# Moduli stacks of vector bundles on curves and the King–Schofield rationality proof

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## Introduction

Let $C$ be a connected smooth projective curve of genus $g \geq 2$ over an algebraically closed field $k$. Consider the coarse moduli scheme $\mathfrak{Bun}_{r,d}$ (resp. $\mathfrak{Bun}_{r,L}$) of stable vector bundles on $C$ with rank $r$ and degree $d \in \mathbb{Z}$ (resp. determinant isomorphic to the line bundle $L$ on $C$).

Motivated by work of A. Tyurin [10, 11] and P. Newstead [7, 8], it has been believed for a long time that $\mathfrak{Bun}_{r,L}$ is rational if $r$ and the degree of $L$ are coprime. Finally, this conjecture was proved in 1999 by A. King and A. Schofield [4]; they deduce it from their following main result:

**Theorem 0.1 (King–Schofield).** *$\mathfrak{Bun}_{r,d}$ is birational to the product of an affine space $\mathbb{A}^n$ and $\mathfrak{Bun}_{h,0}$ where $h$ be the highest common factor of $r$ and $d$.*

The present text contains the complete proof of King and Schofield translated into the language of algebraic stacks. Following their strategy, the moduli stack $\mathcal{B}un_{r,d}$ of rank $r$, degree $d$ vector bundles is shown to be birational to a Grassmannian bundle over $\mathcal{B}un_{r_1,d_1}$ for some $r_1 < r$; then induction is used. However, this Grassmannian bundle is in some sense twisted. Mainly for that reason, King and Schofield need a stronger induction hypothesis than 0.1: They add the condition that their birational map preserves a certain Brauer class $\psi_{r,d}$ on $\mathfrak{Bun}_{r,d}$. One main advantage of the stack language here is that this extra condition is not needed: The stack analogue of theorem 0.1 is proved by a direct induction.

(In more abstract terms, this can be understood roughly as follows: A Brauer class corresponds to a gerbe with band $\mathbb{G}_m$. But the gerbe on $\mathfrak{Bun}_{r,d}$ corresponding to $\psi_{r,d}$ is just the moduli stack $\mathcal{B}un_{r,d}$. Thus a rational map of coarse moduli schemes preserving this Brauer class corresponds to a rational map of the moduli stacks.)

This paper consists of four parts. Section 1 contains the precise formulation of the stack analogue 1.2 to theorem 0.1; then the original results of King and Schofield are deduced. Section 2 deals with Grassmannian bundles over stacks because they are the main tool for the proof of theorem 1.2 in section 3. Finally, appendix A summarizes the basic properties of the moduli stack $\mathcal{B}un_{r,d}$ that we use. In particular, a proof of Hirschowitz' theorem about the tensor product of general vector bundles on $C$ is given here, following Russo and Teixidor [9].

The present article has grown out of a talk in the joint seminar of U. Stuhler and Y. Tschinkel in Göttingen. I would like to thank them for the opportunity to speak and for encouraging me to write this text. I would also like to thank J. Heinloth for some valuable suggestions and for many useful discussions about these stacks.

## 1 The King-Schofield theorem in stack form

We denote by $\mathcal{B}un_{r,d}$ the moduli stack of vector bundles of rank $r$ and degree $d$ on our smooth projective curve $C$ of genus $g \geq 2$ over $k = \bar{k}$. This stack is algebraic in the sense of Artin, smooth of dimension $(g-1)r^2$ over $k$ and irreducible; these properties are discussed in more detail in the appendix.

Our main subject here is the birational type of $\mathcal{B}un_{r,d}$. We will frequently use the notion of a rational map between algebraic stacks; it is defined in the usual way as an equivalence class of morphisms defined on dense open substacks. A birational map is a rational map that admits a two-sided inverse.

**Definition 1.1.** *A rational map of algebraic stacks* $\mathcal{M} \dashrightarrow \mathcal{M}'$ *is* birationally linear *if it admits a factorization*

$$\mathcal{M} \overset{\sim}{\dashrightarrow} \mathcal{M}' \times \mathbb{A}^n \xrightarrow{\mathrm{pr}_1} \mathcal{M}'$$

*into a birational map followed by the projection onto the first factor.*

Now we can formulate the stack analogue of the King-Schofield theorem 0.1; its proof will be given in section 3.

**Theorem 1.2.** *Let $h$ be the highest common factor of the rank $r \geq 1$ and the degree $d \in \mathbb{Z}$. There is a birationally linear map of stacks*

$$\mu : \mathcal{B}un_{r,d} \dashrightarrow \mathcal{B}un_{h,0}$$

*and an isomorphism between the Picard* schemes $\mathrm{Pic}^d(C)$ *and* $\mathrm{Pic}^0(C)$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{B}un_{r,d} & \overset{\mu}{\dashrightarrow} & \mathcal{B}un_{h,0} \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ \mathrm{Pic}^d(C) & \xrightarrow{\sim} & \mathrm{Pic}^0(C) \end{array} \tag{1}$$

**Remark 1.3.** One cannot expect an isomorphism of Picard *stacks* here: If (1) were a commutative diagram of stacks, then choosing a general vector bundle

$E$ of rank $r$ and degree $d$ would yield a commutative diagram of automorphism groups

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\sim} & \mathbb{G}_m \\ {\scriptstyle (\_)^r}\downarrow & & \downarrow{\scriptstyle (\_)^h} \\ \mathbb{G}_m & \xrightarrow{\sim} & \mathbb{G}_m \end{array}$$

which is impossible for $r \neq h$.

**Remark 1.4.** In the theorem, we can furthermore achieve that $\mu$ preserves scalar automorphisms in the following sense:

Let $E$ and $E' = \mu(E)$ be vector bundles over $C$ that correspond to a general point in $\mathcal{B}un_{r,d}$ and its image in $\mathcal{B}un_{h,0}$. Then $E$ and $E'$ are stable (because we have assumed $g \geq 2$) and hence simple. The rational map $\mu$ induces a morphism of algebraic groups

$$\mu^E : \mathbb{G}_m = \mathbb{A}\mathrm{ut}(E) \longrightarrow \mathbb{A}\mathrm{ut}(E') = \mathbb{G}_m$$

which is an isomorphism because $\mu$ is birationally linear. Thus $\mu^E$ is either the identity or $\lambda \mapsto \lambda^{-1}$; it is independent of $E$ because $\mathcal{B}un_{r,d}$ is irreducible. Modifying $\mu$ by the automorphism $E' \mapsto E'^{\mathrm{dual}}$ of $\mathcal{B}un_{h,0}$ if necessary, we can achieve that $\mu^E$ is the identity for every general $E$.

Clearly, the map $\mu$ in the theorem restricts to a birationally linear map between the dense open substacks of *stable* vector bundles. But any rational (resp. birational, resp. birationally linear) map between these induces a rational (resp. birational, resp. birationally linear) map between the corresponding coarse moduli schemes; cf. proposition A.6 in the appendix for details. Hence the original theorem of King and Schofield follows:

**Corollary 1.5 (King–Schofield).** *Let $\mathfrak{Bun}_{r,d}$ be the coarse moduli scheme of stable vector bundles of rank $r$ and degree $d$ on $C$. Then there is a birationally linear map of schemes*

$$\mu : \mathfrak{Bun}_{r,d} \dashrightarrow \mathfrak{Bun}_{h,0}.$$

Of course, this is just a reformulation of the theorem 0.1 mentioned in the introduction.

**Remark 1.6.** As mentioned before, King and Schofield also prove that the rational map $\mu : \mathfrak{Bun}_{r,d} \dashrightarrow \mathfrak{Bun}_{h,0}$ preserves their Brauer class $\psi_{r,d}$. This is equivalent to the condition that $\mu$ induces a rational map between the corresponding $\mathbb{G}_m$-gerbes, i. e. a rational map $\mathcal{B}un_{r,d} \dashrightarrow \mathcal{B}un_{h,0}$ that preserves scalar automorphisms in the sense of remark 1.4.

We recall the consequences concerning the rationality of $\mathfrak{Bun}_{r,L}$. Because the diagram (1) commutes, $\mu$ restricts to a rational map between fixed determinant moduli schemes; thus one obtains:

**Corollary 1.7 (King–Schofield).** *Let $L$ be a line bundle on $C$, and let $\mathfrak{Bun}_{r,L}$ be the coarse moduli scheme of stable vector bundles of rank $r$ and determinant $L$ on $C$. Then there is a birationally linear map of schemes*

$$\mu : \mathfrak{Bun}_{r,L} \dashrightarrow \mathfrak{Bun}_{h,\mathcal{O}}$$

*where $h$ is the highest common factor of $r$ and* $\deg(L)$.

In particular, $\mathfrak{Bun}_{r,L}$ is rational if the rank $r$ and the degree $\deg(L)$ are coprime; this proves the conjecture mentioned in the introduction. More generally, it follows that $\mathfrak{Bun}_{r,L}$ is rational if $\mathfrak{Bun}_{h,\mathcal{O}}$ is. For $h \geq 2$, it seems to be still an open question whether $\mathfrak{Bun}_{h,\mathcal{O}}$ is rational or not.

# 2 Grassmannian bundles

Let $\mathcal{V}$ be a vector bundle over a dense open substack $\mathcal{U} \subseteq \mathcal{B}un_{r,d}$. Recall that a part of this datum is a functor from the groupoid $\mathcal{U}(k)$ to the groupoid of vector spaces over $k$. So for each appropriate vector bundle $E$ over $C$, we do not only get a vector space $\mathcal{V}_E$ over $k$, but also a group homomorphism $\mathrm{Aut}_{\mathcal{O}_C}(E) \to \mathrm{Aut}_k(\mathcal{V}_E)$. Note that both groups contain the scalars $k^*$.

**Definition 2.1.** *A vector bundle $\mathcal{V}$ over a dense open substack $\mathcal{U} \subseteq \mathcal{B}un_{r,d}$* has weight *$w \in \mathbb{Z}$ if the diagram*

$$\begin{array}{ccc} k^* & \hookrightarrow & \mathrm{Aut}_{\mathcal{O}_C}(E) \\ \downarrow{\scriptstyle (\_)^w} & & \downarrow \\ k^* & \hookrightarrow & \mathrm{Aut}_k(\mathcal{V}_E) \end{array}$$

*commutes for all vector bundles $E$ over $C$ that are objects of the groupoid $\mathcal{U}(k)$.*

**Example 2.2.** *The trivial vector bundle $\mathcal{O}^n$ over $\mathcal{B}un_{r,d}$ has weight $0$.*

We denote by $\mathcal{E}^{\mathrm{univ}}$ the universal vector bundle over $C \times \mathcal{B}un_{r,d}$, and by $\mathcal{E}_p^{\mathrm{univ}}$ its restriction to $\{p\} \times \mathcal{B}un_{r,d}$ for some point $p \in C(k)$.

**Example 2.3.** *$\mathcal{E}_p^{\mathrm{univ}}$ is a vector bundle of weight $1$ on $\mathcal{B}un_{r,d}$, and its dual $(\mathcal{E}_p^{\mathrm{univ}})^{\mathrm{dual}}$ is a vector bundle of weight $-1$.*

For another example, we fix a vector bundle $F$ over $C$. By semicontinuity, there is an open substack $\mathcal{U} \subseteq \mathcal{B}un_{r,d}$ that parameterizes vector bundles $E$ of rank $r$ and degree $d$ over $C$ with $\mathrm{Ext}^1(F, E) = 0$; we assume $\mathcal{U} \neq \emptyset$. The vector spaces $\mathrm{Hom}(F, E)$ are the fibres of a vector bundle $\mathrm{Hom}(F, \mathcal{E}^{\mathrm{univ}})$ over $\mathcal{U}$ according to Grothendieck's theory of cohomology and base change in EGA III.

Similarly, there is a vector bundle $\mathrm{Hom}(\mathcal{E}^{\mathrm{univ}}, F)$ defined over an open substack of $\mathcal{B}un_{r,d}$ whose fibre over any point $[E]$ with $\mathrm{Ext}^1(E, F) = 0$ is the vector space $\mathrm{Hom}(E, F)$.

**Example 2.4.** $\mathrm{Hom}(F, \mathcal{E}^{\mathrm{univ}})$ *is a vector bundle of weight* $1$*, and* $\mathrm{Hom}(\mathcal{E}^{\mathrm{univ}}, F)$ *is a vector bundle of weight* $-1$*.*

Note that any vector bundle of weight 0 over an open substack $\mathcal{U} \subseteq \mathcal{B}un_{r,d}$ contained in the stable locus descends to a vector bundle over the corresponding open subscheme $\mathfrak{U} \subseteq \mathfrak{Bun}_{r,d}$ of the coarse moduli scheme, cf. proposition A.6. Vector bundles of nonzero weight do not descend to the coarse moduli scheme.

**Proposition 2.5.** *Consider all vector bundles* $\mathcal{V}$ *of fixed weight* $w$ *over dense open substacks of a fixed stack* $\mathcal{B}un_{r,d}$*. Assume that* $\mathcal{V}_0$ *has minimal rank among them. Then every such* $\mathcal{V}$ *is generically isomorphic to* $\mathcal{V}_0^n$ *for some* $n$*.*

*Proof.* The homomorphism bundles $End(\mathcal{V}_0)$ and $Hom(\mathcal{V}_0, \mathcal{V})$ are vector bundles of weight 0 over dense open substacks of $\mathcal{B}un_{r,d}$. Hence they descend to vector bundles $A$ and $M$ over dense open subschemes of $\mathfrak{Bun}_{r,d}$, cf. proposition A.6. The algebra structure on $End(\mathcal{V}_0)$ and its right(!) action on $Hom(\mathcal{V}_0, \mathcal{V})$ also descend; they turn $A$ into an Azumaya algebra and $M$ into a right $A$-module.

In particular, the generic fibre $M_K$ is a right module under the central simple algebra $A_K$ over the function field $K := k(\mathfrak{Bun}_{r,d})$. By our choice of $\mathcal{V}_0$, there are no nontrivial idempotent elements in $A_K$; hence $A_K$ is a skew field.

We have just constructed a functor $\mathcal{V} \mapsto M_K$ from the category in question to the category of finite-dimensional right vector spaces over $A_K$. This functor is a Morita equivalence; its inverse is defined as follows:

Given such a right vector space $M_K$ over $A_K$, we can extend it to a right $A$-module $M$ over a dense open subscheme of $\mathfrak{Bun}_{r,d}$, i. e. to a right $End(\mathcal{V}_0)$-module of weight 0 over a dense open substack of $\mathcal{B}un_{r,d}$; we send $M_K$ to the vector bundle of weight $w$

$$\mathcal{V} := M \otimes_{End(\mathcal{V}_0)} \mathcal{V}_0.$$

Using this Morita equivalence, the proposition follows from the corresponding statement for right vector spaces over $A_K$. □

**Corollary 2.6.** *There is a vector bundle of weight* $w = 1$ *(resp.* $w = -1$*) and rank* $h = \mathrm{hcf}(r, d)$ *over a dense open substack of* $\mathcal{B}un_{r,d}$*.*

*Proof.* Because the case of weight $w = -1$ follows by dualizing the vector bundles, we only consider vector bundles of weight $w = 1$. Here $\mathcal{E}_p^{\mathrm{univ}}$ is a vector bundle of rank $r$ over $\mathcal{B}un_{r,d}$, and $\mathrm{Hom}(L^{\mathrm{dual}}, \mathcal{E}^{\mathrm{univ}})$ is a vector bundle of rank $r(1 - g + \deg(L)) + d$ over a dense open substack if $L$ is a sufficiently ample line bundle on $C$. Consequently, the rank of $\mathcal{V}_0$ divides $r$ and $r(1 - g + \deg(L)) + d$; hence it also divides their highest common factor $h$. □

To each vector bundle $\mathcal{V}$ over a dense open substack $\mathcal{U} \subseteq \mathcal{B}un_{r,d}$, we can associate a Grassmannian bundle

$$\mathrm{Gr}_j(\mathcal{V}) \longrightarrow \mathcal{U} \subseteq \mathcal{B}un_{r,d}.$$

By definition, $\mathrm{Gr}_j(\mathcal{V})$ is the moduli stack of those vector bundles $E$ over $C$ which are parameterized by $\mathcal{U}$, endowed with a $j$-dimensional vector subspace of $\mathcal{V}_E$. $\mathrm{Gr}_j(\mathcal{V})$ is again a smooth Artin stack locally of finite type over $k$, and its canonical morphism to $\mathcal{U}$ is representable by Grassmannian bundles of schemes.

If $\mathcal{V}$ is a vector bundle of some weight, then all scalar automorphisms of $E$ preserve all vector subspaces of $\mathcal{V}_E$. This means that the automorphism groups of the groupoid $\mathrm{Gr}_j(\mathcal{V})(k)$ also contain the scalars $k^*$. In particular, it makes sense to say that a vector bundle over $\mathrm{Gr}_j(\mathcal{V})$ has weight $w \in \mathbb{Z}$: There is an obvious way to generalize definition 2.1 to this situation.

To give some examples, we fix a point $p \in C(k)$. Let $\mathcal{P}ar_{r,d}^m$ be the moduli stack of rank $r$, degree $d$ vector bundles $E$ over $C$ endowed with a quasiparabolic structure of multiplicity $m$ over $p$. Recall that such a quasiparabolic structure is just a coherent subsheaf $E' \subseteq E$ with the property that $E/E'$ is isomorphic to the skyscraper sheaf $\mathcal{O}_p^m$.

**Example 2.7.** *$\mathcal{P}ar_{r,d}^m$ is canonically isomorphic to the Grassmannian bundle $\mathrm{Gr}_m((\mathcal{E}_p^{\mathrm{univ}})^{\mathrm{dual}})$ over $\mathcal{B}un_{r,d}$.*

Here we have regarded a quasiparabolic vector bundle $E^\bullet = (E' \subseteq E)$ as the vector bundle $E$ together with a dimension $m$ quotient of the fibre $E_p$. But we can also regard it as the vector bundle $E'$ together with a dimension $m$ vector subspace in the fibre at $p$ of the twisted vector bundle $E'(p)$. Choosing a trivialization of the line bundle $\mathcal{O}_C(p)$ over $p$, we can identify the fibres of $E'(p)$ and $E'$ at $p$; hence we also obtain:

**Example 2.8.** *$\mathcal{P}ar_{r,d}^m$ is isomorphic to the Grassmannian bundle $\mathrm{Gr}_m(\mathcal{E}_p'^{\mathrm{univ}})$ over $\mathcal{B}un_{r,d-m}$ where $\mathcal{E}'^{\mathrm{univ}}$ is the universal vector bundle over $C \times \mathcal{B}un_{r,d-m}$.*

These two Grassmannian bundles

$$\mathcal{B}un_{r,d} \xleftarrow{\theta_1} \mathcal{P}ar_{r,d}^m \xrightarrow{\theta_2} \mathcal{B}un_{r,d-m}$$

form a correspondence between $\mathcal{B}un_{r,d}$ and $\mathcal{B}un_{r,d-m}$, the *Hecke correspondence*. Its effect on the determinant line bundles is given by

$$\det \theta_1(E^\bullet) = \det(E) \cong \mathcal{O}_C(mp) \otimes \det(E') = \mathcal{O}_C(mp) \otimes \det \theta_2(E^\bullet) \qquad (2)$$

for each parabolic vector bundle $E^\bullet = (E' \subseteq E)$ with multiplicity $m$ at $p$.

**Proposition 2.9.** *Let $\mathcal{V}$ and $\mathcal{W}$ be two vector bundles of the same weight $w$ over dense open substacks of $\mathcal{B}un_{r,d}$. If $j \leq \mathrm{rk}(\mathcal{W}) \leq \mathrm{rk}(\mathcal{V})$, then there is a birationally linear map*

$$\rho : \mathrm{Gr}_j(\mathcal{V}) \dashrightarrow \mathrm{Gr}_j(\mathcal{W})$$

*over $\mathcal{B}un_{r,d}$.*

*Proof.* According to proposition 2.5, there is a vector bundle $\mathcal{W}'$ of weight $w$ such that $\mathcal{V} \cong \mathcal{W} \oplus \mathcal{W}'$ over some dense open substack $\mathcal{U} \subseteq \mathcal{B}un_{r,d}$. We may assume without loss of generality that $\mathcal{U}$ is contained in the stable locus and denote by $\mathfrak{U} \subseteq \mathfrak{Bun}_{r,d}$ the corresponding open subscheme, cf. proposition A.6.

We use the following simple fact from linear algebra: If $W$ and $W'$ are vector spaces over $k$ with $\dim(W) \geq j$, then every $j$-dimensional vector subspace of $W \oplus W'$ whose image $S$ in $W$ also has dimension $j$ is the graph of a unique linear map $S \to W'$.

This means that $\mathrm{Gr}_j(W \oplus W')$ contains as a dense open subscheme the total space of the vector bundle $\mathrm{Hom}(S^{\mathrm{univ}}, W')$ over $\mathrm{Gr}_j(W)$ where $S^{\mathrm{univ}}$ is the universal subbundle of the constant vector bundle $W$ over $\mathrm{Gr}_j(W)$.

In our stack situation, these considerations imply that $\mathrm{Gr}_j(\mathcal{V})$ is birational to the total space of the vector bundle $\mathcal{H}om(\mathcal{S}^{\mathrm{univ}}, \mathcal{W}')$ over $\mathrm{Gr}_j(\mathcal{W})$ where $\mathcal{S}^{\mathrm{univ}}$ is the universal subbundle of the pullback of $\mathcal{W}$ over $\mathrm{Gr}_j(\mathcal{W})$. This defines the rational map $\rho$.

The vector bundle $\mathcal{H}om(\mathcal{S}^{\mathrm{univ}}, \mathcal{W}')$ has weight 0 because $\mathcal{S}^{\mathrm{univ}}$ and $\mathcal{W}'$ both have weight $w$. Since the scalars act trivially, we can descend $\mathrm{Gr}_j(\mathcal{W})$ and this vector bundle over it to a Grassmannian bundle over $\mathfrak{U}$ and a vector bundle over it, cf. proposition A.6. In particular, our homomorphism bundle is trivial over a dense open substack of $\mathrm{Gr}_j(\mathcal{W})$. This proves that $\rho$ is birationally linear. □

**Corollary 2.10.** *Let $\mathcal{V}$ be a vector bundle of weight $w = \pm 1$ over a dense open substack of $\mathcal{B}un_{r,d}$. If $j$ is divisible by $\mathrm{hcf}(r,d)$, then the Grassmannian bundle*

$$\mathrm{Gr}_j(\mathcal{V}) \longrightarrow \mathcal{B}un_{r,d}$$

*is birationally linear.*

*Proof.* By corollary 2.6, there is a vector bundle $\mathcal{W}$ of weight $w$ and rank $j$. Due to the proposition, $\mathrm{Gr}_j(\mathcal{V})$ is birationally linear over $\mathrm{Gr}_j(\mathcal{W}) \simeq \mathcal{B}un_{r,d}$. □

# 3 Proof of theorem 1.2

The aim of this section is to prove theorem 1.2, i. e. to construct the birationally linear map $\mu : \mathcal{B}un_{r,d} \dashrightarrow \mathcal{B}un_{h,0}$ where $h$ is the highest common factor of the rank $r$ and the degree $d$. We proceed by induction on $r/h$.

For $r = h$, the theorem is trivial: Tensoring with an appropriate line bundle defines even an isomorphism of stacks $\mathcal{B}un_{r,d} \xrightarrow{\sim} \mathcal{B}un_{h,0}$ with the required properties. Thus we may assume $r > h$.

**Lemma 3.1.** *There are unique integers $r_F$ and $d_F$ that satisfy*

$$(1-g)r_F r + r_F d - r d_F = h \tag{3}$$

*and*

$$r < h r_F < 2r. \tag{4}$$

*Proof.* (3) has an integer solution $r_F, d_F$ because $h$ is also the highest common factor of $r$ and $(1-g)r+d$; here $r_F$ is unique modulo $r/h$. Furthermore, $r_F$ is nonzero modulo $r/h$ since $-rd_F = h$ has no solution. Hence precisely one of the solutions $r_F, d_F$ of (3) also satisfies (4). □

We fix $r_F$, $d_F$ and define

$$r_1 := hr_F - r, \qquad d_1 := hd_F - d, \qquad h_1 := \mathrm{hcf}(r_1, d_1).$$

Then $r_1 < r$, and $h_1$ is a multiple of $h$. In particular, $r_1/h_1 < r/h$.

**Lemma 3.2.** *There is an exact sequence*

$$0 \longrightarrow E_1 \longrightarrow F \otimes_k V \longrightarrow E \longrightarrow 0 \tag{5}$$

*where $E_1$, $F$, $E$ are vector bundles over $C$ and $V$ is a vector space over $k$ with*

$$\begin{array}{cccc} \mathrm{rk}(E_1) = r_1, & \mathrm{rk}(F) = r_F, & \mathrm{rk}(E) = r, & \dim(V) = h \\ \deg(E_1) = d_1, & \deg(F) = d_F, & \deg(E) = d & \end{array}$$

*such that the following two conditions are satisfied:*

*i)* $\mathrm{Ext}^1(F, E) = 0$, *and the induced map* $V \to \mathrm{Hom}(F, E)$ *is bijective.*

*ii)* $\mathrm{Ext}^1(E_1, F) = 0$, *and the induced map* $V^{\mathrm{dual}} \to \mathrm{Hom}(E_1, F)$ *is injective.*

*Proof.* We may assume $h = 1$ without loss of generality: If there is such a sequence for $r/h$ and $d/h$ instead of $r$ and $d$, then the direct sum of $h$ copies is the required sequence for $r$ and $d$.

By our choice of $r_F$ and $d_F$ and Riemann-Roch, all vector bundles $F$ and $E$ of these ranks and degrees satisfy

$$\chi(F, E) := \dim_k \mathrm{Hom}(F, E) - \dim_k \mathrm{Ext}^1(F, E) = h = 1.$$

If $F$ and $E$ are general, then

$$\mathrm{Hom}(F, E) \cong k \qquad \text{and} \qquad \mathrm{Ext}^1(F, E) = 0$$

according to a theorem of Hirschowitz [2, section 4.6], and there is a surjective map $F \to E$ by an argument of Russo and Teixidor [9]. Thus we obtain an exact sequence

$$0 \longrightarrow E_1 \longrightarrow F \longrightarrow E \longrightarrow 0 \tag{6}$$

that satisfies condition i (with $V = k$).

(For the convenience of the reader, a proof of the cited results is given in the appendix, cf. theorem A.7.)

Furthermore, all vector bundles of the given ranks and degrees satisfy

$$\chi(E_1, F) = \chi(F, E) - \chi(E, E) + \chi(E_1, E_1) > \chi(F, E) = h = 1$$

because $r_1 < r$. Now we can argue as above: For general $E_1$ and $F$, we have $\mathrm{Ext}^1(E_1, F) = 0$ by Hirschowitz, and there is an injective map $E_1 \to F$ with torsionfree cokernel by Russo-Teixidor; cf. also theorem A.7 in the appendix. Thus we obtain an exact sequence (6) that satisfies condition ii (with $V = k$).

Finally, we consider the moduli stack of all exact sequences (6) of vector bundles with the given ranks and degrees. As explained in the appendix (cf. corollary A.5), it is an *irreducible* algebraic stack locally of finite type over $k$. But i and ii are open conditions, so there is a sequence that satisfies both. $\square$

From now on, let $F$ be a fixed vector bundle of rank $r_F$ and degree $d_F$ that occurs in such an exact sequence (5).

**Definition 3.3.** *The rational map of stacks*

$$\lambda_F : \mathcal{B}un_{r,d} \dashrightarrow \mathcal{B}un_{r_1,d_1}$$

*is defined by sending a general rank $r$, degree $d$ vector bundle $E$ over $C$ to the kernel of the natural evaluation map*

$$\epsilon_E : \mathrm{Hom}(F, E) \otimes_k F \longrightarrow E.$$

We check that this does define a rational map. Let $\mathcal{U}_F \subseteq \mathcal{B}un_{r,d}$ be the open substack that parameterizes all $E$ for which $\mathrm{Ext}^1(F, E) = 0$ and $\epsilon_E$ is surjective. Then the $\epsilon_E$ are the restrictions of a surjective morphism $\epsilon^{\mathrm{univ}}$ of vector bundles over $C \times \mathcal{U}_F$. So the kernel of $\epsilon^{\mathrm{univ}}$ is also a vector bundle; it defines a morphism $\lambda_F : \mathcal{U}_F \to \mathcal{B}un_{r_1,d_1}$. This gives the required rational map because $\mathcal{U}_F$ is nonempty by our choice of $F$.

For later use, we record the effect of $\lambda_F$ on determinant line bundles:

$$\det \lambda_F(E) \cong \det(F)^{\otimes h} \otimes \det(E)^{\mathrm{dual}}. \tag{7}$$

Following [4], the next step is to understand the fibres of $\lambda_F$. We denote by $\mathrm{Hom}(\mathcal{E}_1^{\mathrm{univ}}, F)$ the vector bundle over an open substack of $\mathcal{B}un_{r_1,d_1}$ whose fibre over any point $[E_1]$ with $\mathrm{Ext}^1(E_1, F) = 0$ is the vector space $\mathrm{Hom}(E_1, F)$.

**Proposition 3.4.** *$\mathcal{B}un_{r,d}$ is over $\mathcal{B}un_{r_1,d_1}$ naturally birational to the Grassmannian bundle $\mathrm{Gr}_h(\mathrm{Hom}(\mathcal{E}_1^{\mathrm{univ}}, F))$.*

*Proof.* If $E$ is a rank $r$, degree $d$ vector bundle over $C$ for which $\mathrm{Ext}^1(F, E) = 0$ and the above map $\epsilon := \epsilon_E$ is surjective, then the exact sequence

$$0 \longrightarrow \ker(\epsilon) \longrightarrow \mathrm{Hom}(F, E) \otimes_k F \xrightarrow{\epsilon} E \longrightarrow 0$$

satisfies the condition i of the previous lemma. This identifies the above open substack $\mathcal{U}_F \subseteq \mathcal{B}un_{r,d}$ with the moduli stack of all exact sequences (5) that satisfy i.

Similarly, let $\mathcal{U}'_F \subseteq \mathrm{Gr}_h(\mathrm{Hom}(\mathcal{E}_1^{\mathrm{univ}}, F))$ be the open substack that parameterizes all pairs $(E_1, S \subseteq \mathrm{Hom}(E_1, F))$ for which $\mathrm{Ext}^1(E_1, F) = 0$ and the natural map $\alpha : E_1 \to S^{\mathrm{dual}} \otimes_k F$ is injective with torsionfree cokernel. For such a pair $(E_1, S)$, the exact sequence

$$0 \longrightarrow E_1 \xrightarrow{\alpha} S^{\mathrm{dual}} \otimes_k F \longrightarrow \mathrm{coker}(\alpha) \longrightarrow 0$$

satisfies the condition ii of the previous lemma. This identifies $\mathcal{U}'_F$ with the moduli stack of all exact sequences (5) that satisfy ii.

Hence both $\mathcal{B}un_{r,d}$ and $\mathrm{Gr}_h(\mathrm{Hom}(\mathcal{E}_1^{\mathrm{univ}}, F))$ contain as an open substack the moduli stack $\mathcal{U}''_F$ of all exact sequences (5) that satisfy both conditions i and ii. But $\mathcal{U}''_F$ is nonempty by our choice of $F$, so it is dense in both stacks; thus they are birational over $\mathcal{B}un_{r_1,d_1}$. $\square$

Still following [4], the proof of theorem 1.2 can now be summarized in the following diagram; it is explained below.

$$\begin{array}{ccccccccc}
\mathcal{B}un_{r,d} & \overset{\rho}{\dashrightarrow} & \mathrm{Gr}_h(\mathcal{W}) & \overset{\tilde{\mu_1}}{\dashrightarrow} & \mathcal{P}ar^h_{h_1,0} & \xrightarrow{\theta_2} & \mathcal{B}un_{h_1,-h} & \overset{\mu_2}{\dashrightarrow} & \mathcal{B}un_{h,0} \\
 & \searrow^{\lambda_F} & \downarrow & & \downarrow{\scriptstyle\theta_1} & & & & \\
 & & \mathcal{B}un_{r_1,d_1} & \underset{\mu_1}{\dashrightarrow} & \mathcal{B}un_{h_1,0} & & & &
\end{array}$$

Here $\mu_1$ and $\mu_2$ are the birationally linear maps given by the induction hypothesis. $(\theta_1, \theta_2)$ is the Hecke correspondence explained in the previous section; note that $\theta_2$ is birationally linear by corollary 2.10.

The square in this diagram is cartesian, so $\tilde{\mu_1}$ is the pullback of $\mu_1$, and $\mathcal{W} := \mu_1^*(\mathcal{E}_p^{\mathrm{univ}})^{\mathrm{dual}}$ is the pullback of the vector bundle $(\mathcal{E}_p^{\mathrm{univ}})^{\mathrm{dual}}$ over $\mathcal{B}un_{h_1,0}$ to which $\theta_1$ is the associated Grassmannian bundle. Using remark 1.4, we may assume that $\mu_1$ preserves scalar automorphisms, i. e. that $\mathcal{W}$ has the same weight $-1$ as $(\mathcal{E}_p^{\mathrm{univ}})^{\mathrm{dual}}$. Then we can apply proposition 2.9 to obtain the birationally linear map $\rho$. Now we have the required birationally linear map

$$\mu := \mu_2 \circ \theta_2 \circ \tilde{\mu_1} \circ \rho : \mathcal{B}un_{r,d} \dashrightarrow \mathcal{B}un_{h,0};$$

it satisfies the determinant condition in theorem 1.2 due to equations (7), (2) and the corresponding induction hypothesis on $\mu_1, \mu_2$.

# A Moduli stacks of sheaves on curves

This section summarizes some well-known basic properties of moduli stacks of vector bundles and more generally coherent sheaves on curves. For the general theory of algebraic stacks, we refer the reader to [5] or the appendix of [12]. We prove that the moduli stacks in question are algebraic, smooth and irreducible. Then we discuss descent to the coarse moduli scheme. Finally, we deduce Hirschowitz' theorem [2] and a refinement by Russo and Teixidor [9] about morphisms between general vector bundles.

Recall that we have fixed an algebraically closed field $k$ and a connected smooth projective curve $C/k$ of genus $g$. We say that a coherent sheaf $F$ on $C$ has *type* $t = (r, d)$ if its rank $\mathrm{rk}(F)$ (at the generic point of $C$) equals $r$ and its degree $\deg(F)$ equals $d$.

If $F'$ and $F$ are coherent sheaves of types $t = (r, d)$ and $t' = (r', d')$ on $C$, then the Euler characteristic

$$\chi(F', F) := \dim_k \mathrm{Hom}(F', F) - \dim_k \mathrm{Ext}^1(F', F)$$

satisfies the Riemann-Roch theorem $\chi(F', F) = \chi(t', t)$ with

$$\chi(t', t) := (1-g)r'r + r'd - rd'.$$

Note that $Ext^n(F', F)$ vanishes for all $n \geq 2$ since $\dim(C) = 1$.

We denote by $\mathcal{C}oh_t$ the moduli stack of coherent sheaves $F$ of type $t$ on $C$. More precisely, $\mathcal{C}oh_t(S)$ is for each $k$-scheme $S$ the groupoid of all coherent sheaves on $C \times S$ which are flat over $S$ and whose fibre over every point of $S$ has type $t$.

Now assume $t = t_1 + t_2$. We denote by $\mathcal{E}xt(t_2, t_1)$ the moduli stack of exact sequences of coherent sheaves on $C$

$$0 \to F_1 \to F \to F_2 \to 0$$

where $F_i$ has type $t_i = (r_i, d_i)$ for $i = 1, 2$. This means that $\mathcal{E}xt(t_2, t_1)(S)$ is for each $k$-scheme $S$ the groupoid of short exact sequences of coherent sheaves on $C \times S$ which are flat over $S$ and fibrewise of the given types.

**Proposition A.1.** *The stacks $\mathcal{C}oh_t$ and $\mathcal{E}xt(t_2, t_1)$ are algebraic in the sense of Artin and locally of finite type over $k$.*

*Proof.* Let $\mathcal{O}(1)$ be an ample line bundle on $C$. For $n \in \mathbb{Z}$, we denote by

$$\mathcal{C}oh_t^n \subseteq \mathcal{C}oh_t \qquad (\text{resp. } \mathcal{E}xt(t_2, t_1)^n \subseteq \mathcal{E}xt(t_2, t_1))$$

the open substack that parameterizes coherent sheaves $F$ on $C$ (resp. exact sequences $0 \to F_1 \to F \to F_2 \to 0$ of coherent sheaves on $C$) such that the twist $F(n) = F \otimes \mathcal{O}(1)^{\otimes n}$ is generated by global sections and $\mathrm{H}^1(F(n)) = 0$.

By Grothendieck's theory of Quot-schemes, there is a scheme $\mathrm{Quot}_t^n$ of finite type over $k$ that parameterizes such coherent sheaves $F$ together with a basis of the $k$-vector space $\mathrm{H}^0(F(n))$. Moreover, there is a relative Quot-scheme $\mathrm{Flag}(t_2, t_1)^n$ of finite type over $\mathrm{Quot}_t^n$ that parameterizes such exact sequences $0 \to F_1 \to F \to F_2 \to 0$ together with a basis of $\mathrm{H}^0(F(n))$.

Let $N$ denote the dimension of $\mathrm{H}^0(F(n))$. According to Riemann-Roch, $N$ depends only on $t$, $n$ and the ample line bundle $\mathcal{O}(1)$, but not on $F$.

Changing the chosen basis defines an action of $\mathrm{GL}(N)$ on $\mathrm{Quot}_t^n$, and $\mathcal{C}oh_t^n$ is precisely the stack quotient $\mathrm{Quot}_t^n/\mathrm{GL}(N)$. Similarly, $\mathcal{E}xt(t_2, t_1)^n$ is precisely the stack quotient $\mathrm{Flag}(t_2, t_1)^n/\mathrm{GL}(N)$. Hence these two stacks are algebraic and of finite type over $k$.

By the ampleness of $\mathcal{O}(1)$, the $\mathcal{C}oh_t^n$ (resp. $\mathcal{E}xt(t_2, t_1)^n$) form an open covering of $\mathcal{C}oh_t$ (resp. $\mathcal{E}xt(t_2, t_1)$). $\square$

**Remark A.2.** In general, $\mathcal{C}oh_t$ is not quasi-compact because the family of all coherent sheaves on $C$ of type $t$ is not bounded.

**Proposition A.3.** *i) $\mathcal{C}oh_t$ is smooth of dimension $-\chi(t, t)$ over $k$.*

ii) *If we assign to each exact sequence $0 \to F_1 \to F \to F_2 \to 0$ the two sheaves $F_1, F_2$, then the resulting morphism of algebraic stacks*

$$\mathcal{E}xt(t_2, t_1) \longrightarrow \mathcal{C}oh(t_1) \times \mathcal{C}oh(t_2)$$

*is smooth of relative dimension $-\chi(t_2, t_1)$, and all its fibres are connected.*

iii) *$\mathcal{E}xt(t_2, t_1)$ is smooth of dimension $-\chi(t_2, t_2) - \chi(t_2, t_1) - \chi(t_1, t_1)$ over $k$.*

*Proof.* i) By standard deformation theory, $\mathrm{Hom}(F, F)$ is the automorphism group of any infinitesimal deformation of the coherent sheaf $F$, $\mathrm{Ext}^1(F, F)$ classifies such deformations, and $\mathrm{Ext}^2(F, F)$ contains the obstructions against extending deformations infinitesimally, cf. [3, 2.A.6]. Because $\mathrm{Ext}^2$ vanishes, deformations of $F$ are unobstructed and hence $\mathcal{C}oh_t$ is smooth; its dimension at $F$ is then $\dim \mathrm{Ext}^1(F, F) - \dim \mathrm{Hom}(F, F) = -\chi(t, t)$.

ii) The fibre of this morphism over $[F_1, F_2]$ is the moduli stack of all extensions of $F_2$ by $F_1$; it is the stack quotient of the affine space $\mathrm{Ext}^1(F_2, F_1)$ modulo the trivial action of the algebraic group $\mathrm{Hom}(F_2, F_1)$. Hence this fibre is smooth of dimension $-\chi(t_2, t_1)$ and connected.

More generally, consider a scheme $S$ of finite type over $k$ and a morphism $S \to \mathcal{C}oh(t_1) \times \mathcal{C}oh(t_2)$; let $F_1$ and $F_2$ be the corresponding coherent sheaves over $C \times S$. By EGA III, the object $\mathrm{RHom}(F_2, F_1)$ in the derived category of coherent sheaves on $S$ can locally be represented by a complex of length one $V^0 \xrightarrow{\delta} V^1$ where $V^0, V^1$ are vector bundles. This means that the pullback of $\mathcal{E}xt(t_2, t_1)$ to $S$ is locally the stack quotient of the total space of $V^1$ modulo the action of the algebraic group $V^0/S$ given by $\delta$. Hence this pullback is smooth over $S$; this proves ii.

iii) follows from i and ii. □

**Proposition A.4.** *The stacks $\mathcal{C}oh_t$ and $\mathcal{E}xt(t_2, t_1)$ are connected.*

*Proof.* Proposition A.3 implies that $\mathcal{E}xt(t_2, t_1)$ is connected if $\mathcal{C}oh_{t_1}$ and $\mathcal{C}oh_{t_2}$ are. We prove the connectedness of the latter by induction on the rank (and on the degree for rank zero).

$\mathcal{C}oh_t$ is connected for $t = (0, 1)$ because there is a canonical surjection $C \to \mathcal{C}oh_t$; it sends a point $P$ to the skyscraper sheaf $\mathcal{O}_P$. Now consider $t = (0, d)$ with $d \geq 2$ and write $t = t_1 + t_2$. By induction hypothesis and A.3, $\mathcal{E}xt(t_1, t_2)$ is connected. But there is a canonical surjection $\mathcal{E}xt(t_1, t_2) \to \mathcal{C}oh_t$; it sends an extension $0 \to F_1 \to F \to F_2 \to 0$ to the sheaf $F$. This shows that $\mathcal{C}oh_t$ is also connected; now we have proved all connectedness assertions in rank zero.

If $F$ and $F'$ are two coherent sheaves on $C$ of type $t = (r, d)$ with $r \geq 1$, then there is a line bundle $L$ on $C$ such that both $L^{\mathrm{dual}} \otimes F$ and $L^{\mathrm{dual}} \otimes F'$ have a generically nonzero section. In other words, there are injective morphisms $L \hookrightarrow F$ and $L \hookrightarrow F'$. Let $t_L$ be the type of $L$; then $F$ and $F'$ are both in the image of the canonical morphism $\mathcal{E}xt(t - t_L, t_L) \to \mathcal{C}oh_t$. But $\mathcal{E}xt(t - t_L, t_L)$ is connected by the induction hypothesis and A.3. This shows that any two points $F$ and $F'$ lie in the same connected component of $\mathcal{C}oh_t$, i.e. $\mathcal{C}oh_t$ is connected. □

**Corollary A.5.** *The stacks $\mathcal{C}oh_t$ and $\mathcal{E}xt(t_2, t_1)$ are reduced and irreducible.*

The moduli stack $\mathcal{B}un_t$ of vector bundles, the moduli stack $\mathcal{S}stab_t$ of semistable vector bundles and the moduli stack $\mathcal{S}tab_t$ of (geometrically) stable vector bundles on $C$ of type $t = (r, d)$ are open substacks

$$\mathcal{S}tab_t \subseteq \mathcal{S}stab_t \subseteq \mathcal{B}un_t \subseteq \mathcal{C}oh_t.$$

Hence these stacks are all irreducible and smooth of the same dimension $-\chi(t, t)$ if they are nonempty. $\mathcal{S}tab_t$ is known to be nonempty for $g \geq 2$ and $r \geq 1$. Moreover, $\mathcal{S}stab_t$ and $\mathcal{S}tab_t$ are quasi-compact (and thus of finite type) because the family of (semi-)stable vector bundles of given type $t$ is bounded.

**Proposition A.6.** *Assume $g \geq 2$. Let $\mathcal{S}tab_t \longrightarrow \mathfrak{Bun}_t$ be the coarse moduli scheme of stable vector bundles of type t, and let $\mathcal{V}$ be a vector bundle of some weight $w \in \mathbb{Z}$ over an open substack $\mathcal{U} \subseteq \mathcal{S}tab_t$.*

*i) $\mathcal{U}$ descends to an open subscheme $\mathfrak{U} \subseteq \mathfrak{Bun}_t$.*

*ii) $\mathrm{Gr}_j(\mathcal{V})$ descends to a (twisted) Grassmannian scheme $\mathfrak{Gr}_j(\mathcal{V})$ over $\mathfrak{U}$.*

*iii) If $\mathcal{V}$ has weight $w = 0$, then it descends to a vector bundle over $\mathfrak{U}$.*

*iv) More generally, any vector bundle of weight 0 over $\mathrm{Gr}_j(\mathcal{V})$ descends to a vector bundle over $\mathfrak{Gr}_j(\mathcal{V})$.*

*v) Any birationally linear map of stacks $\mathcal{S}tab_{t'} \dashrightarrow \mathcal{S}tab_t$ induces a birationally linear map of schemes $\mathfrak{Bun}_{t'} \dashrightarrow \mathfrak{Bun}_t$ .*

*Proof.* We continue to use the notation introduced in the proof of proposition A.1. By boundedness, there is an integer $n$ such that $\mathcal{S}tab_t$ is contained in $\mathcal{C}oh_t^n$; hence $\mathcal{S}tab_t = \mathrm{Quot}_t^{\mathrm{stab}}/\mathrm{GL}(N)$ where $\mathrm{Quot}_t^{\mathrm{stab}} \subseteq \mathrm{Quot}_t^n$ is the stable locus. Here the center of $\mathrm{GL}(N)$ acts trivially; by Geometric Invariant Theory [6], $\mathrm{Quot}_t^{\mathrm{stab}}$ is a principal $\mathrm{PGL}(N)$-bundle over $\mathfrak{Bun}_t$.

i) Let $U \subseteq \mathrm{Quot}_t^{\mathrm{stab}}$ be the inverse image of $\mathcal{U}$. Then $U$ is a $\mathrm{PGL}(N)$-stable open subscheme in the total space of this principal bundle and hence the inverse image of a unique open subscheme $\mathfrak{U} \subseteq \mathfrak{Bun}_t$.

ii) Let $V$ be the pullback of $\mathcal{V}$ to $U$; it is a vector bundle with $\mathrm{GL}(N)$-action. Hence its Grassmannian scheme $\mathrm{Gr}_j(V) \to U$ also carries an action of $\mathrm{GL}(N)$. But here the center acts trivially: $\lambda \cdot \mathrm{id} \in \mathrm{GL}(N)$ acts as the scalar $\lambda^w$ on the fibres of $V$ and hence acts trivially on $\mathrm{Gr}_j(V)$. Thus we obtain an action of $\mathrm{PGL}(N)$ on our Grassmannian scheme $\mathrm{Gr}_j(V)$ over $U$. Since this action is free, $\mathrm{Gr}_j(V)$ descends to a Grassmannian bundle $\mathfrak{Gr}_j(\mathcal{V})$ over $\mathfrak{U}$ (which may be twisted, i. e. not Zariski-locally trivial).

iii) The assumption $w = 0$ means that the scalars in $\mathrm{GL}(N)$ act trivially on the fibres of $V$. Hence $\mathrm{PGL}(N)$ acts on $V$ over $U$ here. Again since this action is free, $V$ descends to a vector bundle over $\mathfrak{U}$.

iv) Here weight 0 means that the scalars in $\mathrm{GL}(N)$ act trivially on the pullback of the given vector bundle to $\mathrm{Gr}_j(V)$. Hence $\mathrm{PGL}(N)$ acts on this pullback; but it acts freely on the base $\mathrm{Gr}_j(V)$, so the vector bundle descends to $\mathfrak{Gr}_j(\mathcal{V})$.

v) Such a birationally linear map can be represented by an isomorphism $\varphi : \mathcal{U}' \to \mathcal{U}$ between dense open substacks $\mathcal{U}' \subseteq \mathcal{S}tab_{t'} \times \mathbb{A}^?$ and $\mathcal{U} \subseteq \mathcal{S}tab_t$. By i, $\mathcal{U}$ corresponds to an open subscheme $\mathfrak{U} \subseteq \mathfrak{Bun}_t$; the proof of i shows that $\mathfrak{U}$ is a coarse moduli scheme for the stack $\mathcal{U}$. A straightforward generalization of this argument shows that $\mathcal{U}'$ corresponds to an open subscheme $\mathfrak{U}' \subseteq \mathfrak{Bun}_{t'} \times \mathbb{A}^?$ and that $\mathfrak{U}'$ is again a coarse moduli scheme for $\mathcal{U}'$. By the universal property of coarse moduli schemes, $\varphi$ induces an isomorphism $\mathfrak{U}' \to \mathfrak{U}$ and thus the required birationally linear map of schemes. □

**Theorem A.7 (Hirschowitz, Russo-Teixidor).** *Assume $g \geq 2$. Let $F_1$ and $F_2$ be a general pair of vector bundles on $C$ with given types $t_1 = (r_1, d_1)$ and $t_2 = (r_2, d_2)$.*

- *i) If $\chi(t_1, t_2) \geq 0$, then $\dim \mathrm{Hom}(F_1, F_2) = \chi(t_1, t_2)$ and $\mathrm{Ext}^1(F_1, F_2) = 0$.*
- *ii) If $\chi(t_1, t_2) \geq 1$ and $r_1 > r_2$ (resp. $r_1 = r_2$, resp. $r_1 < r_2$), then a general morphism $F_1 \to F_2$ is surjective (resp. injective, resp. injective with torsionfree cokernel).*

*Proof.* The cases $r_1 = 0$ and $r_2 = 0$ are trivial, so we may assume $r_1, r_2 \geq 1$; then $\mathcal{S}tab_{t_1} \neq \emptyset \neq \mathcal{S}tab_{t_2}$. By semicontinuity, there is a dense open substack $\mathcal{U} \subseteq \mathcal{S}tab_{t_1} \times \mathcal{S}tab_{t_2}$ of stable vector bundles $F_1, F_2$ with $\dim \mathrm{Hom}(F_1, F_2)$ minimal, say equal to $m$. According to Riemann-Roch, $m \geq \chi(t_1, t_2)$; part i of the theorem precisely claims that we have equality here.

Let $\mathrm{Hom}(\mathcal{F}_1^{\mathrm{univ}}, \mathcal{F}_2^{\mathrm{univ}})$ be the vector bundle of rank $m$ over $\mathcal{U}$ whose fibre over $F_1, F_2$ is $\mathrm{Hom}(F_1, F_2)$. By generic flatness (cf. EGA IV, §6.9), there is a dense open substack $\mathcal{V}$ in the total space of $\mathrm{Hom}(\mathcal{F}_1^{\mathrm{univ}}, \mathcal{F}_2^{\mathrm{univ}})$ such that the cokernel of the universal family of morphisms $F_1 \to F_2$ is flat over $\mathcal{V}$. Then its image and kernel are also flat over $\mathcal{V}$; we denote the types of cokernel, image and kernel by $t_Q = (r_Q, d_Q)$, $t = (r, d)$ and $t_K = (r_K, d_K)$.

If $r = 0$, then the theorem clearly holds: In this case, the general morphism $\varphi : F_1 \to F_2$ has generic rank zero, so $\varphi = 0$; this means $m = 0$. Together with $m \geq \chi(t_1, t_2)$, this proves i and shows that the hypothesis of ii cannot hold here. Henceforth, we may thus assume $r \neq 0$.

Note that $t_1 = t_K + t$ and $t_2 = t + t_Q$; moreover, we have a canonical morphism of moduli stacks

$$\Phi : \mathcal{V} \longrightarrow \mathcal{E}xt(t, t_K) \times_{\mathcal{C}oh_t} \mathcal{E}xt(t_Q, t)$$

that sends a morphism $\varphi : F_1 \to F_2$ to the extensions

$$0 \to \ker(\varphi) \to F_1 \to \mathrm{im}(\varphi) \to 0 \quad \text{and} \quad 0 \to \mathrm{im}(\varphi) \to F_2 \to \mathrm{coker}(\varphi) \to 0.$$

Conversely, two extensions $0 \to K \to F_1 \to I \to 0$ and $0 \to J \to F_2 \to Q \to 0$ together with an isomorphism $I \to J$ determine a morphism $\varphi : F_1 \to F_2$. Thus $\Phi$ is an isomorphism onto the open locus in $\mathcal{E}xt \times_{\mathcal{C}oh} \mathcal{E}xt$ where both extension sheaves $F_1, F_2$ are stable vector bundles and $\dim \mathrm{Hom}(F_1, F_2) = m$. Hence the stack dimensions coincide, i. e.

$$m - \chi(t_1, t_1) - \chi(t_2, t_2) = -\chi(t_1, t_K) - \chi(t, t) - \chi(t_Q, t_2).$$

Since $\chi$ is biadditive, this is equivalent to

$$m - \chi(t_1, t_2) = -\chi(t_K, t_Q). \tag{8}$$

In particular, $\chi(t_K, t_Q) \leq 0$ follows.

Now suppose that $t_K$ and $t_Q$ were both nonzero. Since the general vector bundles $F_1$ and $F_2$ are stable, we then have

$$\frac{d_K}{r_K} < \frac{d_1}{r_1} < \frac{d}{r} < \frac{d_2}{r_2} < \frac{d_Q}{r_Q}.$$

Using the assumption $\chi(t_1, t_2) \geq 0$, we get

$$\frac{\chi(t_K, t_Q)}{r_K r_Q} = 1 - g - \frac{d_K}{r_K} + \frac{d_Q}{r_Q} > 1 - g - \frac{d_1}{r_1} + \frac{d_2}{r_2} = \frac{\chi(t_1, t_2)}{r_1 r_2} \geq 0$$

and hence $\chi(t_K, t_Q) > 0$. This contradiction proves $t_K = 0$ or $t_Q = 0$.

(In some sense, this argument also covers the cases $r_K = 0$ and $r_Q = 0$. More precisely, $r_K = 0$ implies $t_K = 0$ because every rank zero coherent subsheaf of a vector bundle $F_1$ is trivial. On the other hand, $r_K \neq 0$ and $t_Q = (0, d_Q) \neq 0$ would imply $\chi(t_K, t_Q) = r_K d_Q > 0$ which is again a contradiction.)

In particular, we get $\chi(t_K, t_Q) = 0$; together with equation (8), this proves part i of the theorem.

If $r_1 > r_2$ (resp. $r_1 \leq r_2$), then $r_K > r_Q$ (resp. $r_K \leq r_Q$) and hence $r_K \neq 0 = r_Q$ (resp. $r_K = 0$); we have just seen that this implies $t_Q = 0$ (resp. $t_K = 0$), i. e. the general morphism $\varphi : F_1 \to F_2$ is surjective (resp. injective).

Furthermore, the morphism of stacks $\mathcal{V} \to \mathcal{C}oh_{t_Q}$ that sends a morphism $\varphi : F_1 \to F_2$ to its cokernel is smooth (due to the open embedding $\Phi$ and proposition A.3). If $r_1 < r_2$, then $r_Q \geq 1$, so $\mathcal{B}un_{t_Q}$ is open and dense in $\mathcal{C}oh_{t_Q}$; this implies that the inverse image of $\mathcal{B}un_{t_Q}$ is open and dense in $\mathcal{V}$, i. e. the general morphism $\varphi : F_1 \to F_2$ has torsionfree cokernel. □

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