THE RATIONALITY OF CERTAIN MODULI SPACES OF CURVES OF GENUS 3

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Introduction

The aim of this paper is to give an explicit geometric description of the birational structure of the moduli space of pairs (C, η), where C is a general curve of genus 3 over an algebraically closed field k of arbitrary characteristic and $\eta \in Pic^0(C)_3$ is a non trivial divisor class of 3-torsion on C.

As it was observed in [B-C04] lemma (2.18), if C is a general curve of genus 3 and $\eta \in Pic^0(C)_3$ is a non trivial 3 - torsion divisor class, then we have a morphism $\varphi_\eta := \varphi_{|K_C+\eta|} \times \varphi_{|K_C-\eta|} : C \to \mathbb{P}^1 \times \mathbb{P}^1$, corresponding to the sum of the linear systems $|K_C + \eta|$ and $|K_C - \eta|$, which is birational onto a curve $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(4, 4)$. Moreover, Γ has exactly six ordinary double points as singularities, located in the six points of the set $\mathcal{S} := \{(x, y) | x \neq y,\ x, y \in \{0, 1, \infty\}\}$.

In [B-C04] we only gave an outline of the proof (and there is also a minor inaccuracy). Therefore we dedicate the first section of this article to a detailed geometrical description of such pairs (C, η), where C is a general curve of genus 3 and $\eta \in Pic^0(C)_3 \setminus \{0\}$.

The main result of the first section is the following:

Theorem 0.1. *Let C be a general (in particular, non hyperelliptic) curve of genus 3 over an algebraically closed field k (of arbitrary characteristic) and $\eta \in Pic^0(C)_3 \setminus \{0\}$.*

Then the rational map $\varphi_\eta : C \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$\varphi_\eta := \varphi_{|K_C+\eta|} \times \varphi_{|K_C-\eta|} : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

is a morphism, birational onto its image Γ, which is a curve of bidegree $(4, 4)$ having exactly six ordinary double points as singularities. We can assume, up to composing φ_η with a transformation of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}GL(2, k)^2$, that the singular set of Γ is the set

$$\mathcal{S} := \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 | x \neq y\ ; x, y \in \{0, 1, \infty\}\}.$$

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Conversely, if Γ *is a curve of bidegree* $(4,4)$ *in* $\mathbb{P}^1 \times \mathbb{P}^1$*, whose singularities consist of exactly six ordinary double points at the points of* $\mathcal{S}$*, its normalization* C *is a curve of genus* 3*, s.t.* $\mathcal{O}_C(H_2 - H_1) =: \mathcal{O}_C(\eta)$ *(where* H_1*,* H_2 *are the respective pull backs of the rulings of* $\mathbb{P}^1 \times \mathbb{P}^1$*) yields a non trivial* 3 *- torsion divisor class, and* $\mathcal{O}_C(H_1) \cong \mathcal{O}_C(K_C + \eta)$*,* $\mathcal{O}_C(H_2) \cong \mathcal{O}_C(K_C - \eta)$*.*

From theorem (0.1) it follows that
$\mathcal{M}_{3,\eta} := \{(C, \eta) : C \text{ is a general curve of genus } 3,\ \eta \in Pic^0(C)_3 \setminus \{0\}\}$ is birational to $\mathbb{P}(V(4,4,-\mathcal{S}))/\mathfrak{S}_3$, where

$$V(4,4,-\mathcal{S}) := H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)(-2 \sum_{a\neq b, a,b\in\{\infty,0,1\}} (a,b)).$$

In fact, the permutation action of the symmetric group
$\mathfrak{S}_3 := \mathfrak{S}(\{\infty, 0, 1\})$ extends to an action on $\mathbb{P}^1$, so $\mathfrak{S}_3$ is naturally a subgroup of $\mathbb{P}GL(2,k)$. We consider then the diagonal action of $\mathfrak{S}_3$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and observe that $\mathfrak{S}_3$ is exactly the subgroup of $\mathbb{P}GL(2,k)^2$ leaving the set $\mathcal{S}$ invariant. The action of $\mathfrak{S}_3$ on $V(4,4,-\mathcal{S})$ is naturally induced by the diagonal inclusion $\mathfrak{S}_3 \subset \mathbb{P}GL(2,k)^2$.

On the other hand, if we consider only the subgroup of order three of $Pic^0(C)$ generated by a non trivial 3 - torsion element η, we see from theorem (0.1) that we have to allow the exchange of η with $-\eta$, which corresponds to exchanging the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$.

Therefore $\mathcal{M}_{3,\langle\eta\rangle} := \{(C, \langle\eta\rangle) : C \text{ general curve of genus } 3,\ \langle\eta\rangle \cong \mathbb{Z}/3\mathbb{Z} \subset Pic^0(C)\}$ is birational to $\mathbb{P}(V(4,4,-\mathcal{S})/(\mathfrak{S}_3 \times \mathbb{Z}/2)$, where the action of the generator σ (of $\mathbb{Z}/2\mathbb{Z}$) on $V(4,4,-\mathcal{S})$ is induced by the action on $\mathbb{P}^1 \times \mathbb{P}^1$ obtained by exchanging the two coordinates.

Our main result is the following:

Theorem 0.2. *Let* k *be an algebraically closed field of arbitrary characteristic. We have:*
1) the moduli space $\mathcal{M}_{3,\eta}$ *is rational;*
2) the moduli space $\mathcal{M}_{3,\langle\eta\rangle}$ *is rational.*

One could obtain the above result abstractly from the method of Bogomolov and Katsylo (cf. [B-K85]), but we prefer to prove the theorem while explicitly calculating the field of invariant functions. It mainly suffices to decompose the vector representation of $\mathfrak{S}_3$ on $V(4,4,-\mathcal{S})$ into irreducible factors. Of course, if the characteristic of k equals two or three, it is no longer possible to decompose the $\mathfrak{S}_3$ - module $V(4,4,-\mathcal{S})$ as a direct sum of irreducible submodules. Nevertheless, we can write down the field of invariants and see that it is rational.

1. The geometric description of pairs (C, η).

In this section we give a geometric description of pairs (C, η), where C is a general curve of genus 3 and η is a non trivial element of $Pic^0(C)_3$, and we prove theorem (0.1).

Let k be an algebraically closed field of arbitrary characteristic. We recall the following observation from [B-C04], p.374.

Lemma 1.1. *Let C be a general curve of genus 3 and $\eta \in Pic^0(C)_3$ a non trivial divisor class (i.e., η is not linearly equivalent to 0). Then the linear system $|K_C + \eta|$ is base point free. This holds more precisely under the assumption that the canonical system $|K_C|$ does not contain two divisors of the form $Q + 3P$, $Q + 3P'$, and where the 3-torsion divisor class $P - P'$ is the class of η. This condition for all such η is in turn equivalent to the fact that C is either hyperelliptic or it is non hyperelliptic but the canonical image Σ of C does not admit two inflexional tangents meeting in a point Q of Σ.*

Proof. Note that P is a base point of the linear system $|K_C + \eta|$ if and only if

$$H^0(C, \mathcal{O}_C(K_C + \eta)) = H^0(C, \mathcal{O}_C(K_C + \eta - P)).$$

Since $dim H^0(C, \mathcal{O}_C(K_C + \eta)) = 2$ this is equivalent to

$$dim H^1(C, \mathcal{O}_C(K_C + \eta - P)) = 1.$$

Since $H^1(C, \mathcal{O}_C(K_C + \eta - P)) \cong H^0(C, \mathcal{O}_C(P - \eta))^*$, this is equivalent to the existence of a point P' such that $P - \eta \equiv P'$ (note that we denote linear equivalence by the classical notation "$\equiv$".) Therefore $3P \equiv 3P'$ and $P \neq P'$, whence in particular $H^0(C, \mathcal{O}_C(3P)) \geq 2$. By Riemann - Roch we have

$$dim H^0(C, \mathcal{O}_C(K_C - 3P)) =$$

$$deg(K_C - 3P) + 1 - g(C) + dim H^0(C, \mathcal{O}_C(3P) \geq 1.$$

In particular, there is a point Q such that $Q \equiv K_C - 3P \equiv K_C - 3P'$.

Going backwards, we see that this condition is not only necessary, but sufficient. If C is hyperelliptic, then $Q + 3P, Q + 3P' \in |K_C|$ hence P, P' are Weierstrass points, whence $2P \equiv 2P'$, hence $P - P'$ yields a divisor class η of 2-torsion, contradicting the nontriviality of η.

Consider now the canonical embedding of C as a plane quartic Σ. Our condition means, geometrically, that C has two inflection points P, P', such that the tangent lines to these points intersect in $Q \in C$.

We shall show now that the (non hyperelliptic) curves of genus three whose canonical image is a quartic Σ with the above properties are contained in a five dimesnional family, whence are special in the moduli space $\mathcal{M}_3$ of curves of genus three.

Let now p, q, p' be three non collinear points in $\mathbb{P}^2$. The quartics in $\mathbb{P}^2$ form a linear system of dimension 14. Imposing that a plane quartic contains the point q is one linear condition. Moreover, the condition that the line containing p and q has intersection multiplicity equal to 3 with the quartic in the point p gives three further linear conditions. Similarly for the point p', and it is easy to see that the above seven linear conditions are independent. Therefore the linear subsystem of quartics Σ having two inflection points p, p', such that the tangent lines to these points intersect in $q \in \Sigma$ has dimension $14 - 3 - 3 - 1 = 7$. The group of automorphisms of $\mathbb{P}^2$ leaving the three points p, q, p' fixed has dimension 2 and therefore the above quartics give rise to a five dimensional algebraic subset of $\mathcal{M}_3$.

Finally, if the points P, P', Q are not distinct, we have (w.l.o.g.) $P = Q$ and a similar calculation shows that we have a family of dimension $7 - 3 = 4$. □

Consider now the morphism

$$\varphi_\eta (:= \varphi_{|K_C+\eta|} \times \varphi_{|K_C-\eta|}) : C \to \mathbb{P}^1 \times \mathbb{P}^1,$$

and denote by $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ the image of C under φ_η.

Remark 1.2. 1) Since η is non trivial, either Γ is of bidegree $(4, 4)$, or $\deg\varphi_\eta = 2$ and Γ is of bidegree $(2, 2)$. In fact deg $\varphi_\eta = 4$ implies $\eta \equiv -\eta$.

2) We shall assume in the following that φ_η is birational, since otherwise C is either hyperelliptic (if Γ is singular) or C is a double cover of an elliptic curve Γ (branched in 4 points).

In both cases C lies in a 5 - dimensional subfamily of the moduli space $\mathcal{M}_3$ of curves of genus 3.

Let $P_1, \ldots, P_m$ be the (possibly infinitely near) singular points of Γ, and let r_i be the multiplicity in P_i of the proper transform of Γ. Then, denoting by H_1, respectively H_2, the divisors of a vertical, respectively of a horizontal line in $\mathbb{P}^1 \times \mathbb{P}^1$, we have that $\Gamma \in |4H_1 + 4H_2 - \sum_{i=1}^m r_i P_i|$. By adjunction, the canonical system of Γ is cut out by $|2H_1 + 2H_2 - \sum_{i=1}^m (r_i - 1)P_i|$, and therefore

$$4 = degK_C = \Gamma \cdot (2H_1 + 2H_2 - \sum_{i=1}^{m} (r_i - 1)P_i) = 16 - \sum_{i=1}^{m} r_i(r_i - 1).$$

Hence $\sum_{i=1}^m r_i(r_i - 1) = 12$, and we have the following possibilities

	m	$(r_1, \dots, r_m)$
i)	1	(4)
ii)	2	(3,3)
iii)	4	(3,2,2,2)
iv)	6	(2,2,2,2,2,2)

We will show now that for a general curve only the last case occurs, i.e., Γ has exactly 6 singular points of multiplicity 2.

We denote by S the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in $P_1, \dots, P_m$, and let E_i be the exceptional divisor of the first kind, total transform of the point P_i.

We shall first show that the first case (i.e., $m = 1$) corresponds to the case $\eta \equiv 0$.

Proposition 1.3. *Let $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ a curve of bidegree $(4,4)$ having a point P of multiplicity 4, such that its normalization $C \in |4H_1 + 4H_2 - 4E|$ has genus 3 (here, E is the exceptional divisor of the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in P.) Then*

$$\mathcal{O}_C(H_1) \cong \mathcal{O}_C(H_2) \cong \mathcal{O}_C(K_C).$$

In particular, if $\Gamma = \varphi_\eta(C)$, (i.e., we are in the case $m = 1$) then $\eta \equiv 0$.

Remark 1.4. Let Γ be as in the proposition. Then the rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ given by $|H_1 + H_2 - E|$ maps Γ to a plane quartic. Viceversa, given a plane quartic C', blowing up two points $p_1, p_2 \in (\mathbb{P}^1 \times \mathbb{P}^1) \setminus C'$, and then contracting the strict transform of the line through p_1, p_2, yields a curve Γ of bidegree $(4,4)$ having a singular point of multiplicity 4.

Proof (of the proposition). Let H_1 be the full transform of a vertical line through P. Then there is an effective divisor H_1' on the blow up S of $\mathbb{P}^1 \times \mathbb{P}^1$ in P such that $H_1 \equiv H_1' + E$. Since $H_1 \cdot C = E \cdot C = 4$, H_1' is disjoint from C, whence $\mathcal{O}_C(H_1) \cong \mathcal{O}_C(E)$. The same argument for a horizontal line through P obviously shows that $\mathcal{O}_C(H_2) \cong \mathcal{O}_C(E)$. If $h^0(C, \mathcal{O}_C(H_1)) = 2$, then the two projections $p_1, p_2 : \Gamma \to \mathbb{P}^1$ induce the same linear series on C, thus $\varphi_{|H_1|}$ and $\varphi_{|H_2|}$ are related by a projectivity of $\mathbb{P}^1$, hence Γ is the graph of a projectivity of $\mathbb{P}^1$, contradicting the fact that the bidegree of Γ is $(4,4)$.

Therefore we have a smooth curve of genus three and a divisor of degree 4 such that $h^0(C, \mathcal{O}_C(H_1)) \geq 3$. Hence $h^0(C, \mathcal{O}_C(K_C - H_1)) \geq 1$, which implies that $K_C \equiv H_1$. Analogously, $K_C \equiv H_2$. $\square$

The next step is to show that for a general curve C of genus 3, cases ii) and iii) do not occur. In fact, we show:

Lemma 1.5. *Let C be a curve of genus 3 and $\eta \in Pic^0(C)_3 \setminus \{0\}$ such that φ_η is birational and the image $\varphi_\eta(C) = \Gamma$ has a singular point P of multiplicity 3. Then C belongs to an algebraic subset of $\mathcal{M}_3$ of dimension ≤ 5.*

Proof. Let S again be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in P, and denote by E the exceptional divisor. Then $\mathcal{O}_C(E)$ has degree 3 and arguing as in prop. (1.3), we see that there are points Q_1, Q_2 on C such that $\mathcal{O}_C(H_i) \cong \mathcal{O}_C(Q_i + E)$. Therefore $\mathcal{O}_C(Q_2 - Q_1) \cong \mathcal{O}_C(H_2 - H_1) \cong \mathcal{O}_C(K_C - \eta - (K_C + \eta)) \cong \mathcal{O}_C(\eta)$, whence $3Q_1 \equiv 3Q_2$, $Q_1 \neq Q_2$. This implies that there is a morphism $f : C \to \mathbb{P}^1$ of degree 3, having double ramification in Q_1 and Q_2. By Hurwitz' formula the degree of the ramification divisor R is 10 and since $R \geq Q_1 + Q_2$ f has at most 8 branch points in $\mathbb{P}^1$. Fixing three of these points to be $\infty, 0, 1$, we obatain (by Riemann's existence theorem) a finite number of families of dimension at most 5. □

From now on, we shall make the following
Assumptions.
C is a curve of genus 3, $\eta \in Pic^0(C)_3 \setminus \{0\}$, and

1) $|K_C + \eta|$ and $|K_C - \eta|$ are base point free;
2) $\varphi_\eta : C \to \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ is birational;
3) $\Gamma \in |4H_1 + 4H_2|$ has only double points as singularities (possibly infinitely near).

Remark 1.6. By the considerations so far, we know that a general curve of genus 3 fulfills the assumptions for any $\eta \in Pic^0(C)_3 \setminus \{0\}$.

We use the notation introduced above: we have $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ and $C \subset S$, $C \in |4H_1 + 4H_2 - 2\sum_{i=1}^6 E_i|$.

Remark 1.7. Since S is a regular surface, we have an easy case of Ramanujam's vanishing theorem: if D is an effective divisor which is 1-connected (i.e., for every decomposition $D = A + B$ with $A, B > 0$, we have $A \cdot B \geq 1$), then $H^1(S, \mathcal{O}_S(-D)) = 0$.

This follows immediately from Ramanujam's lemma ensuring that $H^0(D, \mathcal{O}_D) = k$, and the long exact cohomology sequence associated to

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0.$$

In most of our applications we shall show that D is linearly equivalent to a reduced and connected divisor (this is a stronger property than 1-connectedness).

□

We know now that $\mathcal{O}_C(H_1+H_2) \cong \mathcal{O}_C(2K_C)$, i.e.,

$$\mathcal{O}_C \cong \mathcal{O}_C(3H_1+3H_2-\sum_{i=1}^{6} 2E_i).$$

Since $h^1(S,\mathcal{O}_S(-H_1-H_2))=0$, the exact sequence

$$(1)\quad 0 \to \mathcal{O}_S(-H_1-H_2) \to \mathcal{O}_S(3H_1+3H_2-\sum_{i=1}^{6} 2E_i) \to$$
$$\to \mathcal{O}_C(3H_1+3H_2-\sum_{i=1}^{6} 2E_i) \cong \mathcal{O}_C \to 0,$$

is exact on global sections.

In particular, $h^0(S,\mathcal{O}_S(3H_1+3H_2-\sum_{i=1}^{6} 2E_i))=1$. We denote by G the unique divisor in the linear system $|3H_1+3H_2-\sum_{i=1}^{6} 2E_i|$. Note that $C \cap G = \emptyset$ (since $\mathcal{O}_C \cong \mathcal{O}_C(G)$).

Remark 1.8. There is no effective divisor $\tilde{G}$ on S such that $G = \tilde{G} + E_i$, since otherwise $\tilde{G} \cdot C = -2$, contradicting that $\tilde{G}$ and C have no common component.

This means that $G + 2\sum_{i=1}^{6} E_i$ is the total transform of a curve $G' \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (3,3).

Lemma 1.9. *$h^0(G,\mathcal{O}_G)=3$, $h^1(G,\mathcal{O}_G)=0$.*

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S+G) \to \mathcal{O}_G(K_G) \to 0.$$

Since $h^0(S,\mathcal{O}_S(K_S)) = h^1(S,\mathcal{O}_S(K_S))=0$, we get

$$h^0(S,\mathcal{O}_S(K_S+G)) = h^0(G,\mathcal{O}_G(K_G)).$$

Now, $K_S+G \equiv H_1+H_2-\sum_{i=1}^{6} E_i$, therefore $(K_S+G)\cdot C = -4$, whence $h^0(G,\mathcal{O}_G(K_G)) \cong h^0(S,\mathcal{O}_S(K_S+G))=0$.

Moreover, $h^1(G,\mathcal{O}_G(K_G)) = h^1(S,\mathcal{O}_S(K_S+G))+1$, and by Riemann - Roch we infer that, since $h^1(S,\mathcal{O}_S(K_S+G)) = h^0(S,\mathcal{O}_S(-G))=0$, that $h^1(S,\mathcal{O}_S(K_S+G))=2$. □

We will show now that G is reduced, hence, by the above lemma, we shall obtain that G has exactly 3 connected components.

Proposition 1.10. *G is reduced.*

Proof. By remark (1.8) it is sufficient to show that the image of G in $\mathbb{P}^1 \times \mathbb{P}^1$, which we denoted by G', is reduced.

Assume that there is an effective divisor A' on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $3A' \leq G'$. We clearly have $A' \cap \Gamma \neq \emptyset$ but, after blowing up the six points $P_1, \dots, P_6$, the strict transforms of A' and of Γ are disjoint, whence A' and G' must intersect in one of the P_i's, contradicting remark (1.8).

If G' is not reduced, we may uniquely write $G' = 2D_1 + D_2$ with D_1, D_2 reduced and having no common component. Up to exchanging the factors of $\mathbb{P}^1 \times \mathbb{P}^1$, we have the following two possibilities:

i) $D_1 \in |H_1 + H_2|$;
ii) $D_1 \in |H_1|$.

In the first case also $D_2 \in |H_1 + H_2|$ and its strict transform is disjoint from C. Remark (1.8) implies that D_2 meets Γ in points which do not belong to D_1, whence D_2 has double points where it intersects Γ. Since $D_2 \cdot \Gamma = 8$ we see that D_2 has two points of multiplicity 2, a contradiction (D_2 has bidegree $(1, 1)$).

Assume now that $D_1 \in |H_1|$. Then, since $2D_1 \cdot \Gamma = 8$, D_1 contains 4 of the P_i's and D_2 passes through the other two, say P_1, P_2. This implies that for the strict transform of D_2 we have: $\hat{D}_2 \equiv H_1 + 3H_2 - 2E_1 - 2E_2$, whence $\hat{D}_2 \cdot C = 8$, a contradiction. □

We write now $G = G_1 + G_2 + G_3$ as a sum of its connected components, and accordingly $G' = G'_1 + G'_2 + G'_3$.

Lemma 1.11. *The bidegree of G'_j, ($j \in \{1, 2, 3\}$) is $(1, 1)$.*

Up to renumbering $P_1, \dots, P_6$ we have $G'_1 \cap G'_2 = \{P_1, P_2\}$, $G'_1 \cap G'_3 = \{P_3, P_4\}$ and $G'_2 \cap G'_3 = \{P_5, P_6\}$.

More precisely, $G_1 \in |H_1 + H_2 - E_1 - E_2 - E_3 - E_4|$, $G_2 \in |H_1 + H_2 - E_1 - E_2 - E_5 - E_6|$, $G_3 \in |H_1 + H_2 - E_3 - E_4 - E_5 - E_6|$.

Proof. Assume for instance that G'_1 has bidegree $(1, 0)$. Then there is a subset $I \subset \{1, \dots, 6\}$ such that $G_1 = H_1 - \sum_{i \in I} E_i$. Since $G_1 \cdot C = 0$, it follows that $|I| = 2$. But then $G_1 \cdot (G - G_1) = 1$, contradicting the fact that G_1 is a connected component of G.

Let (a_j, b_j) be the bidegree of G_j: then $a_j, b_j \geq 1$ since a reduced divisor of bidegree $(m, 0)$ is not connected for $m \geq 2$. Since $\sum a_j = \sum b_j = 3$, it follows that $a_j = b_j = 1$.

Writing now $G_j \equiv H_1 + H_2 - \sum_{i=1}^6 \mu(j, i) E_i$ we obtain

$$\sum_{j=1}^{3} \mu(j, i) = 2, \ \sum_{i=1}^{6} \mu(j, i) = 4, \ \sum_{i=1}^{6} \mu(k, i)\mu(j, i) = 2$$

since $G_j \cdot C = 0$) and $G_k \cdot G_j = 0$). We get the second claim of the lemma provided that we show: $\mu(j,i) = 1, \forall i, j$.

The first formula shows that if $\mu(j,i) \geq 2$, then $\mu(j,i) = 2$ and $\mu(h,i) = 0$ for $h \neq j$. Hence the second formula shows that

$$\sum_{h,k \neq j} \sum_{i=1}^{6} \mu(j,i)(\mu(h,i) + \mu(k,i)) \leq 2,$$

contradicting the third formulae. □

In the remaining part of the section we will show that each G_i' consists of the union of a vertical and a horizontal line in $\mathbb{P}^1 \times \mathbb{P}^1$.

Since $\mathcal{O}_C(K_C + \eta) \cong \mathcal{O}_C(H_1)$ and $\mathcal{O}_C(K_C - \eta) \cong \mathcal{O}_C(H_2)$ weget:

$$\mathcal{O}_C(2H_2 - H_1) \cong \mathcal{O}_C(K_C) \cong \mathcal{O}_C(2H_1 + 2H_2 - \sum_{i=1}^{6} E_i),$$

whence the exact sequence

$$(2) \quad 0 \to \mathcal{O}_S(-H_1 - 4H_2 + \sum_{i=1}^{6} E_i) \to \mathcal{O}_S(3H_1 - \sum_{i=1}^{6} E_i) \to$$
$$\to \mathcal{O}_C(3H_1 - \sum_{i=1}^{6} E_i) \cong \mathcal{O}_C \to 0,$$

Proposition 1.12. $H^1(S, \mathcal{O}_S(-(H_1 + 4H_2 - \sum_{i=1}^{6} E_i))) = 0$.

Proof. The result follows immediately by Ramanujam's vanishing theorem, but we can also give an elementary proof using remark 1.7.

It suffices to show that the linear system $|H_1 + 4H_2 - \sum_{i=1}^{6} E_i|$ contains a reduced and connected divisor.

Note that $G_1 + |3H_2 - E_5 - E_6| \subset |H_1 + 4H_2 - \sum_{i=1}^{6} E_i|$, and that $|3H_2 - E_5 - E_6|$ contains $|H_2 - E_5 - E_6| + |2H_2|$, if there is a line H_2 containing P_1, P_2, else it contains $|H_2 - E_5| + |H_2 - E_6| + |H_2|$. Since $G_1 \cdot H_2 = G_1 \cdot (H_2 - E_5) = G_1 \cdot (H_2 - E_6) = G_1 \cdot (H_2 - E_5 - E_6) = 1$, we have obtained in both cases a reduced and connected divisor. □

Remark 1.13. One can indeed show, using $G_2 + |3H_2 - E_3 - E_4| \subset |H_1 + 4H_2 - \sum_{i=1}^{6} E_i|$ and $G_3 + |3H_2 - E_1 - E_2| \subset |H_1 + 4H_2 - \sum_{i=1}^{6} E_i|$ that $|H_1 + 4H_2 - \sum_{i=1}^{6} E_i|$ has no fixed part, and then by Bertini's theorem, since $(H_1 + 4H_2 - \sum_{i=1}^{6} E_i)^2 = 8 - 6 = 2 > 0$, a general curve in $|H_1 + 4H_2 - \sum_{i=1}^{6} E_i|$ is irreducible.

In view of proposition 1.12 the above exact sequence (and the one where the roles of H_1, H_2 are exchanged) yields the following:

Corollary 1.14. *For* $j \in \{1,2\}$ *there is exactly one divisor* $N_j \in |3H_j - \sum_{i=1}^6 E_i|$.

By the uniqueness of G, we see that $G = N_1 + N_2$. Denote by N'_j the curve in $\mathbb{P}^1 \times \mathbb{P}^1$ whose total transform is $N_j + \sum_{i=1}^6 E_i$.

We have just seen that G is the strict transform of three vertical and three horizontal lines in $\mathbb{P}^1 \times \mathbb{P}^1$. Hence each connected component G_j splits into the strict transform of a vertical and a horizontal line. Since G is reduced, the lines are distinct (and there are no infinitely near points).

We can choose coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $G'_1 = (\{\infty\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{\infty\})$, $G'_2 = (\{0\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{0\})$ and $G'_3 = (\{1\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{1\})$.

Remark 1.15. The points $P_1, \dots, P_6$ are then the points of the set $\mathcal{S}$ previously defined.

Conversely, consider in $\mathbb{P}^1 \times \mathbb{P}^1$ the set $\mathcal{S} := \{P_1, \dots, P_6\} = (\{\infty, 0, 1\} \times \{\infty, 0, 1\}) \setminus \{(\infty, \infty), (0,0), (1,1)\}$. Let $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow up of the points $P_1, \dots, P_6$ and suppose (denoting the exceptional divisor over P_i by E_i) that $C \in |4H_1 + 4H_2 - \sum 2E_i|$ is a smooth curve. Then C has genus 3, $\mathcal{O}_C(3H_1) \cong \mathcal{O}_C(\sum E_i) \cong \mathcal{O}_C(3H_2))$. Setting $\mathcal{O}_C(\eta) := \mathcal{O}_C(H_2 - H_1)$, we obtain therefore $3\eta \equiv 0$.

It remains to show that $\mathcal{O}_C(\eta)$ is not isomorphic to $\mathcal{O}_C$.

Lemma 1.16. η *is not trivial.*

Proof. Assume $\eta \equiv 0$. Then $\mathcal{O}_C(H_1) \cong \mathcal{O}_C(H_2)$ and, since Γ has bidegree $(4,4)$, we argue as in the proof of proposition 1.3) that $h^0(\mathcal{O}_C(H_i)) \geq 3$, whence $\mathcal{O}_C(H_i) \cong \mathcal{O}_C(K_C)$.

The same argument shows that the two peojections of Γ to $\mathbb{P}^1$ yield two different pencils in the canonical system. It follows that the canonical map of C factors as the composition of $C \to \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ with the rational map $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ which blows up one point and contracts the vertical and horizontal line through it. Since Γ has six singular points, the canonical map sends C birationally onto a singular quartic curve in $\mathbb{P}^2$, absurd. □

2. Rationality of the moduli spaces

In this section we will use the geometric despcription of pairs (C, η), where C is a genus 3 curve and η a non trivial 3 - torsion divisor class, and study the birational structure of their moduli space.

More precisely, we shall prove the following

Theorem 2.1. *1) The moduli space* $\mathcal{M}_{3,\eta} := \{(C,\eta) : C$ *a general curve of genus* $3,\ \eta \in Pic^0(C)_3 \setminus \{0\}\}$ *is rational.*

2) The moduli space $\mathcal{M}_{3,\langle\eta\rangle} := \{(C,\langle\eta\rangle) : C$ *a general curve of genus* $3,\ \langle\eta\rangle \cong \mathbb{Z}/3\mathbb{Z} \subset Pic^0(C)\}$ *is rational.*

Remark 2.2. By the result of the previous section, and since any automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which sends the set $\mathcal{S}$ to itself belongs to the group $\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$, follows immediately that, if we set

$$V(4,4,-\mathcal{S}) := H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)(-2\sum_{i\neq j, i,j\in\{\infty,0,1\}} P_{ij})),$$

$\mathcal{M}_{3,\eta}$ is birational to $\mathbb{P}(V(4,4,-\mathcal{S}))/\mathfrak{S}_3$, while $\mathcal{M}_{3,\langle\eta\rangle}$ is birational to $\mathbb{P}(V(4,4,-\mathcal{S}))/(\mathfrak{S}_3\times\mathbb{Z}/2\mathbb{Z})$, where the generator σ of $\mathbb{Z}/2\mathbb{Z}$ acts by coordinate exchange on $\mathbb{P}^1 \times \mathbb{P}^1$, whence on $V(4,4,-\mathcal{S})$.

In order to prove the above theorem we will explicitly calculate the respective subfields of invariants of the function field of $\mathbb{P}(V(4,4,-\mathcal{S}))$ and show that they are generated by purely transcendental elements.

Consider the following polynomials of $\mathbb{V} := V(4,4,-\mathcal{S})$, which are invariant under the action of $\mathbb{Z}/2\mathbb{Z}$:

$$f_{11}(x,y) := x_0^2 x_1^2 y_0^2 y_1^2,$$

$$f_{\infty\infty}(x,y) := x_1^2(x_1-x_0)^2 y_0^2 (y_1-y_0)^2,$$

$$f_{00}(x,y) := x_0^2(x_1-x_0)^2 y_0^2 (y_1-y_0)^2.$$

Let $ev : \mathbb{V} \to \bigoplus_{i=0,1,\infty} k_{(i,i)} =: \mathbb{W}$ be the evaluation map at the three standard diagonal points, i.e., $ev(f) := (f(0,0), f(1,1), f(\infty,\infty))$.

Since $f_{ii}(j,j) = \delta_{i,j}$, we can decompose $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$, where $\mathbb{U} := ker(ev)$ and $\mathbb{W}$ is the subspace generated by the three above polynomials, which is easily shown to be an invariant subspace using the following formulae $(*)$:

- $(1,3)$ exchanges x_0 with x_1, multiplies $x_1 - x_0$ by -1,
- $(1,2)$ exchanges $x_1 - x_0$ with x_1, multiplies x_0 by -1,
- $(2,3)$ exchanges $x_0 - x_1$ with x_0, multiplies x_1 by -1.

In fact, 'the permutation' representation $\mathbb{W}$ of the symmetric group splits (in characteristic $\neq 3$) as the direct sum of the trivial representation (generated by $e_1+e_2+e_3$) and the standard representation, generated by $x_0 := e_1 - e_2, x_1 := -e_2 + e_3$, which is isomorphic to the representation on $V(1) := H^0(\mathcal{O}_{\mathbb{P}^1}(1))$.

Note that $\mathbb{U} = x_0x_1(x_1-x_0)y_0y_1(y_0-y_1)H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$.

We write $V(1,1) := H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)) = V(1) \otimes V(1)$, where $V(1) := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, is as above the standard representation of $\mathfrak{S}_3$.

Now $V(1) \otimes V(1)$ splits, in characteristic $\neq 2, 3$, as a sum of irreducible representations $\mathbb{I} \oplus \mathfrak{A} \oplus W$, where the three factors are the *trivial*, the *alternating* and the *standard* representation of $\mathfrak{S}_3$.

Explicitly, $V(1) \otimes V(1) \cong \wedge^2(V(1)) \oplus Sym^2(V(1))$, and $Sym^2(V(1))$ is isomorphic to $\mathbb{W}$, since it has the following basis $x_0y_0, x_1y_1, (x_1 - x_0)(y_1 - y_0)$. We observe for further use that $\mathbb{Z}/2\mathbb{Z}$ acts as the identity on $Sym^2(V(1))$, while it acts on $\wedge^2(V(1))$, spanned by $x_1y_0 - x_0y_1$ via multiplication by -1.

We have thus seen

Lemma 2.3. *If $char(k) \neq 2, 3$ then the $\mathfrak{S}_3$ -module $\mathbb{V}$ splits as a sum of irreducible modules as follows:*

$$\mathbb{V} \cong 2(\mathbb{I} \oplus W) \oplus \mathfrak{A}.$$

Choose now a basis $(z_1, z_2, z_3, w_1, w_2, w_3, u)$ of $\mathbb{V}$, such that the z_i's and the w_i's are respective bases of $\mathbb{I} \oplus W$ consisting of eigenvectors of $\sigma = (123)$, and u is a basis element of $\mathfrak{A}$. The eigenvalue of z_i, w_i with respect to $\sigma = (123)$ is ϵ^{i-1}, u is σ-invariant and $(12)(u) = -u$.

Note that if (v_1, v_2, v_3) is a basis of $\mathbb{I} \oplus W$, such that $\mathfrak{S}_3$ acts by permutation of the indices, then $z_1 = v_1 + v_2 + v_3$, $z_2 = v_1 + \epsilon v_2 + \epsilon^2 v_3$, $z_3 = v_1 + \epsilon^2 v_2 + \epsilon v_3$, where ϵ is a primitive third root of unity.

Remark 2.4. Since z_1, w_1 are $\mathfrak{S}_3$ - invariant, $\mathbb{P}(V(4,4,-\mathcal{S}))/\mathfrak{S}_3$ is birational to a product of the affine line with $Spec(k[z_2, z_3, w_2, w_3, u]^{\mathfrak{S}_3})$, and therefore it suffices to compute $k[z_2, z_3, w_2, w_3, u]^{\mathfrak{S}_3}$.

Part 1 of the theorem follows now from the following

Proposition 2.5. *Let $T := z_2z_3$, $S := z_2^3$, $A_1 := z_2w_3 + z_3w_2$, $A_2 := z_2w_3 - z_3w_2$. Then*

$$k(z_2, z_3, w_2, w_3, u)^{\mathfrak{S}_3} \supset K := k(A_1, T, S + \frac{T^3}{S}, u(S - \frac{T^3}{S}), A_2(S - \frac{T^3}{S})),$$

and $[k(z_2, z_3, w_2, w_3, u) : K] = 6$, hence $k(z_2, z_3, w_2, w_3, u)^{\mathfrak{S}_3} = K$.

Proof. We first calculate the invariants under the action of $\sigma = (123)$, i.e., $k(z_2, z_3, w_2, w_3, u)^\sigma$. Note that u, z_2z_3, z_2w_3, w_2w_3, z_2^3 are σ-invariant, and $[k(z_2, z_3, w_2, w_3, u) : k(u, z_2z_3, z_2w_3, w_2w_3, z_2^3)] = 3$. In particular, $k(z_2, z_3, w_2, w_3, u)^\sigma = k(u, z_2z_3, z_2w_3, w_2w_3, z_2^3) =: L$.

Now, we calculate L^τ, with $\tau = (12)$. We first observe that $L = k(T, A_1, A_2, S, u)$. Since $\tau(z_2) = \epsilon z_3$, $\tau(z_3) = \epsilon^2 z_2$ (and similarly for

w_2, w_3), we see that $\tau(A_1) = A_1$ and $\tau(T) = T$. On the other hand, $\tau(u) = -u$, $\tau(A_2) = -A_2$, $\tau(S) = \frac{T^3}{S}$.

Claim.
$L^\tau = k(A_1, T, S + \frac{T^3}{S}, u(S - \frac{T^3}{S}), A_2(S - \frac{T^3}{S})) =: E.$

Proof of the Claim. Obviously A_1,T,$S + \frac{T^3}{S}$,$u(S - \frac{T^3}{S})$, $A_2(S - \frac{T^3}{S})$ are invariant under τ, whence $E \subset L^\tau$. Since $L = E(S)$, using the equation $B \cdot S = S^2 + T^3$ for $B := S + \frac{T^3}{S}$, we get that $[E(S) : E] \leq 2$.

This proves the claim and the proposition. □

There remains to show the second part of the theorem.

We denote by τ' the involution on $k(z_1, z_2, z_3, w_1, w_2, w_3, u)$ induced by the involution $(x, y) \mapsto (y, x)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. It suffices to prove the following

Proposition 2.6. $E^{\tau'} = k(A_1, T, S + \frac{T^3}{S}, (u(S - \frac{T^3}{S}))^2, A_2(S - \frac{T^3}{S}))$.

Proof. Since $[E : k(A_1, T, S + \frac{T^3}{S}, (u(S - \frac{T^3}{S}))^2, A_2(S - \frac{T^3}{S}))] \leq 2$, it suffices to show that the 5 generators A_1, T,$S+\frac{T^3}{S}$,$(u(S-\frac{T^3}{S}))^2$, $A_2(S-\frac{T^3}{S})$ are τ' - invariant. This will now be proven in lemma (2.7). □

Lemma 2.7. *τ' acts as the identity on $(z_1, z_2, z_3, w_1, w_2, w_3)$ and sends $u \mapsto -u$.*

Proof. We note first that τ' acts trivially on the subspace $\mathbb{W}$ generated by the polynomials f_{ii}.

Since $\mathbb{U} = x_0x_1(x_1-x_0)y_0y_1(y_1-y_0)V(1,1)$ and $x_0x_1(x_1-x_0)y_0y_1(y_1-y_0)$ is invariant under exchanging x and y, it suffices to recall that the action of τ' on $V(1,1) = V(1) \otimes V(1)$ is the identity on the subspace $Sym^2(V(1))$, while the action on the alternating $\mathfrak{S}_3$ - submodule $\mathfrak{A}$ sends the generator u to $-u$. □

2.1. $Char(k) = 3$**.** In order to prove theorem (2.1) if the characteristic of k is equal to 3 we describe the $\mathfrak{S}_3$ -module $\mathbb{V}$ as follows:

$$\mathbb{V} \cong 2\mathbb{W} \oplus \mathfrak{A},$$

where $\mathbb{W}$ is the (3-dimensional) permutation representation of $\mathfrak{S}_3$.

Let now $z_1, z_2, z_3, w_1, w_2, w_3, u$ be a basis of $\mathbb{V}$ such that the action of $\mathfrak{S}_3$ permutes z_1, z_2, z_3 (resp. w_1, w_2, w_3), and $(123) : u \mapsto u$, $(12)u \mapsto -u$. Then we have:

Proposition 2.8. *The $\mathfrak{S}_3$- invariant subfield $k(\mathbb{V})^{\mathfrak{S}_3}$ of $k(\mathbb{V})$ is rational.*

More precisely, the seven $\mathfrak{S}_3$ - invariant functions

$$\sigma_1 = z_1 + z_2 + z_3,$$

$$\sigma_2 = z_1z_2 + z_1z_3 + z_2z_3,$$
$$\sigma_3 = z_1z_2z_3,$$
$$\sigma_4 = z_1w_1 + z_2w_2 + z_3w_3,$$
$$\sigma_5 = w_1z_2z_3 + w_2z_1z_3 + w_3z_1z_2,$$
$$\sigma_6 = w_1(z_2 + z_3) + w_2(z_1 + z_3) + w_3(z_1 + z_2),$$
$$\sigma_7 = u(z_1(w_2 - w_3) + z_2(w_3 - w_1) + z_3(w_1 - w_2))$$

form a basis of the purely transcendental extension over k.

Proof. $\sigma_1, \dots, \sigma_7$ determine a morphism $\psi : \mathbb{V} \to \mathbb{A}^7_k$. We will show that ψ induces a birational map $\bar{\psi} : \mathbb{V}/\mathfrak{S}_3 \to \mathbb{A}^7_k$, i.e., for a Zariski open set of $\mathbb{V}$ we have: $\psi(x) = \psi(x')$ if and only if there is a $\tau \in \mathfrak{S}_3$ such that $x = \tau(x')$. By [Cat], lemma (2.2), we can assume (after acting on x with a suitable $\tau \in \mathfrak{S}_3$), that $x_i = x'_i$ for $1 \leq i \leq 6$, and we know that (setting $u := x_7$, $u' := x'_7$)

$$u(x_1(x_5 - x_6) + x_2(x_6 - x_4) + x_3(x_4 - x_5)) = u'(x_1(x_5 - x_6) + x_2(x_6 - x_4) + x_3(x_4 - x_5)).$$

Therefore, if $B(x_1, \dots, x_6) := x_1(x_5-x_6)+x_2(x_6-x_4)+x_3(x_4-x_5) \neq 0$, this implies that $u = u'$. □

Therefore, we have shown part 1 of theorem (2.1).

We denote again by τ' the involution on $k(z_1, z_2, z_3, w_1, w_2, w_3, u)$ induced by the involution $(x, y) \mapsto (y, x)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. In order to prove part 2) of thm. (2.1), it sufficies to observe that $\sigma_1, \dots, \sigma_6, \sigma_7^2$ are invariant under τ' and $[k(\sigma_1, \dots, \sigma_7) : k(\sigma_1, \dots, \sigma_7^2)] \leq 2$, whence $(k(\mathbb{V})^{\mathfrak{S}_3})^{(\mathbb{Z}/2\mathbb{Z})} = k(\sigma_1, \dots, \sigma_7^2)$. This proves theorem (2.1).

2.2. *Char*$(k) = 2$**.** Let k be an algebraically closed field of characteristic 2. Then we can describe the $\mathfrak{S}_3$ - module $\mathbb{V}$ as follows:

$$\mathbb{V} \cong \mathbb{W} \oplus V(1, 1),$$

where $\mathbb{W}$ is the (3 - dimensional) permutation representation of $\mathfrak{S}_3$. We denote a basis of $\mathbb{W}$ by z_1, z_2, z_3. As in the beginning of the chapter, $V(1, 1) = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$. We choose the following basis of $V(1, 1)$: $w_1 := x_1y_1$, $w_2 := (x_0 + x_1)(y_0 + y_1)$, $w_3 := x_0y_0$, $w := x_0y_1$. Then $\mathfrak{S}_3$ acts on w_1, w_2, w_3 by permutation of the indices and

$$(1, 2) : w \mapsto w + w_3,$$
$$(1, 2, 3) : w \mapsto w + w_2 + w_3.$$

Let $\epsilon \in k$ be a nontrivial third root of unity. Then Theorem (2.1) (in characteristic 2) follows from the following result:

Proposition 2.9. *Let k be an algebraically closed field of characteristic 2. Let $\sigma_1, \dots, \sigma_6$ be as defined in (2.6) and set*

$$v := (w + w_2)(w_1 + \epsilon w_2 + \epsilon^2 w_3) + (w + w_1 + w_3)(w_1 + \epsilon^2 w_2 + \epsilon w_3),$$
$$t := (w + w_2)(w + w_1 + w_3).$$

Then

1) $k(z_1, z_2, z_3, w_1, w_2, w_3, w)^{\mathfrak{S}_3} = k(\sigma_1, \dots, \sigma_6, v)$;

2) $k(z_1, z_2, z_3, w_1, w_2, w_3, w)^{\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}} = k(\sigma_1, \dots, \sigma_6, t)$.

In particular, the respective invariant subfields of $k(\mathbb{V})$ are generated by purely transcendental elements, and this proves theorem (2.1).

Proof of (2.9). 2) We observe that $\mathbb{Z}/2\mathbb{Z}$ ($x_i \mapsto y_i$) acts trivially on $z_1, z_2, z_3, w_1, w_2, w_3$ and maps w to $w + w_1 + w_2 + w_3$. It is now easy to see that t is invariant under the action of $\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$. Therefore $k(\sigma_1, \dots, \sigma_6, t) \subset K := k(z_1, z_2, z_3, w_1, w_2, w_3, w)^{\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}}$. By [Cat], lemma (2.8), $[k(z_1, z_2, z_3, w_1, w_2, w_3, t) : k(\sigma_1, \dots, \sigma_6, t)] = 6$, and obviously, $[k(z_1, z_2, z_3, w_1, w_2, w_3, w) : k(z_1, z_2, z_3, w_1, w_2, w_3, t)] = 2$. Therefore $[k(z_1, z_2, z_3, w_1, w_2, w_3, w) : k(\sigma_1, \dots, \sigma_6, t)] = 12$, whence $K = k(\sigma_1, \dots, \sigma_6, t)$.

1) Note that for $W_2 := w_1 + \epsilon w_2 + \epsilon^2 w_3$, $W_3 := w_1 + \epsilon^2 w_2 + \epsilon^3 w_3$, we have: W_2^3 and W_3^3 are invariant under $(1, 2, 3)$ and are exchanged under $(1, 2)$. Therefore v is invariant under the action of $\mathfrak{S}_3$ and we have seen that $k(\sigma_1, \dots, \sigma_6, v) \subset L := k(z_1, z_2, z_3, w_1, w_2, w_3, w)^{\mathfrak{S}_3}$, in particular $[k(z_1, z_2, z_3, w_1, w_2, w_3, w) : k(\sigma_1, \dots, \sigma_6, v)] \geq 6$. On the other hand, note that $k(z_1, z_2, z_3, w_1, w_2, w_3, w) = k(z_1, z_2, z_3, w_1, w_2, w_3, v)]$ (since v is linear in w) and again, by [Cat], lemma (2.8), $[k(z_i, w_i, v) : k(\sigma_1, \dots, \sigma_6, v)] = 6$. This implies that $L = k(\sigma_1, \dots, \sigma_6, v)$. □

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