The rationality problem and birational rigidity

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Abstract

In this survey paper birational geometry of higher-dimensional rationally connected varieties is discussed. In higher dimensions the classical rationality problem generalizes to the problem of description of the structures of a rationally connected fiber space on a given variety. We discuss the key concept of birational rigidity and present examples of Fano fiber spaces with finitely many rationally connected structures.

Introduction

0.1. The Lüroth problem. The modern age in birational geometry started with the negative solution of the Lüroth problem: does unirationality imply rationality? In [3,12] negative answers were given for dimension three, in [2] for arbitrary dimension ≥ 3. The unirationality of the produced examples was proved by direct (sometimes almost obvious) constructions and the hardest part was to prove their non-rationality. The paper of Iskovskikh and Manin on the three-dimensional quartics [12] started a whole new field of research in the framework of which new methods of proving non-rationality were developed, the methods that work effectively for a large class of higher-dimensional algebraic varieties. The aim of this survey is to describe and explain by examples some of the main ideas in this field.

In [12] the following fact was shown.

Theorem 0.1. *Let $\chi\colon V \dashrightarrow V'$ be a birational map between smooth three-dimensional quartics $V, V' \subset \mathbb{P}^4$. Then χ is a biregular (projective) isomorphism. In particular, the group of birational self-maps $\operatorname{Bir} V = \operatorname{Aut} V$ is finite (for a generic quartic V it is trivial).*

Corollary 0.1. *The smooth three-dimensional quartic $V \subset \mathbb{P}^4$ is non-rational.*

Proof of the corollary. The group of birational self-maps of an algebraic variety X is a birational invariant. However, by Theorem 0.1 the group $\operatorname{Bir} V$ is finite, whereas the Cremona group $\operatorname{Bir} \mathbb{P}^3$ is infinite. Therefore, V cannot be birational to $\mathbb{P}^3$, which is what we need. Q.E.D.

Remark 0.1. The argument above is obvious. For a long time (for more than 20 years after the paper [12] was published) quite a few people believed that this was the only way to deduce non-rationality of the three-dimensional quartic. However, with all simplicity and brevity of this argument, there is a disadvantage, namely, if the group $\mathrm{Bir}\, X$ is "of the same size" as the Cremona group $\mathrm{Bir}\,\mathbb{P}^3$ in the sense of cardinality, one cannot prove non-rationality of the variety X in this way. In particular, this method does not work for the complete intersection $V_{2\cdot 3} \subset \mathbb{P}^5$ of a quadric and a cubic (a description of the group $\mathrm{Bir}\, V_{2\cdot 3}$ is given below following [13]). It is almost certain that the groups $\mathrm{Bir}\, V_{2\cdot 3}$ and $\mathrm{Bir}\,\mathbb{P}^3$ are non-isomorphic, but today we cannot even approach this problem.

However, there are two (very close to each other) ways to derive Corollary 0.1 from the constructions of the paper [12], although not directly from Theorem 0.1. Their advantage is in the fact that they work for other Fano varieties, in particular, for $V_{2\cdot 3}$. Let us describe these arguments.

A second proof of Corollary 0.1. In [12] the following fact was actually shown.

Proposition 0.1. *Let $\chi\colon V \dashrightarrow X$ be a birational map of a smooth three-dimensional quartic V onto a smooth projective variety X, $|R|$ a movable complete linear system on X, $\Sigma \subset |nH| = |-nK_V|$ its strict transform on V with respect to χ, where $H \in \mathrm{Pic}\, V$ is the class of a hyperplane section of $V \subset \mathbb{P}^4$. Then, if for some positive integers $a, b \in \mathbb{Z}_+$ the linear system $|aR + bK_X|$ is empty, then the linear system $|anH + bK_V|$ is empty, either, that is, $b > an$.*

Corollary 0.2. *Let $\alpha\colon X \to S$ be a morphism. Assume that one of the following two cases holds:*

- *$S = \mathbb{P}^1$, the general fiber $\alpha^{-1}(s)$, $s \in S$, is a rational surface,*
- *$\dim S = 2$, and the general fiber $\alpha^{-1}(s)$, $s \in S$, is an irreducible rational curve.*

Then there is no birational map $\chi\colon V \dashrightarrow X$, where $V \subset \mathbb{P}^4$ is a smooth quartic.

Since a linear projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ or $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ realizes $\mathbb{P}^3$ as a $\mathbb{P}^1$- or $\mathbb{P}^2$-bundle, respectively, Corollary 0.2 implies non-rationality of the three-dimensional quartic.

Proof of Corollary 0.2. Assume the converse: there is a birational map $\chi\colon V \dashrightarrow X$. Let Λ be a complete very ample linear system on S. Let $|R| = \alpha^*\Lambda$ be its pull back on X. Obviously, the class R is trivial on the fibers of α, so that for any $a, b > 0$ we get

$$|aR + bK_X| = \emptyset,$$

since the fiber $\alpha^{-1}(s)$ has the negative Kodaira dimension. Let $\Sigma \subset |nH|$ be the strict transform of the system $|R|$ on V with respect to χ. By Proposition 0.1, we get $b > an$. Since a, b are arbitrary, we get $n = 0$. But Σ is a movable linear system, so that $n \geq 1$. A contradiction. Q.E.D. for Corollary 0.2.

Remark 0.2. We have just obtained a much stronger fact than non-rationality of V. Corollary 0.2 asserts that there is no rational map $\gamma\colon V \dashrightarrow S$ onto a variety S of positive dimension, the generic fiber of which is a rational surface or a rational curve. In the modern terminology, on V there are no *structures* of a fiber space into rational curves or rational surfaces. Since on $\mathbb{P}^3$ there are infinitely many such structures, the quartic V is non-rational. Although the argument above is much less obvious than the first proof of Corollary 0.1, its potential is much greater: it shows in which direction one should generalize the rationality problem and what class of algebraic varieties should be involved into consideration. These generalizations will be considered below. Completing our discussion of the three-dimensional quartic, let us give

A third proof of Corollary 0.1. The argument given below is also based on Proposition 0.1, however it is more direct than the previous one. Assume that $\chi: V \dashrightarrow \mathbb{P}^3$ is a birational map and $|R|$ is the complete linear system of planes in $\mathbb{P}^3$. The linear system $|aR + bK_{\mathbb{P}^3}| = |(a-4b)R|$ is empty if and only if $a < 4b$. Let $\Sigma \subset |nH|$ be the strict transform of the system $|R|$ on V. By Proposition 0.1, for any positive integers a, b, satisfying the inequality $a < 4b$, we get $b > an$. Thus $n \leq \frac{1}{4}$, that is, $n = 0$, which is impossible. A contradiction. Q.E.D. for non-rationality of the three-dimensional quartic.

Keeping in mind the three proofs of non-rationality of the three-dimensional quartic, we will show in this paper, what class of varieties it is natural to consider in general, what questions it is natural to ask, and what answers it is natural to expect.

0.2. Rationally connected varieties. Recall [14,15] that an algebraic variety X is said to be *rationally connected*, if any two (generic) points $x, y \in X$ can be joined by an irreducible rational curve, that is, there exists a morphism $f: \mathbb{P}^1 \to X$ such that $x, y \in f(\mathbb{P}^1)$. The projective space $\mathbb{P}^M$ and smooth Fano varieties are rationally connected. In [5] the following fundamental fact was proved.

Theorem 0.2. *Let $\pi: X \to S$ be a fiber space (that is, a surjective morphism of projective varieties with connected fibers), the base S and generic fiber $\pi^{-1}(s)$, $s \in S$, of which are rationally connected. Then the variety X itself is rationally connected.*

The fiber spaces $\pi: X \to S$ described in the theorem above are called *rationally connected fiber spaces*. From the viewpoint of classification of algebraic varieties, rationally connected varieties are the most natural generalization of rational varieties in dimension three and higher. Obviously, the rationality problem makes sense for rationally connected varieties only.

Definition 0.1. A *structure of a rationally connected fiber space* on a rationally connected variety X is an arbitrary rational dominant map $\varphi: X \dashrightarrow S$, the fiber of general position of which $\varphi^{-1}(s)$, $s \in S$, is irreducible and rationally connected. If the base S is a point, then the structure is said to be trivial.

An alternative definition: a structure of a rationally connected fiber space on a variety X is a birational map $\chi: X \dashrightarrow X^\sharp$ onto a variety $X^\sharp$ equipped with a surjective morphism $\pi: X^\sharp \to S$ realizing $X^\sharp$ as a rationally connected fiber space. We identify the structures of a rationally connected fiber space $\varphi_1: X \dashrightarrow S_1$ and $\varphi_2: X \dashrightarrow S_2$, if there exists a birational map $\alpha: S_1 \dashrightarrow S_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} & X & \stackrel{\text{id}}{\leftrightarrow} & X & \\ \varphi_1 & \downarrow & & \downarrow & \varphi_2 \\ & S_1 & \stackrel{\alpha}{\dashrightarrow} & S_2, & \end{array} \qquad (1)$$

that is, $\varphi_2 = \alpha \circ \varphi_1$. In other words, φ_1 and φ_2 have the same fibers. The set of non-trivial structures of a rationally connected fiber space on the variety X (modulo the identification above) is denoted by $RC(X)$.

On the set $RC(X)$ there is a natural relation of partial order: for $\varphi_1, \varphi_2 \in RC(X)$ we have $\varphi_1 \leq \varphi_2$, if there is a rational dominant map $\alpha\colon S_1 \dashrightarrow S_2$ such that the diagram (1) commutes. In other words, the fibers of φ_1 are contained in the fibers of φ_2. For a general point $s \in S_2$ we have

$$\alpha^{-1}(s) = \varphi_1(\varphi_2^{-1}(s)),$$

therefore $\alpha \in RC(S_1)$ is a structure of a rationally connected fiber space on S_1. It is easy to see that the correspondence $\varphi_2 \mapsto \alpha$ determines a bijection of the sets $\{\psi \in RC(X) | \psi \geq \varphi_1\}$ and $RC(S_1)$. Therefore from the geometric viewpoint of primary interest are the *minimal* elements of the ordered set $RC(X)$. Denote the set of minimal elements by $RC_{\min}(X)$. Set also $RC_d(X) \subset RC(X)$ to be the set of structures, the generic fiber of which is of dimension d. Obviously, if $d = \min\{e \in \mathbb{Z}_+ | RC_e \neq \emptyset\}$, then $RC_d \subset RC_{\min}$.

For each $d \in \{1, \dots, \dim X - 1\}$ on the set $RC_d(X)$ there is a natural relation of *fiber-wise birational equivalence*: $\varphi_1 \sim \varphi_2$ if there exists a birational transformation $\chi \in \operatorname{Bir} X$ and a birational map $\alpha\colon S_1 \dashrightarrow S_2$ such that the diagram

$$\begin{array}{ccccc} & X & \stackrel{\chi}{\dashrightarrow} & X & \\ \varphi_1 & \downarrow & & \downarrow & \varphi_2 \\ & S_1 & \stackrel{\alpha}{\dashrightarrow} & S_2, & \end{array}$$

commutes, that is, $\varphi_2 \circ \chi = \alpha \circ \varphi_1$. In other words, the birational self-map χ transforms the fibers of φ_1 into the fibers of φ_2. The quotient set $RC_d(X)/\sim$ we denote by the symbol $\overline{RC}_d(X)$.

For instance, any two linear projections $\varphi_1, \varphi_2\colon \mathbb{P}^M \dashrightarrow \mathbb{P}^{M-d}$ are fiber-wise birationally equivalent and realize the same element in $\overline{RC}_d(\mathbb{P}^M)$. On the other hand, let $V \subset \mathbb{P}^M$ be a smooth Fano hypersurface of index two, that is, a hypersurface of degree $M-1$.

Proposition 0.2. *Any two distinct generic linear projections $\varphi_1, \varphi_2\colon \mathbb{P}^M \dashrightarrow \mathbb{P}^1$ determine the structures of a rationally connected fiber space on V, $\varphi_i|_V\colon V \dashrightarrow \mathbb{P}^1$, which are not fiber-wise birationally equivalent.*

For the **proof,** see Sec. 3.

The fibers of the structures $\varphi_i|_V$ are Fano hypersurfaces of index 1, that is, hypersurfaces of degree $M-1$ in $\mathbb{P}^{M-1}$. Since for a general hypersurface V, a general projection $\varphi\colon \mathbb{P}^M \dashrightarrow \mathbb{P}^1$ and a general point $p \in \mathbb{P}^1$ for $M \geq 5$ we have $RC(\varphi|_V^{-1}(p)) = \emptyset$ (see [18] and Sec. 1 of the present paper), the structures $\varphi|_V$ are minimal elements of the set $RC(V)$.

Conjecture 0.1. *For $e \leq M-2$ and a general hypersurface $V \subset \mathbb{P}^M$ of degree $M-1$ we have $RC_e(V) = \emptyset$.*

For a general four-dimensional quartic $V = V_4 \subset \mathbb{P}^5$ Conjecture 0.1 asserts that V has no structures of a rationally connected fiber space with the base of dimension two or three. The assumption of genericity is essential: if $V \supset P$, where $P \subset \mathbb{P}^5$ is a two-dimensional plane, then the projection from that plane $\pi_P \colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ fibers V into cubic surfaces, that is, $\pi_P|_V \in RC_2(V)$.

Proposition 0.2 shows that the set $\overline{RC}_d(X)$ can be quite big and possess a natural structure of an algebraic variety.

The second proof of Corollary 0.1 now can be formulated in the following way: Proposition 0.1 implies that for a smooth three-dimensional quartic $V \subset \mathbb{P}^4$ we have $RC(V) = \emptyset$. Since $RC(\mathbb{P}^3) \neq \emptyset$, the quartic V is non-rational.

The arguments of Sec. 0.1 show that the rationality problem generalizes to the following questions concerning birational geometry of a rationally connected variety X:

- compute the sets $RC(X)$, $RC_{\min}(X)$, $RC_d(X)$ and $\overline{RC}_d(X)$,
- compute the group of birational self-maps $\operatorname{Bir} X$.

We single out computing the group of birational self-maps as a separate problem, since it is of independent interest. In fact, it is necessary to compute this group to describe the quotient set $\overline{RC}_d(X)$; moreover, one should know the action of the group $\operatorname{Bir} X$ on the set $RC_d(X)$. Besides, the interest to the problem of computing the group $\operatorname{Bir} X$ (like the special interest to the rationality problem) comes from tradition.

0.3. The structure of the paper. The aim of this paper is to explain the main ideas connected with the problems that were set up above, for certain natural classes of rationally connected varieties. Sec. 1 is devoted to discussing the key concept of birational rigidity. We give the necessary definitions and describe the main steps in proving birational rigidity (that is, excluding and "untwisting" maximal singularities). As an example of description of a group of birational self-maps we give (following [13]) a proof of the theorem on generators and relations in the group $\operatorname{Bir} V_{2\cdot 3}$ for the three-dimensional complete intersection of a quadric and a cubic in $\mathbb{P}^5$. Here we follow [13], giving all details of the proof, since the paper [13] is not easily accessible. This group by its "size" is comparable with the Cremona group $\operatorname{Bir} \mathbb{P}^3$, so that the cardinality argument is insufficient to prove non-rationality of the variety $V_{2\cdot 3}$ (which at the same time automatically follows from birational rigidity: $RC(V_{2\cdot 3}) = \emptyset$). Description of the group $\operatorname{Bir} V_{2\cdot 3}$ presents an exceptionally visual example of "untwisting" maximal singularities.

In Sec. 2 we consider examples of rationally connected varieties, the set of rationally connected structures on which is non-empty but finite: the direct products of divisorially canonical Fano varieties (Sec. 2.1), Fano fiber spaces $V/\mathbb{P}^1$ with a non-trivial group of birational self-maps $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, permuting the two elements in $RC(V)$, so that $\sharp\overline{RC}(V)=1$ (Sec. 2.2) and Fano fiber spaces $V/\mathbb{P}^1$ with no non-trivial birational self-maps, $\mathrm{Bir}\, V = \mathrm{Aut}\, V$ and $\sharp RC(V)=\sharp\overline{RC}(V)=2$ (Sec. 2.3). The varieties, considered in Sec. 2.2 and 2.3, present examples of flops in higher dimensions. These are the first examples of non-trivial untwisting of maximal singularities in dimensions higher than three; the varieties of the type of Sec. 2.3 are the first examples of non-trivial links in higher dimensions (in the terminology of Sarkisov program [4,33]).

In Sec. 3, following [21], we prove Proposition 0.2. Computation of the group of birational self-maps of a rationally connected variety V, which is the total space of a rationally connected fiber space $\pi\colon V\to S$, $\dim S\geq 1$, naturally breaks into two separate problems: that of comparison of the group $\mathrm{Bir}\, V$ with the group of fiber-wise (with respect to π) birational self-maps $\mathrm{Bir}(V/S)$ and that of computation of the group $\mathrm{Bir}(V/S)$. In Sec. 3 we consider the problem of computing the group $\mathrm{Bir}(V/S)$, where C is a curve, for an essentially bigger class of fiber spaces. Proposition 0.2 follows from the main theorem of Sec. 3 in a straightforward way. A birational correspondence between two rationally connected structures described in Proposition 0.2 turns out to be a biregular map, for a generic V it is identical.

If a rationally connected fiber space V/S determines a unique non-trivial rationally connected structure on V, then the exact sequence

$$1\to \mathrm{Bir}(V/S)\to \mathrm{Bir}\, V\to \mathrm{Bir}\, S$$

reduces computation of the group of birational self-maps to computation of the group of the *proper* birational self-maps, preserving the fibers of π:

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V & \\ \pi & \downarrow & & \downarrow & \pi \\ & S & \longleftrightarrow & S, & \end{array}$$

or, equivalently, the group $\mathrm{Bir}\, F_\eta$ of birational self-maps of the fiber F_η over the generic (non-closed) point of the base S. We can also look at χ as a continuous family of birational self-maps of fibers

$$S\ni s\mapsto \chi_s\in \mathrm{Bir}\, F_s.$$

If V/S is a Fano fiber space, the general fiber of which is birationally superrigid, then the results of Sec. 3 make it possible to give a complete description of the group of birational self-maps of the variety V, like it is done below in Sec. 2.2 and 2.3 for Fano fiber spaces over $\mathbb{P}^1$.

1 Birational rigidity

1.1. Termination of canonical adjunction. A rationally connected variety X satisfies the classical condition of termination of canonical adjunction: for any effective divisor D the linear system $|D+nK_X|$ is empty for $n\gg 0$, since K_X is negative on some family of rational curves sweeping out X. The classical proof of the Castelnuovo rationality criterion [1] makes use of this condition, fixing the precise step n_* of canonical adjunction when $|D+n_*K_X|$ is still non-empty, but $|D+(n_*+1)K_X|=\emptyset$: it turns out that the linear system $|D+n_*K_X|$ has very useful properties. To formalize this idea, for a smooth rationally connected variety X consider the Chow group A^iX of algebraic cycles of codimension i modulo rational equivalence, $A^1X\cong \operatorname{Pic} X$, and set $A^i_{\mathbb{R}}X=A^iX\otimes\mathbb{R}$. Let $A^i_+X\subset A^i_{\mathbb{R}}X$ be the closed cone generated by effective classes, that is, the cone of pseudoeffective classes. Set also $A^1_{\mathrm{mov}}X\subset A^1_{\mathbb{R}}X$ to be the closed cone generated by the classes of movable divisors (that is, divisors in movable linear systems).

Definition 1.1. *The threshold of canonical adjunction* of a divisor D on the variety X is the number $c(D,X)=\sup\{\varepsilon\in\mathbb{Q}_+|D+\varepsilon K_X\in A^1_+X\}$. If Σ is a non-empty linear system on X, then we set $c(\Sigma,X)=c(D,X)$, where $D\in\Sigma$ is an arbitrary divisor.

Example 1.1. (i) Let X be a primitive Fano variety, that is, a smooth projective variety with the ample anticanonical class and $\operatorname{Pic} X=\mathbb{Z}K_X$. For any effective divisor D we have $D\in|-nK_X|$ for some $n\geq 1$, so that $c(D,X)=n$. If we replace the condition $\operatorname{Pic} X=\mathbb{Z}K_X$ by the weaker one $\operatorname{rk}\operatorname{Pic} X=1$, that is, $K_X=-rH$, where $\operatorname{Pic} X=\mathbb{Z}H$, $r\geq 2$ is the index of the variety X, then for $D\in|nH|$ we get $c(D,X)=\frac{n}{r}$.

(ii) Let $\pi\colon V\to S$ be a rationally connected fiber space with $\dim V>\dim S\geq 1$, Δ an effective divisor on the base S. Obviously, $c(\pi^*\Delta,V)=0$. If $\operatorname{Pic} V=\mathbb{Z}K_V\oplus\pi^*\operatorname{Pic} S$, that is, V/S is a primitive Fano fiber space, and D is an effective divisor on V, which is not a pull back of a divisor on the base S, then

$$D\in|-nK_V+\pi^*R|$$

for some divisor R on S, where $n\geq 1$. Obviously, $c(D,V)\leq n$, and moreover, if the divisor R is effective, then $c(D,V)=n$.

(iii) Let $F_1,\dots,F_K$ be primitive Fano varieties, $V=F_1\times\dots\times F_K$ their direct product. Let $H_i=-K_{F_i}$ be the positive generator of the group $\operatorname{Pic} F_i$. Set

$$S_i=\prod_{j\neq i}F_i,$$

so that $V\cong F_i\times S_i$. Let $\rho_i\colon V\to F_i$ and $\pi_i\colon V\to S_i$ be the projections onto the factors. Abusing our notations, we write H_i instead of $\rho_i^*H_i$, so that

$$\operatorname{Pic} V=\bigoplus_{i=1}^{K}\mathbb{Z}H_i$$

and $K_V=-H_1-\ldots-H_K$. For any effective divisor D on V we get

$$D\in|n_1H_1+\ldots+n_KH_K|$$

for some non-negative $n_1,\dots,n_K\in\mathbb{Z}_+$, and obviously

$$c(D,V)=\min\{n_1,\dots,n_K\}.$$

This example can be reduced to the previous one: assume that $c(D,V)=n_1$ and set $n=n_1$, $\pi=\pi_1$, $F=F_1$, $S=S_1$. We get

$$\Sigma\subset|-nK_V+\pi^*Y|,$$

where $Y=\sum\limits_{i=2}^{K}(n_i-n)H_i$ is an effective class on the base S of the fiber space $\pi\colon V\to S$. This is the case of Example 1.1 (ii) above.

The threshold of canonical adjunction is easy to compute, but the main disadvantage of this concept is that it is not a birational invariant.

Example 1.2. Let $\pi\colon\mathbb{P}^M\dashrightarrow\mathbb{P}^m$ be a linear projection from a $(M-m-1)$-dimensional plane $P\subset\mathbb{P}^M$. Consider a movable linear system Λ of hypersurfaces of degree n in $\mathbb{P}^m$ and let Σ be its pull back via π. Obviously, $c(\Sigma,\mathbb{P}^M)=\frac{n}{M+1}$. However, let us blow up the plane P, say $\sigma\colon\mathbb{P}^+\to\mathbb{P}^M$, so that the composite map $\pi\circ\sigma\colon\mathbb{P}^+\to\mathbb{P}^m$ is a $\mathbb{P}^{M-m}$-bundle. Let Σ^+ be the strict transform of Σ on $\mathbb{P}^+$. Since $\pi\circ\sigma$ is a morphism with rationally connected fibers, we get $c(\Sigma^+,\mathbb{P}^+)=0$. This example can be easily generalized to linear projections of Fano complete intersections $V\subset\mathbb{P}^M$ of index 2 or higher, similar to the case considered in Proposition 0.2.

1.2. Birationally rigid varieties. In order to overcome birational non-invariance of the threshold of canonical adjunction, we give

Definition 1.2. For a movable linear system Σ on the variety X define the *virtual threshold of canonical adjunction* by the formula

$$c_{\mathrm{virt}}(\Sigma)=\inf_{X^\sharp\to X}\{c(\Sigma^\sharp,X^\sharp)\},$$

where the infimum is taken over all birational morphisms $X^\sharp\to X$, $X^\sharp$ is a smooth projective model of $\mathbb{C}(X)$, $\Sigma^\sharp$ the strict transform of the system Σ on $X^\sharp$.

The virtual threshold is obviously a birational invariant of the pair (X,Σ): if $\chi\colon X\dashrightarrow X^+$ is a birational map, $\Sigma^+=\chi_*\Sigma$ is the strict transform of the system Σ with respect to χ^{-1}, we get $c_{\mathrm{virt}}(\Sigma)=c_{\mathrm{virt}}(\Sigma^+)$.

Proposition 1.1. (i) *Assume that on the variety V there are no movable linear systems with the zero virtual threshold of canonical adjunction. Then on V there are no structures of a non-trivial fibration into varieties of negative Kodaira dimension, that is, there is no rational dominant map $\rho\colon V\dashrightarrow S$, $\dim S\geq 1$, the generic fiber of which has negative Kodaira dimension.*

(ii) *Let $\pi\colon V\to S$ be a rationally connected fiber space. Assume that every movable linear system Σ on V with the zero virtual threshold of canonical adjunction, $c_{\mathrm{virt}}(\Sigma)=0$, is the pull back of a system on the base: $\Sigma=\pi^*\Lambda$, where Λ is some movable linear system on S. Then any birational map*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V^\sharp & \\ \pi & \downarrow & & \downarrow & \pi^\sharp \\ & S & & S^\sharp, & \end{array}\qquad(2)$$

where $\pi^\sharp\colon V^\sharp\to S^\sharp$ is a fibration into varieties of negative Kodaira dimension, is fiber-wise, that is, there exists a rational dominant map $\rho\colon S\dashrightarrow S^\sharp$, making the *diagram (2) commutative, $\pi^\sharp \circ \chi = \rho \circ \pi$. In other words, $\pi^\sharp \geq \pi$ in the sense of the order on the set of rationally connected structures: π is the least element of $RC(V)$.*

Thus for certain rationally connected varieties the virtual threshold of canonical adjunction reduces the problem of describing the set $RC(V)$ to the same problem for the base S. This is a crucial step that in many cases leads to an exhaustive description of the set $RC(V)$. But the main disadvantage of the virtual thresholds is that they are extremely hard to compute.

To be precise, the only known way to compute them is by reduction to the ordinary thresholds.

Definition 1.3. (i) The variety V is said to be *birationally superrigid*, if for any movable linear system Σ on V the equality

$$c_{\rm virt}(\Sigma) = c(\Sigma, V)$$

holds.

(ii) The variety V (respectively, the Fano fiber space V/S) is said to be *birationally rigid*, if for any movable linear system Σ on V there exists a birational self-map $\chi \in \operatorname{Bir} V$ (respectively, a fiber-wise birational self-map $\chi \in \operatorname{Bir}(V/S)$), providing the equality

$$c_{\rm virt}(\Sigma) = c(\chi_* \Sigma, V).$$

In the following examples the main classes of Fano varieties and Fano fiber spaces, for which birational rigidity or superrigidity is known today, are listed.

Example 1.3. Smooth three-dimensional quartics $V = V_4 \subset \mathbb{P}^4$ are birationally superrigid: this follows immediately from the arguments of [12]. Generic smooth complete intersections $V_{2\cdot 3} \subset \mathbb{P}^5$ of a cubic and a quadric hypersurfaces are birationally rigid, but not superrigid. For description of their groups of birational self-maps (which also demonstrates how the thresholds of canonical adjunction are decreased by means of birational automorphisms), see Sec. 1.3 below.

Example 1.4. Generic hypersurfaces of index one $V_M \subset \mathbb{P}^M$ are birationally superrigid [18]. The same is true for generic complete intersections $V \subset \mathbb{P}^{M+k}$ of index one and codimension k, provided that $M \geq 2k+1$ [22].

Example 1.5. Let $\sigma \colon V \to Q \subset \mathbb{P}^{M+1}$ be a double cover, where $Q = Q_m \subset \mathbb{P}^{M+1}$ is a smooth hypersurface of degree m, and the branch divisor $W \subset Q$ is cut out on Q by a hypersurface $W^*_{2l} \subset \mathbb{P}^{M+1}$, where $m + l = M + 1$. The Fano variety V is birationally superrigid for general Q, W^* [19]. Instead of a double cover an arbitrary cyclic cover could be considered, instead of a hypersurface $Q \subset \mathbb{P}^{M+1}$ a smooth complete intersection $Q \subset \mathbb{P}^{M+k}$ of appropriate index and codimension $k < \frac{1}{2}M$. A general variety in each of these classes is birationally superrigid [23,27]. Another example is given by iterated double covers [24].

All varieties mentioned in Examples 1.4 and 1.5 can be realized as Fano complete intersections in weighted projective spaces.

Conjecture 1.1. *A smooth Fano complete intersection of index one and dimension ≥ 4 in a weighted projective space is birationally rigid, of dimension ≥ 5 birationally superrigid.*

Now let us consider the known examples of fiber spaces.

Example 1.6. (V.G.Sarkisov, [31,32]) Let $\pi: V \to S$ be a conic bundle with a sufficiently positive discriminant divisor D, satisfying the Sarkisov condition $|4K_S + D| \neq \emptyset$. Then $\sharp RC_1(V) = 1$, that is, there is exactly one structure of a conic bundle on V, namely the projection π.

Example 1.7. Let $\mathcal{F}$ be any of the classes of Fano varieties listed in Examples 1.3-1.5. Let $\pi: V \to \mathbb{P}^1$ be a smooth Fano fiber space, such that every fiber $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$, is in $\mathcal{F}$. Assume furthermore that the strong K^2-condition is satisfied: $K_V^2 \notin \operatorname{Int} A_+^2 V$. In a certain natural sense almost all fiber spaces $V/\mathbb{P}^1$ satisfy the strong K^2-condition, which can be considered as a characteristic of "twistedness" over the base. In these assumptions, a general fiber space $V/\mathbb{P}^1$ is birationally superrigid [17,20,25,29].

Example 1.8. Three-dimensional del Pezzo fibrations, satisfying strong K^2-condition, are birationally rigid [17]. In fact, the strong K^2-condition can be considerably relaxed [7,8,35].

1.3. The method of maximal singularities. In order to prove birational (super)rigidity of a smooth projective rationally connected variety V, fix a movable linear system Σ on V and set $n = c(\Sigma) \in \mathbb{Z}_+$. Assume that the inequality

$$c_{\text{virt}}(\Sigma) < n$$

holds (otherwise no work is required). In particular, $n > 0$. By definition, there exists a birational morphism $\sigma: \widetilde{V} \to V$ of smooth varieties such that

$$c(\widetilde{\Sigma}, \widetilde{V}) < n,$$

where $\widetilde{\Sigma}$ is the strict transform of Σ on $\widetilde{V}$.

Definition 1.4. An exceptional divisor $E \subset \widetilde{V}$ is called *a maximal singularity* of the system Σ, if *the Noether-Fano inequality*

$$\nu_E(\varphi^* \Sigma) > na(E) \tag{3}$$

holds, where $\nu_E(\cdot)$ is the multiplicity of the pull back of Σ on $\widetilde{V}$ along E and $a(E)$ is the discrepancy of E.

Proposition 1.2. *In the assumptions above, a maximal singularity of Σ does exist.*

For a (very simple) **proof**, see [12,18,20].

It turns out that maximal singularities of movable linear systems are a very special phenomenon. For many classes of Fano varieties and Fano fiber spaces a movable linear system cannot have a maximal singularity which in view of Proposition 1.2 implies superrigidity.

In this section we present one of the most sophisticated examples of a birationally rigid, but not superrigid, Fano three-fold, known today, namely the complete intersection of a quadric and a cubic in $\mathbb{P}^5$. The proof was started in [11] and completed in [16]. A detailed exposition can be found in [13].

Here we concentrate on the "untwisting" procedure.

Let us fix notations. We study the complete intersection $V = Q \cap F \subset \mathbb{P}^5$, where Q is a quadric and F is a cubic hypersurface. The variety V is assumed to be smooth and, moreover, generic in the sense described below, in particular, $\operatorname{Pic} V = \mathbb{Z}H$, where $H = -K_V$ is the class of a hyperplane section of $V \subset \mathbb{P}^5$.

1.3.1. Lines on the complete intersection V. Let $L \subset V$ be a line in $\mathbb{P}^5$.

Proposition 1.3. *For the normal sheaf $\mathcal{N}_{L/V}$ there are two possible cases:*

- *either $\mathcal{N}_{L/V} \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L$; in this case the line L is said to be of general type,*
- *or $\mathcal{N}_{L/V} \cong \mathcal{O}_L(-2) \oplus \mathcal{O}_L(1)$; in this case the line L is said to be of non-general type.*

Moreover, the line L is of non-general type if and only if any of the following two equivalent conditions holds:

- *there exists a plane $P \subset \mathbb{P}^5$ such that $L \subset P$ and the scheme-theoretic intersection $V \cap P$ is not reduced everywhere along L,*
- *let $\sigma: \widetilde{V} \to V$ be the blow up of L, $E = \sigma^{-1}(L)$ the exceptional divisor. Then restricting to E the strict transform on $\widetilde{V}$ of a generic hyperplane section containing L, we get a non-ample divisor on E.*

Proof is straightforward and left to the reader.

We will consider the general complete intersections $V = Q \cap F$, satisfying the following conditions:

- V does not contain lines of non-general type (it is easy to check by the usual dimension count that this condition is justified, that is, a general complete intersection satisfies it),
- there are no three lines on V lying in one plane and having a common point,
- the quadric Q is non-degenerate.

Let $L \subset V$ be a line. The projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^3$ from L defines a rational map $\pi_L: V \dashrightarrow \mathbb{P}^3$ of degree two. Set $\alpha_L \in \operatorname{Bir} V$ to be the corresponding Galois involution.

More formally, let $\sigma: \widetilde{V} \to V$ be the blow up of L, $E = \sigma^{-1}(L) \subset \widetilde{V}$ the exceptional divisor. The map π_L extends to a morphism $p = \pi_L \circ \sigma: \widetilde{V} \to \mathbb{P}^3$.

Lemma 1.1. *The morphism p is a finite morphism of degree 2 outside a closed subset $W \subset \widetilde{V}$ of codimension two, and $p(W) \subset \mathbb{P}^3$ is a finite set of points. The involution α_L extends to a biregular involution of $\widetilde{V} \setminus W$. Its action on $\operatorname{Pic} \widetilde{V} = \mathbb{Z}H \oplus \mathbb{Z}E$ is given by the formulas*

$$\alpha_L^*(H) = 4H - 5E, \quad \alpha_L^*(E) = 3H - 4E.$$

Proof. The projection $p\colon \widetilde{V} \to \mathbb{P}^3$ is a finite morphism outside the set $W \subset \widetilde{V}$ that consists of curves that are contracted by the morphism p. We will show there are finitely many of them. Set $H' = nH - \nu E$ and $E' = mH - \mu E$ to be the classes in $\operatorname{Pic}\widetilde{V}$ of the strict transform of a general hyperplane section and the divisor E with respect to α_L. The linear system $|H - E|$ is clearly invariant under α_L. Take a general surface $S \in |H - E|$.

Since $K_S = 0$, the birational involution $\alpha_L\,|\,_S$ extends to a biregular involution of this surface. Denote it by α_S, and the restrictions of H and E to S by H_S and E_S, respectively. We get

$$\alpha_S^* H_S = nH_S - \nu E_S, \quad \alpha_S^* E_S = mH_S - \mu E_S$$

and the class $H_S - E_S$ is α_S^*-invariant, whence we get $n = m + 1$, $\nu = \mu + 1$. Since α_S is an automorphism,

$$(\alpha_S^* H_S \cdot (H_S - E_S)) = (H_S \cdot (H_S - E_S)) = 5$$

and $(\alpha_S^* H_S)^2 = (H_S)^2 = 6$, whence by the obvious equalities $(H_S \cdot E_S) = 1$, $(E_S^2) = -2$ we get the following two possibilities for n, m, ν, μ:

- either $H' = 4H - 5E$, $E' = 3H - 4E$,
- or $H' = H$, $E' = E$,

the latter being clearly impossible because α_L can not be extended to a biregular automorphism of V.

By construction, the system $|4H - 5E|$ is movable. However, if a curve C is contracted by the morphism p, then $(C \cdot (H - E)) = 0$ and therefore $(C \cdot H') < 0$. We conclude that there can be only finitely many such curves. Q.E.D.

Now let $P \subset \mathbb{P}^5$ be a 2-plane such that $P \cap V$ is a union of three lines, $P \cap V = L \cup L_1 \cup L_2$. This is possible only if $P \subset Q$. Let $\sigma\colon \widetilde{V} \to V$ be the composition of three blow ups: first, we blow up L, then the strict transform of L_1, then the strict transform of L_2.

We denote the exceptional divisors on $\widetilde{V}$, corresponding to the lines L, L_1, L_2, by the symbols E, E_1, E_2, respectively.

Lemma 1.2. *The involution α_L extends to a biregular involution on $\widetilde{V} \setminus W$, where W is a closed subset of codimension two. The action of α_L on $\operatorname{Pic}\widetilde{V} = \mathbb{Z}H \oplus \mathbb{Z}E \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$ is given by the formulae:*

$$\begin{array}{rl} \alpha_L^* = & 4H - 5E - 2E_1 - 2E_2, \\ \alpha_L^* E = & 3H - 4E - 2E_1 - 2E_2, \\ \alpha_L^* E_i = & E_j, \end{array}$$

where $\{i, j\} = \{1, 2\}$.

Proof is obtained in the same way as for the previous lemma; one has to consider, along with the projection π_L, the projection $\pi_P\colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ from the plane P. The considerations are more subtle but essentially similar.

1.3.2. Conics on the complete intersection V. It is easy to see that there is a one-dimensional family of irreducible conics $Y \subset V$ such that the plane $P(Y) =< Y >$ is contained entirely in the quadric Q. Obviously,

$$P(Y) \cap V = Y \cup L(Y),$$

where $L(Y)$ is the residual line. We will call the conics described above the *special conics*.

Every special conic Y generates the following construction. Set $P = P(Y)$. Consider the projection $\pi_P: \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ from the plane P. The fibres of π_P are 3-planes $S \supset P$, so that $S \cap Q = P \cup P(S)$, where $P(S)$ is the residual plane. Therefore, π_P fibers V over $\mathbb{P}^2$ into elliptic curves $C_S = P(S) \cap F$, that is, plane cubics. A general curve C_S intersects the residual line $L(Y)$ an one point, which is $L(Y) \cap P(S)$. We define the involution $\beta_Y \in \mathrm{Bir}\, V$ as a fiber-wise map, setting $\beta_Y\,|_{C_S}$ to be the elliptic reflection, where the group law on C_S is defined by the point $L(Y) \cap P(S)$ as the zero.

Let $\sigma: \widetilde{V} \to V$ be the composition of the blow up of the conic Y and the blow up of the strict transform of the line $L(Y)$, E and E^+ be the corresponding exceptional divisors. Obviously, $\pi_P \circ \sigma: V \to \mathbb{P}^2$ is a morphism, the general fiber of which is an elliptic curve C_t, $t \in \mathbb{P}^2$. The divisor E^+ is a section of this elliptic fibration, $(E^+ \cdot C_t) = 1$.

Lemma 1.3. *The birational involution β_Y extends to a biregular involution on the complement $\widetilde{V} \backslash W$, where W is a closed subset of codimension two, and moreover, $\pi_P \circ \sigma(W) \subset \mathbb{P}^2$ is a finite set. The action of β_Y on $\mathrm{Pic}\, \widetilde{V} = \mathbb{Z}H \oplus \mathbb{Z}E \oplus \mathbb{Z}E^+$ is given by the formulas*

$$\begin{aligned} \beta_Y^* H = &\quad 13H - 14E - 8E^+, \\ \beta_Y^* E = &\quad 12H - 13E - 8E^+, \\ \beta_Y^* E^+ = &\quad E^+. \end{aligned}$$

Proof is quite similar to the proof of Lemma 1.1. Let $H', E', E^\sharp \in \mathrm{Pic}\, \widetilde{V}$ be the classes of the strict transforms of a general hyperplane section and the divisors E and E^+, respectively. On the general curve C_t, $t \in \mathbb{P}^2$, the involution β_Y maps a point $x \in C_t$ to the point $\beta_Y(x) \in C_t$ satisfying the relation

$$\beta_Y(x) + x \sim 2(C_t \cap E^+)$$

as divisors on C_t. The kernel of the restriction of $\mathrm{Pic}\, \widetilde{V}$ onto a general fiber C_t is $\mathbb{Z}(H - E - E^+) = (\pi_P \circ \sigma)^* \, \mathrm{Pic}\, \mathbb{P}^2$, so that

$$H' + H = 6E^+ + m(H - E - E^+), \quad E' + E = 4E^+ + l(H - E - E^+)$$

and $E^\sharp = E^+ + k(H - E - E^+)$. Now we proceed exactly as in the proof of Lemma 1.1: we restrict β_Y and all the classes involved onto a general surface $S \in |H - E - E^+|$ (that is, S is the inverse image of a general line in $\mathbb{P}^2$ via $\pi_P \circ \sigma$). Since $K_S = 0$,

$\beta_Y|_S$ extends to a biregular involution of S. Comparing intersection indices, we get $m = 14$, $l = 12$.

Now β_Y is well defined on irreducible fibers, and it is easy to see that any reducible fiber C_t contains a component which intersects H' negatively. Therefore, there are only finitely many of them. Now $k = 0$ and the proof is complete. Q.E.D.

1.3.3. Relations between the involutions α_L. Let $P \subset \mathbb{P}^5$ be a plane such that $P \subset Q$ and $P \cap F = L_1 \cup L_2 \cup L_3$ is a union of three lines.

Lemma 1.4. *The following relation holds:*

$$(\alpha_{L_1} \circ \alpha_{L_2} \circ \alpha_{L_3})^2 = \mathrm{id}_V .$$

Proof. Obviously, each of the three involutions α_{L_i} preserves the fibers of the projection $\pi_P: V \dashrightarrow \mathbb{P}^2$ from the plane P. Recall that a general fiber $\pi_P^{-1}(t)$ is a cubic curve C_t, where $C_t \cap P = \{x_1, x_2, x_3\}$, $x_i = C_t \cap L_i$. Take a point $x \in C_t$; obviously,

$$\alpha_{L_i}(x) + x + x_i \sim x_1 + x_2 + x_3$$

on C_t. Therefore we compute:

$$\begin{array}{rcl} \alpha_{L_3}(x) & \sim & x_1 + x_2 - x, \\ \alpha_{L_2} \circ \alpha_{L_3}(x) & \sim & x_3 - x_2 + x, \\ \alpha_{L_1} \circ \alpha_{L_2} \circ \alpha_{L_3}(x) & \sim & 2x_2 - x, \\ (\alpha_{L_1} \circ \alpha_{L_2} \circ \alpha_{L_3})^2(x) & \sim & x, \end{array}$$

which is what we need. Q.E.D.

1.3.4. Copresentation of the group $\mathrm{Bir}\, V$**.** After this preparatory work we can formulate the main theorem describing birational geometry of V.

Set $\mathcal{L}$ and $\mathcal{C}$ to be the sets of lines and special conics on V, respectively. Let G^+ be the free group generated by symbols A_L and B_Y for all $L \in \mathcal{L}$ and $Y \in \mathcal{C}$, respectively. Let $R^+ \subset G^+$ be the normal subgroup, generated by the words A_L^2 for all $L \in \mathcal{L}$, B_Y^2 for all $Y \in \mathcal{C}$ and, finally, $(A_{L_1} A_{L_2} A_{L_3})^2$ for all triples of distinct lines $L_1, L_2, L_3 \in \mathcal{L}$ such that $< L_1 \cup L_2 \cup L_3 >= \mathbb{P}^2$.

Set $G = G^+/R^+$ to be the quotient group. We construct a semi-direct product $G\, \mathrm{Aut}\, V$ using the obvious action of $\mathrm{Aut}\, V$ on G: for $\rho \in \mathrm{Aut}\, V$ set

$$\rho A_L \rho^{-1} = A_{\rho(L)}, \quad \rho\, B_Y \rho^{-1} = B_{\rho(Y)}.$$

Let $\varepsilon: \mathrm{Aut}\, V \rightarrow \mathrm{Bir}\, V$ be the homomorphism, sending A_L to α_L, B_Y to β_Y and identical on $\mathrm{Aut}\, V$.

Theorem 1.1. *V is birationally rigid and ε is an isomorphism of groups.*

Proof. Set $\mathcal{B} = \mathcal{L} \cup \mathcal{C}$. Take any movable linear system $\Sigma \subset |nH|$ on V. Obviously, $c(\Sigma, V) = n$. In order to prove that ε is a bijection, we take Σ to be the strict transform of the linear system $|H|$ of hyperplane sections with respect to a fixed birational self-map $\chi \in \mathrm{Bir}\, V$. Clearly, in that case $n = 1$ if and only if $\chi \in \mathrm{Aut}\, V$ (and by construction biregular automorphisms are in the image of ε).

We will prove birational rigidity and surjectivity of ε simultaneously, using the following crucial technical fact.

Proposition 1.4. *Assume that* $c_{\rm virt}(\Sigma) < n$. *Then there exist a subvariety* $B \in \mathcal{B}$ *(that is, a line or a special conic) such that* $\mathrm{mult}_B \Sigma > n$. *Moreover, there are at most two subvarieties in* $\mathcal{B}$ *with that property, and if there are two, say* $B_1, B_2 \in \mathcal{B}$, *then they are lines,* $B_1, B_2 \in \mathcal{L}$, *their span* $< B_1, B_2 >$ *is a plane* $P = \mathbb{P}^2$, *and* $P \subset Q$.

Proof is very technical and represents the main step in the study of birational geometry of V. A subvariety $B \in \mathcal{B}$ satisfying the inequality $\mathrm{mult}_B \Sigma > n$ is called *a maximal subvariety* of the linear system Σ. By Proposition 1.2, we know that a maximal singularity exists. Now the hard part of work is to show that this implies existence of a maximal curve and this curve is necessarily a line or a special conic. For the details, see [13].

1.3.5. The untwisting procedure. Now we derive Theorem 1.1 from Proposition 1.4.

Lemma 1.5. (i) *Let* $L \subset V$ *be a line,* $\Sigma^+ \subset |n_+ H|$ *the strict transform of the linear system* Σ *with respect to* α_L. *The following equalities hold:*

$$n_+ = 4n - 3\,\mathrm{mult}_L \Sigma, \quad \mathrm{mult}_L \Sigma^+ = 5n - 4\,\mathrm{mult}_L \Sigma.$$

(ii) *Let* $Y \in \mathcal{C}$ *be a special conic,* $L = L(Y) \in \mathcal{L}$ *the residual line,* $\Sigma^+ \subset |n_+ H|$ *the strict transform of the linear system* Σ *with respect to* β_Y. *The following equalities hold:*

$$n_+ = 13n - 12\,\mathrm{mult}_Y \Sigma, \quad \mathrm{mult}_Y \Sigma^+ = 14n - 13\,\mathrm{mult}_Y \Sigma,$$
$$\mathrm{mult}_L \Sigma^+ = 8n - 8\,\mathrm{mult}_Y \Sigma + \mathrm{mult}_L \Sigma.$$

(iii) *Let* $P \subset \mathbb{P}^5$ *be a 2-plane such that* $P \cap V = L \cup L_1 \cup L_2$, Σ^+ *as in (i) above. Then for* $\{i, j\} = \{1, 2\}$ *we have*

$$\mathrm{mult}_{L_i} \Sigma^+ = 2n - 2\,\mathrm{mult}_L \Sigma + \mathrm{mult}_{L_j} \Sigma.$$

Proof is a straightforward application of Lemmas 1.1-1.3. Q.E.D.

Corollary 1.1. *An involution* $\tau = \alpha_L$ *or* β_Y *satisfies the inequality* $n_+ < n$ *if and only if* L *or* Y *is a maximal curve of the linear system* Σ, *respectively, where* $\Sigma^+ \subset |n_+ H|$ *is the strict transform of* Σ *with respect to* τ.

Corollary 1.2. *In the notations of the previous corollary assume that* $n_+ = n$. *Then* $\tau = \alpha_L$ *for line* $L \in \mathcal{L}$ *and there exist lines* $L_1, L_2 \in \mathcal{L}$, *such that* $L \cup L_1 \cup L_2 = P \cap V$, *where* $P \subset Q$ *is a plane.*

Now let us prove birational rigidity of V and surjectivity of ε. Assume that $c_{\rm virt}(\Sigma) < n$ for a movable linear system Σ. By Proposition 1.4, there exists a curve $B \in \mathcal{B}$ such that $\mathrm{mult}_B \Sigma > n$. Let $\tau \in \varepsilon(G)$ be the corresponding involution (that is, $\tau = \alpha_L$ if $B = L \in \mathcal{L}$ and $\tau = \beta_Y$ if $B = Y \in \mathcal{C}$). By Corollary 1.1, $\Sigma^+ \subset |n_+ H|$ with $n_+ < n$, where Σ^+ is the strict transform of Σ with respect to τ. Iterating this procedure, we construct a sequence of involutions $\tau_i \in \varepsilon(G)$ such that the strict transforms $\Sigma^{(i)} \subset |n_i H|$ of the system Σ with respect to the compositions $\tau_i \dots \tau_1$

satisfy the inequalities $n_i < n_{i-1}$. Since $n_i \in \mathbb{Z}_+$, at some step we cannot decrease the threshold $c(\Sigma^{(i)}, V)$ any longer. Therefore, for some $k \geq 1$ we get

$$c(\Sigma^{(k)}, V) = c_{\mathrm{virt}}(\Sigma^{(k)}, V) = c_{\mathrm{virt}}(\Sigma, V),$$

which is birational rigidity. Moreover, if we fix a birational self-map $\chi \in \operatorname{Bir} V$ and take Σ to be the strict transform of the system $|H|$ via χ, then the procedure described above gives $n_k = 1$ for some k, that is, $\Sigma^{(k)} \subset |H|$. Comparing dimensions, we get $\Sigma^{(k)} = |H|$, which implies that $\tau_k \dots \tau_1 \chi \in \operatorname{Aut} V$ is a biregular map. This proves surjectivity of ε.

The last step in the proof of Theorem 1.1 is to show that ε has the trivial kernel.

1.3.6. The set of relations is complete. For convenience of notations, we write down words in A_L, B_Y, using capital letters and corresponding birational self-maps using small letters, say $t = \varepsilon(T)$ etc. For a self-map $t \in \operatorname{Bir} V$ we define the integer $n(t) \in \mathbb{Z}_+$ by the formula $\Sigma \subset |n(t)H|$, where Σ is the strict transform of the system $|H|$ via t; obviously, $n(t) = 1$ if and only if $t \in \operatorname{Aut} V$. Theorem 1.1 immediately follows from

Proposition 1.5. *Let $W = T_1 \dots T_l$ be an arbitrary word in the alphabet $\{A_L, B_Y \,|\, L \in \mathcal{L}, Y \in \mathcal{C}\}$. If $w \in \operatorname{Aut} V$ then using the relations in R^+ one can transform the word W into the empty word.*

Proof. Denote by W_i, $i \leq l(W) = l$, the left segment of the word W of length i, that is, $W_i = T_1 \dots T_i$. Set

$$n^*(W) = \max\{n(w_i) \,|\, 1 \leq i \leq l(W)\},$$

$$\omega(W) = \sharp\{i \,|\, n(w_i) = n^*(W), 1 \leq i \leq l(W)\}.$$

Now we associate with every word W the ordered triple

$$(n^*(W), \omega(W), l(W)).$$

We order the set of words, setting $W > W'$, if either $n^*(W) > n^*(W')$, or $n^*(W) = n^*(W')$ and $\omega(W) > \omega(W')$, or $n^*(W = n^*(W')$, $\omega(W) = \omega(W')$ and $l(W) > l(W')$. It is easy to see that every decreasing chain of words $W^{(1)} > W^{(2)} > \dots$ breaks. Therefore, it is sufficient to show that if $w \in \operatorname{Aut} V$, then the word W can be transformed into a word W' such that $W > W'$, $w = w'$.

If the word W contains the subword $A_L A_L$ or $B_Y B_Y$, then, eliminating this subword, we get a smaller word W' (because the image of each left segment of the word W' coincides with the image of some left segment of the word W and the map of the set of left segments of W' into the set of left segments of W is injective).

So we can assume that W does not contain subwords $A_L A_L$ or $B_Y B_Y$.

Since $n(w) = 1$, we can assume that $n^*(W) \geq 2$ (otherwise there is nothing to prove). Let $s = \min\{i \,|\, n(w_i) = n^*(W)\} \leq l(W) - 1$. Let us consider the two cases $T_s = A_L$ and $T_s = B_Y$ separately.

Case 1. $T_s = B_Y$. In this case $n(w_{s-1}) = n(w_s\beta_Y) < n(w_s)$, by the choice of s. By Corollary 1.1, $\mathrm{mult}_Y \Sigma_s > n(w_s)$, where Σ_s is the strict transform of $|H|$ via w_s. Since by construction $n(w_{s+1}) \leq n(w_s)$, we get $T_{s+1} = T_s = B_Y$. A contradiction to our assumption that W does not contain subwords $A_L A_L$ and $B_Y B_Y$.

Case 2. Let $T_s = A_L$. By the choice of s we get

$$\mathrm{mult}_L \Sigma_s > n(w_s).$$

By assumption, $T_{s+1} \neq T_s$ and $n(w_{s+1}) \leq n(w_s)$. By Corollary 1.2, $T_{s+1} = A_L$, where $L' \subset V$ is a line such that there exists a third line $Z \subset V$,

$$L \cup L' \cup Z = P \cap V$$

for some plane $P \subset Q$.

Lemma 1.6. (i) *Z is a maximal line of the map w_{s-1}, that is,* $\mathrm{mult}_Z \Sigma_{s-1} > n(w_{s-1})$. *Therefore,*

$$n(w_{s-1}\alpha_Z) < n(w_{s-1}).$$

(ii) *The equality*

$$n(w_{s-1}\alpha_Z) - \mathrm{mult}_{L'} \Sigma' = n(w_s) - \mathrm{mult}_{L'} \Sigma_s \leq 0$$

holds, where Σ' is the strict transform of $|H|$ with respect to $w_{s-1}\alpha_Z$. Therefore,

$$n(w_{s-1}\alpha_Z\alpha_{L'}) \leq n(w_{s-1}\alpha_Z).$$

Proof: straightforward computations based on Lemma 1.5. We will consider the claim (i) only, leaving (ii) to the reader. Since $w_s = w_{s-1}\alpha_L$, we get $w_{s-1} = w_s\alpha_L$ and by Lemma 1.5,

$$n(w_{s-1}) = n(w_s\alpha_L) = 4n(w_s) - 3\,\mathrm{mult}_L \Sigma_s,$$

$$\mathrm{mult}_Z \Sigma_{s-1} = 2n(w_s) - 2\,\mathrm{mult}_L \Sigma_s + \mathrm{mult}_{L'} \Sigma_s.$$

Therefore, $n(w_{s-1}) - \mathrm{mult}_Z \Sigma_{s-1} = 2n(w_s) - \mathrm{mult}_L \Sigma_s - \mathrm{mult}_{L'} \Sigma_s < 0$, which is what we need. For the claim (ii), the arguments are similar. Q.E.D.

Now let us complete the proof of Theorem 1.1. Consider first the case when $\mathrm{mult}_{L'} \Sigma_s > n(w_s)$. Using the relations $A_Z^2 = e$ and $A_Z A_{L'} A_L = A_L A_{L'} A_Z$, we can replace the subword $A_L A_{L'}$ by the subword $A_Z A_{L'} A_L A_Z$. This operation increases the length. Denote the new word by W^+.

Obviously, $W_i^+ = W_i$ for $i \leq s-1$. Furthermore,

$$w_s^+ = w_{s-1}\alpha_Z, \quad w_{s+1}^+ = w_{s-1}\alpha_Z\alpha_{L'}$$

and $w_{s+2}^+ = w_{s-1}\alpha_Z\alpha_{L'}\alpha_L = w_{s+1}\alpha_Z$, whereas

$$w_{s+i}^+ = w_{s+i-2}$$

for $i \geq 3$. By the lemma above, $n(w_i^+) < n(w_s) = n^*(W)$ for $i = s, s+1, s+2$ (and by construction this is true for the smaller values $i < s$, either). Therefore, if $\omega(W) \geq 2$, then $n^*(W^+) = n^*(W)$ and $\omega(W^+) = \omega(W) - 1$. If $\omega(W) = 1$, then $n^*(W^+) < n^*(W)$. In any case, $W^+ < W$.

It remains to consider the case $\mathrm{mult}_{L'} \Sigma_s = n(w_s)$. In this case $n(w_{s+1}) = n(w_s)$, $\mathrm{mult}_{L'} \Sigma_{s+1} = n(w_{s+1})$. Since by assumption there are no subwords $A_{L'}A_{L'}$, we must have $T_{s+2} = A_Z$. Now let us replace the subword

$$T_s T_{s+1} T_{s+2} = A_L A_{L'} A_Z$$

by the subword $A_Z A_{L'} A_L$. Denote the new word by W^+. Now the length is the same, and by Lemma 1.5 we obtain the inequalities $n(w_i^+) < n^*(W)$ for $i = s, s+1, s+2$. Arguing as in the previous case, we complete the proof.

2 Varieties with finitely many structures

In this section, we discuss three types of rationally connected varieties with finitely many (but more than just one) structures of a rationally connected fiber space: Fano direct products and two classes of varieties with a pencil of Fano double covers. Our considerations are based on [26,28,30]. For other examples, see [6,9,10,34,35].

2.1. Fano direct products. Recall that a smooth projective variety F is a *primitive Fano variety*, if $\mathrm{Pic}\, F = \mathbb{Z}K_F$, the anticanonical class is ample and $\dim F \geq 3$.

Definition 2.1. We say that a primitive Fano variety F is *divisorially canonical*, or satisfies the condition (C) (respectively, is *divisorially log canonical*, or satisfies the condition (L)), if for any effective divisor $D \in |-nK_F|$, $n \geq 1$, the pair

$$(F, \frac{1}{n}D) \tag{4}$$

has canonical (respectively, log canonical) singularities. If the pair (4) has canonical singularities for a general divisor $D \in \Sigma \subset |-nK_F|$ of any *movable* linear system Σ, then we say that F satisfies the condition of *movable canonicity*, or the condition (M).

Explicitly, the condition (C) is formulated in the following way: for any birational morphism $\varphi\colon \widetilde{F} \to F$ and any exceptional divisor $E \subset \widetilde{F}$ the following inequality

$$\nu_E(D) \leq na(E) \tag{5}$$

holds. The inequality (5) is opposite to the Noether-Fano inequality (3). The condition (L) is weaker: the inequality

$$\nu_E(D) \leq n(a(E) + 1) \tag{6}$$

is required. It is well known (essentially starting from the classical paper of V.A.Iskovskikh and Yu.I.Manin [12]) that the condition (M) ensures birational superrigidity. This condition is proved for many classes of primitive Fano varieties, see [12,18,22,24]. Note also that the condition (C) is stronger than both (L) and (M).

The following fact was proved in [28].

Theorem 2.1. *Assume that primitive Fano varieties $F_1,\dots,F_K$, $K\geq 2$, satisfy the conditions (L) and (M). Then their direct product*

$$V=F_1\times\ldots\times F_K$$

is birationally superrigid.

Now let us show how birational superrigidity makes it possible to describe rationally connected structures on V.

Corollary 2.1. (i) *Every structure of a rationally connected fiber space on the variety V is given by a projection onto a direct factor. More precisely, let $\beta\colon V^\sharp\to S^\sharp$ be a rationally connected fiber space and $\chi\colon V--\to V^\sharp$ a birational map. Then there exists a subset of indices*

$$I=\{i_1,\dots,i_k\}\subset\{1,\dots,K\}$$

and a birational map $\alpha\colon F_I=\prod\limits_{i\in I}F_i \dashrightarrow S^\sharp$, such that the diagram

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V^\sharp & \\ \pi_I & \downarrow & & \downarrow & \beta \\ & F_I & \stackrel{\alpha}{\dashrightarrow} & S^\sharp & \end{array}$$

commutes, that is, $\beta\circ\chi=\alpha\circ\pi_I$, where $\pi_I\colon\prod\limits_{i=1}^{K}F_i\to\prod\limits_{i\in I}F_i$ is the natural projection onto a direct factor. In particular, the variety V admits no structures of a fibration into rationally connected varieties of dimension smaller than $\min\{\dim F_i\}$. *In particular, V admits no structures of a conic bundle or a fibration into rational surfaces.*

(ii) *The groups of birational and biregular self-maps of the variety V coincide:* $\mathrm{Bir}\,V=\mathrm{Aut}\,V$. *In particular, the group* $\mathrm{Bir}\,V$ *is finite.*

(iii) *The variety V is non-rational.*

Proof. Let us prove the claim (i). Let $\beta\colon V^\sharp\to S^\sharp$ be a rationally connected fiber space, $\chi\colon V\dashrightarrow V^\sharp$ a birational map. Take a very ample linear system $\Sigma^\sharp_S$ on the base $S^\sharp$ and let $\Sigma^\sharp=\beta^*\Sigma^\sharp_S$ be a movable linear system on $V^\sharp$. As we have mentioned above (Example 1.1, (ii)), $c(\Sigma^\sharp)=0$. Let Σ be the strict transform of the system $\Sigma^\sharp$ on V. By our remark, $c_{\mathrm{virt}}(\Sigma)=0$, so that by Theorem 2.1 we conclude that $c(\Sigma)=0$. Therefore, in the presentation

$$\Sigma\subset|-n_1H_1-\ldots-n_KH_K|$$

some coefficient $n_e=0$. We may assume that $e=1$. Setting $S=F_2\times\ldots\times F_K$ and $\pi\colon V\to S$ to be the projection, we get $\Sigma\subset|\pi^*Y|$ for a non-negative class Y on

S. But this means that the birational map χ of the fiber space V/S onto the fiber space $V^\sharp/S^\sharp$ is fiber-wise: there exists a rational dominant map $\gamma\colon S \dashrightarrow S^\sharp$, making the diagram

$$\begin{array}{rcccl} & V & \stackrel{\chi}{\dashrightarrow} & V^\sharp & \\ \pi & \downarrow & & \downarrow & \beta \\ & S & \stackrel{\gamma}{\dashrightarrow} & S^\sharp & \end{array}$$

commutative. For a point $z\in S^\sharp$ of general position let $F^\sharp_z=\beta^{-1}(z)$ be the corresponding fiber, $F^\chi_z\subset V$ its strict transform with respect to χ. By assumption, the variety F^χ_z is rationally connected. On the other hand,

$$F^\chi_z=\pi^{-1}(\gamma^{-1}(z))=F\times\gamma^{-1}(z),$$

where $F=F_1$ is the fiber of π. Therefore, the fiber $\gamma^{-1}(z)$ is also rationally connected.

Thus we have reduced the problem of description of rationally connected structures on V to the same problem for S. Now the claim (i) of Corollary 2.1 is easy to obtain by induction on the number of direct factors K. For $K=1$ it is obvious that there are no non-trivial rationally connected structures (see Proposition 1.1, (i)). The second part of the claim (i) (about the structures of conic bundles and fibrations into rational surfaces) is obvious since $\dim F_i\geq 3$ for all $i=1,\dots,K$. Non-rationality of V is now obvious, either.

Let us prove the claim (ii) of Corollary 2.1. Set $RC(V)$ to be the set of all structures of a rationally connected fiber space on V with a non-trivial base. By the part (i) we have

$$RC(V)=\{\pi_I\colon V\to F_I=\prod_{i\in I}F_i\,|\,\emptyset\neq I\subset\{1,\dots,K\}\}.$$

Now recall (Sec. 0.2) that the set $RC(V)$ has a natural structure of an ordered set: $\alpha\leq\beta$ if β factors through α. Obviously, $\pi_I\leq\pi_J$ if and only if $J\subset I$. For $I=\{1\dots,K\}\setminus\{e\}$ set $\pi_I=\pi_e$, $F_I=S_e$. It is obvious that $\pi_1\dots,\pi_K$ are the minimal elements of $RC(V)$.

Let $\chi\in\operatorname{Bir}V$ be a birational self-map. The map

$$\chi^*\colon RC(V)\to RC(V),$$

$$\chi^*\colon\alpha\longmapsto\alpha\circ\chi,$$

is a bijection preserving the relation $\leq$. From here it is easy to conclude that χ^* is of the form

$$\chi^*\colon\pi_I\longmapsto\pi_{I^\sigma},$$

where $\sigma\in S_K$ is a permutation of K elements and for $I=\{i_1,\dots,i_k\}$ we define $I^\sigma=\{\sigma(i_1),\dots,\sigma(i_k)\}$. Furthermore, for each $I\subset\{1,\dots,K\}$ we get the diagram

$$\begin{array}{rcccl} & V & \stackrel{\chi}{\dashrightarrow} & V & \\ \pi_I & \downarrow & & \downarrow & \pi_{I^\sigma} \\ & F_I & \stackrel{\chi_I}{\dashrightarrow} & F_{I^\sigma}, & \end{array}$$

where χ_I is a birational map. In particular, χ induces birational isomorphisms

$$\chi_e \colon F_e \dashrightarrow F_{\sigma(e)},$$

$e = 1, \dots, K$. However, all the varieties F_e are birationally superrigid, so that all the maps χ_e are biregular isomorphisms. Thus

$$\chi = (\chi_1, \dots, \chi_K) \in \operatorname{Bir} V$$

is a biregular isomorphism, too: $\chi \in \operatorname{Aut} V$. Q.E.D. for Corollary 2.1.

Remark 2.1. The group of biregular automorphisms $\operatorname{Aut} V$ is easy to compute. Let us break the set $F_1, \dots, F_K$ into subsets of pair-wise isomorphic varieties:

$$I = \{1, \dots, K\} = \bigvee_{k=1}^{l} I_k,$$

where $F_i \cong F_j$ if and only if $\{i, j\} \subset I_k$ for some $k \in \{1, \dots, l\}$. It is easy to see that

$$\operatorname{Aut} V = \prod_{j=1}^{l} \operatorname{Aut}(\prod_{i \in I_j} F_i).$$

In particular, if the varieties $F_1, \dots, F_K$ are pair-wise non-isomorphic, we get

$$\operatorname{Aut} V = \prod_{i=1}^{K} \operatorname{Aut} F_i$$

(and this group acts on V component-wise). In the opposite case, if

$$F_1 \cong F_2 \cong \dots \cong F_K \cong F,$$

we obtain the exact sequence

$$1 \to (\operatorname{Aut} F)^{\times K} \to \operatorname{Aut} V \to S_K \to 1,$$

where S_K is the symmetric group of permutations of K elements. In fact, in this case $\operatorname{Aut} V$ contains a subgroup isomorphic to S_K which permutes direct factors of V, so that $\operatorname{Aut} V$ is a semi-direct product of the normal subgroup $(\operatorname{Aut} F)^{\times K}$ and the symmetric group S_K.

It seems that the following generalization of Theorem 2.1 is true.

Conjecture 2.1. *Assume that $F_1, \dots, F_K$ are birationally (super)rigid primitive Fano varieties. Then their direct product $V = F_1 \times \dots \times F_K$ is birationally (super)rigid.*

Of course, Theorem 2.1 is meaningful only provided that we are able to prove the condition (C) for some particular Fano varieties. Certain examples were shown in [28]: generic Fano hypersurfaces $F = F_M \subset \mathbb{P}^M$ for $M \geq 6$ and generic Fano double spaces of index 1. More examples (Fano complete intersections) were given in [29].

2.2. Varieties with an involution. Following [26], let us construct a series of rationally connected varieties with exactly two non-trivial structures of a rationally connected fiber space. Fix positive integers m, l, satisfying the equality $m + l = M + 1$, $M \geq 4$. Set $\mathbb{P} = \mathbb{P}^{M+1}$ and take a hypersurface $W_{\mathbb{P}} \subset \mathbb{P}$ of degree $2l$. Let $\sigma_{\mathbb{Y}}: \mathbb{Y} \to \mathbb{P}$ the double cover branched over the divisor $W_{\mathbb{P}}$. Consider the variety $Y = \mathbb{P}^1 \times \mathbb{Y}$, which is realized as the double cover $\sigma_Y: Y \to X = \mathbb{P}^1 \times \mathbb{P}$ branched over the divisor $W = \mathbb{P}^1 \times W_{\mathbb{P}}$. Set $V = \sigma_Y^{-1}(Q)$, where $Q \subset X = \mathbb{P}^1 \times \mathbb{P}$ is a smooth divisor of the type $(2, m)$, that is, it is given by the equation

$$A(x_*)u^2 + 2B(x_*)uv + C(x_*)v^2 = 0,$$

where $A(\cdot), B(\cdot), C(\cdot)$ are homogeneous of degree m. Here $(u : v)$ and $(x_*) = (x_0: \ldots : x_{M+1})$ are homogeneous coordinates on $\mathbb{P}^1$ and $\mathbb{P}$, respectively.

Furthermore, let $H_{\mathbb{P}}$ be the class of a hyperplane in $\mathbb{P}$, $L_X = p_X^* H_{\mathbb{P}}$ the tautological class on X, where $p_X: X \to \mathbb{P}$ is the projection onto the second factor, $L_V = \sigma_Y^* L_X|_V$. It is easy to see that $K_V = -L_V$, so that the anticanonical linear system $|-K_V|$ is free and determines the projection $p_V = p_X \circ \sigma: V \to \mathbb{P}$.

On the other hand, the projection $\pi: V \to \mathbb{P}^1$, which is the composition of $\sigma_Y|_V$ and the projection of $\mathbb{P}^1 \times \mathbb{P}$ onto the first factor, realizes V as a primitive Fano fiber space, the fiber of which is a Fano double hypersurface of index 1 [19]: $\mathrm{Pic}\, V = \mathbb{Z}L_V \oplus \mathbb{Z}F$, where F is the class of a fiber of π.

Lemma 2.1. *The projection p_V factors through the double cover $\sigma_{\mathbb{Y}}: \mathbb{Y} \to \mathbb{P}$. More precisely, there is a morphism $p: V \to \mathbb{Y}$ such that*

$$p_V = \sigma_{\mathbb{Y}} \circ p.$$

The degree of the morphism p at a general point is equal to 2.

Proof. Consider a point $x \in \mathbb{P} \backslash W_{\mathbb{P}}$ of general position. Set $\{y^+, y^-\} = \sigma_{\mathbb{Y}}^{-1}(x) \subset \mathbb{Y}$. Set also

$$L_x = \mathbb{P}^1 \times \{x\} \subset X, \quad L_x^{\pm} = \mathbb{P}^1 \times \{y^{\pm}\} \subset Y.$$

It is obvious that the inverse image $\sigma_Y^{-1}(L_x)$ is the disjoint union of the lines L_x^+ and L_x^-, whereas

$$p_Y(L_x^{\pm}) = y^{\pm},$$

where $p_Y: Y \to \mathbb{Y}$ is the projection onto the second factor. The divisor Q intersects L_x at two distinct (for a general point x) points q_1, q_2. Set

$$\sigma^{-1}(q_i) = \{o_i^+, o_i^-\} \subset V, \quad o_i^{\pm} \in L_x^{\pm}.$$

The morphism p is the restriction $p_Y|_V$. Obviously,

$$p^{-1}(y^{\pm}) = \{o_1^{\pm}, o_2^{\pm}\},$$

where the sign + or − is the same in the right hand and left hand side. This proves the lemma.

Let $\Delta \subset V$ be a subvariety of codimension 2, given by the system of equations $A = B = C = 0$. The subvariety Δ is swept out by the lines $L_y = \mathbb{P}^1 \times \{y\}$ which are contracted by the morphism p. Set $\Delta_{\mathbb{Y}} = p(\Delta)$. Obviously,

$$p: V \setminus \Delta \to \mathbb{Y} \setminus \Delta_{\mathbb{Y}}$$

is a finite morphism of degree 2. Let $\tau \in \operatorname{Bir} V$ be the corresponding Galois involution. It is easy to see that τ commutes with the Galois involution $\alpha \in \operatorname{Aut} V$ of the double cover $\sigma: V \to Q$, so that τ and α generate a group of four elements. Since the involution τ is biregular outside the invariant closed subset Δ of codimension 2, that is, $\tau \in \operatorname{Aut}(V \setminus \Delta)$, the action of τ on the Picard group $\operatorname{Pic} V$ is well defined.

Let $\Sigma \subset |-nK_V + lF|$ be a movable linear system.

Lemma 2.2. (i) *The involution τ transforms the pencil $|F|$ of fibers of the morphism π into the pencil $|mL_V - F|$.*

(ii) *If $l < 0$, then the involution τ transforms the linear system Σ into the linear system $\Sigma^+ \subset |n^+ L_V + l^+ F|$, where $n^+ = n + lm \geq 0$, $l^+ = -l > 0$.*

Proof. Obviously, $\tau^* L_V = L_V$. Let $F_t = \pi^{-1}(t)$ be a fiber. We get

$$p^{-1}(p(F_t)) = F_t \cup \tau(F_t).$$

However, $p(F_t) \sim mH_{\mathbb{Y}} = m\sigma_{\mathbb{Y}}^* H_{\mathbb{P}}$ by the construction of the variety V. Since $p^* H_{\mathbb{Y}} = L_V$, we obtain the claim (i). Thus $\tau^* F = mL_V - F$. This directly implies the second claim of the lemma.

Now let us formulate the main result on birational geometry of the variety V.

Theorem 2.2. *The variety V is birationally superrigid. The group* $\operatorname{Bir} V$ *of birational self-maps is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with α and τ as generators. On the variety V there are exactly two non-trivial structures of a rationally connected fiber space, the projection $\pi: V \to \mathbb{P}^1$ and the map $\pi\tau: V \dashrightarrow \mathbb{P}^1$.*

For the **proof,** see [26]. Let us just remind the scheme of the arguments modulo the hardest technical part. Let $\Sigma \subset |-nK_V + lF|$ be a movable linear system. If $l \in \mathbb{Z}_+$, then the general constructions of [26, Theorem 2] imply that $c_{\text{virt}}(\Sigma) = c(\Sigma)$, which is what we need. If $l < 0$, then consider the system $\Sigma^+ = \tau_* \Sigma$. Since τ is an isomorphism in codimension one, we have $c(\Sigma^+) = c(\Sigma)$. Since the virtual threshold is a birational invariant, $c_{\text{virt}}(\Sigma^+) = c_{\text{virt}}(\Sigma)$. However, $\Sigma^+ \subset |-n_+ K_V + l_+ F|$, where by Lemma 2.2 $n_+ = n + lm$, $l_+ = -l \geq 1$. Applying to Σ^+ the general theory ([26, Theorem 2]), we get $c_{\text{virt}}(\Sigma^+) = c(\Sigma^+)$, which implies birational rigidity by what has been said above.

The very same arguments prove that there are exactly two non-trivial structures of a fiber space into varieties of negative Kodaira dimension on V, that is, the projection π and $\pi\tau$.

Finally, if $\chi \in \operatorname{Bir} V$, then twisting by τ if necessary, one may assume that χ preserves the structure π, that is, transforms the fibers of F_t into the fibers $F_{\gamma(t)}$ for some isomorphism $\gamma: \mathbb{P}^1 \to \mathbb{P}^1$. However, for a generic variety V a general fiber F_t has the trivial group of birational (= biregular) self-maps and moreover, a general fiber F_t is not isomorphic to any other fiber F_s, $s \neq t$, which implies that $\chi \in \operatorname{Aut} V$ is either the identity map, or the Galois involution α. Therefore, $\operatorname{Bir} V = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{\mathrm{id}_V, \tau, \alpha, \alpha\tau\}$. Q.E.D. for the Theorem.

2.3. Varieties with two non-equivalent structures. Following [30], let us construct a family of rationally connected varieties with exactly two non-trivial structures of a rationally connected fiber space and this time the trivial group of birational self-maps. Let X be a projective bundle, $X = \mathbb{P}(\mathcal{E})$, where the locally free sheaf $\mathcal{E}$ is of the form $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus M} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Thus X is a $\mathbb{P} = \mathbb{P}^{M+1}$-bundle over $\mathbb{P}^1$. Let $L_X \in \operatorname{Pic} X = \mathbb{Z}L_X \oplus \mathbb{Z}R$ be the class of the tautological sheaf, R the class of a fiber of the fiber space $X/\mathbb{P}^1$. Let $Q \sim mL_X$ be a smooth divisor, $\sigma\colon V \to Q$ the double cover branched over a smooth hypersurface $W \cap Q$, where $W \sim 2lL_X$, $m + l = M + 1$. Obviously, $\pi\colon V \to \mathbb{P}^1$ is a Fano fiber space, the fiber of which is a Fano double hypersurface of index 1. We get $\operatorname{Pic} V = \mathbb{Z}L_V \oplus \mathbb{Z}F$, where $L_V = \sigma^*(L_X\,|_Q)$ and F is the class of a fiber of π. It is easy to see that $-K_V = L_V$ and thus the linear system

$$|-K_V - F| = \sigma^*(|L_X - R|\,\Big|_Q)$$

is movable. Let $\varphi\colon V \dashrightarrow \mathbb{P}^1$ be the rational map, given by the pencil $|-K_V - F|$. Birational geometry of the variety V is completely described by

Theorem 2.3. (i) *The variety V is birationally superrigid: for any movable linear system Σ on V its virtual and actual thresholds of canonical adjunction coincide,*

$$c_{\mathrm{virt}}(\Sigma) = c(\Sigma).$$

(ii) *On the variety V there are exactly two non-trivial structures of a rationally connected fiber space, namely $\pi\colon V \to \mathbb{P}^1$ and $\varphi\colon V \dashrightarrow \mathbb{P}^1$. These structures are birationally distinct, that is, there is no birational self-map $\chi \in \operatorname{Bir} V$, transforming the fibers of π into the fibers of φ. The groups of birational and biregular self-maps of the variety V coincide:* $\operatorname{Bir} V = \operatorname{Aut} V$.

(iii) *There is a unique, up to a fiber-wise isomorphism, Fano fiber space $\pi^+\colon V^+ \to \mathbb{P}^1$ of the same type* $((1,1),(0,0))$, *such that the following diagram commutes:*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V^+ & \\ \varphi & \downarrow & & \downarrow & \pi^+ \\ & \mathbb{P}^1 & = & \mathbb{P}^1, & \end{array}$$

where χ is a birational map. The construction $V \to V^+$ is involutive, that is, $(V^+)^+ = V$.

Proof. The space $H^0(X, \mathcal{L}_X \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))$ is two-dimensional and defines a pencil of divisors $|L_X - R|$. Its base set $\Delta_X = \operatorname{Bs}|L_X - R|$ is of codimension 2: it is easy to see that

$$\Delta_X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus M}) \cong \mathbb{P}^{M-1} \times \mathbb{P}^1.$$

Set $\Delta_Q = \Delta_X \cap Q, \quad \Delta = \sigma^{-1}(\Delta_Q) \subset V$. Obviously, Δ_Q is a smooth divisor of bidegree $(m,0)$ on $\Delta_X = \mathbb{P}^{M-1} \times \mathbb{P}^1$, $\Delta \subset V$ is a smooth irreducible subvariety of codimension 2.

Lemma 2.3. *The base set of the movable linear system $|-K_V - F|$ is equal to* $\mathrm{Bs}|-K_V - F| = \Delta$. *Furthermore, $-K_V - F \in \partial A^1_{\mathrm{mov}} V$. More precisely, $|-nK_V + lF| = \emptyset$ for $l < -n$.*

Proof is straightforward (see [30]).

Now let us study the rational map $\varphi\colon V \dashrightarrow \mathbb{P}^1$. In order to do this, we need an explicit coordinate presentation of the varieties X, Q and W, participating in the construction of the Fano fiber space $V/\mathbb{P}^1$.

Consider the locally free subsheaves

$$\mathcal{E}_0 = \mathcal{O}_{\mathbb{P}^1}^{\oplus M} \hookrightarrow \mathcal{E} \quad \text{and} \quad \mathcal{E}_1 = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \hookrightarrow \mathcal{E}.$$

Obviously, $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$. Let $\Pi_0 \subset H^0(X, \mathcal{L}_X)$ be the subspace, corresponding to the space of sections of the sheaf $H^0(\mathbb{P}^1, \mathcal{E}_0) \hookrightarrow H^0(\mathbb{P}^1, \mathcal{E})$. Set also

$$\Pi_1 = H^0(X, \mathcal{L}_X \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\mathbb{P}^1, \mathcal{E}_1(-1)).$$

Let $x_0, \ldots, x_{M-1}$ be a basis of the space Π_0, y_0, y_1 a basis of the space Π_1. Then the sections

$$x_0, \ldots, x_{M-1}, y_0 t_0, y_0 t_1, y_1 t_0, y_1 t_1, \tag{7}$$

where t_0, t_1 is a system of homogeneous coordinates on $\mathbb{P}^1$, make a basis of the space $H^0(X, \mathcal{L}_X)$. It is easy to see that the complete linear system (7) defines a morphism

$$\xi\colon X \to \bar{X} \subset \mathbb{P}^{M+3},$$

the image X of which is a quadratic cone with the vertex space $\mathbb{P}^{M-1} = \xi(\Delta_X)$ and a smooth quadric in $\mathbb{P}^3$, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, as a base. The morphism ξ is birational, more precisely, $\xi\colon X \setminus \Delta_X \to \bar{X} \setminus \xi(\Delta_X)$ is an isomorphism and ξ contracts $\Delta_X = \mathbb{P}^{M-1} \times \mathbb{P}^1$ onto the vertex space of the cone. Let

$$u_0, \ldots, u_{M-1}, u_{00}, u_{01}, u_{10}, u_{11}$$

be the homogeneous coordinates on $\mathbb{P}^{M+3}$, corresponding to the ordered set of sections (7). The cone $\bar{X}$ is given by the equation

$$u_{00} u_{11} = u_{01} u_{10}.$$

On the cone $\bar{X}$ there are two pencils of $(M+1)$-planes, corresponding to the two pencils of lines on a smooth quadric in $\mathbb{P}^3$. Let $\tau \in \mathrm{Aut}\, \mathbb{P}^{M+3}$ be the automorphism permuting the coordinates u_{01} and u_{10} and not changing the other coordinates. Obviously, $\tau \in \mathrm{Aut}\, \bar{X}$ is an automorphism of the cone $\bar{X}$, permuting the above-mentioned pencils of $(M+1)$-planes. One of these pencils is the image of the pencil of fibers of the projection π, that is, the pencil $\xi(|R|)$. For the other pencil we get the equality

$$\tau \xi(|R|) = \xi(|L_X - R|).$$

The automorphism τ induces an involutive birational self-map $\tau^+ \in \mathrm{Bir}\, X$. More precisely, τ^+ is a biregular automorphism outside a closed subset Δ_X of codimension 2. Let $\varepsilon\colon \tilde{X} \to X$ be the blow up of the smooth subvariety Δ_X. Obviously, the variety $\tilde{X}$ is isomorphic to the blow up of the cone $\bar{X}$ at its vertex space $\xi(\Delta_X)$. It is easy to check that τ^+ extends to a biregular automorphism of the smooth variety $\tilde{X}$.

Set $Q^+ = \tau^+(Q) \subset X$, $W^+ = \tau^+(W) \subset X$. The divisors Q^+ and W^+ are well defined because τ^+ is an isomorphism in codimension 1.

Lemma 2.4. *The divisors Q^+ and W^+ are divisors of general position in the linear systems $|mL_X|$ and $|2lL_X|$, respectively. In particular, Q^+, W^+ and $Q^+ \cap W^+$ are smooth varieties.*

Proof. The claim follows immediately from the fact that the linear systems $|kL_X|$, $k \in \mathbb{Z}_+$, are invariant under τ^+, whereas Q and W are sufficiently general divisors of the corresponding linear systems. Note that if a divisor $D \in |kL_X|$ is given by a polynomial

$$h(u_0, \dots, u_{M-1}, u_{00}, u_{01}, u_{10}, u_{11}),$$

of degree k, then its image $\tau^+(D)$ is given by the polynomial

$$h^+(u_*) = h(u_0, \dots, u_{M-1}, u_{00}, u_{10}, u_{01}, u_{11})$$

with permuted coordinates u_{01} and u_{10}. Q.E.D. for the lemma.

Let $\sigma^+\colon V^+ \to Q^+$ be the double cover, branched over a smooth divisor $Q^+ \cap W^+$. Obviously, $V^+/\mathbb{P}^1$ is a general Fano fiber space of type $((1,1),(0,0))$.

Lemma 2.5. (i) *The map τ^+ lifts to a birational map $\chi\colon V \dashrightarrow V^+$, biregular in codimension 1.*

(ii) *The action of χ on the Picard group is given by the formulas*

$$\chi^* K_{V^+} = K_V, \quad \chi^* F^+ = -K_V - F,$$

where F^+ is the class of the fiber of the projection $V^+ \to \mathbb{P}^1$, so that $\mathrm{Pic}\, V^+ = \mathbb{Z}K_{V^+} \oplus \mathbb{Z}F^+$.

(iii) *The construction of the variety V^+ is involutive: $(V^+)^+ \cong V$.*

Proof: the claims (i)-(iii) are obvious. Just note that the following presentation holds: $\chi = q^+ \circ q^{-1}$, where $q\colon \tilde{V} \to V$ and $q^+\colon \tilde{V} \to V^+$ are blow ups of the smooth subvarieties of codimension two $\Delta \subset V$ and $\Delta^+ \subset V^+$, respectively. Furthermore, $E = q^{-1}(\Delta)$ is the exceptional divisor of both blow ups, $E = \Delta \times \mathbb{P}^1 = \Delta_F \times \mathbb{P}^1 \times \mathbb{P}^1$, whereas the projections $q\,|\,_E$ and $q^+\,|\,_E$ are projections with respect to the second and third direct factors, respectively.

Finally, let us prove Theorem 2.3. Let $\Sigma \subset |-nK_{V^+} + lF|$ be a movable linear system. If $l \in \mathbb{Z}_+$, then by Theorem 2 of the paper [26] we get the desired coincidence of the thresholds: $c_{\mathrm{virt}}(\Sigma) = c(\Sigma)$. Assume that $l < 0$. Consider the linear system $\Sigma^+ = \tau^+(\Sigma)$ on V^+. By Lemma 2.5, $\Sigma^+ \subset |-n_+ K_{V^+} + l_+ F^+|$, where $l_+ = -l \geq 1$. Since τ^+ is an isomorphism in codimension 1, we get $c(\Sigma) = c(\Sigma^+)$. Again applying Theorem 2 of the paper [26], we obtain the desired coincidence of thresholds

$$c_{\mathrm{virt}}(\Sigma^+) = c_{\mathrm{virt}}(\Sigma) = c(\Sigma^+) = c(\Sigma) = n_+ = n + l.$$

This proves birational superrigidity.

Let us prove the claim (ii). Arguing as in Sec. 2.2, we show that on V there are exactly two non-trivial structures of a rationally connected fiber space (the arguments above imply that if a movable linear system Σ satisfies the equality $c_{\rm virt}(\Sigma)=0$, then either Σ is composed from the pencil $|F|$, or Σ is composed from the pencil $|-K_V-F|$, which gives a description of the existing structures). For a general variety V these structures cannot be birationally equivalent. Indeed, by birational superrigidity of Fano double hypersurfaces of index 1, any birational map $\chi^+\in\mathrm{Bir}\,V$, which transforms the pencil $|F|$ into the pencil $|-K_V-F|$, induces a biregular isomorphism of the fibers of general position in the pencils $|F|$ and $|F^+|$ (the latter is taken on the variety V^+). Therefore, χ^+ induces a biregular isomorphism of the fibers of general position of the fiber spaces $Q/\mathbb{P}^1$ and $Q^+/\mathbb{P}^1$. Now by Theorem 3.1 below for $m\geq 3$ we get that these fiber spaces are globally fiber-wise isomorphic. It checks easily that for a sufficiently general divisor $Q\subset X$ this is impossible. For $m=2$ we argue in a similar way, using the branch divisor W.

Finally, the claim (iii) follows from the arguments above.

Q.E.D. for Theorem 2.3.

3 Fiber-wise birational correspondences

In this section, following [21], we study fiber-wise birational correspondences of fiber spaces, the fiber of which is a hypersurface.

3.1. Fibrations into complete intersections. Let C be a smooth algebraic curve with a marked point $p\in C$, and $C^*=C\setminus\{p\}$ a "punctured" curve. In what follows our arguments remain correct if we replace C by a smooth germ of a curve $p\in C$, or a small disk $\Delta_\varepsilon=\{|z|<\varepsilon\}\subset\mathbb{C}$. The symbol $\mathbb{P}$ stands for the complex projective space $\mathbb{P}^M$, $M\geq 3$. Let $\mathcal{V}(d)$ be the set of smooth divisors $V\subset X=C\times\mathbb{P}$, each fiber of which $F_x=V\cap\{x\}\times\mathbb{P}$, $x\in C$, is a hypersurface of degree $d\geq 2$. Set

$$X^*=C^*\times\mathbb{P},\quad V^*=V\cap X^*,$$

so that V^* is obtained from V by throwing away the fiber F_p over the marked point.

Theorem 3.1. *Assume that $d\geq 3$. Take $V_1,V_2\in\mathcal{V}(d)$ and let $\chi^*\colon V_1^*\to V_2^*$ be a fiber-wise isomorphism. Then χ^* extends to a fiber-wise isomorphism $\chi\colon V_1\to V_2$.*

In other words, within the limits of the class $\mathcal{V}(d)$ these varieties do not permit non-trivial birational transforms of the fibers.

Let $\mathbb{Z}_{\geq 2}$ be the set of integers $m\geq 2$.

Conjecture 3.1. *For a given $k\geq 2$ there exist an integer $M_*\geq k+2$ and a finite set $S\subset\mathbb{Z}^k_{\geq 2}$ (which may occur to be empty) such that for each $M\geq M_*$ and each set $(d_1,\dots,d_k)\in\mathbb{Z}^k_{\geq 2}\setminus S$ the statement of Theorem 3.1 is true for the class $\mathcal{V}(d_1,\dots,d_k)$ of smooth complete intersections of the type $(d_1,\dots,d_k)$ in $C\times\mathbb{P}^M$.*

Theorem 3.1 implies the following global fact.

Corollary 3.1. *Let $V/\mathbb{P}^1$ and $V'/\mathbb{P}^1$ be smooth fibrations into Fano hypersurfaces of index 1. Assume that $V/\mathbb{P}^1$ is sufficiently twisted over the base [20,26]. Then any birational map $\chi\colon V \dashrightarrow V'$ is a fiber-wise biregular isomorphism.*

Let us start with the following question: which singularities can acquire a special fiber if the total space is smooth?

Let $(d_1, \dots, d_k) \in \mathbb{Z}^k_{\geq 2}$ be a fixed type of complete intersection. Consider the class of subvarieties in $C \times \mathbb{P}$, which can be represented locally over C as

$$f_1 = \ldots = f_k = 0,$$

where the equations f_i with respect to a system $(x_0 : \ldots : x_M)$ of homogeneous coordinates on $\mathbb{P}$ are of the form

$$f_i = \sum_{|I|=d_i} a_I x^I,$$

$I = (j_0, \dots, j_M)$ are multi-indices of degree $j_0 + \ldots + j_M = d_i$, and the coefficients a_I are regular functions on C, whereas for each point $y \in C$ the set of equations $\{f_*\}$, restricted on the fiber $X_y = \{y\} \times \mathbb{P} \cong \mathbb{P}$, defines a complete intersection of codimension k in $\mathbb{P}$. Let us denote the class of these varieties by $\mathcal{Z}(d_1, \dots, d_k)$.

Take $V \in \mathcal{Z}(d_1, \dots, d_k)$. Let $F = V \cap X_p$ be the fiber over the marked point. Fix a system of equations $\{f_*\}$ for V near the point $p \in C$ and a local parameter t on the curve C at the point p. Now the equations f_i can be expanded into their Taylor series

$$f_i = f_i^{(0)} + t f_i^{(1)} + \ldots + t^j f_i^{(j)} + \ldots,$$

where $f_i^{(j)}$ are homogeneous polynomials of degree d_i in (x_*). The fiber $F \subset \mathbb{P}$ is given by the system of equations $\{f_*^{(0)} = 0\}$.

Lemma 3.1. *The following estimate holds*

$$\dim(X_p \cap \operatorname{Sing} V) \geq \dim \operatorname{Sing} F - 1.$$

Proof is similar to the proof of Lemma 3.4.2 in [13]. The set $\operatorname{Sing} F$ is given on F by the condition

$$\operatorname{rk} \| \frac{\partial f_i^{(0)}}{\partial x_j} \| \leq k - 1.$$

If $\dim \operatorname{Sing} F \leq 0$, then there is nothing to prove. Otherwise, let $Y \subset \operatorname{Sing} F$ be a component of maximal dimension, $\dim Y \geq 1$. The set $X_p \cap \operatorname{Sing} V$ is given on F by the condition

$$\operatorname{rk} \| \frac{\partial f_i^{(0)}}{\partial x_j} \mid f_i^{(1)} \| \leq k - 1. \tag{8}$$

If the set $D = \{x \in Y \mid \operatorname{rk} \|\partial f_i^{(0)}/\partial x_j\| \leq k - 2\}$ is of codimension 1 in Y, then the lemma is proved, since $D \subset X_p \cap \operatorname{Sing} V$. Assume the converse: $\operatorname{codim}_Y D \geq 2$.

Take a general curve $\Gamma \subset Y$ disjoint from D. At each point of the curve Γ the rank of the matrix $\|\partial f_i^{(0)}/\partial x_j\|$ is equal to $k-1$. Consider the morphisms of sheaves

$$\mu_j\colon \bigoplus_{i=1}^{k} \mathcal{O}_\Gamma(1-d_i) \to \mathcal{O}_\Gamma,$$

that are defined locally on the sets of sections $(s_1,\dots,s_k)$ by the formula

$$\mu_j\colon (s_1,\dots,s_k) \mapsto \sum_{i=1}^{k} s_i \frac{\partial f_0^{(i)}}{\partial x_j}$$

with respect to a fixed isomorphism $\mathcal{O}(-a)\otimes\mathcal{O}(a) \cong \mathcal{O}$. By assumption the subsheaf

$$\operatorname{Ker}(\mu_*) = \bigcap_{j=0}^{M} \operatorname{Ker}\mu_j \subset \bigoplus_{i=1}^{k} \mathcal{O}_\Gamma(1-d_i)$$

is of constant rank 1. Now consider the morphism of sheaves

$$\begin{array}{lcl} \lambda\colon \operatorname{Ker}(\mu_*) & \to & \mathcal{O}_\Gamma(1), \\ \lambda\colon (s_1,\dots,s_k) & \mapsto & \sum_{i=1}^{k} s_i f_i^{(1)}. \end{array}$$

Assume that the condition (8) is not true at each point of the curve Γ. Then λ is an isomorphism of invertible sheaves, which means that

$$\mathcal{O}_\Gamma(1) \hookrightarrow \bigoplus_{i=1}^{k} \mathcal{O}_\Gamma(1-d_i).$$

But this is impossible. Q.E.D. for Lemma 3.1.

Let us consider fibrations into hypersurfaces. In accordance with Lemma 3.1, a variety $V \in \mathcal{Z}(d)$ with a local equation $f = f^{(0)} + tf^{(1)} + \dots$ is smooth, that is, $V \in \mathcal{V}(d)$, if and only if the following two conditions hold:

(i) the hypersurface $F = \{f^{(0)} = 0\}$ has at most zero-dimensional singularities;

(ii) for each point $x \in \operatorname{Sing} F$ we have $f^{(1)}(x) \neq 0$.

3.2. The diagonal presentation. Take $V_1, V_2 \in \mathcal{V}(d)$, $d \geq 2$, and let $\chi^*\colon V_1^* \to V_2^*$ be a fiber-wise isomorphism outside the marked point $p \in C$. Since the fibers over generic points $y \in C$ are smooth hypersurfaces of degree $d \geq 2$, the isomoprhisms χ_y^* over the points $y \in C^*$ are induced by automorphisms of the ambient projective space $\xi_y \in \operatorname{Aut}\mathbb{P}$. Thus $\chi^* = \xi^*|_{V_1}$, where $\xi_y^* = \xi_y$ is an algebraic curve

$$\xi^*\colon C^* \to \operatorname{Aut}\mathbb{P}$$

of projective automorphisms. Let $\mathbb{P} = \mathbb{P}(L)$ be the projectivization of a linear space $L \cong \mathbb{C}^{M+1}$. The curve ξ^* can be lifted to a curve $\xi\colon C \to \operatorname{End} L$, where $\xi(C^*) \subset \operatorname{Aut} L$. If $\xi(p) \in \operatorname{Aut} L$, then χ^* extends to the fiber-wise (biregular) isomorphism $\chi = \xi|_{V_1}$, and the varieties V_1 and V_2 are fiber-wise isomorphic. Assume the converse: $\det \xi(p) = 0$.

Fix a local parameter t on the curve C at the point p, and let

$$\sum_{i=0}^{\infty} t^i \xi^{(i)}$$

be the Taylor series of the curve ξ. We may assume that $\xi^{(0)} \neq 0$.

Lemma 3.2. *There exist curves of linear self-maps $\beta, \gamma: C \to \operatorname{End} L$, $\beta(p), \gamma(p) \in \operatorname{Aut} L$, and a basis $(e_0, \ldots, e_M)$ of the space L such that with respect to this basis the curve $\beta\xi\gamma^{-1}: C \to \operatorname{End} L$ has a diagonal form:*

$$\beta\xi\gamma^{-1}: e_i \mapsto t^{w(e_i)} e_i, \tag{9}$$

where $w(e_i) \in \mathbb{Z}_+$.

Proof. This is a well-known fact of elementary linear algebra.

Now replace V_1 by $\gamma(V_1)$, V_2 by $\beta(V_2)$. We may simply assume that the fiber-wise birational correspondence ξ has the form (9) from the beginning. We claim that if $m = \max\{w(e_i)\} \geq 1$, then this is impossible.

Let $\{a_0 = 0 < a_1 < \ldots < a_k\} = \{w(e_i), i = 0, \ldots, M\} \subset \mathbb{Z}_+$ be the set of weights of the diagonal transform (9), $k \leq M$, $m = a_k$ the maximal weight. Take the system of homogeneous coordinates $(x_0 : \ldots : x_M)$, dual to the basis (e_*). We define the weight of monomials in x_*, setting

$$w(x_0^{n_0} x_1^{n_1} \ldots x_M^{n_M}) = \sum_{i=0}^{M} n_i w(e_i).$$

Set $\mathcal{A}_i = \{x_j | w(e_j) = a_i\} \subset \mathcal{A} = \{x_0, \ldots, x_M\}$ to be the collection of coordinates of the weight a_i. The distinguished sets of coordinates of the maximal and minimal weight we denote by $\mathcal{A}_* = \mathcal{A}_0$ and $\mathcal{A}^* = \mathcal{A}_k$.

3.3. Birational = biregular. Let $f = f^{(0)}(x) + tf^{(1)} + \ldots$ be a local (over the base C) equation of the hypersurface $V_2 \subset C \times \mathbb{P}$, where $f^{(i)}$ are homogeneous polynomials of degree $d \geq 3$ in the coordinates x_*. The series

$$f_\xi = \sum_{l=0}^{\infty} t^l f_\xi^{(l)}(x) = \sum_{l=0}^{\infty} t^l f^{(l)}(t^{w(x_0)} x_0, \ldots, t^{w(x_M)} x_M)$$

vanishes on V_1, and outside the marked fiber F_1, that is, for $t \neq 0$, gives an equation of V_1. Let $b \in \mathbb{Z}_+$ be the maximal degree of the parameter t, dividing f_ξ. Then

$$t^{-b} f_\xi = g = \sum_{l=0}^{\infty} t^l g^{(l)}(x_0, \ldots, x_M)$$

gives an equation of the hypersurface V_1 at the marked fiber X_p, too.

Lemma 3.3. *For each $l \in \mathbb{Z}_+$ the polynomial $f^{(l)}$ belongs to the linear span of monomials of weight $\geq b - l$, whereas the polynomial $g^{(l)}$ belongs to the linear span of monomials of weight $\leq b + l$.*

Proof. Assume that the monomial x^I comes into the polynomial $f^{(l)}$ with a non-zero coefficient. Then it generates the component $t^{l+w(x^I)}x^I$ of the series f_ξ and, moreover, this component comes from this monomial of $f^{(l)}$ *only.* Therefore $l + w(x^I) \geq b$, which is what we need.

Assume that the monomial x^I comes into $g^{(l)}$ with a non-zero coefficient. It is generated by the monomial $t^{l+b}x^I$ of the series f_ξ, which, in its turn, can be generated by the monomial x^I from the polynomial f^α only, where $\alpha + w(x^I) = l + b$. Q.E.D. for the lemma.

Let

$$\begin{aligned} P_* &= \{x_j = 0 | w(x_j) \geq 1\} = \mathbb{P}\langle e_j | w(x_j) = 0\rangle, \\ P^* &= \{x_j = 0 | w(x_j) \leq m - 1\} = \mathbb{P}\langle e_j | w(x_j) = m\rangle \end{aligned}$$

be the subspaces of the minimal and the maximal weight, respectively.

Lemma 3.4. *If $b \geq m + 1$, then $P_* \subset \operatorname{Sing} F_2$. If $m(d-1) \geq b + 1$, then $P^* \subset \operatorname{Sing} F_1$.*

Proof. Assume that $b \geq m + 1$. The fiber $F_2 \subset \mathbb{P}$ over the marked point is given by the equation $f^{(0)} = 0$. By assumption $f^{(0)}$ belongs to the linear span of monomials of weight $\geq m + 1$. If a monomial x^I comes into $f^{(0)}$ with a non-zero coefficient, then x^I contains a quadratic monomial in the variables $\mathcal{A} \setminus \mathcal{A}_*$ (otherwise $w(x^I) \leq m$).Thus all the first partial derivatives of the polynomial $f^{(0)}$ vanish on P_*. Thus $P \subset \operatorname{Sing} F_2$.

Similarly, if $b \leq m(d-1) - 1$, then each monomial x^I in $g^{(0)}$ contains a quadratic monomial in $\mathcal{A} \setminus \mathcal{A}^*$, otherwise we get $w(x^I) \geq m(d-1)$, which gives a contradiction with our assumption and Lemma 3.3. Q.E.D. for Lemma 3.4.

Now take into account that for $d \geq 3$ the inequalities

$$b \leq m \quad \text{and} \quad b \geq m(d-1)$$

can not both be true. Consequently, at least one of the two inequalities of Lemma 3.4 holds. Suppose that $b \geq m+1$. Since V_2 is smooth, P_* is a point. Let $\mathcal{A}_* = \{x_0\}$, so that $P_* = (1, 0, \ldots, 0)$. Again we use the fact that V_2 is smooth and conclude that

$$f^{(1)}(1, 0, \ldots, 0) \neq 0.$$

Consequently, the monomial x_0^d comes into $f^{(1)}$ with a non-zero coefficient. By Lemma 3.3 $b \leq 1$.

Therefore $m = 0$, which is a contradiction.

In the case $b \leq m(d-1) - 1$ the arguments are symmetric: V_1 is smooth, P^* is the point $(0, \ldots, 0, 1)$, $\mathcal{A}^* = \{x_M\}$ and $g^{(1)}(0, \ldots, 0, 1) \neq 0$, so that $md \leq b + 1$, whence we get $m = 0$ again, a contradiction.

Therefore, non-trivial weights cannot occur and ξ is a fiber-wise biregular isomorphism. Consequently $\chi = \xi|_{V_1}$ is a fiber-wise isomorphism, too. Proof of Theorem 3.1 is complete.

Finally, let us prove Proposition 0.2. Let $\varphi_1, \varphi_2 : \mathbb{P}^M \dashrightarrow \mathbb{P}^1$ be two generic projections. Assume that the structures

$$\pi_1 = \varphi_1 |_V : V \dashrightarrow \mathbb{P}^1 \text{ and } \pi_2 = \varphi_2 |_V : V \dashrightarrow \mathbb{P}^1$$

are fiber-wise birationally equivalent, where $V \subset \mathbb{P}^M$ is a generic smooth hypersurface of degree $M - 1 \geq 4$, that is, there exists a birational self-map $\chi \in \operatorname{Bir} V$ such that $\pi_2 \circ \chi = \pi_1$. Let $P_1, P_2 \subset \mathbb{P}^M$ be the centres of the projections φ_1, φ_2, respectively. By genericity we may assume that $V \cap P_i$ is smooth. Let us blow up $V \cap P_i$:

$$\sigma_i : V_i \to V,$$

$E_i = \sigma_i^{-1}(V \cap P_i) \subset V_i$ being the exceptional divisor. The projections π_i extend to the morphisms $\pi_i^+ : V_i \to \mathbb{P}^1$, the map χ extends to a birational map $\chi^+ : V_1 \dashrightarrow V_2$. We get the commutative diagram

$$\begin{array}{ccccc} & V_1 & \stackrel{\chi^+}{\dashrightarrow} & V_2 & \\ \pi_1^+ & \downarrow & & \downarrow & \pi_2^+ \\ & \mathbb{P}^1 & = & \mathbb{P}^1 & \end{array}$$

Now a general fiber of π_i^+ is birationally superrigid. Applying Theorem 3.1, we see that χ^+ extends to an isomorphism between V_1 and V_2, which maps every fiber $(\pi_1^+)^{-1}(t)$ isomorphically onto the fiber $(\pi_2^+)^{-1}(t)$, $t \in \mathbb{P}^1$. Now an easy dimension count shows that for a generic plane $P \subset \mathbb{P}^M$ of codimension 2 there are at most finitely many planes $S \subset \mathbb{P}^M$ such that $P \cap V \cong S \cap V$. Since $E_1 \cap (\pi_1^+)^{-1}(t) \cong P \cap V$, we obtain that

$$\chi^+(E_1 \cap (\pi_1^+)^{-1}(t)) = E_2 \cap (\pi_2^+)^{-1}(t)$$

(otherwise, there would have been a one-dimensional family of planes $S \subset \mathbb{P}^M$ with the property $S \cap V \cong P_1 \cap V$). Therefore, $\chi^+(E_1) = E_2$ and the original map $\chi \in \operatorname{Bir} V$ is biregular outside $P_1 \cap V$ and $P_2 \cap V$, respectively. Therefore, $\chi \in \operatorname{Aut} V = \{\mathrm{id}_V\}$. Q.E.D. for Proposition 0.2.

References

1. Algebraic surfaces. By the members of the seminar of I.R.Shafarevich. I.R.Shafarevich ed. Proc. Steklov Math. Inst. **75**. 1965. English transl. by AMS, 1965. 281 p.

2. Artin M., Mumford D. Some elementary examples of unirational varieties which are not rational, Proc. London Math. Soc. **25** (1972), 75-95.

3. Clemens H., Griffiths Ph. The intermediate Jacobian of the cubic three-fold. Ann. of Math. **95** (1972), 73-100.

4. Corti A., Factoring birational maps of threefolds after Sarkisov. J. Algebraic Geom. **4** (1995), no. 2, 223-254.

5. Graber T., Harris J. and Starr J. Families of rationally connected varieties. J. Amer. Math. Soc. **16** (2002), no. 1, 57-67.

6. Grinenko M.M., Birational automorphisms of a three-dimensional double cone. Sbornik: Mathematics. **189** (1998), no. 7-8, 991-1007.

7. Grinenko M.M., Birational properties of pencils of del Pezzo surfaces of degrees 1 and 2. Sbornik: Mathematics. **191** (2000), no. 5-6, 633-653.

8. Grinenko M.M., Birational properties of pencils of del Pezzo surfaces of degrees 1 and 2. II. Sbornik: Mathematics. **194** (2003), no. 5-6, 669-696.

9. Grinenko M.M., On the double cone over the Veronese surface. Izvestiya: Mathematics. **67** (2003), no. 3, 421-438.

10. Grinenko M.M., Mori structures on a Fano three-fold of index 2 and degree 1. Proc. Steklov Inst. Math. **2004**, no. 3 (246), 103-128.

11. Iskovskikh V.A., Birational automorphisms of three-dimensional algebraic varieties, J. Soviet Math. **13** (1980), 815-868.

12. Iskovskikh V.A. and Manin Yu.I., Three-dimensional quartics and counterexamples to the Lüroth problem, Math. USSR Sbornik. **86** (1971), no. 1, 140-166.

13. Iskovskikh V.A. and Pukhlikov A.V., Birational automorphisms of multi-dimensional algebraic varieties, Journal of Math. Sciences, **82** (1996), no. 4, 3528-3613.

14. Kollár J., Rational curves on algebraic varieties. Springer-Verlag, Berlin, 1996.

15. Kollár J., Miyaoka Y. and Mori S., Rationally connected varieties, J. Alg. Geom. V. 1 (1992), 429-448.

16. Pukhlikov A.V., Maximal singularities on a Fano variety V_6^3, Moscow Univ. Math. Bulletin, **44** (1989), 70-75.

17. Pukhlikov A.V., Birational automorphisms of three-dimensional algebraic varieties with a pencil of del Pezzo surfaces, Izvestiya: Mathematics **62**:1 (1998), 115-155.

18. Pukhlikov A.V., Birational automorphisms of Fano hypersurfaces, Invent. Math. **134** (1998), no. 2, 401-426.

19. Pukhlikov A.V., Birationally rigid Fano double hypersurfaces, Sbornik: Mathematics **191** (2000), No. 6, 101-126.

20. Pukhlikov A.V., Birationally rigid Fano fibrations, Izvestiya: Mathematics **64** (2000), no. 3, 563-581.

21. Pukhlikov A.V., Fiber-wise birational correspondences, Mathematical Notes, **68** (2000), no. 1, 103-112.

22. Pukhlikov A.V., Birationally rigid Fano complete intersections, Crelle J. für die reine und angew. Math. **541** (2001), 55-79.

23. Pukhlikov A.V., Birationally rigid Fano hypersurfaces with isolated singularities, Sbornik: Mathematics. **193** (2002), no. 3-4, 445-471.

24. Pukhlikov A.V., Birationally rigid iterated Fano double covers. Izvestiya: Mathematics. **67** (2003), no. 3, 555-596.

25. Pukhlikov A.V., Birationally rigid varieties with a pencil of Fano double covers. I. Sbornik: Mathematics **195** (2004), no. 7, 1039-1071.

26. Pukhlikov A.V., Birationally rigid varieties with a pencil of Fano double covers. II. Sbornik: Mathematics **195** (2004), no. 11, 1665-1702.

27. Pukhlikov A.V., Birationally rigid Fano cyclic covers, arXiv:math.AG/0411198.

28. Pukhlikov A.V., Birational geometry of Fano direct products, Izvestiya: Mathematics, **69** (2005), no. 6, 1225-1255.

29. Pukhlikov A.V., Birational geometry of algebraic varieties with a pencil of Fano complete intersections, arXiv:math.AG/0511090.

30. Pukhlikov A.V., Birationally rigid varieties with a pencil of Fano double covers. III. Sbornik: Mathematics **197** (2006), no. 3.

31. Sarkisov V.G., Birational automorphisms of conic bundles. Math. USSR Izvestiya, **17** (1981), 177-202.

32. Sarkisov V.G., On conic bundle structures. Math. USSR Izvestiya, **20** (1982), no. 2, 354-390.

33. Sarkisov V.G., Birational maps of standard $\mathbb{Q}$-Fano fibrations, Preprint, Kurchatov Institute of Atomic Energy, 1989.

34. Sobolev I. V., On a series of birationally rigid varieties with a pencil of Fano hypersurfaces. Sbornik: Mathematics, **192** (2001), no. 9-10, 1543-1551.

35. Sobolev I. V., Birational automorphisms of a class of varieties fibered into cubic surfaces. Izvestiya: Mathematics, **66** (2002), no. 1, 203-224.

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