}(E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}) / \text{Im}(E_r^{p-r,q+r-1} \rightarrow E_r^{p,q}).$$

Lemma 3.4 *We have $d_r = 0$ for any $1 \leq r \leq r_0 - 1$ if and only if $A(r_0)$ holds.*

Proof We shall use an induction on r_0 . Suppose that $d_r = 0$ ($1 \leq r \leq r_0 - 2$) and that $A(r_0 - 1)$ holds. We have $E_r^{p,q} = E_1^{p,q} = H^{p+q}(F^{p,p+1}(K^\bullet))$ for $1 \leq r \leq r_0 - 1$. We consider

$$\begin{array}{ccccccc} & & H^j(F^{p,p+1}) & & & & \\ & & a \downarrow & & & & \\ 0 & \longrightarrow & H^{j+1}(F^{p+r_0-1,p+r_0}) & \xrightarrow{c} & H^{j+1}(F^{p+1,p+r_0}) & \xrightarrow{b} & H^{j+1}(F^{p+1,p+r_0-1}) \longrightarrow 0. \end{array}$$

By $A(r_0 - 1)$, c is a monomorphism, and we have $b \circ a = 0$. There exists a unique morphism $\varphi : H^j(F^{p,p+1}) \rightarrow H^{j+1}(F^{p+r_0-1,p+r_0})$ such that $c \circ \varphi = b$. By the construction, $\varphi = d_{r_0-1}$. Hence, $A(r_0)$ holds if and only if $d_{r_0-1} = 0$. \blacksquare

We say that the spectral sequence degenerates at the E_1 -level if $d_r = 0$ for any $r \geq 1$. We obtain the following proposition.

Proposition 3.5 *The E_1 -degeneration holds if and only if one of the following equivalent conditions holds.*

- $H^j(F^{p,p+r}(K^\bullet)) \rightarrow H^j(F^{p,p+r-1}(K^\bullet))$ are epimorphisms for any $j, p \in \mathbb{Z}$ and $1 \leq r$. \blacksquare

3.2 Refinement of the Theorem 1.6

3.2.1 Main Theorem

Let us explain a refined statement of Theorem 1.6. We define the filtration $\mathcal{F}_\lambda^k(\mathcal{M} \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet) = \lambda^k(\mathcal{M} \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)$ for $k \in \mathbb{Z}_{\geq 0}$. Note that $\text{Gr}_{\mathcal{F}_\lambda}^k(\mathcal{M} \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)$ is isomorphic to $\iota_\lambda^*(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)$ for any $k \geq 0$. Let us state the main theorem of this paper.

Theorem 3.6 *The E_1 -degeneration holds for the filtered complex $\mathcal{F}_\lambda(\mathcal{M} \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)$ with respect to the push-forward by the projection $\mathcal{X} \rightarrow \mathbb{C}_\lambda$. In other words, we obtain the following exact sequences for any ℓ and j :*

$$0 \rightarrow H^j\left(X, \mathrm{Gr}_\ell^{\mathcal{F}_\lambda}(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)\right) \rightarrow H^j\left(X, (\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet) / \mathcal{F}_\lambda^\ell(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)\right) \rightarrow H^j\left(X, (\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet) / \mathcal{F}_\lambda^{\ell+1}(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)\right) \rightarrow 0. \quad (3)$$

We obtain Theorem 1.6 from Theorem 3.6 as the special case $\mathcal{M} = \mathcal{O}_\mathcal{X}$. We shall also prove the following.

Theorem 3.7 *We have $\dim H^j(X, \iota_\lambda^*(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)) = \dim H^j(X, \phi_f(\Xi_{\mathrm{DR}}\mathcal{M}) \otimes \Omega_X^\bullet)$ for any j .*

Corollary 3.8 *In particular, we have $\dim H^j(X, (\Omega_X^\bullet, df)) = \dim H^j(X, \Omega_X^\bullet \otimes \phi_f(\mathcal{O}_\mathcal{X}))$.*

Corollary 3.9 *There exists an isomorphism*

$$\varprojlim_\ell H^j\left(X, (\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet) / \mathcal{F}_\lambda^\ell(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)\right) \simeq H^j\left(X, \iota_\lambda^*(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)\right) \otimes \mathbb{C}[[\lambda]]. \quad (4)$$

3.2.2 Completion

We naturally regard $(\mathcal{M}_f / \mathcal{F}_\lambda^j \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet$ as the complexes of sheaves on X . We obtain the following complex of sheaves on X :

$$\widehat{\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet} = \varprojlim_j (\mathcal{M}_f / \mathcal{F}_\lambda^j \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet.$$

Corollary 3.10 *There exists an isomorphism*

$$H^j\left(X, \widehat{\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet}\right) \simeq H^j\left(X, \iota_\lambda^*(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)\right) \otimes \mathbb{C}[[\lambda]].$$

In particular, $H^j\left(X, \widehat{\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet}\right)$ are free $\mathbb{C}[[\lambda]]$ -modules of finite rank.

Proof Let $\Omega_X^{0,q}$ denote the sheaf of smooth $(0, q)$ -forms on X . We obtain the following double complex

$$(\widehat{\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet}) \hat{\otimes}_{\mathcal{O}_X} \Omega_X^{0,\bullet} := \varprojlim_j \left((\mathcal{M}_f / \mathcal{F}_\lambda^j \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \otimes_{\mathcal{O}_X} \Omega_X^{0,\bullet} \right).$$

Let $\mathrm{Tot}\left((\widehat{\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet}) \hat{\otimes}_{\mathcal{O}_X} \Omega_X^{0,\bullet}\right)$ denote the total complex.

Lemma 3.11 *The natural morphism is a fine resolution:*

$$\widehat{\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet} \longrightarrow \mathrm{Tot}\left((\widehat{\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet}) \hat{\otimes}_{\mathcal{O}_X} \Omega_X^{0,\bullet}\right). \quad (5)$$

Proof It is enough to prove that (5) is a quasi-isomorphism locally around any point of X . Recall that the sheaf of C^∞ -functions on X is flat over \mathcal{O}_X according to [8].

Let G be any pseudo-coherent \mathcal{O}_X -module. (See [5, Appendix A] for pseudo-coherent sheaves.) We have $G \otimes_{\mathcal{O}_X}^L \Omega_X^{0,q} \simeq G \otimes_{\mathcal{O}_X} \Omega_X^{0,q}$. Let $F_\bullet \rightarrow G$ be a free resolution of G . The natural morphisms $F_\bullet \rightarrow \mathrm{Tot}(F_\bullet \otimes_{\mathcal{O}_X} \Omega_X^{0,\bullet}) \rightarrow G \otimes_{\mathcal{O}_X} \Omega_X^{0,\bullet}$ are quasi-isomorphisms. Hence, $G \rightarrow G \otimes_{\mathcal{O}_X} \Omega_X^{0,\bullet}$ is a quasi-isomorphism.

Let \mathcal{G} be a pseudo-coherent $\mathcal{O}_\mathcal{X}$ -module flat over \mathbb{C}_λ . We naturally regard $\mathcal{G}/\lambda^j \mathcal{G}$ as \mathcal{O}_X -modules. Let $\pi_{j+1} : \mathcal{G}/\lambda^{j+1} \mathcal{G} \rightarrow \mathcal{G}/\lambda^j \mathcal{G}$ denote the projections. We obtain the quasi-isomorphisms

$$\mathcal{G}/\lambda^j \mathcal{G} \rightarrow (\mathcal{G}/\lambda^j \mathcal{G}) \otimes_{\mathcal{O}_X} \Omega_X^{0,\bullet}.$$

For any open subset $U \subset X$, the morphisms $H^0(U, (\mathcal{G}/\lambda^{j+1} \mathcal{G}) \otimes \Omega_X^{0,q}) \rightarrow H^0(U, (\mathcal{G}/\lambda^j \mathcal{G}) \otimes \Omega_X^{0,q})$ are surjective because $\iota_\lambda^*(\mathcal{G}) \otimes \Omega_X^{0,q}$ is fine. Let $a_j^{q+1} \in H^0(U, (\mathcal{G}/\lambda^j \mathcal{G}) \otimes \Omega_X^{0,q+1})$ ($j = 1, 2, \dots$) be sections such that $\bar{\partial}(a_j^{q+1}) = 0$ and $\pi_{j+1}(a_{j+1}^{q+1}) = a_j^{q+1}$. Let us construct $b_j^q \in H^0(U, (\mathcal{G}/\lambda^j \mathcal{G}) \otimes \Omega_X^{0,q})$ ($j = 1, 2, \dots$) such that $\bar{\partial} b_j^q = a_j^q$ and $\pi_{j+1}(b_{j+1}^q) = b_j^q$.

inductively on j . Suppose that we have already constructed b_j^q . There exists $c_{j+1}^q \in H^0(U, (\mathcal{G}/\lambda^{j+1}\mathcal{G}) \otimes \Omega_X^{0,q})$ such that $\pi_{j+1}(c_{j+1}^q) = b_j^q$. We obtain

$$d_{j+1}^{q+1} = a_{j+1}^{q+1} - \bar{\partial}c_{j+1}^q \in H^0(U, (\lambda^j\mathcal{G}/\lambda^{j+1}\mathcal{G}) \otimes \Omega^{0,q+1}) \simeq H^0(U, \ell_\lambda^*\mathcal{G} \otimes \Omega^{0,q+1})$$

such that $\bar{\partial}(d_{j+1}^{q+1}) = 0$. There exists $e_{j+1}^q \in H^0(U, (\lambda^j\mathcal{G}/\lambda^{j+1}\mathcal{G}) \otimes \Omega_X^{0,q})$ such that $\bar{\partial}e_{j+1}^q = d_{j+1}^{q+1}$. By setting $b_{j+1}^q = c_{j+1}^q + e_{j+1}^q$, the induction can proceed.

As a result, the natural morphism

$$\varprojlim_j (\mathcal{G}/\lambda^j\mathcal{G}) \rightarrow \varprojlim_j ((\mathcal{G}/\lambda^j\mathcal{G}) \otimes \Omega_X^{0,\bullet})$$

is a quasi-isomorphism. Then, we obtain the claim of Lemma 3.11. ■

We obtain Corollary 3.10 from the following lemma.

Lemma 3.12 *The natural morphisms*

$$H^j\left(X, \widehat{\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet}\right) \rightarrow \varprojlim_\ell H^j\left(X, (\mathcal{M}_f/\mathcal{F}_\lambda^\ell \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet\right)$$

are isomorphisms.

Proof Let $\pi_{\ell+1}$ denote the projection induced by $\mathcal{M}_f/\mathcal{F}_\lambda^{\ell+1}\mathcal{M}_f \rightarrow \mathcal{M}_f/\mathcal{F}_\lambda^\ell \mathcal{M}_f$. Let d denote the differential of the complex $\text{Tot}^q\left((\mathcal{M}_f/\mathcal{F}_\lambda^\ell \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \otimes \Omega_X^{0,\bullet}\right)$.

Let us consider cohomology classes

$$\alpha_\ell \in H^q\left(X, (\mathcal{M}_f/\mathcal{F}_\lambda^\ell \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet\right) \quad (\ell = 1, 2, \dots)$$

such that $\pi_{\ell+1}(\alpha_{\ell+1}) = \alpha_\ell$. We shall construct cocycles

$$a_\ell \in H^0\left(X, \text{Tot}^q\left((\mathcal{M}_f/\mathcal{F}_\lambda^\ell \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \otimes \Omega_X^{0,\bullet}\right)\right) \quad (\ell = 1, 2, \dots)$$

such that a_ℓ are representatives of α_ℓ and that $\pi_{\ell+1}(a_{\ell+1}) = a_\ell$ inductively on ℓ . Suppose that we have already constructed a_ℓ . There exists $a'_{\ell+1} \in H^0\left(X, \text{Tot}^q\left((\mathcal{M}_f/\mathcal{F}_\lambda^{\ell+1} \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \otimes \Omega_X^{0,\bullet}\right)\right)$ such that $\pi_{\ell+1}(a'_{\ell+1}) = a_\ell$. We obtain a cocycle

$$d(a'_{\ell+1}) \in H^0\left(X, \text{Tot}^{q+1}\left((\mathcal{F}_\lambda^\ell \mathcal{M}_f/\mathcal{F}_\lambda^{\ell+1} \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \otimes \Omega_X^{0,\bullet}\right)\right).$$

Because $\pi_{\ell+1}(\alpha_{\ell+1}) = \alpha_\ell$, there exists

$$b_{\ell+1} \in H^0\left(X, \text{Tot}^q\left((\mathcal{F}_\lambda^\ell \mathcal{M}_f/\mathcal{F}_\lambda^{\ell+1} \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \otimes \Omega_X^{0,\bullet}\right)\right)$$

such that $d(b_{\ell+1}) = d(a'_{\ell+1})$. Let $\alpha'_{\ell+1}$ denote the cohomology class of $a'_{\ell+1} - b_{\ell+1}$. Because $\alpha_{\ell+1} - \alpha'_{\ell+1}$ comes from $H^q\left(X, \text{Tot}\left((\mathcal{F}_\lambda^\ell \mathcal{M}_f/\mathcal{F}_\lambda^{\ell+1} \mathcal{M}_f) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \otimes \Omega_X^{0,\bullet}\right)\right)$, there exists $a_{\ell+1}$ with the desired property.

Let us consider coboundaries

$$a_\ell \in H^0\left(X, \text{Tot}^q\left((\mathcal{M}_f/\mathcal{F}_\lambda^\ell \mathcal{M}_f) \otimes \tilde{\Omega}_X^\bullet \otimes \Omega_X^{0,\bullet}\right)\right) \quad (\ell = 1, 2, \dots)$$

such that $\pi_{\ell+1}(a_{\ell+1}) = a_\ell$. Let us construct

$$b_\ell \in H^0\left(X, \text{Tot}^{q-1}\left((\mathcal{M}_f/\mathcal{F}_\lambda^\ell \mathcal{M}_f) \otimes \tilde{\Omega}_X^\bullet \otimes \Omega_X^{0,\bullet}\right)\right) \quad (\ell = 1, 2, \dots)$$

such that $d(b_\ell) = a_\ell$ and $\pi_{\ell+1}(b_{\ell+1}) = b_\ell$ inductively on ℓ . Suppose that we have already constructed b_ℓ . There exists $b'_{\ell+1} \in H^0\left(X, \text{Tot}^{q-1}\left((\mathcal{M}_f/\mathcal{F}_\lambda^{\ell+1} \mathcal{M}_f) \otimes \tilde{\Omega}_X^\bullet \otimes \Omega_X^{0,\bullet}\right)\right)$ such that $\pi_{\ell+1}(b'_{\ell+1}) = b_\ell$. We obtain a cocycle

$$a_{\ell+1} - d(b'_{\ell+1}) \in H^0\left(X, \text{Tot}^q\left((\mathcal{F}_\lambda^\ell \mathcal{M}_f/\mathcal{F}_\lambda^{\ell+1} \mathcal{M}_f) \otimes \tilde{\Omega}_X^\bullet \otimes \Omega_X^{0,\bullet}\right)\right).$$

By Theorem 3.6, the natural morphism

$$H^q(X, \mathrm{Gr}_{\mathcal{F}_\lambda}^\ell \mathcal{M} \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet) \longrightarrow H^q(X, (\mathcal{M}/\mathcal{F}_\lambda^{\ell+1} \mathcal{M}) \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)$$

is injective. Hence, there exists

$$c_{\ell+1} \in H^0\left(X, \mathrm{Tot}^{q-1}\left((\mathcal{F}_\lambda^\ell \mathcal{M}_f / \mathcal{F}_\lambda^{\ell+1} \mathcal{M}_f) \otimes \tilde{\Omega}_X^\bullet \otimes \Omega_X^{0,\bullet}\right)\right)$$

such that $d(c_{\ell+1}) = a_{\ell+1} - d(b'_{\ell+1})$. By setting $b_{\ell+1} = b'_{\ell+1} + c_{\ell+1}$, the induction can proceed. ■

3.2.3 Reformulation

Let $\iota_f : X \rightarrow X \times \mathbb{C}_t$ denote the graph embedding. We obtain the $\tilde{\mathcal{R}}_{X \times \mathbb{C}_t}$ -module $\tilde{\mathcal{M}} = \iota_{f*} \mathcal{M}$ on $X \times \mathbb{C}_t$. The following lemma is standard.

Lemma 3.13 *There exists a natural isomorphism $\tilde{\mathcal{M}} \otimes \mathcal{L}(t) \simeq \iota_{f*}(\mathcal{M}_f)$. As a result, there exists a natural quasi-isomorphism:*

$$\tilde{\mathcal{M}} \otimes \mathcal{L}(t) \otimes \tilde{\Omega}_{\mathbb{C}_\lambda \times X \times \mathbb{C}_t / \mathbb{C}_\lambda}^\bullet \simeq \iota_{f*}(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet)[-1].$$

It induces the quasi-isomorphisms of the subcomplexes

$$\mathcal{F}_\lambda^k(\tilde{\mathcal{M}} \otimes \mathcal{L}(t) \otimes \tilde{\Omega}_{\mathbb{C}_\lambda \times X \times \mathbb{C}_t / \mathbb{C}_\lambda}^\bullet) \simeq \iota_{f*}(\mathcal{F}_\lambda^k(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet))[-1]$$

and the quotient complexes:

$$\mathrm{Gr}_{\mathcal{F}_\lambda}^k(\tilde{\mathcal{M}} \otimes \mathcal{L}(t) \otimes \tilde{\Omega}_{\mathbb{C}_\lambda \times X \times \mathbb{C}_t / \mathbb{C}_\lambda}^\bullet) \simeq \iota_{f*}(\mathrm{Gr}_{\mathcal{F}_\lambda}^k(\mathcal{M}_f \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet))[-1].$$
■

Let $\pi_{01} : \mathbb{C}_\lambda \times X \times \mathbb{C}_t \rightarrow \mathbb{C}_\lambda \times X$ denote the projection. Let $V\mathcal{R}_{X \times \mathbb{C}_t} \subset \mathcal{R}_{X \times \mathbb{C}_t}$ denote the sheaf of subalgebras generated by $\pi_{01}^* \mathcal{R}_X$ and $t\partial_t$. Because $\tilde{\mathcal{M}} \in \mathcal{C}_{\mathrm{reg}}(X \times \mathbb{C}_t)$, $\tilde{\mathcal{M}}$ has a V -filtration along t , that is an increasing and exhaustive filtration $V_a(\tilde{\mathcal{M}})$ ($a \in \mathbb{R}$) of $\tilde{\mathcal{M}}$ by coherent $V\mathcal{R}_{X \times \mathbb{C}_t}$ -submodules satisfying the following conditions.

- For any $a \in \mathbb{R}$, there exists $\epsilon > 0$ such that $V_a(\tilde{\mathcal{M}}) = V_{a+\epsilon}(\tilde{\mathcal{M}})$.
- $\mathrm{Gr}_a^V = V_a / V_{<a}$ are flat over $\mathcal{O}_{\mathbb{C}_\lambda}$.
- $tV_a \subset V_{a-1}$, and $tV_a = V_{a-1}$ if $a < 0$.
- $\partial_t V_a \subset V_{a+1}$, and the induced morphisms $\partial_t : \mathrm{Gr}_a^V \rightarrow \mathrm{Gr}_{a+1}^V$ are isomorphisms if $a > -1$.
- The induced actions of $-\partial_t t - \lambda a$ on $\mathrm{Gr}_a^V(\tilde{\mathcal{M}})$ are locally nilpotent.

Because $\tilde{\mathcal{M}}$ is induced by a mixed Hodge module, $V_a(\tilde{\mathcal{M}})$ are coherent over $\pi_{01}^* \mathcal{R}_X$.

We obtain the following complex:

$$V_{-1}(\tilde{\mathcal{M}}) \longrightarrow \lambda^{-1} V_0(\tilde{\mathcal{M}}) \otimes dt, \quad s \longmapsto (\partial_t + \lambda^{-1})s dt.$$

The first term sits in the degree 0. It extends to the following double complex:

$$V_{-1}(\tilde{\mathcal{M}}) \otimes \pi_{01}^* \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \longrightarrow \lambda^{-1} V_0(\tilde{\mathcal{M}}) \otimes dt \otimes \pi_{01}^* \tilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet. \quad (6)$$

Let $\mathfrak{C}_1(\mathcal{M})$ denote the complex of sheaves on $\mathbb{C}_\lambda \times X \times \mathbb{C}_t$ obtained as the total complex of (6). We have the subcomplexes $\mathcal{F}_\lambda^j \mathfrak{C}_1(\mathcal{M}) = \lambda^j \mathfrak{C}_1(\mathcal{M})$. We may naturally regard $\mathfrak{C}_1(\mathcal{M})$ as a subcomplex of $\tilde{\mathcal{M}} \otimes \mathcal{L}(t) \otimes \tilde{\Omega}_{\mathbb{C}_\lambda \times X \times \mathbb{C}_t / \mathbb{C}_\lambda}^\bullet$.

Lemma 3.14 *The inclusion $\mathfrak{C}_1(\mathcal{M}) \rightarrow \tilde{\mathcal{M}} \otimes \mathcal{L}(t) \otimes \tilde{\Omega}_{\mathbb{C}_\lambda \times X \times \mathbb{C}_t / \mathbb{C}_\lambda}^\bullet$ is a quasi-isomorphism. It induces quasi-isomorphisms of subcomplexes*

$$\mathcal{F}_\lambda^k \mathfrak{C}_1(\mathcal{M}) \longrightarrow \mathcal{F}_\lambda^k(\tilde{\mathcal{M}} \otimes \mathcal{L}(t) \otimes \tilde{\Omega}_{\mathbb{C}_\lambda \times X \times \mathbb{C}_t / \mathbb{C}_\lambda}^\bullet)$$

and the quotient complexes:

$$\mathrm{Gr}_{\mathcal{F}_\lambda}^k \mathfrak{C}_1(\mathcal{M}) \longrightarrow \mathrm{Gr}_{\mathcal{F}_\lambda}^k(\tilde{\mathcal{M}} \otimes \mathcal{L}(t) \otimes \tilde{\Omega}_{\mathbb{C}_\lambda \times X \times \mathbb{C}_t / \mathbb{C}_\lambda}^\bullet).$$

Proof For any $a > -1$, the induced morphisms

$$\mathrm{Gr}_a^V(\widetilde{\mathcal{M}}) \rightarrow \lambda^{-1} \mathrm{Gr}_{a+1}^V(\widetilde{\mathcal{M}}) \otimes dt \quad s \mapsto \partial_t(s) \otimes dt$$

are isomorphisms. We obtain that the quotient of $\mathfrak{C}_1(\mathcal{M}) \rightarrow \widetilde{\mathcal{M}} \otimes \mathcal{L}(t) \otimes \widetilde{\Omega}_{\mathbb{C}_\lambda \times X \times \mathbb{C}_t/\mathbb{C}_\lambda}^\bullet$ is acyclic, that is the first claim. The second claim is equivalent to the first claim. The third claim follows from the second claim. \blacksquare

By Lemma 3.14, Theorem 3.6 is reduced to the following theorem.

Theorem 3.15 *The E_1 -degeneration property holds for the filtered complex $\mathcal{F}_\lambda^\bullet \mathfrak{C}_1(\mathcal{M})$ with respect to the push-forward by the projection $\mathbb{C}_\lambda \times X \times \mathbb{C}_t \rightarrow \mathbb{C}_\lambda$.*

3.2.4 Refinement

Let $\pi_{012} : \mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \mathbb{C}_u \rightarrow \mathbb{C}_\lambda \times X \times \mathbb{C}_t$ and $\pi_{01} : \mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \mathbb{C}_u \rightarrow \mathbb{C}_\lambda \times X$ denote the projections. We obtain the following complex on $\mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \mathbb{C}_u$:

$$\pi_{012}^* V_{-1}(\widetilde{\mathcal{M}}) \longrightarrow \lambda^{-1} \pi_{012}^* V_0(\widetilde{\mathcal{M}}) \otimes dt, \quad s \mapsto (u\partial_t + \lambda^{-1})s dt. \quad (7)$$

The first term sits in the degree 0. We extend it to the following double complex:

$$\pi_{012}^* V_{-1}(\widetilde{\mathcal{M}}) \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \longrightarrow \lambda^{-1} \pi_{012}^* V_0(\widetilde{\mathcal{M}}) \otimes dt \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet.$$

Let $\mathfrak{C}(\mathcal{M})$ denote the associated complex on $\mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \mathbb{C}_u$. We have the subcomplexes $\mathcal{F}_\lambda^j \mathfrak{C}(\mathcal{M}) = \lambda^j \mathfrak{C}(\mathcal{M})$.

We consider the \mathbb{C}^* -action on $\mathbb{C}_\lambda \times \mathbb{C}_u$ given by $a(\lambda, u) = (a^{-1}\lambda, au)$. It induces a \mathbb{C}^* -action on $\mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \mathbb{C}_u$. Because \mathcal{M} is induced by a Hodge module, $\pi_{012}^* V_{-1}(\widetilde{\mathcal{M}})$ and $\lambda^{-1} \pi_{012}^* V_0(\widetilde{\mathcal{M}})$ are naturally \mathbb{C}^* -equivariant. The differential $u\partial_t + \lambda^{-1}$ is \mathbb{C}^* -equivariant.

Let $\pi_{03} : \mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \mathbb{C}_u \rightarrow \mathbb{C}_\lambda \times \mathbb{C}_u$ denote the projection. The direct image sheaves $R^\ell \pi_{03*} \mathfrak{C}(\mathcal{M})$ are \mathbb{C}^* -equivariant.

Theorem 3.16 *The E_1 -degeneration holds for the filtered complex $\mathcal{F}_\lambda^\bullet \mathfrak{C}(\mathcal{M})$ with respect to the push-forward by π_{03} , and $R^\ell \pi_{03*} \mathrm{Gr}_{\mathcal{F}_\lambda}^j \mathfrak{C}(\mathcal{M})$ are locally free \mathbb{C}^* -equivariant $\mathcal{O}_{\{0\} \times \mathbb{C}_u}$ -modules of finite rank.*

We obtain Theorem 3.15 from Theorem 3.16 by specializing along $u = 1$.

3.3 Formal neighbourhood along $u = 0$

3.3.1 Basic strictness

Let $\pi_0 : \mathbb{C}_\lambda \times X \times \mathbb{C}_t \rightarrow \mathbb{C}_\lambda$ denote the projection. Let $\iota_u : \mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \{0\} \rightarrow \mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \mathbb{C}_u$ and $\iota_{01} : \mathbb{C}_\lambda \times X \times \{0\} \rightarrow \mathbb{C}_\lambda \times X \times \mathbb{C}_t$ denote the inclusion maps.

Lemma 3.17 *$R^\ell \pi_{0*}(\iota_u^* \mathfrak{C}(\mathcal{M}))$ are locally free $\mathcal{O}_{\mathbb{C}_\lambda}$ -modules of finite rank for any ℓ .*

Proof Because $\mathrm{Gr}_a^V(\widetilde{\mathcal{M}})$ ($-1 < a \leq 0$) underlie regular mixed twistor \mathcal{D} -modules on a Kähler manifold X whose supports are compact, the sheaves $R^j \pi_{0*}(\mathrm{Gr}_a^V(\widetilde{\mathcal{M}}) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet)$ are locally free $\mathcal{O}_{\mathbb{C}_\lambda}$ -modules of finite rank. (See Corollary 2.5.)

Note that $\iota_u^* \mathfrak{C}(\mathcal{M})$ is quasi-isomorphic to

$$\lambda^{-1} \iota_{01*} \left((V_0(\widetilde{\mathcal{M}})/V_{-1}(\widetilde{\mathcal{M}})) \otimes dt \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \right) [-1].$$

For any $b < a$, there exists the following canonical splitting of $V\mathcal{R}_{X \times \mathbb{C}_t | \mathbb{C}_\lambda^* \times X \times \mathbb{C}_t}$ -modules

$$(V_a(\widetilde{\mathcal{M}})/V_b(\widetilde{\mathcal{M}}))|_{\mathbb{C}_\lambda^* \times X \times \mathbb{C}_t} = \bigoplus_{b < c \leq a} \mathrm{Gr}_c^V(\widetilde{\mathcal{M}})|_{\mathbb{C}_\lambda^* \times X \times \mathbb{C}_t}. \quad (8)$$

Here, the actions of $-\partial_t t - c$ on $\mathrm{Gr}_c^V(\widetilde{\mathcal{M}})|_{\mathbb{C}_\lambda^* \times X \times \mathbb{C}_t}$ are nilpotent. For any $-1 < b < a \leq 0$, we obtain the following exact sequence for any ℓ :

$$\begin{aligned} 0 \longrightarrow R^\ell \pi_{0*}(\mathrm{Gr}_b^V(\widetilde{\mathcal{M}}) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet)|_{\mathbb{C}_\lambda^*} &\longrightarrow R^\ell \pi_{0*} \left((V_a(\widetilde{\mathcal{M}})/V_{<b}(\widetilde{\mathcal{M}})) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet \right)|_{\mathbb{C}_\lambda^*} \\ &\longrightarrow R^\ell \pi_{0*} \left((V_a(\widetilde{\mathcal{M}})/V_b(\widetilde{\mathcal{M}})) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet \right)|_{\mathbb{C}_\lambda^*} \longrightarrow 0. \end{aligned} \quad (9)$$

Because $R^k \pi_{0*}(\mathrm{Gr}_b^V(\widetilde{\mathcal{M}}) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet)$ are locally free $\mathcal{O}_{\mathbb{C}_\lambda}$ -modules for any k , we obtain the vanishing of the following morphisms:

$$R^\ell \pi_{0*} \left((V_a(\widetilde{\mathcal{M}})/V_b(\widetilde{\mathcal{M}})) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet \right) \longrightarrow R^{\ell+1} \pi_{0*}(\mathrm{Gr}_b^V(\widetilde{\mathcal{M}}) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet).$$

We obtain the following exact sequences:

$$\begin{aligned} 0 \longrightarrow R^\ell \pi_{0*}(\mathrm{Gr}_b^V(\widetilde{\mathcal{M}}) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet) &\longrightarrow R^\ell \pi_{0*} \left((V_a(\widetilde{\mathcal{M}})/V_{<b}(\widetilde{\mathcal{M}})) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet \right) \\ &\longrightarrow R^\ell \pi_{0*} \left((V_a(\widetilde{\mathcal{M}})/V_b(\widetilde{\mathcal{M}})) \otimes \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet \right) \longrightarrow 0. \end{aligned} \quad (10)$$

Then, we obtain the claim of Lemma 3.17 by an easy induction. \blacksquare

3.3.2 The E_1 -degeneration for another filtration \mathcal{F}_u

We consider the subcomplexes $\mathcal{F}_u^j = u^j \mathfrak{C}(\mathcal{M})$. There exist natural isomorphisms $\mathrm{Gr}_{\mathcal{F}_u}^j \mathfrak{C}(\mathcal{M}) \simeq \iota_{u*} \iota_u^* \mathfrak{C}(\mathcal{M})$. There exists the following exact sequence:

$$0 \longrightarrow \mathrm{Gr}_{\mathcal{F}_u}^j \mathfrak{C}(\mathcal{M}) \longrightarrow \mathfrak{C}(\mathcal{M})/\mathcal{F}_u^{j+1} \mathfrak{C}(\mathcal{M}) \longrightarrow \mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}) \longrightarrow 0. \quad (11)$$

Proposition 3.18 *We obtain the following exact sequences:*

$$0 \longrightarrow R^\ell \pi_{03*} \mathrm{Gr}_{\mathcal{F}_u}^j \mathfrak{C}(\mathcal{M}) \longrightarrow R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^{j+1} \mathfrak{C}(\mathcal{M})) \longrightarrow R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M})) \longrightarrow 0. \quad (12)$$

Proof Let us consider the following exact sequence:

$$0 \longrightarrow \mathcal{F}_u^1 \mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}) \longrightarrow \mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}) \longrightarrow \mathrm{Gr}_{\mathcal{F}_u}^0 \mathfrak{C}(\mathcal{M}) \longrightarrow 0.$$

Lemma 3.19 *We obtain the following exact sequences for any $\ell \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}_{\geq 1}$:*

$$\begin{aligned} 0 \longrightarrow R^\ell \pi_{03*} (\mathcal{F}_u^1 \mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}))|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u} &\longrightarrow R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}))|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u} \\ &\longrightarrow R^\ell \pi_{03*} (\mathrm{Gr}_{\mathcal{F}_u}^0 \mathfrak{C}(\mathcal{M}))|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u} \longrightarrow 0. \end{aligned} \quad (13)$$

Proof Recall that $\mathrm{Gr}_{\mathcal{F}_u}^0 \mathfrak{C}(\mathcal{M}) = \iota_{u*} \iota_u^* \mathfrak{C}(\mathcal{M})$ is quasi-isomorphic to

$$\iota_{u*} \left(\lambda^{-1} \iota_{01*} \left(\pi_{012}^* (V_0(\widetilde{\mathcal{M}})/V_{-1}(\widetilde{\mathcal{M}})) \otimes dt \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet \right) [-1] \right).$$

Let $\pi_1 : \mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \mathbb{C}_u \rightarrow X$ denote the projection. Let \mathcal{C}_2^\bullet denote the Dolbeault resolution, i.e., the associated complex of the double complex

$$\iota_{u*} \left(\lambda^{-1} \iota_{01*} \left(\pi_{012}^* (V_0(\widetilde{\mathcal{M}})/V_{-1}(\widetilde{\mathcal{M}})) \otimes dt \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet \right) [-1] \right) \otimes \pi_1^* \Omega_X^{0,\bullet}.$$

For any $N \geq 0$, there exists the following subcomplex $V_{-N-1} \mathfrak{C}(\mathcal{M})$ of $\mathfrak{C}(\mathcal{M})$:

$$\pi_{012}^* V_{-N-1}(\widetilde{\mathcal{M}}) \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \longrightarrow \lambda^{-1} \pi_{012}^* V_{-N}(\widetilde{\mathcal{M}}) \otimes dt \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet.$$

Let U be an open subset of $\mathbb{C}_\lambda^* \times \mathbb{C}_u$. Let α be a section of \mathcal{C}_2^\bullet on $U \times X \times \mathbb{C}_t$ such that $d\alpha = 0$. Let $N > j + 10$. By using the splitting (8), we can construct a section $\tilde{\alpha}$ of $\left(\lambda^{-1} \pi_{012}^* V_0(\widetilde{\mathcal{M}}) \otimes dt \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet [-1] \right) \otimes \pi_1^* \Omega_X^{0,\bullet}$ such that (i) $\tilde{\alpha}$ induces α , (ii) $\beta^{(0)} := d\tilde{\alpha}$ is contained in $\left(\lambda^{-1} \pi_{012}^* V_{-N}(\widetilde{\mathcal{M}}) \otimes dt \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \right) \otimes \pi_1^* \Omega_X^{0,\bullet}$ on $U \times X \times \mathbb{C}_t$. Note that $d\beta^{(0)} = 0$. There exists a section $\gamma^{(1)}$ of $\left(\pi_{012}^* V_{-N}(\widetilde{\mathcal{M}}) \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \right) \otimes \pi_1^* \Omega_X^{0,\bullet}$ on $U \times X \times \mathbb{C}_t$ such that $\gamma^{(1)} \otimes \lambda^{-1} dt = \beta^{(0)}$. We set $\beta^{(1)} = \beta^{(0)} - d\gamma^{(1)}$ which is contained in

$$u \left(\lambda^{-1} \pi_{012}^* V_{-N+1}(\widetilde{\mathcal{M}}) \otimes dt \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \right) \otimes \pi_1^* \Omega_X^{0,\bullet}$$

on $U \times X \times \mathbb{C}_t$. We have $d\beta^{(1)} = 0$. Inductively, for $m = 1, \dots, j+1$, we can construct sections $\beta^{(m)}$ of

$$u^m \left(\lambda^{-1} \pi_{012}^* V_{-N+m}(\widetilde{\mathcal{M}}) \otimes dt \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \right) \otimes \pi_1^* \Omega_X^{0,\bullet}$$

and sections $\gamma^{(m)}$ of $u^{m-1} \left(\pi_{012}^* V_{-N+m-1}(\widetilde{\mathcal{M}}) \otimes \pi_{01}^* \widetilde{\Omega}_{\mathcal{X}/\mathbb{C}}^\bullet \right) \otimes \pi_1^* \Omega_X^{0,\bullet}$ such that $\beta^{(m)} = \beta^{(m-1)} - d\gamma^{(m)}$ and $\gamma^{(m)} \otimes \lambda dt = \beta^{(m-1)}$. Then, $\hat{\alpha} = \tilde{\alpha} - \sum_{m=1}^{j+1} \gamma^{(m)}$ is a section of $\mathfrak{C}(\mathcal{M}) \otimes \pi_1^* \Omega_X^{0,\bullet}$ such that (i) $\hat{\alpha}$ induces α , (ii) $d\hat{\alpha}$ is a section of $\mathcal{F}_u^{j+1} \mathfrak{C}(\mathcal{M}) \otimes \pi_1^* \Omega_X^{0,\bullet}[1]$. Then, we obtain the claim of Lemma 3.19. \blacksquare

Lemma 3.20 *We obtain the following exact sequences for any $\ell, j \in \mathbb{Z}_{\geq 0}$.*

$$\begin{aligned} 0 \longrightarrow R^\ell \pi_{03*} \operatorname{Gr}_{\mathcal{F}_u}^j \mathfrak{C}(\mathcal{M})|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u} &\longrightarrow R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^{j+1} \mathfrak{C}(\mathcal{M}))|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u} \\ &\longrightarrow R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}))|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u} \longrightarrow 0. \end{aligned} \quad (14)$$

Proof It is enough to prove that

$$R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^{j+1} \mathfrak{C}(\mathcal{M}))|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u} \longrightarrow R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}))|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u} \quad (15)$$

are epimorphisms for any ℓ and j . We use an induction on j . Lemma 3.19 implies the claim in the case $j = 1$. By the hypothesis of the induction, we may assume that

$$R^\ell \pi_{03*} (\mathcal{F}_u^1 \mathfrak{C}(\mathcal{M})/\mathcal{F}_u^{j+1} \mathfrak{C}(\mathcal{M}))|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u} \longrightarrow R^\ell \pi_{03*} (\mathcal{F}_u^1 \mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}))|_{\mathbb{C}_\lambda^* \times \mathbb{C}_u}$$

are epimorphisms. Then, by Lemma 3.19, we obtain that (15) are also epimorphisms. ■

We obtain the exactness of (12) from the exactness of (14) and the strictness of $R^\ell \pi_{03*} \operatorname{Gr}_{\mathcal{F}_u}^j \mathfrak{C}(\mathcal{M})$ for any ℓ and j as in the proof of Lemma 3.17. Thus, we obtain Proposition 3.18. ■

The supports of $R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}))$ are contained in $\mathbb{C}_\lambda \times \{0\} \subset \mathbb{C}_\lambda \times \mathbb{C}_u$. We may naturally regard them as $\mathcal{O}_{\mathbb{C}_\lambda}[u]/u^j$ -modules.

Corollary 3.21 *$R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^j \mathfrak{C}(\mathcal{M}))$ are locally free $\mathcal{O}_{\mathbb{C}_\lambda}[u]/u^j$ -modules.* ■

3.4 Coherence of the specialization along $\lambda = 0$

Let $\tilde{\iota}_\lambda : \{0\} \times X \times \mathbb{C}_t \times \mathbb{C}_u \longrightarrow \mathbb{C}_\lambda \times X \times \mathbb{C}_t \times \mathbb{C}_u$ denote the inclusion. Let $\pi_3 : X \times \mathbb{C}_t \times \mathbb{C}_u \rightarrow \mathbb{C}_u$ denote the projection.

Proposition 3.22 *$R^\ell \pi_{3*} (\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))$ are coherent $\mathcal{O}_{\mathbb{C}_u}$ -modules.*

Proof Let $\pi_{12} : X \times \mathbb{C}_t \times \mathbb{C}_u \rightarrow X \times \mathbb{C}_t$ denote the projection. Note that $\tilde{\iota}_\lambda^* \pi_{12}^* \mathcal{R}_{X \times \mathbb{C}_t}$ is isomorphic to the algebra of the symmetric product of $\pi_{12}^* \Theta_{X \times \mathbb{C}_t}$, where $\Theta_{X \times \mathbb{C}_t}$ denote the tangent sheaf of $X \times \mathbb{C}_t$. Because $\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M})$ is a complex of coherent $\tilde{\iota}_\lambda^* \mathcal{R}_{X \times \mathbb{C}_t}$ -modules, it induces a complex of coherent $\mathcal{O}_{T^*(X \times \mathbb{C}_t) \times \mathbb{C}_u}$ -modules $(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))^\sim$. Let Z denote the cohomological support of $(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))^\sim$.

Lemma 3.23 *Let $0_{X \times \mathbb{C}_t} : X \times \mathbb{C}_t \rightarrow T^*(X \times \mathbb{C}_t)$ denote the 0-section. Then, $Z \subset (0_{X \times \mathbb{C}_t} \circ \iota_f)(\operatorname{Cr}(f)) \times \mathbb{C}_u$.*

Proof Let $(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))_u$ denote the pull back of $\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M})$ by the inclusion $X \times \mathbb{C}_t \times \{u\} \rightarrow X \times \mathbb{C}_t \times \mathbb{C}_u$. Let $(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))_u^\sim$ denote the induced complex of coherent $\mathcal{O}_{T^*(X \times \mathbb{C}_t) \times \mathbb{C}_u}$ -modules. It equals the pull back of $(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))^\sim$ by the inclusion $T^*(X \times \mathbb{C}_t) \times \{u\} \rightarrow T^*(X \times \mathbb{C}_t) \times \mathbb{C}_u$. Let Z_u denote the cohomological support of $(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))_u^\sim$. Because $Z \cap (T^*(X \times \mathbb{C}_t) \times \{u\}) = Z_u$, it is enough to prove that $Z_u \subset (0_{X \times \mathbb{C}_t} \circ \iota_f)(\operatorname{Cr}(f))$.

Let $\iota_{0,\lambda} : \{0\} \times X \times \mathbb{C}_t \rightarrow \mathbb{C}_\lambda \times X \times \mathbb{C}_t$ denote the inclusion. By Lemma 3.14, if $u \neq 0$, $(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))_u$ is quasi-isomorphic to $\iota_{0,\lambda}^* (\widetilde{\mathcal{M}} \otimes \mathcal{L}(u^{-1}t) \otimes \tilde{\Omega}_{\mathbb{C}_\lambda \times X \times \mathbb{C}_t/\mathbb{C}_\lambda}^\bullet)$. It is quasi-isomorphic to $\iota_{f*} \iota_\lambda^* (\mathcal{M}_{u^{-1}f} \otimes \tilde{\Omega}_{\mathcal{X}/\mathbb{C}_\lambda}^\bullet[-1])$. Hence, by Lemma 3.1, the cohomological support of $(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))_u^\sim$ is contained in $(0_{X \times \mathbb{C}_t} \circ \iota_f)(\operatorname{Cr}(f))$.

Let us consider the case $u = 0$. There exists the following quasi-isomorphism

$$(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))_0 \simeq \iota_{\lambda*} (V_0(\widetilde{\mathcal{M}})/V_{-1}(\widetilde{\mathcal{M}})) \otimes dt \otimes \Omega_X^\bullet[-1].$$

Because $df(X) \cap \operatorname{Ch}(\Xi_{\operatorname{DR}}(\mathcal{M})) \subset 0_X(\operatorname{Cr}(f))$, the support of the $\operatorname{Sym} \Theta_X$ -module $\iota_{\lambda*} (V_0(\widetilde{\mathcal{M}})/V_{-1}(\widetilde{\mathcal{M}}))$ is contained in $\iota_f(\operatorname{Cr}(f))$ as in the case of Lemma 3.2. By a similar argument to the proof of Lemma 3.1, we obtain that the cohomological support of $(\tilde{\iota}_\lambda^* \mathfrak{C}(\mathcal{M}))_0$ is contained in $(0_{X \times \mathbb{C}_t} \circ \iota_f)(\operatorname{Cr}(f))$. ■

Because Z is proper over \mathbb{C}_u , we obtain the claim of Proposition 3.22. ■

Note that $\operatorname{Gr}_{\mathcal{F}_\lambda}^j \mathfrak{C}(\mathcal{M})$ are isomorphic to $\iota_{\lambda*} \iota_\lambda^* \mathfrak{C}(\mathcal{M})$ for any $j \in \mathbb{Z}_{\geq 0}$. The supports of $R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_\lambda^j \mathfrak{C}(\mathcal{M}))$ are contained in $\{0\} \times \mathbb{C}_u$. We may naturally regard $R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_\lambda^j \mathfrak{C}(\mathcal{M}))$ as $\mathcal{O}_{\mathbb{C}_u}$ -modules.

Corollary 3.24 *$R^\ell \pi_{03*} (\mathfrak{C}(\mathcal{M})/\mathcal{F}_\lambda^j \mathfrak{C}(\mathcal{M}))$ are coherent $\mathcal{O}_{\mathbb{C}_u}$ -modules for any ℓ and j .* ■

3.5 Proof of Theorem 3.16

There exist the following exact sequences:

$$\begin{aligned} 0 \longrightarrow \lambda^j(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) / \lambda^{j+1}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) \\ \longrightarrow (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) / \lambda^{j+1}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) \\ \longrightarrow (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) / \lambda^j(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) \longrightarrow 0. \end{aligned} \quad (16)$$

We obtain the following morphisms of the stalks at $(0, 0) \in \mathbb{C}_\lambda \times \mathbb{C}_u$:

$$\begin{aligned} R^\ell \pi_{03*} \left((\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) / \lambda^j(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) \right)_{(0,0)} \longrightarrow \\ R^{\ell+1} \pi_{03*} \left(\lambda^j(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) / \lambda^{j+1}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) \right)_{(0,0)}. \end{aligned} \quad (17)$$

Lemma 3.25 *The morphisms (17) are 0.*

Proof Because the multiplication of λ^j on $R^\ell \pi_{03*}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M}))$ is a monomorphism by Corollary 3.21, the induced morphism

$$R^\ell \pi_{03*} \left((\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) / \lambda^j(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) \right)_{(0,0)} \longrightarrow R^{\ell+1} \pi_{03*} \left(\lambda^j(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) \right)_{(0,0)}$$

is 0. Hence, the morphism (17) is 0. ■

There exist the following exact sequences:

$$0 \longrightarrow \mathrm{Gr}_{\mathcal{F}_\lambda}^j \mathfrak{C}(\mathcal{M}) \longrightarrow \mathfrak{C}(\mathcal{M})/\mathcal{F}_\lambda^{j+1} \mathfrak{C}(\mathcal{M}) \longrightarrow \mathfrak{C}(\mathcal{M})/\mathcal{F}_\lambda^j \mathfrak{C}(\mathcal{M}) \longrightarrow 0.$$

We obtain the following induced morphisms of the stalks:

$$R^\ell \pi_{03*}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_\lambda^j \mathfrak{C}(\mathcal{M}))_{(0,0)} \longrightarrow R^{\ell+1} \pi_{03*}(\mathrm{Gr}_{\mathcal{F}_\lambda}^j \mathfrak{C}(\mathcal{M}))_{(0,0)}. \quad (18)$$

Lemma 3.26 *The morphisms (18) are 0.*

Proof There exist the following commutative diagrams of stalks at $(\lambda, u) = (0, 0)$ for any $k \geq 0$:

$$\begin{array}{ccc} R^\ell \pi_{03*}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_\lambda^j \mathfrak{C}(\mathcal{M}))_{(0,0)} & \xrightarrow{b^\ell} & R^{\ell+1} \pi_{03*}(\mathrm{Gr}_{\mathcal{F}_\lambda}^j \mathfrak{C}(\mathcal{M}))_{(0,0)} \\ \downarrow & & \downarrow \\ R^\ell \pi_{03*}((\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) / \lambda^j(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})))_{(0,0)} & \xrightarrow{0} & R^{\ell+1} \pi_{03*}(\lambda^j(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) / \lambda^{j+1}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})))_{(0,0)}. \end{array}$$

The lower horizontal arrow is 0 by Lemma 3.25. By the construction, $\mathfrak{C}(\mathcal{M})$ is flat over $\mathcal{O}_{\mathbb{C}_\lambda \times \mathbb{C}_u}$, and hence $\lambda^p \mathfrak{C}(\mathcal{M}) \cap u^q \mathfrak{C}(\mathcal{M}) = \lambda^p u^q \mathfrak{C}(\mathcal{M})$ for any $p, q \in \mathbb{Z}_{\geq 0}$. The right vertical arrows are identified as follows:

$$\begin{array}{ccc} R^{\ell+1} \pi_{03*} \iota_{\lambda*} \iota_\lambda^* \mathfrak{C}(\mathcal{M})_{(0,0)} & \xrightarrow{\simeq} & R^{\ell+1} \pi_{03*}(\mathrm{Gr}_{\mathcal{F}_\lambda}^j \mathfrak{C}(\mathcal{M}))_{(0,0)} \\ \downarrow & & \downarrow \\ R^{\ell+1} \pi_{03*} \iota_{\lambda*} \iota_\lambda^* (\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M}))_{(0,0)} & \xrightarrow{\simeq} & R^{\ell+1} \pi_{03*}(\lambda^j(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})) / \lambda^{j+1}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_u^k \mathfrak{C}(\mathcal{M})))_{(0,0)}. \end{array}$$

Under the identification, the image of b^ℓ is contained in the image of the morphism

$$R^{\ell+1} \pi_{03*} \iota_{\lambda*} \iota_\lambda^* \mathcal{F}_u^k \mathfrak{C}(\mathcal{M})_{(0,0)} \longrightarrow R^{\ell+1} \pi_{03*} \iota_{\lambda*} \iota_\lambda^* \mathfrak{C}(\mathcal{M})_{(0,0)}$$

for any $k \geq 0$. It equals the image of the morphism

$$u^k : R^{\ell+1} \pi_{03*} \iota_{\lambda*} \iota_\lambda^* \mathfrak{C}(\mathcal{M})_{(0,0)} \longrightarrow R^{\ell+1} \pi_{03*} \iota_{\lambda*} \iota_\lambda^* \mathfrak{C}(\mathcal{M})_{(0,0)}$$

for any $k \geq 0$. Because $R^{\ell+1} \pi_{03*} \iota_{\lambda*} \iota_\lambda^* \mathfrak{C}(\mathcal{M})$ is $\mathcal{O}_{\mathbb{C}_\lambda}$ -coherent by Proposition 3.22, we obtain $b^\ell = 0$. ■

For any $j \geq 0$, there exists $\epsilon > 0$ such that the following morphism is an epimorphism on $\{|u| < \epsilon\}$:

$$R^\ell \pi_{03*}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_\lambda^{j+1} \mathfrak{C}(\mathcal{M})) \longrightarrow R^\ell \pi_{03*}(\mathfrak{C}(\mathcal{M})/\mathcal{F}_\lambda^j \mathfrak{C}(\mathcal{M})).$$

By using the \mathbb{C}^* -equivariance, we obtain that it is an epimorphism on \mathbb{C}_u . Thus, we obtain Theorem 3.16. ■

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