THE RATIONALITY OF THE MODULI SPACE OF CURVES OF GENUS 3 AFTER P. KATSYLO

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ABSTRACT. This article is a survey of P. Katsylo's proof that the moduli space $\mathfrak{M}_3$ of smooth projective complex curves of genus 3 is rational. We hope to make the argument more comprehensible and transparent by emphasizing the underlying geometry in the proof and its key structural features.

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1. INTRODUCTION

The question whether $\mathfrak{M}_3$ is a rational variety or not had been open for a long time until an affirmative answer was finally given by P. Katsylo in 1996. There is a well known transition in the behaviour of the moduli spaces $\mathfrak{M}_g$ of smooth projective complex curves of genus g from unirational for small g to general type for larger values of g; the moral reason that $\mathfrak{M}_3$ should have a good chance to be rational is that it is birational to a quotient of a projective space by a *connected* linear algebraic group. No variety of this form has been proved irrational up to now. More precisely, $\mathfrak{M}_3$ is birational to the moduli space of plane quartic curves for $\mathrm{PGL}_3\,\mathbb{C}$-equivalence. All the moduli spaces $C(d)$ of plane curves of given degree d are conjectured to be rational (see [Dol2], p.162; in fact, there it is conjectured that all the moduli spaces of hypersurfaces of given degree d in $\mathbb{P}^n$ for the $\mathrm{PGL}_{n+1}\,\mathbb{C}$-action are rational. I do not know if this conjecture should be attributed to Dolgachev or someone else).

There are heuristic reasons that the spaces $C(d)$ should be rational at least for all large enough values for d. Maybe it is not completely out of reach to prove this rigorously. We hope to return to this problem in the future. In any case one might hazard the guess that irregular behaviour of $C(d)$ is most likely to be found for small values of d, and showing rationality for $C(4)$ turned out to be exceptionally hard.

Katsylo's proof is long and computational, and, due to the importance of the result, it seems desirable to give a more accessible and geometric treatment of the argument.

This paper is divided into two main sections (sections 2 and 3) which are further divided into subsections. Section 2 treats roughly the contents of Katsylo's first paper [Kat1] and section 3 deals with his second paper [Kat2].

Finally I would like to thank Professor Yuri Tschinkel for proposing the project and many useful discussions. Moreover, I am especially grateful to Professor Fedor Bogomolov with whom I discussed parts of the project and who provided a wealth of helpful ideas.

2. A remarkable $(\mathrm{SL}_3\,\mathbb{C}, \mathrm{SO}_3\,\mathbb{C})$-section

2.1. **(G, H)-sections and covariants.** A general, i.e. nonhyperelliptic, smooth projective curve C of genus 3 is realized as a smooth plane quartic curve via the canonical embedding, whence $\mathfrak{M}_3$ is birational to the orbit space $C(4) := \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(4))/\mathrm{SL}_3\,\mathbb{C}$. We remark that whenever one has an affine algebraic group G acting on an irreducible variety X, then, according to a result of Rosenlicht, there exists a nonempty invariant open subset $X_0 \subset X$ such that there is a geometric quotient for the action of G on X_0 (cf. [Po-Vi], thm. 4.4). In the following we denote by X/G any birational model of this quotient, i.e. any model of the field $\mathbb{C}(X)^G$ of invariant rational functions.

The number of methods to prove rationality of quotients of projective spaces by connected reductive groups is quite limited (cf. [Dol1] for an excellent survey). The only approach which our problem is immediately amenable to seems to be the method of (G, H)-sections. (There are two other points of view I know of: The first is based on the remark

that if we have a nonsingular plane quartic curve C, the double cover of $\mathbb{P}^2$ branched along C is a Del Pezzo surface of degree 2, and conversely, given a Del Pezzo surface S of degree 2, then $|-K_S|$ is a regular map which exhibits S as a double cover of $\mathbb{P}^2$ branched along a plane quartic C; this sets up a birational isomorphism between $\mathfrak{M}_3$ and $\mathfrak{DP}(2)$, the moduli space of Del Pezzo surfaces of degree 2. We can obtain such an S by blowing up 7 points in $\mathbb{P}^2$, and one can prove that $\mathfrak{DP}(2)$ is birational to the quotient of an open subset of $P_2^7 := (\mathbb{P}^2)^7/\mathrm{PGL}_3\,\mathbb{C}$, the configuration space of 7 points in $\mathbb{P}^2$ (which is visibly rational), modulo an action of the Weyl group $W(E_7)$ of the root system of type E_7 by Cremona transformations (note that $W(E_7)$ coincides with the permutation group of the (-1)-curves on S that preserves the incidence relations between them). This group is a rather large finite group, in fact, it has order $2^{10}\cdot 3^4\cdot 5\cdot 7$. This approach does not seem to have led to anything definite in the direction of proving rationality of $\mathfrak{M}_3$ by now, but see [D-O] for more information.
The second alternative, pointed out by I. Dolgachev, is to remark that $\mathfrak{M}_3$ is birational to $\mathfrak{M}_3^{\mathrm{ev}}$, the moduli space of genus 3 curves together with an even theta-characteristic; this is the content of the classical theorem due to G. Scorza. The latter space is birational to the space of nets of quadrics in $\mathbb{P}^3$ modulo the action of $\mathrm{SL}_4\,\mathbb{C}$, i.e. $\mathrm{Grass}(3, \mathrm{Sym}^2\,(\mathbb{C}^4)^\vee)/\mathrm{SL}_4\,\mathbb{C}$. See [Dol3], 6.4.2, for more on this. Compare also [Kat0], where the rationality of the related space $\mathrm{Grass}(3, \mathrm{Sym}^2\,(\mathbb{C}^5)^\vee)/\mathrm{SL}_5\,\mathbb{C}$ is proven; this proof, however, cannot be readily adapted to our situation, the difficulty seems to come down to that 4, in contrast to 5, is even).

Definition 2.1.1. Let X be an irreducible variety with an action of a linear algebraic group G, $H < G$ a subgroup. An irreducible subvariety $Y \subset X$ is called a (G,H)-*section* of the action of G on X if

(1) $\overline{G\cdot Y} = X$;
(2) $H\cdot Y \subset Y$;
(3) $g\in G,\ gY\cap Y \neq \emptyset \implies g\in H$.

In this situation H is the normalizer $N_G(Y) := \{g\in G \mid gY\subset Y\}$ of Y in G. The following proposition collects some properties of (G,H)-sections.

Proposition 2.1.2. (1) *The field* $\mathbb{C}(X)^G$ *is isomorphic to the field* $\mathbb{C}(Y)^H$ *via restriction of functions to* Y.
(2) *Let* Z *and* X *be* G*-varieties,* $f: Z\to Y$ *a dominant* G*-morphism,* Y *a* (G,H)*-section of* X*, and* Y' *an irreducible component of*

$f^{-1}(Y)$ that is H-invariant and dominates Y. Then Y' is a (G,H)-section of Z.

Part (2) of the proposition suggests that, to simplify our problem of proving rationality of $C(4)$, we should look at covariants $\mathrm{Sym}^4(\mathbb{C}^3)^\vee \to \mathrm{Sym}^2(\mathbb{C}^3)^\vee$ of low degree ($\mathbb{C}^3$ is the standard representation of $\mathrm{SL}_3\,\mathbb{C}$). The highest weight theory of Cartan-Killing allows us to decompose $\mathrm{Sym}^i(\mathrm{Sym}^4(\mathbb{C}^3)^\vee)$, $i \in \mathbb{N}$, into irreducible subrepresentations (this is best done by a computer algebra system, e.g. `Magma`) and pick the smallest i such that $\mathrm{Sym}^2(\mathbb{C}^3)^\vee$ occurs as an irreducible summand. This turns out to be 5 and $\mathrm{Sym}^2(\mathbb{C}^3)^\vee$ occurs with multiplicity 2.
For nonnegative integers a, b we denote by $V(a,b)$ the irreducible $\mathrm{SL}_3\,\mathbb{C}$-module whose highest weight has numerical labels a, b.
Let us now describe the two resulting independent covariants

$$\alpha_1,\ \alpha_2 : V(0,4) \to V(0,2)$$

of order 2 and degree 5 geometrically. We follow a classical geometric method of Clebsch to pass from invariants of binary forms to contravariants of ternary forms (see [G-Y], §215). The covariants α_1, α_2 are described in Salmon's treatise [Sal], p. 273, l.18-19, and p. 271, l. 32-33, cf. also [Dix], p. 280-282. We start by recalling the structure of the ring of $\mathrm{SL}_2\,\mathbb{C}$-invariants of binary quartics ([Muk], section 1.3, [Po-Vi], section 0.12).

2.2. **Binary quartics.** Let

$$f_4 = \xi_0 x_0^4 + 4\xi_1 x_0^3 x_1 + 6\xi_2 x_0^2 x_1^2 + 4\xi_3 x_0 x_1^3 + \xi_4 x_1^4 \tag{1}$$

be a general binary quartic form. The invariant algebra $R = \mathbb{C}[\xi_0, \dots, \xi_4]^{\mathrm{SL}_2\,\mathbb{C}}$ is freely generated by two homogeneous invariants g_2 and g_3 (where subscripts indicate degrees):

$$g_2(\xi) = \det\begin{pmatrix} \xi_0 & \xi_2 \\ \xi_2 & \xi_4 \end{pmatrix} - 4\det\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_3 \end{pmatrix}, \tag{2}$$

$$g_3(\xi) = \det\begin{pmatrix} \xi_0 & \xi_1 & \xi_2 \\ \xi_1 & \xi_2 & \xi_3 \\ \xi_2 & \xi_3 & \xi_4 \end{pmatrix}. \tag{3}$$

If we identify f_4 with its zeroes $z_1, \dots, z_4 \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and write

$$\lambda = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

for the cross-ratio, then

$$g_2 = 0 \iff \lambda = -1,\ 2,\ \text{or}\ \frac{1}{2}\,,$$
$$g_3 = 0 \iff \lambda = -\omega\ \text{or}\ -\omega^2\ \text{with}\ \omega = e^{\frac{2\pi i}{3}}\,,$$

the first case being commonly referred to as *harmonic cross-ratio*, the second as *equi-anharmonic cross-ratio*.
Clebsch's construction is as follows: Let x, y, z be coordinates in $\mathbb{P}^2$, and let u, v, w be coordinates in the dual projective plane $(\mathbb{P}^2)^\vee$. Let $\varphi = ax^4 + 4bx^3y^4 + \dots$ be a general ternary quartic. We want to consider those lines in $\mathbb{P}^2$ such that their intersection with the associated quartic curve C_φ is a set of points whose cross-ratio is harmonic resp. anharmonic. Writing a line as $ux+vy+wz = 0$ and substituting in (2) resp. (3), we see that in the harmonic case we get a quartic in $(\mathbb{P}^2)^\vee$, and in the equi-anharmonic case a sextic. More precisely this gives us two $\mathrm{SL}_3\,\mathbb{C}$-equivariant polynomial maps

$$\sigma : V(0,4) \to V(0,4)^\vee\,, \tag{4}$$
$$\psi : V(0,4) \to V(0,6)^\vee\,, \tag{5}$$

and σ is homogeneous of degree 2 in the coefficients of φ whereas ψ is homogeneous of degree 3 in the coefficients of φ (we say σ is a *contravariant* of degree 2 on $V(0,4)$ with values in $V(0,6)$, and analogously for ψ). Finally we have the *Hessian covariant* of φ:

$$\mathrm{Hess} : V(0,4) \to V(0,6) \tag{6}$$

which associates to φ the determinant of the matrix of second partial derivatives of φ. It is of degree 3 in the coefficients of φ.
We will now cook up α_1, α_2 from φ, σ, ψ, Hess: Let φ operate on ψ; by this we mean that if $\varphi = ax^4 + 4bx^3y + \dots$ then we act on ψ by the differential operator

$$a\frac{\partial^4}{\partial u^4} + 4b\frac{\partial^4}{\partial u^3 \partial v} + \dots$$

(i.e. we replace a coordinate by partial differentiation with respect to the dual coordinate). In this way we get a contravariant ρ of degree 4 on $V(0,4)$ with values in $V(0,2)$. If we operate with ρ on φ we get α_1. We obtain α_2 if we operate with σ on Hess.
This is a geometric way to describe α_1, α_2. For every $c = [c_1 : c_2] \in \mathbb{P}^1$ we get in this way a rational map

$$f_c = c_1\alpha_1 + c_2\alpha_2 : \mathbb{P}(V(0,4)) \dashrightarrow \mathbb{P}(V(0,2))\,. \tag{7}$$

For the special quartics

$$\varphi = ax^4 + by^4 + cz^4 + 6fy^2z^2 + 6gz^2x^2 + 6hx^2y^2 \tag{8}$$

the quantities α_1 and α_2 were calculated by Salmon in [Sal], p. 257 ff. We reproduce the results here for the reader's convenience. Put

$$\begin{aligned} L &:= abc\,,\ P := af^2 + bg^2 + ch^2\,, \\ Q &:= bcg^2h^2 + cah^2f^2 + abf^2g^2\,,\ R := fgh\,; \end{aligned} \tag{9}$$

Then

$$\begin{aligned} \alpha_1 = {} & (3L + 9P + 10R)(afx^2 + bgy^2 + chz^2) + \\ & (10L + 2P + 4R)(ghx^2 + hfy^2 + fgz^2) \\ & -12(a^2f^3x^2 + b^2g^3y^2 + c^2h^3z^2)\,; \end{aligned} \tag{10}$$

$$\begin{aligned} \alpha_2 = {} & (L + 3P + 30R)(afx^2 + bgy^2 + chz^2) + \\ & (10L - 6P - 12R)(ghx^2 + hfy^2 + fgz^2) \\ & -4(a^2f^3x^2 + b^2g^3y^2 + c^2h^3z^2)\,. \end{aligned} \tag{11}$$

Note that the covariant conic $-\frac{1}{20}(\alpha_1 - 3\alpha_2)$ looks a little simpler.
Let us see explicitly, using (8)-(11), that f_c is dominant for every $c \in \mathbb{P}^1$; for $a = b = c = f = g = h = 1$ we get $\alpha_1 = 48(x^2 + y^2 + z^2)$, $\alpha_2 = 16(x^2+y^2+z^2)$, so the image of φ under f_c in this case is a nonsingular conic unless $c = [-1 : 3]$. But for $a = 1$, $b = c = 0$, $f = g = h = 1$ we obtain $\alpha_1 = 13x^2 + 6y^2 + 6z^2$, $\alpha_2 = 11x^2 - 18y^2 - 18z^2$, and for these values $-\alpha_1 + 3\alpha_2$ defines a nonsingular conic.
Let $\mathcal{L}_c$ be the linear system generated by 6 quintics which defines f_c and let B_c be its base locus; thus $U_c := \mathbb{P}(V(0,4))\backslash B$ is an $\mathrm{SL}_3\,\mathbb{C}$-invariant open set, and if $f_{c,0} := f_c|_{U_c}$, then $X_c := f_{c,0}^{-1}(\mathbb{C}h_0)$, where h_0 defines a non-singular conic, is a good candidate for an $(\mathrm{SL}_3\,\mathbb{C},\ \mathrm{SO}_3\,\mathbb{C})$-section of U_c. We choose $h_0 = xz - y^2$.

Proposition 2.2.1. *X_c is a smooth irreducible $\mathrm{SO}_3\,\mathbb{C}$-invariant variety, $\overline{\mathrm{SL}_3\,\mathbb{C}\cdot X} = \mathbb{P}(V(0,4))$, and the normalizer of X_c in $\mathrm{SL}_3\,\mathbb{C}$ is exactly $\mathrm{SO}_3\,\mathbb{C}$. In particular, X_c is an $(\mathrm{SL}_3\mathbb{C},\ \mathrm{SO}_3\,\mathbb{C})$-section of U_c.*

Proof. The $\mathrm{SO}_3\,\mathbb{C}$-invariance of X_c follows from its construction. We show that the differential $d(f_{c,0})_x$ is surjective for all $x \in X_c$: In fact,

$$d(f_{c,0})(T_xU_c) \supset d(f_{c,0})(\mathfrak{sl}_3(x)) = \mathfrak{sl}_3(f_{c,0}(x)) = T_{\mathbb{C}h_0}\,.\mathbb{P}V(0,2)$$

Here $\mathfrak{sl}_3(x)$ denotes the tangent space to the $\mathrm{SL}_3\,\mathbb{C}$-orbit of x in U_c, i.e. if $O_x : \mathrm{SL}_3\,\mathbb{C} \to U_c$ is the map with $O_x(g) = gx$, then we get a map $d(O_x)_e : \mathfrak{sl}_3 \to T_xU_c$, and $\mathfrak{sl}_3(x) := \{d(O_x)_e(\xi)\,|\,\xi \in \mathfrak{sl}_3\}$. Hence X_c is

smooth.
Assume X_c were reducible, let X_1 and X_2 be two irreducible components. By prop. 2.1.2 (2) and the irreducibility of the group $\mathrm{SO}_3\,\mathbb{C}$, X_1 and X_2 are $(\mathrm{SL}_3\,\mathbb{C}, \mathrm{SO}_3\,\mathbb{C})$-sections of U_c, so we can find $g \in \mathrm{SL}_3\,\mathbb{C}$, $x_1 \in X_1$, $x_2 \in X_2$, such that $gx_1 = x_2$. But then, by the $\mathrm{SL}_3\,\mathbb{C}$-equivariance of $f_{c,0}$, g stabilizes $\mathbb{C}h_0$ and is thus in $\mathrm{SO}_3\,\mathbb{C}$. But, again by the irreducibility of $\mathrm{SO}_3\,\mathbb{C}$, x_2 is also a point of X_1, i.e. X_1 and X_2 meet. This contradicts the smoothness of X_c. □

The trouble is that, if $\overline{X}_c$ is the closure of X_c in $\mathbb{P}(V(0,4))$, then $\overline{X}_c$ is an irreducible component of the intersection of 5 quintics. To eventually prove rationality, however, we would like to have some equations of lower degree. This can be done for special c.

2.3. **From quintic to cubic equations.** If $\Gamma_{f_c} \subset \mathbb{P}V(0,4) \times \mathbb{P}V(0,2)$ is the graph of f_c, it is natural to look for $\mathrm{SL}_3\,\mathbb{C}$-equivariant maps

$$\vartheta : V(0,4) \times V(0,2) \to V'$$

where V' is another $\mathrm{SL}_3\,\mathbb{C}$-representation, ϑ is a homogeneous polynomial map in both factors $V(0,4)$, $V(0,2)$, *of low degree*, say d, in the first factor, *linear* in the second, and such that Γ_{f_c} is an irreducible component of $\{(x,y) \in \mathbb{P}V(0,4) \times \mathbb{P}V(0,2) \,|\, \vartheta(x,y) = 0\}$. If V' is irreducible, there is an easy way to tell if ϑ vanishes on Γ_{f_c} *for some* $c \in \mathbb{P}^1$: This will be the case if V' occurs with multiplicity one in $\mathrm{Sym}^{d+5}\, V(0,4)$. Here is the result.

Definition 2.3.1. Let $\Psi : V(0,4) \to V(2,2)$ be the up to factor unique $\mathrm{SL}_3\,\mathbb{C}$-equivariant, homogeneous of degree 3 polynomial map with the indicated source and target spaces, and let $\Phi : V(2,2) \times V(0,2) \to V(2,1)$ be the up to factor unique bilinear $\mathrm{SL}_3\,\mathbb{C}$-equivariant map. Define $\Theta : V(0,4) \times V(0,2) \to V(2,1)$ by $\Theta(x,y) := \Phi(\psi(x), y)$.

Remark 2.3.2. The existence and essential uniqueness of the maps of definition 2.3.1 can be easily deduced from known (and implemented in `Magma`) decomposition laws for $\mathrm{SL}_3\,\mathbb{C}$-representations. That they are only determined up to a nonzero constant factor will never bother us, and we admit this ambiguity in notation. The explicit form of Ψ, Φ, Θ will be needed later for checking certain non-degeneracy conditions through explicit computation. They can be found in Appendix A, formulas (64), (65).

Theorem 2.3.3. (1) *The linear map* $\Theta(f, \cdot) : V(0,2) \to V(2,1)$ *has one-dimensional kernel for* f *in an open dense subset* V_0 *of* $V(0,4)$*, and, in particular,* $\ker \Theta(h_0^2, \cdot) = \mathbb{C}h_0$.

(2) *For some* $c_0 \in \mathbb{P}^1$, $\Gamma_{f_{c_0}}$ *is an irreducible component of* $\{\Theta(x,y) = 0\} \subset V(0,4) \times V(0,2)$.

(3) $\overline{X_{c_0}} \subset \mathbb{P}V(0,4)$ *coincides with the closure* $\overline{X}$ *in* $\mathbb{P}V(0,4)$ *of the preimage* X *of* h_0 *under the morphism from* $\mathbb{P}V_0 \to \mathbb{P}V(0,2)$ *given by* $f \mapsto \ker \Theta(F, \cdot)$, *and is thus an irreducible component of the algebraic set* $\{\mathbb{C}f \mid \Phi(\Psi(f), h_0) = 0\} \subset \mathbb{P}V(0,4)$ *defined by* 15 *cubic equations.*

(4) *The rational map* $\Psi : \mathbb{P}V(0,4) \dashrightarrow \overline{\Psi\mathbb{P}V(0,4)} \subset \mathbb{P}V(2,2)$ *as well as its restriction to* X *are birational isomorphisms unto their images.*

Proof. (1): One checks that $V(2,1)$ occurs with multiplicity one in the decomposition of $\mathrm{Sym}^8 V(0,4)$. Thus for some $c_0 \in \mathbb{P}^1$, we have $\Theta(f, c_{0,1}\alpha_1 + c_{0,2}\alpha_2) = 0$ for all $f \in V(0,4)$. The fact that $\ker \Theta(h_0^2, \cdot) = \mathbb{C}h_0$ follows from a direct computation using the explicit form of Θ (the inclusion "$\supset$" also follows from Salmon's equations 2.2 (8)-(11)). Thus, by upper-semicontinuity, (1) follows.
(2): We have seen in (1) that $\Gamma_{f_{c_0}}$ is contained in $\{\Theta(x,y) = 0\}$. Again by (1),

$$\Gamma_{f_{c_0}} \cap ((U_{c_0} \cap \mathbb{P}V_0) \times \mathbb{P}V(0,2)) = \\ \{\Theta(x,y) = 0\} \cap ((U_{c_0} \cap \mathbb{P}V_0) \times \mathbb{P}V(0,2)) \ ,$$

and (2) follows.
(3) follows from to (2) and the definition of X_{c_0}.
(4): Since X is an $(\mathrm{SL}_3\,\mathbb{C}, \mathrm{SO}_3\,\mathbb{C})$-section of $\mathbb{P}V_0$, it suffices to prove that the $\mathrm{SL}_3\,\mathbb{C}$-equivariant rational map $\Psi : \mathbb{P}V(0,4) \dashrightarrow \overline{\Psi\mathbb{P}V(0,4)}$ (defined e.g. in the point $\mathbb{C}h_0^2$) is birational. We will do this by writing down an explicit rational inverse. To do this, remark that $V(a,b)$ sits as an $\mathrm{SL}_3\,\mathbb{C}$-invariant linear subspace inside $\mathrm{Sym}^a\mathbb{C}^3 \otimes \mathrm{Sym}^b(\mathbb{C}^3)^\vee$ (it has multiplicity one in the decomposition into irreducibles), thus elements of $V(a,b)$ may be viewed as tensors $x = (x^{i_1,\dots,i_b}_{j_1,\dots,j_a}) \in T^b_a\,\mathbb{C}^3$, covariant of order b and contravariant of order a, or of type $\binom{b}{a}$. The inverse of the determinant tensor $\det^{-1}$ is thus in $T^0_3\mathbb{C}^3$. For $f \in V(0,4)$ and $g \in V(2,2)$ one defines a bilinear $\mathrm{SL}_3\,\mathbb{C}$-equivariant map $\alpha : V(0,4) \times V(2,2) \to \mathrm{Sym}^2\mathbb{C}^3 \otimes \mathrm{Sym}^3(\mathbb{C}^3)^\vee$, $(f,g) \mapsto \alpha(f,g)$, as the contraction

$$s^{i_1\,i_2\,i_3}_{j_1\,j_2} := f^{i_1\,i_2\,i_4\,i_5} g^{i_6\,i_3}_{i_5\,j_1} \det{}^{-1}_{j_2\,i_4\,i_6} \ ,$$

followed by the symmetrization map. One checks that $\mathrm{Sym}^2\mathbb{C}^3 \otimes \mathrm{Sym}^3(\mathbb{C}^3)^\vee$ decomposes as $V(2,3) \oplus V(1,2) \oplus V(0,1)$, but $\mathrm{Sym}^4 V(0,4)$ does not contain these as subrepresentations (use `Magma`), so $\alpha(f, \Psi(f)) = 0$ for all $f \in V(0,4)$. But the explicit form of Ψ and α show that

$\ker \alpha(\cdot, \Psi(h_0^2)) = \mathbb{C}h_0^2$, whence, by upper-semicontinuity, the dimension of the kernel of $\alpha(\cdot, \Psi(f))$ is one for all f in a dense open subset of $V(0,4)$, and the rational map $\Psi : \mathbb{P}V(0,4) \dashrightarrow \overline{\Psi\mathbb{P}V(0,4)} \subset \mathbb{P}V(2,2)$ has the rational inverse $\Psi(f) \mapsto \ker \alpha(\cdot, \Psi(f))$. $\square$

Remark 2.3.4. It would probably be illuminating to have a geometric interpretation of the covariant $\Psi : V(0,4) \to V(2,2)$ given above similar to the one for α_1, α_2 in subsection 2.2. Though there is a huge amount of classical projective geometry attached to plane quartics, I have been unable to find such a geometric description.
Clearly, Ψ vanishes on the cone of dominant vectors in $V(0,4)$, and one may check, using the explicit formula for Ψ in Appendix A (64), that Ψ also vanishes on the $\mathrm{SL}_3\,\mathbb{C}$-orbit of the degree 4 forms in two variables, x and y, say. However, this is not enough to characterize Ψ since the same holds also for e.g. the Hessian covariant.

2.4. **From cubic to quadratic equations.** We have to fix some further notation.

Definition 2.4.1. (1) Z is the affine cone in $V(2,2)$ over $\overline{\Psi(X)} \subset \mathbb{P}V(2,2)$.
(2) L is the linear subspace $L := \{g \in V(2,2) \,|\, \Phi(g, h_0) = 0\} \subset V(2,2)$.
(3) $\epsilon : V(0,4) \times V(0,2) \to V(2,2)$ is the unique (up to a nonzero factor) nontrivial $\mathrm{SL}_3\,\mathbb{C}$-equivariant bilinear map with the indicated source and target spaces (the explicit form is in Appendix A (66)).
(4) $\zeta : V(0,4) \times V(0,2) \to V(1,1)$ is the unique (up to factor) nontrivial $\mathrm{SL}_3\,\mathbb{C}$-equivariant map with the property that it is homogeneous of degree 2 in both factors of its domain (cf. Appendix A (67) for the explicit description). We put $\Gamma := \zeta(\cdot, h_0) : V(0,4) \to V(1,1)$.

Let us state explicitly what we are heading towards:

> The affine cone Z over the birational modification $\overline{\Psi(X)}$ of our $(\mathrm{SL}_3\,\mathbb{C}, \mathrm{S0}_3\,\mathbb{C})$-section $X \subset \mathbb{P}V_0 \subset \mathbb{P}V(0,4)$ (whose closure in $\mathbb{P}V(0,4)$ was seen to be an irreducible component of an algebraic set defined by 15 *cubic* equations) has the following wonderful properties: Z lies in L, the linear map $\epsilon(\cdot, h_0) : V(0,4) \to V(2,2)$ restricts to an $\mathrm{SO}_3\mathbb{C}$-equivariant isomorphism between $V(0,4)$ and L, and if, via this isomorphism, we transport Z into $V(0,4)$ and call this Y, then the equations for Y are given by

Γ! More precisely, Y is the unique irreducible component of $\Gamma^{-1}(0)$ passing through the point h_0^2, and Γ maps $V(0,4)$ into a five-dimensional $\mathrm{SO}_3\,\mathbb{C}$-invariant subspace of $V(1,1)$!

Thus, if we have carried out this program, Y (or Z) will be proven to be an irreducible component of an algebraic set defined by 5 *quadratic equations*! This seems quite miraculous, but a satisfactory explanation why this happens probably requires an answer to the problem raised in remark 2.3.4.

We start with some preliminary observations: It is clear that $Z \subset L$ and $\mathbb{C}(\mathbb{P}V(0,4))^{\mathrm{SL}_3\,\mathbb{C}} \simeq \mathbb{C}(Z)^{\mathrm{SO}_3\,\mathbb{C}\times\mathbb{C}^*}$, $\mathbb{C}^*$ acting by homotheties. In the following, we need the decomposition into irreducibles of $\mathrm{SL}_3\,\mathbb{C}$-modules such as $V(2,2)$, $V(2,1)$ and $V(1,1)$ as $\mathrm{SO}_3\,\mathbb{C}$-modules. The patterns according to which irreducible representations of a complex semi-simple algebraic group decompose when restricted to a smaller semi-simple subgroup are generally known as *branching rules*. In our case the answer is

$$V(2,2) = V(2,2)_8 \oplus V(2,2)_6 \oplus V(2,2)_4 \oplus V(2,2)'_4 \oplus V(2,2)_0\,, \tag{12}$$

$$V(2,1) = V(2,1)_6 \oplus V(2,1)_4 \oplus V(2,1)_2\,, \tag{13}$$

$$V(1,1) = V(1,1)_4 \oplus V(1,1)_2\,, \tag{14}$$

$$V(0,4) = V(0,4)_8 \oplus V(0,4)_4 \oplus V(0,4)_0\,. \tag{15}$$

Here the subscripts indicate the numerical label of the highest weight of the respective $\mathrm{SO}_3\,\mathbb{C}$-submodule of the ambient $\mathrm{SL}_3\,\mathbb{C}$-module under consideration. Note also that $\mathrm{SO}_3\,\mathbb{C} \simeq \mathrm{PSL}_2\,\mathbb{C}$, so we are really back in the much classically studied theory of *binary forms*. It is not difficult (and fun) to check (12), (13), (14) by hand; let us briefly digress on how this can be done (cf. [Fu-Ha]):

We fix the following notation. Let first $n = 2l+1$ be an odd integer, $\mathfrak{g} = \mathfrak{sl}_3\,\mathbb{C}$ the Lie algebra of $\mathrm{SL}_3\,\mathbb{C}$, and let $\mathfrak{t}_{\mathfrak{g}}$ its standard torus of diagonal matrices of trace 0, and define the standard weights $\epsilon_i \in \mathfrak{t}_{\mathfrak{g}}^{\vee}$, $i = 1, \dots, n$, by $\epsilon_i(\mathrm{diag}(x_1, \dots, x_n)) := x_i$. Inside $\mathfrak{g}$ we find $\mathfrak{h} := \mathfrak{so}_3\,\mathbb{C}$ defined by

$$\mathfrak{h} := \left\{ \begin{pmatrix} X & Y & U \\ Z & -X^t & V \\ -V^t & -U^t & 0 \end{pmatrix} \,|\, X,\, Y,\, Z \in \mathfrak{gl}_l\,\mathbb{C},\; Y^t = -Y^t, \right.$$
$$\left. Z = -Z^t,\; U, V \in \mathbb{C}^l \right\}\,.$$

Then $\mathfrak{t}_{\mathfrak{h}} := \{\mathrm{diag}(x_1, \dots, x_l, -x_1, \dots, -x_l) \,|\, x_i \in \mathbb{C}\}$; by abuse of notation we denote the restrictions of the functions ϵ_i to $\mathfrak{t}_{\mathfrak{h}}$ by the same letters. The fundamental weights of $\mathfrak{g}$ are $\pi_i := \epsilon_1 + \cdots + \epsilon_i$, $i = 1, \dots, n-1$, the fundamental weights of $\mathfrak{h}$ are $\omega_i := \epsilon_1 + \cdots + \epsilon_i$, $(1 \le i \le l)$ and $\omega_l := (\epsilon_1 + \cdots + \epsilon_l)/2$. Let $\Lambda_{\mathfrak{g}}$ and $\Lambda_{\mathfrak{h}}$ the corresponding weight lattices. $\Lambda_{\mathfrak{g}}^+$ and $\Lambda_{\mathfrak{h}}^+$ are the dominant weights. For $\mathfrak{g}$ (and similarly for $\mathfrak{h}$) an irreducible representation $V(\lambda)$ for $\lambda \in \Lambda_{\mathfrak{g}}^+$ comes with its *formal character*

$$\mathrm{ch}_\lambda := \sum_{\mu \in \Pi(\lambda)} m_\lambda(\mu) e(\mu) \in \mathbb{Z}[\Lambda_{\mathfrak{g}}]\,,$$

an element of the group algebra $\mathbb{Z}[\Lambda_{\mathfrak{g}}]$ generated by the symbols $e(\lambda)$ for $\lambda \in \Lambda_{\mathfrak{g}}$, where $\Pi(\lambda)$ means the weights of $V(\lambda)$, and $m_\lambda(\mu)$ is the dimension of the weight space corresponding to μ in $V(\lambda)$. We have a formal character ch_V for any finite-dimensional $\mathfrak{g}$-module $V = V(\lambda_1) \oplus \cdots \oplus V(\lambda_t)$, $\lambda_1, \ldots, \lambda_t \in \Lambda_{\mathfrak{g}}^+$ defined by

$$\mathrm{ch}_V := \sum_{i=1}^{t} \mathrm{ch}_{\lambda_i}\,.$$

The important point is that V (i.e. its irreducible constituents) can be recovered from the formal character ch_V, meaning that in $\mathbb{Z}[\Lambda_{\mathfrak{g}}]$ we can write ch_V uniquely as a $\mathbb{Z}$-linear combination of characters corresponding to dominant weights $\lambda \in \Lambda_{\mathfrak{g}}^+$.

We go back to the case $l = 1$, $n = 3$. We have $\mathfrak{h} = \mathfrak{so}_3\,\mathbb{C} = \mathfrak{sl}_2\,\mathbb{C}$. The character $\mathrm{ch}_{V(a)}$ of the irreducible $\mathfrak{so}_3\,\mathbb{C}$-module $V(a) := V(a\omega_1)$ is not hard: The weights of $V(a)$ are

$$-a\omega_1,\ (-a+2)\omega_1,\ \ldots,\ (a-2)\omega_1,\ a\omega_1$$

(all multiplicities are 1). It remains to understand the weights and their multiplicities in the irreducible $\mathfrak{g} = \mathfrak{sl}_3\,\mathbb{C}$-module $V(a,b) := V(a\pi_1 + b\pi_2)$. In fact noting that π_1 restricted to the diagonal torus of $\mathfrak{so}_3\,\mathbb{C}$ above is $2\omega_1$, and the restriction of π_2 is 0, we see that, once we know the formal character of $V(a,b)$ as $\mathfrak{sl}_3\,\mathbb{C}$-module, we simply substitute $2\omega_1$ for π_1 and 0 for π_2 in the result and obtain in this way the formal character of the $\mathfrak{so}_3\,\mathbb{C}$-*module* $V(a,b)$, and hence its decomposition into irreducible constituents as $\mathfrak{so}_3\,\mathbb{C}$-module.

Let us assume $a \ge b$ (otherwise pass to the dual representation); we describe the weights and their multiplicities of the $\mathfrak{sl}_3\,\mathbb{C}$-module $V(a,b)$ following [Fu-Ha], p. 175ff.: Imagine a plane with a chosen origin from which we draw two vectors of unit length, representing π_1 and π_2, such that the angle measured counterclockwise from π_1 to π_2 is 60°. Thus the points of the lattice spanned by π_1, π_2 are the vertices of a set of equilateral congruent triangles which gives a tiling of the plane.

The weights of $V(a,b)$ are the lattice points which lie on the edges of a sequence of b (not necessarily regular) hexagons H_i with vertices at lattice points, $i = 0, \dots, b-1$, and a sequence of $[(a-b)/3]+1$ triangles T_j, $j = 0, \dots, [(a-b)/3]$. The H_i and T_j are concentric around the origin, and H_i has one vertex at $(a-i)\pi_1 + (b-i)\pi_2$, T_j has one vertex at the point $(a-b-3j)\pi_1$, and H_i and T_j are otherwise determined by the condition that the lines through π_1, π_2, $\pi_2 - \pi_1$ are axes of symmetry for them, i.e. they are preserved by the reflections in these lines (one should make a picture now).

The multiplicities of the weights obtained in this way are as follows: Weights lying on H_i have multiplicity $i+1$, and weights lying on one of the T_j have multiplicity b. This completely determines the formal character of $V(a,b)$.

Let us look at $V(2,2)$ for example. Here we get three concentric regular hexagons (one of them is degenerate and consists of the origin alone). The weights are thus:

$$2\pi_1 + 2\pi_2,\ 3\pi_2,\ -2\pi_1 + 4\pi_2,\ -3\pi_1 + 3\pi_2,\ -4\pi_1 + 2\pi_2,\ -3\pi_1,$$
$$-2\pi_1 - 2\pi_2,\ -3\pi_2,\ 2\pi_1 - 4\pi_2,\ 3\pi_1 - 3\pi_2,\ 4\pi_1 - 2\pi_2,\ 3\pi_1$$

(these are the ones on the outer hexagon, read counterclockwise, and have multiplicity one),

$$\pi_1 + \pi_2,\ -\pi_1 + 2\pi_2,\ -2\pi_1 + \pi_2,\ -\pi_1 - \pi_2,\ \pi_1 - 2\pi_2,\ 2\pi_1 - \pi_2$$

(these lie on the middle hexagon and have multiplicity two), and finally there is 0 with multiplicity 3 corresponding to the origin. Consequently, the formal character of $V(2,2)$ as a representation of $\mathfrak{so}_3\,\mathbb{C}$ is

$$e(-8\omega_1) + 2e(-6\omega_1) + 4e(-4\omega_1) + 4e(-2\omega_1) + 5e(0\omega_1)\,,$$
$$+4e(2\omega_1) + 4e(4\omega_1) + 2e(6\omega_1) + e(8\omega_1)$$

which is equal to $\mathrm{ch}_{V(8)} + \mathrm{ch}_{V(6)} + 2\mathrm{ch}_{V(4)} + \mathrm{ch}_{V(0)}$. This proves (12), and (13), (14) and (15) are similar.

We resume the discussion of the main content of subsection 2.4. Before stating the main theorem, we collect some preliminary facts in the following lemma.

Lemma 2.4.2. (1) *The following deccomposition of $L \subset V(2,2)$ as $\mathrm{SO}_3\,\mathbb{C}$-subspace of $V(2,2)$ holds (possibly after interchanging the roles of $V(2,2)_4$ and $V(2,2)'_4$):*

$$L = V(2,2)_8 \oplus V(2,2)_4 \oplus V(2,2)_0\,.$$

(2) *The map $\epsilon(\cdot, h_0) : V(0,4) \to V(2,2)$ is an SO_3-equivariant isomorphism onto L.*

(3) *Putting $Y := \epsilon(\cdot, h_0)^{-1}(Z) \subset V(0,4)$, we have $h_0^2 \in Y$.*

(4) *One has $\Gamma(V(0,4)) \subset V(1,1)_4 \subset V(1,1)$, and the inclusion $Y \subset \Gamma^{-1}(0)$ holds.*

Proof. (1): Using the explicit form of Φ one calculates that the dimension of the image of the $\mathrm{SO}_3\,\mathbb{C}$-equivariant map $\Phi(\cdot, h_0) : V(2,2) \to V(2,1)$ is 12. Thus, in view of the decomposition (13) of $V(2,1)$ as $\mathrm{SO}_3\,\mathbb{C}$-representation, we must have $\Phi(V(2,2), h_0) = V(2,1)_6 \oplus V(2,1)_4$. Since

$$\dim V(a,b) = \frac{1}{2}(a+1)(b+1)(a+b+2)\,, \tag{16}$$

the dimension of $V(2,2)$ is 27 and the kernel L of $\Phi(\cdot, h_0)$ has dimension 15; in fact, $V(2,2)_8$, $V(2,2)_0$ and (after possibly exchanging $V(2,2)_4$ and $V(2,2)'_4$) $V(2,2)_4$ must all be in the kernel, since these representations do not appear in the decomposition of the image.
(2): Using the explicit form of ϵ given in Appendix A (66), one calculates that the dimension of the image of $\epsilon(\cdot, h_0)$ is 15 whence this linear map is injective. Moreover, its image is contained in L, hence equals L, because the map $V(0,4) \times V(0,2) \to V(2,1)$ given by $(f,g) \mapsto \Phi(\epsilon(f), g)$ is identically zero since there is no $V(2,1)$ in the decomposition of $V(0,4) \otimes \mathrm{Sym}^2 V(0,2)$.
(3): As we saw in theorem 2.3.3 (1), $\mathbb{C}h_0 \in X$, and we have $0 \neq \Psi(h_0^2) \in Z$. From the decomposition (12), we get, $\Psi(h_0^2)$ being invariant, $\langle \Psi(h_0^2)\rangle_{\mathbb{C}} = L^{\mathrm{SO}_3\,\mathbb{C}}$. By the decomposition (15), we get that the preimage under $\epsilon(\cdot, h_0)$ of $\Psi(h_0^2)$ spans the $\mathrm{SO}_3\,\mathbb{C}$-invariants $V(0,4)_0$ which are thus in Y. So in particular, $h_0^2 \in Y$.
(4): The first part is straightforward: Just decompose $\mathrm{Sym}^2 V(0,4)$ as $\mathrm{SO}_3\,\mathbb{C}$-module by the methods explained above, and check that it does not contain any $\mathrm{SO}_3\,\mathbb{C}$-submodule the highest weight of which has numerical label 2 (this suffices by (14)). The second statement of (4) follows from the observation that the map $\zeta : V(0,4) \times V(0,2) \to V(1,1)$ (Def. 2.4.1 (4)) factors:

$$c \cdot \zeta = \tilde{\gamma} \circ \epsilon\,,\ c \in \mathbb{C}^*\,,$$

where $\tilde{\gamma} : V(2,2) \to V(1,1)$ is the unique (up to nonzero scalar) non-trivial $\mathrm{SL}_3\,\mathbb{C}$-equivariant map which is homogeneous of degree 2. This is because $V(1,1)$ occurs in the decomposition of $\mathrm{Sym}^2 V(0,4) \otimes \mathrm{Sym}^2 V(0,2)$ with multiplicity one, and $\tilde{\gamma} \circ \epsilon$ is not identically zero, as follows from the explicit form of these maps (cf. Appendix A, (66), (68)). Thus, defining $\tilde{\Gamma} : V(0,4) \to V(1,1)$ by $\tilde{\Gamma}(\cdot) := (\tilde{\gamma} \circ \epsilon)(\cdot, h_0)$ (which thus differs from Γ just by a nonzero scalar), we must show $\tilde{\Gamma}(Y) = 0$. But recalling the definitions of Y, $\tilde{\Gamma}$ and Z (Def. 2.4.1 (1)), it suffices to show that $\tilde{\gamma} \circ \Psi$ is identically zero; the latter is true since it is an $\mathrm{SL}_3\,\mathbb{C}$-equivariant map from $V(0,4)$ to $V(1,1)$, homogeneous of degree 6, but $\mathrm{Sym}^6 V(0,4)$ does not contain $V(1,1)$. This proves (4). $\square$

Let us now pass from $\mathrm{SO}_3\,\mathbb{C}$ to the $\mathrm{PSL}_2\,\mathbb{C}$-picture and denote by $V(d)$ the space of binary forms of degree d in the variables z_1, z_2. This is of course consistent with our previous notation since, under the isomorphism $\mathfrak{so}_3\,\mathbb{C} \simeq \mathfrak{sl}_2\,\mathbb{C}$, $V(d)$ is just the irreducible $\mathfrak{so}_3\,\mathbb{C}$-module the highest weight of which has numerical label d; since we consider $\mathrm{PSL}_2\,\mathbb{C}$-representations, d is always even.
We will fix a covering $\mathrm{SL}_2\,\mathbb{C} \to \mathrm{SO}_3\,\mathbb{C}$ and thus an isomorphism $\mathrm{PSL}_2\,\mathbb{C} \simeq \mathrm{SO}_3\,\mathbb{C}$, and we will fix isomorphisms $\delta_1 : V(0) \oplus V(4) \oplus V(8) \to V(0,4)$ and $\delta_2 : V(4) \to V(1,1)_4$ such that $(1,0,0)$ maps to h_0^2 under δ_1 and both δ_1 and δ_2 are equivariant with respect to the isomorphism $\mathrm{PSL}_2\,\mathbb{C} \simeq \mathrm{SO}_3\,\mathbb{C}$; we will discuss in a moment how this is done, but for now this is not important. Look at the diagram

$$
\begin{array}{ccccc}
h_0^2 & & \xleftarrow{\quad\delta_1\quad} & (1,0,0) & U := \delta_1^{-1}(Y) \\
 & & & & \cap \\
Y \subset \Gamma^{-1}(0) \subset V(0,4) & & \underset{\delta_1}{\xleftarrow{\ \simeq\ }} & & V(0) \oplus V(4) \oplus V(8) \\
\Gamma|_{\Gamma^{-1}(0)} \Big\downarrow \quad \Big\downarrow \Gamma & & & & \Big\downarrow \delta := \delta_2^{-1} \circ \Gamma \circ \delta_1 \\
0 \in V(1,1)_4 & & \underset{\delta_2}{\xleftarrow{\ \simeq\ }} & & V(4) \\
\cap & & & & \\
V(1,1) \simeq V(1,1)_4 \oplus V(1,1)_2 & & & &
\end{array}
$$

By part (4) of lemma 2.4.2, we have $\delta^{-1}(0) \supset U$, and by part (3) of the same lemma, $(1,0,0) \in U$. Moreover, recalling our construction of X in theorem 2.3.3, we see that $\dim X = \dim \mathbb{P} V(0,4) - \dim \mathbb{P} V(0,2) = 14 - 5 = 9$, whence, chasing through the definitions of Z, Y, U, we get $\dim U = 10$. But the explicit form of δ (we will see this in a moment) allows us to conclude, by explicit calculation of the rank of the differential of δ at the invariant point $(1,0,0)$, that $\dim T_{(1,0,0)} U = 10$, whence $T_{(1,0,0)} U = V(0) \oplus V(8)$. Therefore, as U is irreducible, it is the unique component of the (possibly reducible) variety $\delta^{-1}(0)$ passing through $(1,0,0)$. Moreover, it is clear the condition $\{\delta = 0\}$ amounts to 5 quadratic equations! We have proven

Theorem 2.4.3. *There is an isomorphism*

$$
\mathbb{C}(\mathbb{P} V(0,4))^{\mathrm{SL}_3\,\mathbb{C}} \simeq \mathbb{C}(U)^{\mathrm{PSL}_2\,\mathbb{C} \times \mathbb{C}^*} \tag{17}
$$

where

$$\delta : V(0) \oplus V(4) \oplus V(8) \to V(4)$$

is PSL_2*-equivariant and homogeneous of degree* 2*, and* U *is the unique irreducible component of* $\delta^{-1}(0)$ *passing through* $(1,0,0)$*. Moreover,* $\dim U = 10$ *and* $T_{(1,0,0)}\, U = V(0) \oplus V(8)$*.*

We close this section by describing the explicit form of the covering $\mathrm{SL}_2\,\mathbb{C} \to \mathrm{SO}_3\,\mathbb{C}$ and the maps δ_1, δ_2, and by making some remarks on transvectants and the final formula for the map δ.
Let e_1, e_2, e_3 be the standard basis in $\mathbb{C}^3$, and denote by x_1, x_2, x_3 the dual basis in $(\mathbb{C}^3)^\vee$. In this notation, $h_0^2 = x_1x_3 - x_2^2$. We may view the $x's$ as coordinates on $\mathbb{C}^3$ and identify $\mathbb{C}^3$ with the Lie algebra $\mathfrak{sl}_2\,\mathbb{C}$ by assigning to (x_1, x_2, x_3) the matrix

$$X = \begin{pmatrix} x_2 & -x_1 \\ x_3 & -x_2 \end{pmatrix} \in \mathfrak{sl}_2\,\mathbb{C}\,.$$

Consider the adjoint representation Ad of $\mathrm{SL}_2\,\mathbb{C}$ on $\mathfrak{sl}_2\,\mathbb{C}$. Clearly, for $X \in \mathfrak{sl}_2\,\mathbb{C}$, $A \in \mathrm{SL}_2\,\mathbb{C}$, the map $\mathrm{Ad}(A) : X \mapsto AXA^{-1}$ preserves the determinant of X, which is just our h_0; the kernel of Ad is the center $\{\pm 1\}$ of $\mathrm{SL}_2\,\mathbb{C}$, and since $\mathrm{SL}_2\,\mathbb{C}$ is connected, the image of Ad is $\mathrm{SO}_3\,\mathbb{C}$. This is how we fix the isomorphism $\mathrm{PSL}_2\,\mathbb{C} \simeq \mathrm{SO}_3\,\mathbb{C}$ explicitly, and how we view $\mathrm{SO}_3\,\mathbb{C}$ as a subgroup of $\mathrm{SL}_3\,\mathbb{C}$. Note that the induced isomorphism $\mathfrak{sl}_2\,\mathbb{C} \to \mathfrak{so}_3\,\mathbb{C}$ on the Lie algebra level can be described as follows:

(18)
$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$
$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(where we view $\mathfrak{so}_3\,\mathbb{C}$ as a subalgebra of $\mathfrak{sl}_3\,\mathbb{C}$ in a way consistent with the inclusion on the group level described above). For example,

$$\mathrm{ad}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)(X) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 & -x_1 \\ x_3 & -x_2 \end{pmatrix}$$
$$- \begin{pmatrix} x_2 & -x_1 \\ x_3 & -x_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_3 & -2x_1 \\ 0 & -x_3 \end{pmatrix},$$

so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

To give the isomorphism $\delta_1 : V(0) \oplus V(4) \oplus V(8) \to V(0,4)$ explicitly, we just have to find highest weight vectors inside $V(0)$, $V(4)$, $V(8)$ and corresponding highest weight vectors inside $V(0,4)$. For example, h acts on $z_2^4 \in V(4)$ by multiplication by 4, and z_2^4 is killed by e, so this is a highest weight vector inside $V(4)$. But if we compute

$$\begin{aligned} h \cdot (x_1 x_3^3 - x_2^2 x_3^2) &= (h \cdot x_1) x_3^3 + 3x_1 (h \cdot x_3) x_3^2 - 2(h \cdot x_2) x_2 x_3^2 \\ -2x_2^2 (h \cdot x_3) x_3 &= (-2x_1) x_3^3 + 3x_1 (2x_3) x_3^2 - 2 \cdot 0 \cdot x_2 x_3^2 \\ -2x_2^2 (2x_3) x_3 &= 4(x_1 x_3^3 - x_2^2 x_3^2) \quad \text{and} \\ e \cdot (x_1 x_3^3 - x_2^2 x_3^2) &= (e \cdot x_1) x_3^3 + 3x_1 (e \cdot x_3) x_3^2 - 2(e \cdot x_2) x_2 x_3^2 \\ -2x_2^2 (e \cdot x_3) x_3 &= (-2x_2) \cdot x_3^3 + 3x_1 \cdot 0 \cdot x_3^2 - 2(-x_3) x_2 x_3^2 \\ -2x_2^2 \cdot 0 \cdot x_3 &= 0 \end{aligned}$$

(use (18) and remark that the x's are dual variables, so we have to use the dual action), then we find that a corresponding highest weight vector for the submodule of $V(0,4)$ isomorphic to $V(4)$ is $x_1 x_3^3 - x_2^2 x_3^2$. Proceeding in this way, we see that we can define δ_1 uniquely by the requirements:

$$\delta_1 \, : \, 1 \mapsto h_0^2 \, , \; z_2^4 \mapsto x_1 x_3^3 - x_2^2 x_3^2 \, , \; z_2^8 \mapsto x_3^4 \, , \tag{19}$$

and using the Lie algebra action and linearity, we can compute the values of δ_1 on a set of basis vectors in $V(0) \oplus V(4) \oplus V(8)$.

To write down δ_2 explicitly, remark that $V(1,1)$ may be viewed as the $\mathrm{SL}_3\,\mathbb{C}$-submodule of $\mathbb{C}^3 \otimes (\mathbb{C}^3)^\vee$ consisting of those tensors that are annihilated by

$$\Delta := \frac{\partial}{\partial e_1} \otimes \frac{\partial}{\partial x_1} + \frac{\partial}{\partial e_2} \otimes \frac{\partial}{\partial x_2} + \frac{\partial}{\partial e_3} \otimes \frac{\partial}{\partial x_3} \, .$$

We take again our highest weight vector $z_2^4 \in V(4)$, and all we have to do is to find a vector in $\mathbb{C}^3 \otimes (\mathbb{C}^3)^\vee$ on which h acts by multiplication by 4 and which is annihilated by e and Δ. Indeed, $e_1 x_3$ is one such. Thus we define δ_2 by

$$\delta_2 \, : \, z_2^4 \mapsto e_1 x_3 \, .$$

Then it is easy to compute the values of δ_2 on basis elements of $V(4)$ in the same way as for δ_1.

Let us recall the classical notion of *transvectants* ("Überschiebung " in

German). Let d_1, d_2, n be nonnegative integers such that $0 \le n \le \min(d_1, d_2)$. For $f \in V(d_1)$ and $g \in V(d_2)$ one puts

$$\psi_n(f,g) := \frac{(d_1-n)!}{d!}\frac{(d_2-n)!}{d_2!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{\partial^n f}{\partial z_1^{n-i}\partial z_2^i}\frac{\partial^n g}{\partial z_1^i \partial z_2^{n-i}} \tag{20}$$

(cf. [B-S], p. 122). The map $(f,g) \mapsto \psi_n(f,g)$ is a bilinear and $\mathrm{SL}_2\,\mathbb{C}$-equivariant map from $V(d_1) \times V(d_2)$ onto $V(d_1+d_2-2n)$. The map

$$\begin{aligned} V(d_1)\otimes V(d_2) &\to \bigoplus_{n=0}^{\min(d_1,d_2)} V(d_1+d_2-2n) \\ (f,g) &\mapsto \sum_{n=0}^{\min(d_1,d_2)} \psi_n(f,g) \end{aligned}$$

is an isomorphism of $\mathrm{SL}_2\,\mathbb{C}$-modules ("Clebsch-Gordan decomposition"). Thus transvectants make the decomposition of $V(d_1)\otimes V(d_2)$ into irreducibles explicit; a similar result for $\mathrm{SL}_3\,\mathbb{C}$-representations would be very important in several areas of computational invariant theory and also for the rationality question for moduli spaces of plane curves, but is apparently unknown.

The explicit form of δ that results from the computations is then

$$\begin{aligned} \delta(f_0,f_4,f_8) = &-\frac{6}{1225}\psi_6(f_8,f_8)+\frac{1}{840}\psi_4(f_8,f_4) \\ &+\frac{11}{54}\psi_2(f_4,f_4)-\frac{7}{36}f_4f_0\,, \end{aligned} \tag{21}$$

where $(f_0,f_4,f_8) \in V(0)\oplus V(4)\oplus V(8)$. Note that the fact that δ turns out to be such a linear combination of transvectants is no surprise in view of the Clebsch-Gordan decomposition: In fact, δ may be viewed as map

$$\delta' : (V(0)\oplus V(4)\oplus V(8))\otimes(V(0)\oplus V(4)\oplus V(8)) \to V(4)$$

and using the fact that δ is symmetric and collecting only those tensor products in the preceding formula for which $V(4)$ is a subrepresentation, we see that δ comes from a map

$$\begin{aligned} \delta'' : &(V(0)\otimes V(4))\oplus(V(4)\otimes V(4)) \\ &\oplus(V(8)\otimes V(4))\oplus(V(8)\otimes V(8)) \to V(4)\,. \end{aligned}$$

Thus it is clear from the beginning that δ will be a linear combination of ψ_6, ψ_4, ψ_2, ψ_0 as in formula (21), and the actual coefficients are easily calculated once we know δ explicitly!

In fact, the next lemma shows that the actual coefficients of the transvectants ψ_i's occurring in δ are not very important.

Lemma 2.4.4. *For* $\lambda := (\lambda_0, \lambda_2, \lambda_4, \lambda_6) \in \mathbb{C}^4$ *consider the homogeneous of degree* 2 PSL_2*-equivariant map*

$$\delta_\lambda : V(8) \oplus V(0) \oplus V(4) \to V(4)$$

$$f_8 + f_0 + f_4 \mapsto \lambda_6\psi_6(f_8, f_8) + 2\lambda_4\psi_4(f_8, f_4) + \lambda_2\psi_2(f_4, f_4) + 2\lambda_0 f_4 f_0\,.$$

Suppose that $\lambda_0 \neq 0$*. Then:*

(1) *One has* $1 \in \delta_\lambda^{-1}(0)$ *and* $T_1\,\delta_\lambda^{-1}(0) = V(8) \oplus V(0)$*; thus there is a unique irreducible component* U_λ *of* $\delta_\lambda^{-1}(0)$ *passing through* 1 *on which* 1 *is a smooth point.*

(2) *If furthermore* $\lambda \in (\mathbb{C}^*)^4$*, then* $\mathbb{P}U_\lambda$ *is* $\mathrm{PSL}_2\,\mathbb{C}$*-equivariantly isomorphic to* $\mathbb{P}U_{(1,6\epsilon,1,6)}$ *for some* $\epsilon \neq 0$ *(depending on* λ*).*

Proof. Part (1) is a straightforward calculation, and for part (2) we choose complex numbers μ_0, μ_4, μ_8 with the properties $6\mu_8^2 = \lambda_6$, $\mu_4\mu_8 = \lambda_4$, $\mu_0\mu_4 = \lambda_0$, and compute ϵ from $6\epsilon\mu_4^2 = \lambda_2$. Then the map from $\mathbb{P}U_\lambda$ to $\mathbb{P}U_{(1,6\epsilon,1,6)}$ given by sending $[f_0 + f_4 + f_8]$ to $[\mu_0 f_0 + \mu_4 f_4 + \mu_8 f_8]$ gives the desired isomorphism. □

In the next section we will see that for any $\epsilon \neq 0$, the $\mathrm{PSL}_2\,\mathbb{C}$-quotient of $\mathbb{P}U_{(1,6\epsilon,1,6)}$ is rational, and so the same holds for $\mathbb{P}U_\lambda$ for any $\lambda \in (\mathbb{C}^*)^4$; note however that the reduction step in lemma 2.4.4 (2) just simplifies the subsequent calculations, but is otherwise not substantial.

3. Further sections and inner projections

3.1. **Binary quartics again and a** $(\mathrm{PSL}_2\,\mathbb{C}, \mathfrak{S}_4)$**-section.** All the subsequent constructions and calculations depend very much on the geometry of the $\mathrm{PSL}_2\,\mathbb{C}$-action on the module $V(4)$. In fact, the first main point in the proof that $\mathbb{P}U_\lambda/\mathrm{PSL}_2\,\mathbb{C}$ is rational will be the construction of a $(\mathrm{PSL}_2\,\mathbb{C}, \mathfrak{S}_4)$-section of this variety ($\mathfrak{S}_4$ being the group of permutations of 4 elements); this is done by using proposition 2.1.2 (2) for the projection of $V(8) \oplus V(0) \oplus V(4)$ to $V(4)$ and producing such a section for $V(4)$ via the concept of *stabilizer in general position* which we recall next.

Definition 3.1.1. Let G be a linear algebraic group G acting on an irreducible variety X. A stabilizer in general position (s.g.p.) for the action of G on X is a subgroup H of G such that the stabilizer of a general point in X is conjugate to H in G.

An s.g.p. (if it exists) is well-defined to within conjugacy, but it need not exist in general; however, for the action of a reductive group G on an irreducible smooth affine variety, an s.g.p. always exists by results of Richardson and Luna (cf. [Po-Vi], §7).

Proposition 3.1.2. *For the action of* $\mathrm{PSL}_2\,\mathbb{C}$ *on* $V(4)$*, an s.g.p. is given by the subgroup* H *generated by*

$$\omega := \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \quad \text{and} \quad \rho := \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right].$$

H *is isomorphic to the Klein four-group* $\mathfrak{V}_4 \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *and its normalizer* $N(H)$ *in* $\mathrm{PSL}_2\,\mathbb{C}$ *is isomorphic to* $\mathfrak{S}_4$*; one has* $N(H)/H \simeq \mathfrak{S}_3$*.*
More explicitly, $N(H) = \langle \tau, \sigma \rangle$*, where, putting* $\theta := \exp(2\pi i/8)$*, one has*

$$\tau := \left[\begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix}\right], \quad \sigma := \left[\frac{1}{\sqrt{2}}\begin{pmatrix} \theta^3 & \theta^7 \\ \theta^5 & \theta^5 \end{pmatrix}\right].$$

Proof. We will give a geometric proof due to Bogomolov ([Bog1], p.18). A general homogeneous degree 4 binary form $f \in V(4)$ determines a set of 4 points $\Sigma \subset \mathbb{P}^1$; the double cover of $\mathbb{P}^1$ with branch points Σ is an elliptic curve; it is acted on by its subgroup of 2-torsion points $H_f \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and this action commutes with the sheet exchange map, hence descends to an action of H_f on $\mathbb{P}^1$ which preserves the point set Σ and thus the polynomial f; in general H_f will be the full automorphism group of the point set Σ since a general elliptic curve does not have complex multiplication.
Let us see that H_f is conjugate to H: H_f is generated by two commuting reflections γ_1, γ_2 acting on the Riemann sphere $\mathbb{P}^1$ (with two fixed points each). By applying a suitable projectivity, we see that H_f is conjugate to $\langle \omega, \gamma_2' \rangle$ where γ_2' is another reflection *commuting* with ω; thus ω interchanges the fixed points of γ_2' and also the fixed points of ρ: Thus if we change coordinates via a suitable dilation (a projectivity preserving the fixed points of ω), γ_2' goes over to ρ, and thus H_f is conjugate to H.
One computes that σ and τ normalize H; in fact, $\sigma^{-1}\omega\sigma = \rho$, $\sigma^{-1}\rho\sigma = \omega\rho$, and $\tau^{-1}\omega\tau = \omega\rho$, $\tau^{-1}\rho\tau = \rho$. Moreover, τ has order 4 and σ order 3, $(\tau\sigma)^2 = 1$, thus one has the relations

$$\tau^4 = \sigma^3 = (\tau\sigma)^2 = 1\,.$$

It is known that $\mathfrak{S}_4$ is the group on generators R, S with relations $R^4 = S^2 = (RS)^3 = 1$; mapping $R \mapsto \tau^{-1}$, $S \mapsto \tau\sigma$, we see that the group $\langle \tau, \sigma \rangle < N(H)$ is a quotient of $\mathfrak{S}_4$; since $\langle \tau, \sigma \rangle$ contains elements of order 4 and order 3, its order is at least 12, but since there are no normal subgroups of order 2 in $\mathfrak{S}_4$, $\mathfrak{S}_4 = \langle \tau, \sigma \rangle$. To finish the proof, it therefore suffices to note that the order of $N(H)$ is at most 24: For this one just has to show that the centralizer of H in $\mathrm{PSL}_2\,\mathbb{C}$ is just H, for then $N(H)/H$ is a subgroup of the group of permutations of the three nontrivial elements $H - \{1\}$ in H (in fact equal to it). Elements in $\mathrm{PGL}_2\,\mathbb{C}$ commuting with ω must be of the form

$$\left[\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right] \quad \text{or} \quad \left[\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\right],$$

and if these commute also with ρ, the elements 1, ω, ρ, $\omega\rho$ are the only possibilities. □

Corollary 3.1.3. *The variety $(V(4)^H)^0 \subset V(4)$ consisiting of those points whose stabilizer in $\mathrm{PSL}_2\,\mathbb{C}$ is exactly H is a $(\mathrm{PSL}_2\,\mathbb{C}, N(H))$-section of $V(4)$.*

Proof. The fact that the orbit $\mathrm{PSL}_2\,\mathbb{C}\cdot(V(4)^H)^0$ is dense in $V(4)$ follows since a general point in $V(4)$ has stabilizer conjugate to H; the assertion $\forall g \in \mathrm{PGL}_2\,\mathbb{C}$, $\forall x \in (V(4)^H)^0 : gx \in (V(4)^H)^0 \implies g \in N(H)$ is clear by definition. □

Let us recall the representation theory of $N(H) = \mathfrak{S}_4$ viewed as the group of permutations of four letters $\{a,\, b,\, c,\, d\}$; the character table is as follows (cf. [Se]).

	1	(ab)	$(ab)(cd)$	(abc)	$(abcd)$
χ_0	1	1	1	1	1
ϵ	1	-1	1	1	-1
θ	2	0	2	-1	0
ψ	3	1	-1	0	-1
$\epsilon\psi$	3	-1	-1	0	1

V_{χ_0} is the trivial 1-dimensional representation, V_ϵ is the 1-dimensional representation where $\epsilon(g)$ is the sign of the permutation g; $\mathfrak{S}_4 = N(H)$ being the semidirect product of $N(H)/H = \mathfrak{S}_3$ by the normal subgroup H, V_θ is the irreducible two-dimensional representation induced from the representation of $\mathfrak{S}_3$ acting on the elements of $\mathbb{C}^3$ which satisfy $x+y+z=0$ by permutation of coordinates. V_ψ is the extension to $\mathbb{C}^3$ of the natural representation of $\mathfrak{S}_4$ on $\mathbb{R}^3$ as the group of rigid motions stabilizing a regular tetrahedron; finally, $V_{\epsilon\psi} = V_\epsilon \otimes V_\psi$.
We want to decompose $V(8) \oplus V(0) \oplus V(4)$ as $N(H)$-module; we fix

the notation:

$$
\begin{gathered}
a_0 := 1; \quad a_1 := z_1^4 + z_2^4,\ a_2 := 6z_1^2z_2^2,\ a_3 := z_1^4 - z_2^4, \\
a_4 := 4(z_1^3z_2 - z_1z_2^3),\ a_5 := 4(z_1^3z_2 + z_1z_2^3); \\
e_1 := 28(z_1^6z_2^2 - z_1^2z_2^6),\ e_2 := 56(z_1^7z_2 + z_1^5z_2^3 - z_1^3z_2^5 - z_1z_2^7), \\
e_3 := 56(z_1^7z_2 - z_1^5z_2^3 - z_1^3z_2^5 + z_1z_2^7),\ e_4 := z_1^8 - z_2^8 \\
e_5 := 8(z_1^7z_2 - 7z_1^5z_2^3 + 7z_1^3z_2^5 - z_1z_2^7), \\
e_6 := 8(z_1^7z_2 + 7z_1^5z_2^3 + 7z_1^3z_2^5 + z_1z_2^7), \\
e_7 := z_1^8 + z_2^8,\ e_8 := 28(z_1^6z_2^2 + z_1^2z_2^6),\ e_9 := 70z_1^4z_2^4 .
\end{gathered} \tag{22}
$$

Lemma 3.1.4. *One has the following decompositions as $N(H)$-modules:*

$$
V(0) = V_{\chi_0},\ V(4) = V_\psi \oplus V_\theta,\ V(8) = V_{\epsilon\psi} \oplus V_\psi \oplus V_\theta \oplus V_{\chi_0} . \tag{23}
$$

More explicitly,

$$
\begin{gathered}
V(0) = \langle a_0\rangle,\ V(4) = \langle a_3, a_4, a_5\rangle \oplus \langle a_1, a_2\rangle, \\
V(8) = \langle e_4, e_5, e_6\rangle \oplus \langle e_1, e_2, e_3\rangle \oplus \langle e_8, 7e_7 - e_9\rangle \oplus \langle 5e_7 + e_9\rangle .
\end{gathered} \tag{24}
$$

Here $\langle e_4, e_5, e_6\rangle$ corresponds to $V_{\epsilon\psi}$ and $\langle e_1, e_2, e_3\rangle$ corresponds to V_ψ. Moreover,

$$
V(0)^H = \langle a_0\rangle,\ V(4)^H = \langle a_1, a_2\rangle,\ V(8)^H = \langle e_7, e_8, e_9\rangle . \tag{25}
$$

Proof. We will prove (25) first; one observes that quite generally for $k \geq 0$, $V(2k)^H = (V(2k)^\rho)^\omega$ (ρ and ω commute) and that the monomials $z_1^j z_2^{2k-j}$, $j = 0, \ldots, 2k$, are invariant under ρ if $j + k$ is even, and otherwise anti-invariant, so if $k = 2s$, $\dim V(2k)^\rho = 2s + 1$, and if $k = 2s + 1$, $\dim V(2k)^\rho = 2s + 1$. Since ω is also a reflection, we have $2\dim(V(2k)^\rho)^\omega - \dim V(2k)^\rho = \mathrm{tr}(\omega|_{V(2k)^\rho})$, and the trace is 1 for $k = 2s$, and -1 for $k = 2s + 1$, thus

$$
\dim V(2k)^H = s + 1,\ k = 2s, \quad \dim V(2k)^H = s,\ k = 2s + 1 .
$$

In particular, the H-invariants in $V(0)$, $V(4)$, $V(8)$ have the dimensions as claimed in (25), and one checks that the elements given there are indeed invariant.

To prove (23), we use the Clebsch-Gordan formula $V(2k) \otimes V(2) = V(2k + 2) \oplus V(2k) \oplus V(2k - 2)$ (cf. (20)) iteratively together with the fact that the character of the tensor product of two representations of a finite group is the product of the characters of each of the factors; since $V(2)$ has dimension 3 and $\dim V(2)^H = 0$, $V(2)$ is irreducible; the value of the character of the $N(H)$-module $V(2)$ on τ is 1, so $V(2) = V_{\epsilon\psi}$.

Now $V(2) \otimes V(2) = V(4) \oplus V(2) \oplus V(0)$, and looking at the character table, one checks that

$$(\epsilon\psi)^2 = \chi_0 + (\epsilon\psi) + (\psi) + (\theta)\,.$$

This proves the decomposition in (23) for $V(4)$. The decomposition for $V(8)$ is proven similarly (one proves $V(6) = V_\psi \oplus V_{\epsilon\psi} \oplus V_\epsilon$ first).
The proof of (24) now amounts to checking that the given spaces are invariant under σ and τ; finally note that $V_{\epsilon\psi}$ corresponds to $\langle e_4, e_5, e_6\rangle$ since the value of the character on τ is 1. □

Recall from Lemma 2.4.4 that we want to prove the rationality of $(\mathbb{P}\, U_\lambda)/\mathrm{PSL}_2\,\mathbb{C}$ and we can and will always assume in the sequel that $\lambda = (1, 6\epsilon, 1, 6)$ for $\epsilon \neq 0$. In view of Lemma 3.1.4 it will be convenient for subsequent calculations to write the map $\delta_\lambda : V(8)\oplus V(0)\oplus V(4) \to V(4)$ in terms of the basis $(e_1, \dots, e_9, a_0, a_1, \dots, a_5)$ in the source and the basis $(a_1, \dots, a_5)$ in the target. Denote coordinates in $V(8)\oplus V(0)\oplus V(4)$ with respect to the chosen basis by $(x_1, \dots, x_9, s_0, s_1, \dots, s_5) =: (x, s)$. Then one may write

$$\delta_\lambda(x, s) = \begin{pmatrix} Q_1(x,s) \\ \vdots \\ Q_5(x,s) \end{pmatrix} \tag{26}$$

with $Q_1(x, s), \dots, Q_5(x, s)$ quadratic in (x, s); their values may be computed using formulas (20), (22), and the definition of δ_λ in Lemma 2.4.4, and they can be found in Appendix B.

Theorem 3.1.5. *Let $\tilde{\mathcal{Q}}_\lambda \subset V(8)\oplus V(0)\oplus V(4)$ be the subvariety defined by the equations $Q_1 = \dots = Q_5 = 0$, $s_3 = s_4 = s_5 = 0$. There is exactly one 7-dimensional irreducible component $\mathcal{Q}_\lambda$ of $\tilde{\mathcal{Q}}_\lambda$ passing through the $N(H)$-invariant point $5e_7 + e_9$ in $V(8)$; $\mathcal{Q}_\lambda$ is $N(H)$-invariant and*

$$\mathbb{C}(\mathbb{P}\, U_\lambda)^{\mathrm{PSL}_2\,\mathbb{C}} = \mathbb{C}(\mathbb{P}\,\mathcal{Q}_\lambda)^{N(H)}\,. \tag{27}$$

Proof. We want to use Proposition 2.1.2, (2).
Note that $5e_7 + e_9 \in U_\lambda$: In fact, δ_λ maps the $N(H)$-invariants in $V(8) \oplus V(0) \oplus V(4)$ to the $N(H)$-invariants in $V(4)$ which are 0. Since U_λ is the unique irreducible component of $\delta_\lambda^{-1}(0)$ passing through $a_0 = 1$, U_λ contains the whole plane of invariants $\langle a_0, 5e_7 + e_9\rangle$.
If we denote by $p : V(8) \oplus V(0) \oplus V(4) \to V(4)$ the projection, then $\tilde{\mathcal{Q}}_\lambda = p^{-1}(V(4)^H) \cap \delta_\lambda^{-1}(0)$. Clearly, $\tilde{\mathcal{Q}}_\lambda$ is $N(H)$-invariant, and one only has to check that $5e_7 + e_9$ is a nonsingular point on it with tangent space of dimension 7 by direct calculation: Then there is a unique 7-dimensional irreducible component $\mathcal{Q}_\lambda$ of $\tilde{\mathcal{Q}}_\lambda$ passing through $5e_7+e_9$ which is $N(H)$-invariant (since $5e_7 + e_9$ is an invariant point on it and

this point is nonsingular on $\tilde{\mathcal{Q}}_\lambda$).
It remains to prove (27): $\mathcal{Q}_\lambda$ is an irreducible component of $p^{-1}(V(4)^H) \cap U_\lambda$ and $\mathcal{Q}_\lambda^0 = \mathcal{Q}_\lambda \cap p^{-1}((V(4)^H)^0)$ is a dense $N(H)$-invariant open subset of $\mathcal{Q}_\lambda$ dominating $(V(4)^H)^0$. Thus by Proposition 2.1.1 (2),

$$\mathbb{C}(\mathbb{P}\, U_\lambda)^{\mathrm{PSL}_2 \mathbb{C}} \simeq \mathbb{C}(\mathbb{P}\, \mathcal{Q}_\lambda^0)^{N(H)} \simeq \mathbb{C}(\mathbb{P}\, \mathcal{Q}_\lambda)^{N(H)} .$$

□

3.2. **Dividing by the action of H.** Next we would like to "divide out" the action by H, so that we are left with an invariant theory problem for the group $N(H)/H = \mathfrak{S}_3$. Look back at the action of $N(H)$ on $M := \{s_3 = s_4 = s_5 = 0\} \subset V(8) \oplus V(0) \oplus V(4)$ which is explained in formulas (23), (24); we will adopt the notational convention to denote the irreducible $N(H)$-submodule of $V(8)$ isomorphic to V_ψ by $V(8)_{(\psi)}$ and so forth; thus

(28)
$$M = V(0)_{(\chi_0)} \oplus V(4)_{(\theta)} \oplus V(8)_{(\chi_0)} \oplus V(8)_{(\theta)} \oplus V(8)_{(\psi)} \oplus V(8)_{(\epsilon\psi)} ,$$

and looking at the character table of $\mathfrak{S}_4$, we see that the action of H is nontrivial only on $V(8)_{(\psi)} \oplus V(8)_{(\epsilon\psi)} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle$ where x_1, x_2, x_3 and x_4, x_5, x_6 are coordinates; in terms of these, we have

(29)
$$\begin{aligned} (\omega)(x_1, \dots, x_6) &= (-x_1,\ x_2,\ -x_3,\ -x_4,\ x_5,\ -x_6) , \\ (\rho)(x_1, \dots, x_6) &= (x_1,\ -x_2,\ -x_3,\ x_4,\ -x_5,\ -x_6) , \\ (\omega\rho)(x_1, \dots, x_6) &= (-x_1,\ -x_2,\ x_3,\ -x_4,\ -x_5,\ x_6) , \end{aligned}$$

and

(30)
$$\begin{aligned} \tau(x_1, \dots, x_6) &= (-x_1,\ -ix_3,\ -ix_2,\ x_4,\ -ix_6,\ -ix_5) , \\ \sigma(x_1, \dots, x_6) &= \left(4x_3, -\frac{i}{4}x_1,\ ix_2,\ -8x_6,\ -\frac{i}{8}x_4,\ -ix_5\right) . \end{aligned}$$

Thus we see that the map

$$\begin{aligned} \mathbb{P}(V(8)_{(\psi)} \oplus V(8)_{(\epsilon\psi)}) - \{x_1x_2x_3 = 0\} &\to R \times \mathbb{P}^2 , \\ (x_1, \dots, x_6) &\mapsto \left(\left(\frac{x_4}{x_1}, \frac{x_5}{x_2}, \frac{x_6}{x_3} \right) ,\ \left(\frac{1}{x_1^2} : \frac{1}{x_2^2} : \frac{1}{x_3^2} \right) \right) , \end{aligned}$$

where $R = \mathbb{C}^3$, is dominant with fibres H-orbits, and furthermore $N(H)$-equivariant for a suitable action of $N(H)$ on $R \times \mathbb{P}^2$: In fact, we will agree to write

$$\left(\frac{1}{x_1^2} : \frac{1}{x_2^2} : \frac{1}{x_3^2} \right) = \left(\frac{x_2x_3}{x_1} : \frac{x_3x_1}{x_2} : \frac{x_1x_2}{x_3} \right)$$

and remark that the subspaces

$$R = \left\langle \frac{x_4}{x_1}, \frac{x_5}{x_2}, \frac{x_6}{x_3} \right\rangle , \quad T := \left\langle \frac{x_2x_3}{x_1}, \frac{x_3x_1}{x_2}, \frac{x_1x_2}{x_3} \right\rangle$$

of the field of fractions of $\mathbb{C}\left[V(8)_{(\psi)} \oplus V(8)_{(\epsilon\psi)}\right]$ are invariant under σ and τ (thus $\mathbb{P}^2 = \mathbb{P}(T)$). If we denote the coordinates with respect to the basis vectors in R resp. T given above by r_1, r_2, r_3 resp. y_1, y_2, y_3, then the actions of τ and σ are described by

$$\tau(r_1, r_2, r_3) = (-r_1, r_3, r_2)\, , \ \sigma(r_1, r_2, r_3) = (-2r_3, r_1/2, -r_2)$$
$$\tau(y_1, y_2, y_3) = (y_1, -y_3, -y_2)\, , \ \sigma(y_1, y_2, y_3) = ((1/16)y_3, -16y_1, -y_2)\, .$$

Thus the only $N(H)$-invariant lines in R resp. T are the ones spanned by $(2, 1, -1)$ resp. $(-1, 16, -16)$ on which τ acts by multiplication by -1 resp. by $+1$ and hence

$$R = R_{(\epsilon)} \oplus R_{(\theta)}\, , \ T = T_{(\chi_0)} \oplus T_{(\theta)}\, . \tag{31}$$

We see that the morphism

$$\begin{aligned} &\pi \ : \ \mathbb{P}(M) - \{x_1x_2x_3 = 0\} \\ &\to R \times \mathbb{P}(T \oplus V(8)_{(\chi_0)} \oplus V(8)_{(\theta)} \oplus V(0)_{(\chi_0)} \oplus V(4)_{(\theta)}) \simeq R \times \mathbb{P}^8\, , \\ &\pi(x, s) := \left(\left(\frac{x_4}{x_1}, \frac{x_5}{x_2}, \frac{x_6}{x_3} \right) \, , \ \left(\frac{x_2x_3}{x_1} : \frac{x_3x_1}{x_2} : \frac{x_1x_2}{x_3} \right. \right. \\ &\qquad \left. : x_7 : x_8 : x_9 : s_0 : s_1 : s_2 \right) \end{aligned} \tag{32}$$

is $N(H)$-equivariant, dominant, and all fibres are H-orbits. If we consider $(x_7, x_8, x_9, s_0, s_1, s_2)$ as coordinates in $V(8)_{(\chi_0)} \oplus V(8)_{(\theta)} \oplus V(0)_{(\chi_0)} \oplus V(4)_{(\theta)}$ in the *target* of the map π (as we do in formula (32)) we denote them by $(y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$ to achieve consistency with [Kat2].

How do we get equations which define the image

$$\pi(\mathbb{P}\,\tilde{\mathcal{Q}}_\lambda \cap \{x_1x_2x_3 \neq 0\}) \subset R \times (\mathbb{P}^8 - \{y_1y_2y_3 = 0\})$$

in $\mathbb{P}^8 - \{y_1y_2y_3 = 0\}$ from the quadrics $Q_1(x, s), \dots, Q_5(x, s)$ in formula (26)? We can set $s_3 = s_4 = s_5 = 0$ in $Q_1, \dots, Q_5$ to obtain equations $\bar{Q}_1, \dots, \bar{Q}_5$ for $\mathbb{P}\,\tilde{\mathcal{Q}}_\lambda$ in $\mathbb{P}(M)$; the point is now that the quantities

$$\bar{Q}_1,\ \bar{Q}_2,\ \frac{\bar{Q}_3}{x_1},\ \frac{\bar{Q}_4}{x_2},\ \frac{\bar{Q}_4}{x_3}$$

are H-invariant (as one sees from the equations in Appendix B). Moreover, the map

$$\pi : \mathbb{P}(M) - \{x_1x_2x_3 = 0\} \to R \times (\mathbb{P}^8 - \{y_1y_2y_3 = 0\})$$

is a geometric quotient for the action of H on the source (by [Po-Vi], Thm. 4.2), so we can write

$$\bar{Q}_1 = q_1(r_1, \dots, y_{12}),\ \bar{Q}_2 = q_2(r_1, \dots, y_{12}),\ \frac{\bar{Q}_3}{x_1} = q_3(r_1, \dots, y_{12}),$$
$$\frac{\bar{Q}_4}{x_2} = q_4(r_1, \dots, y_{12}),\ \frac{\bar{Q}_4}{x_3} = q_5(r_1, \dots, y_{12})$$

where $q_1, \dots, q_5$ are polynomials in $(r_1, r_2, r_3),\ (y_1, y_2, y_3, y_7, \dots, y_{12})$ which one may find written out in Appendix B. Here we just want to emphasize their structural properties which will be most important for the subsequent arguments:

(1) The polynomials q_1, q_2 are homogeneous of degree 2 in the set of variables $(y_1, \dots, y_{12})$; the coefficients of the monomials in the y's are (inhomogeneous) polynomials of degrees ≤ 2 in r_1, r_2, r_3. For $r_1 = r_2 = r_3 = 0$, q_1, q_2 do not vanish identically.
(2) The polynomials q_3, q_4, q_5 are homogeneous linear in $(y_1, \dots, y_{12})$; the coefficients of the monomials in the y's are (inhomogeneous) polynomials of degrees ≤ 2 in r_1, r_2, r_3. For $r_1 = r_2 = r_3 = 0$, q_3, q_4, q_5 do not vanish identically.

Theorem 3.2.1. *Let $\tilde{Y}_\lambda$ be the subvariety of $R \times \mathbb{P}^8$ defined by the equations $q_1 = q_2 = q_3 = q_4 = q_5 = 0$. There is an irreducible $N(H)$-invariant component Y_λ of $\tilde{Y}_\lambda$ with $\pi([x^0]) \in Y_\lambda$, where $x^0 := 13i(5e_7 + e_9) + 5(4e_1 - ie_2 + e_3)$, such that*

$$\mathbb{C}(\mathbb{P}\,\mathcal{Q}_\lambda)^{N(H)} \simeq \mathbb{C}(Y_\lambda)^{N(H)}\,. \tag{33}$$

Proof. The variety Y_λ will be the the closure of the image $\pi(\mathbb{P}\,\mathcal{Q}_\lambda \cap \{x_1x_2x_3 \neq 0\})$ in $R \times \mathbb{P}^8$.
It remains to see that $x^0 \in \mathcal{Q}_\lambda$. Recall from Theorem 3.1.5 that $\mathcal{Q}_\lambda$ is the unique irreducible component of $\tilde{\mathcal{Q}}_\lambda$ passing through the $N(H)$-invariant point $5e_7+e_9$, and that this point is a nonsingular point on $\tilde{\mathcal{Q}}_\lambda$; thus, if we can find an irreducible subvariety of $\tilde{\mathcal{Q}}_\lambda$ which contains both $5e_7 + e_9$ and x^0, we are done. The sought-for subvariety is $\tilde{\mathcal{Q}}_\lambda \cap V(8)^\sigma$, where $V(8)^\sigma$ are the elements in $V(8)$ invariant under $\sigma \in N(H)$. One sees that x^0 and $5e_7 + e_9$ lie on it, and computing

$$V(8)^\sigma = \langle 5e_7 + e_9, 8e_4 - ie_5 - e_6, 4e_1 - ie_2 + e_3 \rangle\,,$$
$$V(4)^\sigma = \langle 2(z_1^4 - z_2^4) + 4(z_1^3 z_2 + z_1 z_2^3) + 4i(z_1^3 z_2 - z_1 z_2^3) \rangle\,,$$

and using $\delta_\lambda(V(8)^\sigma) \subset V(4)^\sigma$, we find that $\tilde{\mathcal{Q}}_\lambda \cap V(8)^\sigma$ is a quadric in $V(8)^\sigma$ which is easily checked to be irreducible. □

Thus it remains to prove the rationality of $Y_\lambda/N(H) = Y_\lambda/\mathfrak{S}_3$.

3.3. **Inner projections and the "no-name" method.** The variety $\tilde{Y}_\lambda$ comes with the two projections

$$\begin{array}{ccc} \tilde{Y}_\lambda & \xrightarrow{p_{\mathbb{P}^8}} & \mathbb{P}^8 \\ {\scriptstyle p_R}\downarrow & & \\ R & & \end{array}$$

Recall from (32) that $N := \mathbb{P}(V(8)_\theta \oplus V(4)_\theta) \subset \mathbb{P}^8$ is an $N(H)$-invariant 3-dimensional projective subspace of $\mathbb{P}^8$. We will show $\mathbb{C}(Y_\lambda)^{N(H)} \simeq \mathbb{C}(R \times N)^{N(H)}$ via the following theorem.

Theorem 3.3.1. *There is an open $N(H)$-invariant subset $R_0 \subset R$ containing $0 \in R$ with the following properties:*

(1) *For all $r \in R_0$ the fibre $p_R^{-1}(r) \subset \tilde{Y}_\lambda$ is irreducible of dimension 3, and $p_R^{-1}(R_0)$ is an open $N(H)$-invariant subset of Y_λ.*

(2) *There exist $N(H)$-sections σ_1, σ_2 of the $N(H)$-equivariant projection $R_0 \times \mathbb{P}^8 \to R_0$ such that $N(r) := \langle \sigma_1(r), \sigma_2(r), (1:0:0:\dots:0), (0:1:0:\dots:0), (0:0:1:0:\dots:0)\rangle \subset \mathbb{P}^8$, $r \in R_0$, is an $N(H)$-invariant family of 4-dimensional projective subspaces in $\mathbb{P}^8$ with the properties:*
   - (i) *$N(r)$ is disjoint from N for all $r \in R_0$.*
   - (ii) *The fibre $p_{\mathbb{P}^8}(p_R^{-1}(r)) \subset \mathbb{P}^8$ contains the line $\langle \sigma_1(r), \sigma_2(r)\rangle \subset N(r)$ for all $r \in R_0$.*
   - (iii) *The projection $\pi_r : \mathbb{P}^8 \dashrightarrow N$ from $N(r)$ to N maps the fibre $p_{\mathbb{P}^8}(p_R^{-1}(r)) \subset \mathbb{P}^8$ dominantly onto N for all $r \in R_0$.*

Before turning to the proof, let us note the following corollary.

Corollary 3.3.2. *One has the field isomorphism*

$$\mathbb{C}(Y_\lambda)^{N(H)} \simeq \mathbb{C}(R \times N)^{N(H)},$$

and the latter field is rational. Hence $\mathfrak{M}_3$ is rational.

Proof. (of corollary) The $N(H)$-invariant set $p_R^{-1}(R_0)$ is an open subset of Y_λ. Let us see that the projection $\pi_r : F_r := p_{\mathbb{P}^8}(p_R^{-1}(r)) \dashrightarrow N$ is birational. In fact, F_r is of dimension 3 and irreducible and the intersection of a 3-codimensional linear subspace and two quadrics in $\mathbb{P}^8$. Moreover, $F_r \cap N(r)$ contains a line L_r by Theorem 3.3.1 (2), (ii). Thus for a general point P in N, $F_r \cap \langle L_r, P\rangle$ consists of L_r and a single point (namely the point of intersection of the two lines which are the residual intersections of each of the two quadrics defining F_r with $\langle L_r, P\rangle$, the other component being L_r itself). Thus π_r is generically one-to-one whence birational.

Thus one has a birational $N(H)$-isomorphism $p_R^{-1}(R_0) \dashrightarrow R_0 \times N$, given by sending $(r, [y])$ to $(r, \pi_r([y]))$. Thus one gets the field isomorphism in Corollary 3.3.2.

By the no-name lemma (cf. e.g. [Dol1], section 4), $\mathbb{C}(R \times N)^{N(H)} \simeq \mathbb{C}(N)^{N(H)}(T_1, T_2, T_3)$, where T_1, T_2, T_3 are indeterminates, thus it suffices to show that the quotient of N by $N(H)$ is stably rational of level ≤ 3. This in turn follows from the same lemma, since clearly, if we take the representation of $\mathfrak{S}_3$ in $\mathbb{C}^3$ by permutation of coordinates, the quotient of $\mathbb{P}(\mathbb{C}^3)$ by $\mathfrak{S}_3$, a unirational surface, is rational. $\square$

Proof. (of theorem) The proof will be given in several steps.

Step 1. (Irreducibility of the fibre over 0*)* We have to show that the variety $p_{\mathbb{P}^8} p_R^{-1}(0)) \subset \mathbb{P}^8$ is irreducible and 3-dimensional. We have explicit equations for it (namely the ones that arise if we substitute $r_1 = r_2 = r_3 = 0$ in $q_1, \dots, q_5$, which are thus 3 linear and 2 quadratic equations); the assertions can then be checked with a computer algebra system such as `Macaulay 2`. Recall from Theorem 3.2.1 that Y_λ contains $\pi([x^0])$. In fact,

$$(34) \quad \pi([x^0]) = \left((0,0,0), \left(-\frac{5}{4} : 20 : -20 : 65 : 0 : 13 : 0 : 0 : 0\right)\right),$$

as follows from the definition of x^0 in Theorem 3.2.1 and the definition of π in (32). Thus $\pi([x^0])$ lies in the fibre over 0 of p_R^{-1} and thus, since there is an open subset around 0 in R over which the fibres are irreducible and 3-dimensional, assertion (1) of Theorem 3.3.1 is established.

Step 2. (Construction of σ_1*)* To obtain σ_1, we just assign to $r \in R$ the point $(r, \sigma_1(r))$ with $\sigma_1(r) = (0 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0)$, i.e. $y_{10} = 1$, the other y's being 0. This always is in the fibre $p_{\mathbb{P}^8}(p_R^{-1}(r))$ as one sees on substituting in the equations $q_1, \dots, q_5$. Moreover, this is an $N(H)$-section, since y_{10} is a coordinate in the space $V(0)_{\chi_0}$ in formula (32).

Step 3. (Construction of σ_2*; decomposition of* $V := \mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$ *)* The construction of a section σ_2, $\sigma_2(r) = (\sigma_2^{(1)}(r) : \dots : \sigma_2^{(9)}(r))$, involves a little more work. Let us look back at the construction of Y_λ in subsection 3.2 for this, especially the definition of the projection π in formula (32), and the decomposition of the linear subspace $M \subset V(8) \oplus V(0) \oplus V(4)$. By definition of R, the family of codimension 3 linear subspaces

$$(35) \quad L(r) := \{[(x,s)] \mid x_4 = r_1 x_1,\ x_5 = r_2 x_2,\ x_6 = r_3 x_3\} \subset \mathbb{P}(M),$$

$r = (r_1, r_2, r_3) \in R$, is $N(H)$-invariant, i.e. $gL(r) = L(gr)$, for $g \in N(H)$. It is natural to intersect this family with $\mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$ which, as we will see, has dimension 3 and look for an H-orbit $\mathfrak{O}_r$ in the intersection of $\mathbb{P}(\delta_\lambda^{-1}(0)$ with the open set of $L(r)$ where $x_1x_2x_3 \neq 0$. Moreover, we will see that for $r = 0$, the point $[x^0]$ is in this intersection. Thus passing to the quotient we may put

$$(r, \sigma_2(r)) := \pi(\mathfrak{O}_r) \tag{36}$$

to obtain a σ_2 with the required properties. Indeed, note that we will have $\sigma_2^(7)(r) = \sigma_2^(8) = \sigma_2^(9) = 0$ which ensures that σ_2 and σ_1 span a line. Moreover,

$$\sigma_2(0) = \left(-\frac{5}{4} : 20 : -20 : 65 : 0 : 13 : 0 : 0 : 0 \right) , \tag{37}$$

by formula (34), which allows us to check assertions (2), (i) and (iii) of Theorem 3.3.1, which are open properties on the base R, by explicit computation for the fibre over 0. Property (2), (ii) stated in the theorem is clear by construction. Let us now carry out this program. We will start by explicitly decomposing $V := \mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$ into irreducible components.

To guess what V might be, note that according to the definition of δ_λ in Lemma 2.4.4, δ_λ vanishes on $f_8 \in V(8)$ if for the transvectant ψ_6 one has $\psi_6(f_8, f_8) = 0$; but looking back at the definition of transvectants in formula (20), we see that $\psi_6 : V(8) \times V(8) \to V(4)$ vanishes if f_8 is a linear combination of z_1^8, $z_1^7 z_2$ and $z_1^6 z_2^2$ (since we differentiate at least 3 times with respect to z_2 in one factor in the summands in formula (20)). Thus $X_1 := \overline{\mathrm{PSL}_2\,\mathbb{C} \cdot \langle z_1^8,\ z_1^7 z_2,\ z_1^6 z_2^2 \rangle}$, the variety of forms of degree 8 with a six-fold zero, is contained in V, and one computes that the differential of $\delta_\lambda|_{V(8)}$ in $z_1^6 z_2^2$ is surjective, so that X_1 is an irreducible component of V.

The dimension of X_1 is clearly three. Weyman, in [Wey], Cor. 4, computed the Hilbert function of $X_{p,g}$, the variety of binary forms of degree g having a root of multiplicity $\geq p$ which is

$$H(X_{p,g}, d) = (dp + 1)\binom{g - p + d}{g - p} - (d(p + 1) - 1)\binom{g - p + d - 1}{g - p - 1} .$$

For $d = 6$, $g = 8$, the leading term in d in this expression is $3d^3$, which shows

$$\deg X_1 = 18 . \tag{38}$$

Moreover, we know already that $\overline{5e_7 + e_9}$ is in V from the proof of Theorem 3.1.5; thus set $X_2 := \overline{\mathrm{PSL}_2\,\mathbb{C} \cdot \langle 5e_7 + e_9 \rangle}$. We know that the stabilizer of $5e_7+e_9$ in $\mathrm{PSL}_2\,\mathbb{C}$ contains $N(H)$ because $5e_7+e_9 = 5z_1^8+5z_2^8+70z_1^4z_2^4$ spans the $N(H)$-invariants in $V(8)$ by Lemma 3.1.4. The claim is that the stabilizer is not larger. An easy way to check this is to use the beautiful theory developed in [Ol], p. 188 ff., using differential invariants and signature curves, which allows the explicit determination of the order of the symmetry group of a complex binary form. More precisely we have (cf. [Ol], Cor. 8.68):

Theorem 3.3.3. *Let $Q(p)$ be a binary form of degree n (written in terms of the inhomogeneous coordinate $p=z_1/z_2$) which is not equivalent to a monomial. Then the cardinality k of the symmetry group of $Q(p)$ satisfies*

$$k \le 4n-8\,,$$

provided that U is not a constant multiple of H^2, where U and H are the following polynomials in p: $H := (1/2)(Q,Q)^{(2)}$, $T := (Q,H)^{(1)}$, $U := (Q,T)^{(1)}$ where, if Q_1 is a binary form of degree n_1, and Q_2 is a binary form of degree n_2, we put

$$\begin{aligned}(Q_1,Q_2)^{(1)} &:= n_2Q_1'Q_2 - n_1Q_1Q_2'\,,\\ (Q_1,Q_2)^{(2)} &:= n_2(n_2-1)Q_1''Q_2 - 2(n_2-1)(n_1-1)Q_1'Q_2'\\ &\quad + n_1(n_1-1)Q_1Q_2''\,.\end{aligned}$$

(these are certain transvectants).

Applying this result in our case, we find the upper bound 24 for the symmetry group of $5e_7+e_9$, which is indeed the order of $N(H)=\mathfrak{S}_4$. X_2 is irreducible of dimension 3, and computing that the differential of $\delta_\lambda|_{V(8)}$ is surjective in $5e_7+e_9$, we get that X_2 is another irreducible component of V. But let us intersect X_2 with the codimension 3 linear subspace in $V(8)$ consisting of forms with zeroes $\zeta_1,\zeta_2,\zeta_3\in\mathbb{P}^1$; there is a unique projectivity carrying these two three roots of $5e_7+e_9$, which are all distinct, thus there are $8\cdot7\cdot6$ such projectivities, and $\deg X_2 \ge (8\cdot7\cdot6)/|N(H)|$. But one checks easily that V itself has dimension 3 and is the intersection of 5 quadrics in $\mathbb{P}(V(8))$, thus has degree ≤ 32. Thus we must have

$$\deg X_2 = 14,\ V = X_1\cup X_2,\ \deg V = 32. \tag{39}$$

Note also that

$$[x^0]\in X_2\cap L(0)\,. \tag{40}$$

In fact, from the proof of Theorem 3.2.1, we know $[x^0]\in V$, and $[x^0]\in L(0)$ being clear, we just check that x^0 has no root of multiplicity ≥ 6.

Step 4. (Construction of σ_2; intersecting V with a family of linear spaces in $\mathbb{P}(M)$) Let $L^0(r)$ be the open subset of $L(r) \subset \mathbb{P}(M)$ where $x_1x_2x_3 \neq 0$. According to the strategy outlined at the beginning of Step 3, we would like to compute the cardinalities

$$|L^0(r) \cap X_1|, \quad |L^0(r) \cap X_2|,$$

for r varying in a small neighbourhood of 0 in R. It is, however, easier from a computational point of view to determine the number of intersection points of X_1 resp. X_2 with certain boundary components of $L^0(r)$ in $L(r)$ first; the preceding cardinalities will afterwards fall out as the residual quantities needed to have $\deg X_1 = 18$, $\deg X_2 = 14$. Thus let us introduce the following additional strata of $L(r) \backslash L^0(r)$:

$$\begin{aligned} L_0 &:= \{[(x,s)] \,|\, x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0\}, \\ L_1(r) &:= \{[(x,s)] \,|\, x_1 \neq 0,\ x_4 = r_1x_1,\ x_2 = x_3 = x_5 = x_6 = 0\}, \\ L_2(r) &:= \{[(x,s)] \,|\, x_2 \neq 0,\ x_5 = r_2x_2,\ x_1 = x_3 = x_4 = x_6 = 0\}, \\ L_3(r) &:= \{[(x,s)] \,|\, x_3 \neq 0,\ x_6 = r_3x_3,\ x_1 = x_2 = x_4 = x_5 = 0\}, \\ \tilde{L}_1(r) &:= \{[(x,s)] \,|\, x_2x_3 \neq 0,\ x_5 = r_2x_2,\ x_6 = r_3x_3,\ x_1 = x_4 = 0\} \\ \tilde{L}_2(r) &:= \{[(x,s)] \,|\, x_1x_3 \neq 0,\ x_4 = r_1x_1,\ x_6 = r_3x_3,\ x_2 = x_5 = 0\} \\ \tilde{L}_3(r) &:= \{[(x,s)] \,|\, x_1x_2 \neq 0,\ x_4 = r_1x_1,\ x_5 = r_2x_2,\ x_3 = x_6 = 0\}. \end{aligned} \tag{41}$$

$L(r)$ is the disjoint union of these and $L^0(r)$. From the equations describing δ_λ one sees that V is defined in $\mathbb{P}(V(8))$ with coordinates $x_1, \dots, x_9$ by

$$\begin{aligned} -192x_6^2 - 192x_3x_6 + 384x_3^2 - 192x_5^2 - 192x_2x_5 + 384x_2^2 \\ -12x_1x_4 + 12x_7x_8 + 180x_8x_9 = 0, \end{aligned} \tag{42}$$

$$\begin{aligned} 64x_6^2 - 192x_3x_6 - 128x_3^2 - 64x_5^2 + 192x_2x_5 + 128x_2^2 \\ -2x_4^2 + 16x_1^2 + 2x_7^2 - 16x_8^2 - 50x_9^2 = 0, \end{aligned} \tag{43}$$

$$\begin{aligned} 96x_5x_6 - 672x_3x_5 - 672x_2x_6 + 1248x_2x_3 \\ -12x_1x_7 + 12x_4x_8 + 180x_1x_9 = 0, \end{aligned} \tag{44}$$

$$\begin{aligned} 6x_4x_6 + 42x_3x_4 + 84x_1x_6 + 156x_1x_3 \\ -6x_5x_7 - 42x_2x_7 + 24x_5x_8 - 264x_2x_8 + 30x_5x_9 - 30x_2x_9 = 0, \end{aligned} \tag{45}$$

$$\begin{aligned} -6x_4x_5 - 42x_2x_4 + 84x_1x_5 + 156x_1x_2 \\ +6x_6x_7 + 42x_3x_7 + 24x_6x_8 - 264x_3x_8 - 20x_6x_9 + 30x_3x_9 = 0, \end{aligned} \tag{46}$$

and thus

$$\tilde{L}_i(r) \cap V = \emptyset \quad \forall i = 1, 2, 3 \tag{47}$$

for r in a Zariski open neighbourhood of $0 \in R$ (for $\tilde{L}_1(r)$ consider equation (44) and assume $96r_2r_3 - 672r_2 - 672r_3 + 1248 \neq 0$, for $\tilde{L}_2(r)$ we see that (45) cannot hold if $6r_1r_3 + 42r_1 + 84r_3 + 156 \neq 0$, and for $\tilde{L}_3(r)$ equation (46) is impossible provided that $-6r_1r_2 - 42r_1 + 84r_2 + 156 \neq 0$).

Let us consider the intersection $V \cap L_0$. We have to solve the equations

$$12x_7x_8 + 180x_8x_9 = 0, \quad 2x_7^2 - 16x_8^2 - 50x_9^2 = 0,$$

which have the four distinct solutions $(x_7, x_8, x_9) = (5, 0, \pm 1)$, $(x_7, x_8, x_9) = (15, \pm 5, -1)$, whence

$$L_0 \cap V = \{[5e_7 \pm e_9], [15e_7 \pm 5e_8 - e_9]\}. \tag{48}$$

We will also have to determine the intersection $V \cap L_1(r)$ explicitly. We have to solve the equations

$$\begin{gathered} -12r_1x_1^2 + 12x_7x_8 + 180x_8x_9 = 0, \\ -2r_1^2x_1^2 + 16x_1^2 + 2x_7^2 - 16x_8^2 - 50x_9^2 = 0, \\ -12x_1x_7 + 12r_1x_1x_8 + 180x_1x_9 = 0, \end{gathered}$$

in the variables x_1, x_7, x_8, x_9. We can check (e.g. with `Macaulay 2`) that the subscheme they define has dimension 0 (and degree 8) for $r_1 = 0$. We already know four solutions with $x_1 = 0$, namely the ones given in formula (48). Then it suffices to check that

$$(x_1, x_7, x_8, x_9) = (\pm 1, r_1, 1, 0), \ (x_1, x_7, x_8, x_9) = (\pm a, (90 - 5r_1^2), -5r_1, 6),$$

where a is a square-root of $25(r_1^2 - 36)$, are also solutions (with $x_1 \neq 0$ in a neighbourhood of 0 in R, and obviously all distinct there). Thus

$$\begin{aligned} L_1(r) \cap V = \{&[\pm(e_1 + r_1e_4) + r_1e_7 + e_8], \\ &[\pm(ae_1 + r_1ae_4) + (90 - 5r_1^2)e_7 - 5r_1e_8 + 6e_9]\}. \end{aligned} \tag{49}$$

We still have to see how the intersection points $L_0 \cap V$ and $L_1(r) \cap V$ are distributed among X_1 and X_2: Suppose $f \in V(8)$ is a binary octic such that $[f] \in L_0 \cap \mathbb{P}(V(8))$ or $[f] \in L_1(r) \cap \mathbb{P}(V(8))$; then f is a linear combination of the binary octics e_1, e_4, e_7, e_8, e_9 defined in (22), which involve only even powers of z_1 and z_2; thus if $(a : b) \in \mathbb{P}^1$ is a root of one of them, so is its negative $(a : -b)$ whence

> $[f]$ lies in X_1 if and only if $(1 : 0)$ or $(0 : 1)$ is a root of multiplicity ≥ 6.

Applying this criterion, we get, using (48) and (49)

$$(50)\qquad \begin{gathered} L_0 \cap X_1 = \emptyset,\ L_0 \cap X_2 = \{[5e_7 \pm e_9], [15e_7 \pm 5e_8 - e_9]\}, \\ L_1(r) \cap X_1 = \{[\pm(e_1 + r_1 e_4) + r_1 e_7 + e_8]\}, \\ L_1(r) \cap X_2 = \{[\pm(ae_1 + r_1 ae_4) + (90 - 5r_1^2)e_7 - 5r_1 e_8 + 6e_9]\}\,. \end{gathered}$$

The reader may be glad to hear now that we do not have to repeat this entire procedure for $L_2(r)$ and $L_3(r)$; in fact, $L_1(r)$, $L_2(r)$, $L_3(r)$ are permuted by $N(H)$ in the following way: For the element $\sigma \in N(H)$ we have

$$\sigma \cdot L_1(r) = L_2(\sigma \cdot r), \quad \sigma \cdot L_2(r) = L_3(\sigma \cdot r), \quad \sigma \cdot L_3(r) = L_1(\sigma \cdot r)\,,$$

which follows from (30) (and (28)) and the definition of R. Thus we get that generally for $i = 1, 2, 3$

$$(51)\qquad L_i(r) \cap X_1 = \{P_1(r),\ P_2(r)\},\ L_i(r) \cap X_2 = \{Q_1(r),\ Q_2(r)\}$$

where $P_1(r)$, $P_2(r)$, $Q_1(r)$, $Q_2(r)$ are mutually distinct points, and this is valid in a Zariski open $N(H)$-invariant neighbourhood of $0 \in R$. It remains to check that

$L(0) \cap V$ consists of 32 reduced points.

We check (with `Macaulay 2`) that if we substitute $x_4 = x_5 = x_6 = 0$ in equations (42)-(46), they define a zero-dimensional reduced subscheme of degree 32 in the projective space with coordinates $x_1, x_2, x_3, x_7, x_8, x_9$. Taking into account (47), (50), (51), we see that all the intersections in equations (50), (51) are free of multiplicities in an open $N(H)$-invariant neighbourhood of $0 \in R$ and moreover, since $\deg X_1 = 18$, $\deg X_2 = 14$, we must have there

$L^0(r) \cap X_1$ consists of 12 reduced points, and $L^0(r) \cap X_2$ consists of 4 reduced points.

Now these 4 points make up the H-orbit $\mathfrak{O}_r$ we wanted to find in Step 3: Clearly $L^0(r) \cap X_2$ is H-invariant, and H acts with trivial stabilizers in $L^0(r)$ (as is clear from (29)). Thus we have completed the program outlined at the beginning of Step 3. It just remains to notice that $[x^0] \in X_2 \cap L^0(0)$. This is clear since $[x^0] \in V$, but x^0 does not have a root of multiplicity ≥ 6.

Step 5. (Verification of the properties of $N(r)$) For the completion of the proof of Theorem 3.3.1, it remains to verify the properties of the subspace $N(r)$ in parts (2), (i) and (iii) of that theorem. First of all, it is clear that

$$\begin{gathered} N(r) = \langle \sigma_1(r), \sigma_2(r),\ (1:0:0:\dots:0), \\ (0:1:0:\dots:0),\ (0:0:1:\dots:0)\rangle \end{gathered}$$

is $N(H)$-invariant in the sense that $g \cdot N(r) = N(g \cdot r)$ for $g \in N(H)$ by the construction of σ_1, σ_2 and because the last three vectors in the preceding formula are a basis in the invariant subspace $\mathbb{P}(T) \subset \mathbb{P}^8$ (where by (31) $T = T_{(\chi_0)} \oplus T_{(\theta)}$). Moreover, by the definition of σ_1 in Step 2, and the formula (37) for $\sigma_2(0)$, one has $\dim N(0) = 4$, which thus holds also for $r \in R$ sufficiently close to 0.
Recall that N was defined to be $N := \mathbb{P}(V(8)_{(\theta)} \oplus V(4)_{(\theta)}) \subset \mathbb{P}^8$, and as such can be described in terms of the coordinates $(y_1 : y_2 : y_3 : y_7 : y_8 : \dots : y_{12})$ in $\mathbb{P}^8$ as

$$N = \{y_1 = y_2 = y_3 = y_7 + 7y_9 = y_{10} = 0\}$$

(cf. (24)). Thus we get that $N(0) \cap N = \emptyset$, and the same holds in an open $N(H)$-invariant neighbourhood of 0 in R.
For Theorem 3.3.1, (2), (iii), it suffices to check that π_0 maps the fibre $p_{\mathbb{P}^8}(p_R^{-1}(0))$ dominantly onto N, which can be done by direct calculation. This concludes the proof. □

Appendix A. Collection of formulas for section 2

We start with some remarks on how to calculate equivariant projections, and then we give explicit formulas for the equivariant maps in section 2.
Let a, b be nonnegative integers, $m := \min(a, b)$, and let $G := \mathrm{SL}_3\,\mathbb{C}$. We denote the irreducible G-module whose highest weight has numerical labels a, b by $V(a, b)$. For $k = 0, \dots, m$ we define $V^k := \mathrm{Sym}^{a-k}\,\mathbb{C}^3 \otimes \mathrm{Sym}^{b-k}(\mathbb{C}^3)^\vee$. Let e_1, e_2, e_3 be the standard basis in $\mathbb{C}^3$ and x_1, x_2, x_3 the dual basis in $(\mathbb{C}^3)^\vee$.
There are G-equivariant linear maps $\Delta^k : V^k \to V^{k+1}$ for $k = 0, \dots, m-1$ and $\delta^k : V^k \to V^{k-1}$ for $k = 1, \dots, m$ given by

$$\Delta^k := \sum_{i=1}^{3} \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i}, \quad \delta^k := \sum_{i=1}^{3} e_i \otimes x_i \,. \tag{52}$$

(The superscript k thus only serves as a means to remember the sources and targets of the respective maps). If for some positive integers α, β the G-module V^k contains a G-submodule isomorphic to $V(\alpha, \beta)$ we will denote it by $V^k(\alpha, \beta)$ to indicate the ambient module (this is unambiguous because it is known that all such modules occur with multiplicity one).
It is clear that Δ^k is surjective and δ^k injective; one knows that $\ker(\Delta^k) = V^k(a-k, b-k)$ whence

$$V^k = \bigoplus_{i=k}^{m} V^k(a-i, b-i)\ . \tag{53}$$

We want to find a formula for the G-equivariant projection of $V^0 = \mathrm{Sym}^a\,\mathbb{C}^3 \otimes \mathrm{Sym}^b(\mathbb{C}^3)^\vee$ onto the subspace $V^0(a-i, b-i)$ for $i = 0, \dots, m$. We call this linear map $\pi^i_{a,b}$.

We remark that, by (53), one can decompose each vector $v \in V^0$ as $v = v_0 + \dots + v_m$ where $v_i \in V^0(a-i, b-i)$, and this decomposition is unique. Note that

$$\delta^1 \dots \delta^i(\ker \Delta^i) = V^0(a-i, b-i) \tag{54}$$

so that

$$V^0 = \ker \Delta^0 \oplus \delta^1(\ker \Delta^1) \oplus \delta^1\delta^2(\ker \Delta^2) \oplus \dots \oplus \delta^1 \dots \delta^i(\ker \Delta^i) \\ \oplus \dots \oplus \delta^1 \dots \delta^m(V^m)\ .$$

Of course, $\pi^i_{a,b}(v) = v_i$. It will be convenient to put

$$L^i := \delta^1 \circ \delta^2 \circ \dots \circ \delta^i \circ \Delta^{i-1} \circ \dots \circ \Delta^1 \circ \Delta^0\ , \quad i = 0, \dots, m \tag{55}$$

(whence L^0 is the identity) and

$$U^i := \Delta^{i-1} \circ \Delta^{i-2} \circ \dots \circ \Delta^0 \circ \delta^1 \circ \dots \circ \delta^{i-1} \circ \delta^i\ , \quad i = 0, \dots, m \tag{56}$$

(U^0 being again the identity). By Schur's lemma, we have

$$U^i|_{V^i(a-i,b-i)} = c_i \cdot \mathrm{id}_{V^i(a-i,b-i)}$$

for some nonzero rational number $c_i \in \mathbb{Q}^*$. This is easy to calculate: For example, since $e_1^{a-i} \otimes x_2^{b-i} \in \ker \Delta^i = V^i(a-i, b-i)$, we have that c_i is the unique number such that

$$U^i(e_1^{a-i} \otimes x_2^{b-i}) = c_i \cdot e_1^{a-i} \otimes x_2^{b-i}\ . \tag{57}$$

We will now calculate $\pi^{m-l}_{a,b}$ for $l = 0, \dots, m$ by induction on l; the case $l = 0$ can be dealt with as follows:

Write $v = v_1 + \dots + v_m \in V^0$ as before. Then $v_m = \delta^1\delta^2 \dots \delta^m(u_m)$ for some $u_m \in V^m$. Now

$$L^m(v) = L^m(v_m) = L^m(\delta^1\delta^2\dots\delta^m(u_m))$$
$$= \delta^1\delta^2\dots\delta^m \circ U^m(u_m) = c_m v_m$$

so we set

$$\pi^m_{a,b} := \frac{1}{c_m} L^m\,. \tag{58}$$

Now assume, by induction, that $\pi^{m-l}_{a,b}$, $\pi^{m-l+1}_{a,b}, \dots, \pi^m_{a,b}$ have already been determined. We show how to calculate $\pi^{m-l-1}_{a,b}$.
Now, by (54), $v_{m-l-1} \in \delta^1 \dots \delta^{m-l-1}(\ker \Delta^{m-l-1})$. We write $v_{m-l-1} = \delta^1 \dots \delta^{m-l-1}(u_{m-l-1})$, for some $u_{m-l-1} \in \ker \Delta^{m-l-1} = V^{m-l-1}(a - (m-l-1), b-(m-l-1))$, and using (57) we get

$$L^{m-l-1}\left(v - \sum_{i=0}^{l} \pi^{m-i}_{a,b}(v)\right) = L^{m-l-1}(v_0 + v_1 + \dots + v_{m-l-1})$$
$$= L^{m-l-1}(v_{m-l-1}) = L^{m-l-1}(\delta^1 \dots \delta^{m-l-1}(u_{m-l-1}))$$
$$= \delta^1 \dots \delta^{m-l-1} \circ \Delta^{m-l-2} \dots \Delta^0 \circ \delta^1 \dots \delta^{m-l-1}(u_{m-l-1})$$
$$= \delta^1 \dots \delta^{m-l-1} \circ U^{m-l-1}(u_{m-l-1}) = c_{m-l-1} v_{m-l-1}\,.$$

So we put

$$\pi^{m-l-1}_{a,b} := \frac{1}{c_{m-l-1}} \left(L^{m-l-1}\left(\mathrm{id}_{V^0} - \sum_{i=0}^{l} \pi^{m-i}_{a,b} \right)\right)\,. \tag{59}$$

Formulas (52), (55), (56), (57), (58), (59) contain the algorithm to compute the G-equivariant linear projection

$$\pi^i_{a,b} : V^0 \to V^0(a-i, b-i) \subset V^0$$

and thus to compute the associated G-equivariant bilinear map

$$\beta^i_{a,b} : V(a,0) \times V(0,b) \to V(a-i, b-i)$$

in suitable bases in source and target (remark that $V(a,0) = \mathrm{Sym}^a\,\mathbb{C}^3$ and $V(0,b) = \mathrm{Sym}^b\,(\mathbb{C}^3)^\vee$).

In particular, we obtain for $a = 2$, $b = 1$ the map

$$\pi^0_{2,1} \,:\, V^0 = \mathrm{Sym}^2\mathbb{C}^3 \otimes (\mathbb{C}^3)^\vee \to V(2,1) \subset V^0 \tag{60}$$

$$\pi^0_{2,1} = \mathrm{id} - \frac{1}{4}\delta^1\Delta^0 \,,$$

for $a = b = 2$ the map

$$\pi^0_{2,2} \,:\, V^0 = \mathrm{Sym}^2\mathbb{C}^3 \otimes \mathrm{Sym}^2(\mathbb{C}^3)^\vee \to V(2,2) \subset V^0 \tag{61}$$

$$\pi^0_{2,2} = \mathrm{id} - \frac{1}{5}\delta^1\Delta^0 + \frac{1}{40}\delta^1\delta^2\Delta^1\Delta^0 \,,$$

and for $a = b = 1$ the map

$$\pi^0_{1,1} \,:\, V^0 = \mathbb{C}^3 \otimes (\mathbb{C}^3)^\vee \to V(1,1) \subset V^0 \tag{62}$$

$$\pi^0_{1,1} = \mathrm{id} - \frac{1}{3}\delta^1\Delta^0 \,.$$

In the following, we will often view elements $x \in V(a,b)$ as tensors $x = (x^{i_1,\dots,i_b}_{j_1,\dots,j_a}) \in (\mathbb{C}^3)^{\otimes a} \otimes (\mathbb{C}^{3\vee})^{\otimes b} =: T^b_a\mathbb{C}^3$ (the indices ranging from 1 to 3) which are covariant of order b and contravariant of order a via the natural inclusions

$$V(a,b) \subset \mathrm{Sym}^a\mathbb{C}^3 \otimes \mathrm{Sym}^b(\mathbb{C}^3)^\vee \subset T^b_a\mathbb{C}^3$$

(the first inclusion arises since $V(a,b)$ is the kernel of Δ^0, the second is a tensor product of symmetrization maps). In particular, we have the determinant tensor $\det \in T^3_0\mathbb{C}^3$ and its inverse $\det^{-1} \in T^0_3\mathbb{C}^3$. In formulas involving several tensors, we will also adopt the summation convention throughout. Finally, we define

$$\mathrm{can} \,:\, T^b_a\mathbb{C}^3 \to \mathrm{Sym}^a\,\mathbb{C}^3 \otimes \mathrm{Sym}^b\,(\mathbb{C}^3)^\vee \,, \tag{63}$$

$$e_{j_1} \otimes \dots \otimes e_{j_a} \otimes x_{i_1} \otimes \dots \otimes x_{i_b} \mapsto e_{j_1} \cdot \dots \cdot e_{j_a} \otimes x_{i_1} \cdot \dots \cdot x_{i_b}$$

as the canonical projection.

We write down the explicit formulas for the equivariant maps in section 2. The map $\Psi : V(0,4) \to V(2,2)$ (degree 3) is given by

$$\Psi(f) := \pi^0_{2,2}(\mathrm{can}(g)) \,, \tag{64}$$

$$g^{i_1 i_2}_{j_1 j_2} := f^{i_1 i_2 i_3 i_4} f^{i_5 i_6 i_7 i_8} f^{i_9 i_{10} i_{11} i_{12}} \det{}^{-1}_{i_3 i_5 i_9} \det{}^{-1}_{i_4 i_6 i_{10}} \det{}^{-1}_{j_1 i_7 i_{11}} \det{}^{-1}_{j_2 i_8 i_{12}} \,.$$

The map $\Phi : V(2,2) \times V(0,2) \to V(2,1)$ (bilinear) is given by

$$\Phi(g,h) := \pi^0_{2,1}(\mathrm{can}(r)) \,, \tag{65}$$

$$r^{i_1}_{j_1 j_2} := g^{i_1 i_2}_{j_1 i_3} h^{i_3 i_4} \det{}^{-1}_{i_2 i_4 j_2} \,.$$

The map $\epsilon : V(0,4) \times V(0,2) \to V(2,2)$ (bilinear) is

$$\epsilon(f,h) := \mathrm{can}(g), \; g^{i_1 i_2}_{j_1 j_2} := f^{i_3 i_4 i_1 i_2} h^{i_5 i_6} \det{}^{-1}_{i_3 j_1 i_5} \det{}^{-1}_{i_4 j_2 i_6} \,. \tag{66}$$

The map $\zeta : V(0,4) \times V(0,2) \to V(1,1)$ (homogeneous of degree 2 in both factors) is given by

$$\zeta(f,h) := \pi^0_{1,1}(a)\,, \tag{67}$$
$$a^{i_1}_{j_1} := h^{i_1 i_2} h^{i_3 i_4} f^{i_5 i_6 i_7 i_8} f^{i_9 i_{10} i_{11} i_{12}} \det{}^{-1}_{i_5 i_9 j_1} \det{}^{-1}_{i_6 i_{10} i_2} \det{}^{-1}_{i_7 i_{11} i_3} \det{}^{-1}_{i_8 i_{12} i_4}\,.$$

The map $\tilde{\gamma} : V(2,2) \to V(1,1)$ (homogeneous of degree 2) is given by

$$\tilde{\gamma} := \pi^0_{1,1}(u)\,,\; u^{i_1}_{j_1} := g^{i_1 i_2}_{i_3 i_4} g^{i_3 i_4}_{j_1 i_2}\,. \tag{68}$$

Appendix B. Collection of formulas for section 3

In section 3.1, we saw (formula (26)) that

$$\delta_\lambda = Q_1(x,s)a_1 + Q_2(x,s)a_2 + Q_3(x,s)a_3 + Q_4(x,s)a_4 + Q_5(x,s)a_5\,. \tag{69}$$

We collect here the explicit values of the $Q_i(x,s)$ (recall $\lambda = (1, 6\epsilon, 1, 6)$, $\epsilon \neq 0$):

$$Q_1(x,s) = \hat{Q}_1(x) + 2x_7 s_1 + 12x_8 s_2 + 2x_9 s_1 + \epsilon(12 s_1 s_2) + 2 s_0 s_1$$
$$+48x_2 s_4 - 48 x_3 s_5 - 2x_4 s_3 + 16 x_5 s_4 - 16 x_6 s_5 + \epsilon(-12 s_4^2 - 12 s_5^2)\,, \tag{70}$$

$$Q_2(x,s) = \hat{Q}_2(x) + 4x_8 s_1 + 12 x_9 s_2 + \epsilon(2s_1^2 - 6 s_2^2) + 2 s_0 s_2$$
$$-4x_1 s_3 + 16 x_2 s_4 + 16 x_3 s_5 - 16 x_5 s_4 - 16 x_6 s_5 + \epsilon(-2s_3^2 - 4s_4^2 + 4 s_5^2)\,, \tag{71}$$

$$Q_3(x,s) = \hat{Q}_3(x) + 2x_4 s_1 + 12 x_1 s_2 + 64 x_2 s_5 + 64 x_3 s_4$$
$$-2x_7 s_3 + 2 x_9 s_3 + \epsilon(12 s_2 s_3 - 24 s_4 s_5) + 2 s_0 s_3\,, \tag{72}$$

$$Q_4(x,s) = \hat{Q}_4(x) + 4x_5 s_1 + 12 x_2 s_1 - 12 x_5 s_2 + 12 x_2 s_2$$
$$-8x_1 s_5 - 16 x_3 s_3 + 8 x_8 s_4 - 8 x_9 s_4 + \epsilon(-6 s_1 s_4 - 6 s_2 s_4 + 6 s_3 s_5) + 2 s_0 s_4\,, \tag{73}$$

$$Q_5(x,s) = \hat{Q}_5(x) + 4x_6 s_1 + 12 x_3 s_1 + 12 x_6 s_2 - 12 x_3 s_2$$
$$+8x_1 s_4 - 16 x_2 s_3 - 8 x_8 s_5 - 8 x_9 s_5 + \epsilon(6 s_1 s_5 - 6 s_2 s_5 - 6 s_3 s_4) + 2 s_0 s_5\,, \tag{74}$$

where

$$\hat{Q}_1(x) = -192x_6^2 - 192x_3x_6 + 384x_3^2 - 192x_5^2 - 192x_2x_5 + 384x_2^2 \\ -12x_1x_4 + 12x_7x_8 + 180x_8x_9, \tag{75}$$

$$\hat{Q}_2(x) = 64x_6^2 - 192x_3x_6 - 128x_3^2 - 64x_5^2 + 192x_2x_5 + 128x_2^2 \\ -2x_4^2 + 16x_1^2 + 2x_7^2 - 16x_8^2 - 50x_9^2, \tag{76}$$

$$\hat{Q}_3(x) = 96x_5x_6 - 672x_3x_5 - 672x_2x_6 + 1248x_2x_3 \\ -12x_1x_7 + 12x_4x_8 + 180x_1x_9, \tag{77}$$

$$\hat{Q}_4(x) = 6x_4x_6 + 42x_3x_4 + 84x_1x_6 + 156x_1x_3 \\ -6x_5x_7 - 42x_2x_7 + 24x_5x_8 - 264x_2x_8 + 30x_5x_9 - 30x_2x_9, \tag{78}$$

$$\hat{Q}_5(x) = -6x_4x_5 - 42x_2x_4 + 84x_1x_5 + 156x_1x_2 \\ +6x_6x_7 + 42x_3x_7 + 24x_6x_8 - 264x_3x_8 - 20x_6x_9 + 30x_3x_9\,. \tag{79}$$

The polynomials $q_1, \dots, q_5$ defining $\tilde{Y}_\lambda \subset R \times \mathbb{P}^8$ (cf. Theorem 3.2.1) are:

$$q_1 = (-192r_3^2 - 192r_3 + 384)y_1y_2 + (-192r_2^2 - 192r_2 + 384)y_1y_3 \\ +(-12r_1)y_2y_3 + 12y_7y_8 + 180y_8y_9 + 2y_7y_{11} + 12y_8y_{12} \\ +2y_9y_11 + \epsilon(12y_{11}y_{12}) + 2y_{10}y_{11}\,, \tag{80}$$

$$q_2 = (64r_3^2 - 192r_3 - 128)y_1y_2 + (-64r_2^2 + 192r_2 + 128)y_1y_3 \\ +(-2r_1^2 + 16)y_2y_3 + 2y_7^2 - 16y_8^2 - 50y_9^2 + 4y_8y_{11} + 12y_9y_{12} \\ +\epsilon(2y_{11}^2 - 6y_{12}^2) + 2y_{10}y_{12}\,, \tag{81}$$

$$q_3 = (96r_2r_3 - 672r_2 - 672r_3 + 1248)y_1 \\ -12y_7 + 12r_1y_8 + 180y_9 + 2r_1y_{11} + 12y_{12}\,, \tag{82}$$

(83)

$$q_4 = (6r_1r_3 + 42r_1 + 84r_3 + 156)y_2 + (-6r_2 - 42)y_7 + (24r_2 - 264)y_8 \\ +(30r_2 - 30)y_9 + (4r_2 + 12)y_{11} + (-12r_2 + 12)y_{12}\,,$$

(84)

$$q_5 = (-6r_1r_2 - 42r_1 + 84r_2 + 156)y_3 + (6r_1 + 42)y_7 + (24r_3 - 264)y_8 \\ +(-30r_3 + 30)y_9 + (4r_3 + 12)y_{11} + (12r_3 - 12)y_{12}\,.$$

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