ISOCLINISM AND STABLE COHOMOLOGY OF WREATH PRODUCTS

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Abstract. Using the notion of isoclinism introduced by P. Hall for finite p-groups, we show that many important classes of finite p-groups have stable cohomology detected by abelian subgroups, see Theorem 4.4. Moreover, we show that the stable cohomology of the n-fold wreath product $G_n = \mathbb{Z}/p \wr \dots \wr \mathbb{Z}/p$ of cyclic groups $\mathbb{Z}/p$ is detected by elementary abelian p-subgroups and we describe the resulting cohomology algebra explicitly. Some applications to the computation of unramified and stable cohomology of finite groups of Lie type are given.

1. Introduction

Recall that for a prime p and finite group G, the stable cohomology $H^*_{\mathrm{s}}(G,\, \mathbb{Z}/p)$ is defined as the quotient $H^*(G,\, \mathbb{Z}/p)/N_{G,p}$ where, for some generically free G-representation V with open part V^L with free G-action, $N_{G,p}$ is the kernel of the map

$$H^*(G,\, \mathbb{Z}/p) \to \varinjlim_{U} H^i(U/G,\, \mathbb{Z}/p\mathbb{Z})\,,$$

the direct limit running over all nonempty G-invariant Zariski open subsets $U \subset V^L$. In fact, $N_{G,p}$ is independent of the choice of V. In $H^*_{\mathrm{s}}(G;\, \mathbb{Z}/p)$ we have the subring of unramified elements $H^*_{\mathrm{nr}}(G,\, \mathbb{Z}/p)$; these play a vital role in the study of birational properties of generically free linear quotients V/G and varieties X in general, see [Bogo93] for definitions and background.

The object of this paper is to make $H^*_{\mathrm{s}}(G,\, \mathbb{Z}/p)$ (and $H^*_{\mathrm{nr}}(G,\, \mathbb{Z}/p)$) amenable to effective computation for rather important and large classes of groups. The development of the theory formally parallels that in the ordinary cohomology of finite groups: iterated wreath products of finite cyclic p-groups play an important part because they occur as building blocks of Sylow subgroups of a variety of classes of finite groups, in

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particular many finite groups of Lie type. In ordinary group cohomology, systematic studies along these lines were made in the famous papers by Quillen [Quill71a], [Quill71b], [Quill71c]. Our treatment uses the notion of isoclinism of finite p-groups introduced by Hall in the paper [Hall]. It turns out that generically free linear quotients by isoclinic groups are stably birational, see Theorem 3.2; this answers a question raised in [HKK] (Question 1.11) in the affirmative. A partial result in this direction has been obtained previously in [Mor] where it is proven that isoclinic groups have isomorphic unramified cohomology in degree 2. Moreover, the stable cohomology of isoclinic groups share some important properties, see Proposition 2.2, namely if G_1 and G_2 are isoclinic, then the stable cohomology of G_1 is detected by abelian subgroups if and only if the same is true for G_2.

Here is a further outline of the contents of this paper: in Sections 2 and 3 we prove the afore-mentioned results for isoclinic groups and, as an ingredient of the proof, we show that the notions of isoclinism and being toroidally related coincide for group extensions.

Section 4 then contains the application of the results on isoclinic groups to the computation of the stable cohomology of iterated wreath products of groups $\mathbb{Z}/p$. The main results are Theorem 4.4 and its Corollary 4.5 saying that for the stable cohomology of these groups we have detection by abelian subgroups. Theorem 4.6 shows that for $G_n = \mathbb{Z}/p \wr \dots \wr \mathbb{Z}/p$ (n-factors) we even get detection by elementary abelian p-subgroups. This enables us to determine the structure of the stable cohomology algebra of G_n in Theorem 4.8 completely. In Section 5 we give some applications of the preceding results to the computation of the unramified and stable cohomology of some finite groups of Lie type. A more intensive treatment will be given elsewhere. In particular, we recover (and extend) results from [BPT] by this method.

2. Toroidally related extensions

Definition 2.1. Let G be a finite group and let A_1, A_2 be finite abelian groups. If $e_1 : 1 \to A_1 \to G_1 \to G \to 1$ and $e_2 : 1 \to A_2 \to G_2 \to G \to 1$ are two central extensions of G, we call them (resp. G_1 and G_2) *toroidally related* if there is an algebraic torus $T \simeq (\mathbb{C}^*)^r$ together with embeddings $i_k : A_k \hookrightarrow T$, $k = 1, 2$, such that the images of $e_1 \in H^2(G, A_1)$ and $e_2 \in H^2(G, A_2)$ in the cohomology $H^2(G, T)$ coincide.

Examples. (1) If G is abelian, then one knows for the group homology $H_2(G, \mathbb{Z}) = \Lambda^2 G$, and the universal coefficient sequence for an arbitrary

G-module M reads

$$0 \to \mathrm{Ext}^1(G,\, M) \to H^2(G,\, M) \xrightarrow{c} \mathrm{Hom}(\Lambda^2 G,\, M) \to 0\,.$$

For $M = \mathbb{Q}/\mathbb{Z}$, we have $\mathrm{Ext}^1(G,\, \mathbb{Q}/\mathbb{Z}) = 0$ as $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module; hence an isomorphism

$$c \,:\, H^2(G,\, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(\Lambda^2 G,\, \mathbb{Q}/\mathbb{Z})\,.$$

More concretely, for any $M = A$ with trivial action, the map $c : H^2(G,\, A) \to \mathrm{Hom}(\Lambda^2 G,\, A)$ can be described as follows: to a central extension

$$e \,:\, 1 \to A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \to 1$$

one associates the A-valued skew-form $c(e)$ on G given by the commutator:

$$i(c(e)(g,\, h)) = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}\,, \text{ where } \pi(\tilde{g}) = g,\ \pi(\tilde{h}) = h\,.$$

The kernel $\mathrm{Ext}^1(G,\, A)$ of c in this more general set-up can be identified with the abelian extensions $\mathcal{E}_{\mathrm{ab}}(G,\, A)$ of G by A. From the short exact sequence of coefficients

$$0 \to \mathbb{Q}/\mathbb{Z} \to \mathbb{C}^* \to \mathbb{C}^*/(\mathbb{Q}/\mathbb{Z}) \to 1$$

we get $H^2(G,\, \mathbb{Q}/\mathbb{Z}) \simeq H^2(G,\, \mathbb{C}^*)$ because $\mathbb{C}^*/(\mathbb{Q}/\mathbb{Z})$ is an infinite-dimensional vector space over $\mathbb{Q}$ (with "vector addition" = multiplication and "scalar multiplication"=exponentiation; this uses the algebraic closedness of $\mathbb{C}$), whence $H^i(G,\, \mathbb{C}^*/(\mathbb{Q}/\mathbb{Z})) = 0$ for $i > 0$. If $i \,:\, A \hookrightarrow (\mathbb{C}^*)^r$ is an embedding, the diagram

$$\begin{array}{ccc} H^2(G,\, A) & \xrightarrow{c} & \mathrm{Hom}(\Lambda^2 G,\, A) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i\circ(-)} \\ H^2(G,\, (\mathbb{C}^*)^r) & \xrightarrow{c} & \mathrm{Hom}(\Lambda^2 G,\, (\mathbb{C}^*)^r)\,. \end{array}$$

Thus, for fixed abelian G and A, two central extensions e_1 and e_2 of G by A are toroidally related if and only if they give the same skew form in $\mathrm{Hom}(\Lambda^2 G,\, A)$, or equivalently, their difference $e_1 - e_2 \in H^2(G,\, A)$ represents the class of an abelian extension. If A and G are elementary abelian p-groups, this is the same as saying that $e_1 - e_2$ lies in the subspace of $H^2(G,\, A)$ spanned by the Bocksteins.

(2) In general, when G is not necessarily abelian, we still have a universal coefficient sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_1(G,\, \mathbb{Z}),\, A) \to H^2(G,\, A) \to \mathrm{Hom}(H_2(G,\, \mathbb{Z}),\, A) \to 0$$

and, for an embedding $i : A \hookrightarrow (\mathbb{C}^*)^r$ a diagram

$$\begin{array}{ccc} H^2(G,\, A) & \xrightarrow{c} & \mathrm{Hom}(H_2(G,\, \mathbb{Z}),\, A) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i\circ(-)} \\ H^2(G,\, (\mathbb{C}^*)^r) & \xrightarrow{c} & \mathrm{Hom}(H_2(G,\, \mathbb{Z}),\, (\mathbb{C}^*)^r) \end{array}$$

where the bottom horizontal arrow is again an isomorphism. Thus extensions e_1 and e_2 are toroidally related in this case if and only if their difference $e_1 - e_2$ is in the subspace $\mathrm{Ext}^1_{\mathbb{Z}}(H_1(G,\, \mathbb{Z}),\, A)$. Now $H_1(G,\, \mathbb{Z}) = G^{\mathrm{ab}}$ and the map $\mathrm{Ext}^1_{\mathbb{Z}}(G^{\mathrm{ab}},\, A) \to H^2(G,\, A)$ factors

$$\mathrm{Ext}^1_{\mathbb{Z}}(G^{\mathrm{ab}},\, A) \to H^2(G^{\mathrm{ab}},\, A) \xrightarrow{p^*} H^2(G,\, A)$$

where $p : G \to G^{\mathrm{ab}}$ is the projection and the arrow $\mathrm{Ext}^1_{\mathbb{Z}}(G^{\mathrm{ab}},\, A) \to H^2(G^{\mathrm{ab}},\, A)$ is the one in the universal coefficient sequence

$$0 \to \mathrm{Ext}^1(G^{\mathrm{ab}},\, A) \to H^2(G^{\mathrm{ab}},\, A) \xrightarrow{c} \mathrm{Hom}(\Lambda^2 G^{\mathrm{ab}},\, A) \to 0\,.$$

This means that if $e_1 - e_2$ are toroidally related, then their difference corresponds to an extension $\tilde{G}$ of G by A induced from an abelian extension $\tilde{G}^{\mathrm{ab}}$ of G^{ab} by A:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & A & \longrightarrow & \tilde{G}^{\mathrm{ab}} & \longrightarrow & G^{\mathrm{ab}} & \longrightarrow & 1\,. \end{array}$$

The next Proposition shows that if G_1 and G_2 are toroidally related, then their stable cohomologies have important properties in common.

Proposition 2.2. *Let G_1 and G_2 be toroidally related as in Definition 2.1. Then we have*

(1) $H^*_{\mathrm{nr}}(G_1,\, \mathbb{Z}/p) \simeq H^*_{\mathrm{nr}}(G_2,\, \mathbb{Z}/p)$;
(2) *for toroidally related p-groups G_1 and G_2 one has that if $H^*_{\mathrm{s}}(G_1, \mathbb{Z}/p)$ is detected by abelian subgroups, so is $H^*_{\mathrm{s}}(G_2,\, \mathbb{Z}/p)$.*

Proof. (1) Let G_c be the central extension of G by $T = (\mathbb{C}^*)^r$ determined by G_1 and G_2. Then G_c is an extension $1 \to G_i \to G_c \to T/A_i \to 1$. Then a generically free representation V for G_c gives a generically free representation for both G_i, and $V/G_i \to V/G_c$ is a torus principal bundle, hence locally trivial. In particular, V/G_i, $i = 1,\, 2$, are stably birationally isomorphic and have the same unramified cohomology.

(2) Look at the fiber product of G_1 and G_2 over G:

$$\begin{array}{ccc} G_{12} = G_1 \times_G G_2 & \xrightarrow{\pi_1} & G_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow \\ G_2 & \longrightarrow & G\,. \end{array}$$

Then G_{12} is a toroidally trivial extension of both G_1 and G_2. We subdivide the proof into two auxiliary steps:

(a) If the stable cohomology of G_1 is detected by abelian subgroups, the same holds for the stable cohomology of G_{12}.
(b) $H^*_{\mathrm{s}}(G_2,\,\mathbb{Z}/p)$ injects into $H^*_{\mathrm{s}}(G_{12},\,\mathbb{Z}/p)$.

Given (a) and (b), one may conclude as follows: a nontrivial element $\alpha \in H^*_{\mathrm{s}}(G_2,\,\mathbb{Z}/p)$ is still nonzero in $H^*_{\mathrm{s}}(G_{12},\,\mathbb{Z}/p)$, hence is nonzero on some abelian subgroup $A \subset G_{12}$ by (a). Hence α will be nontrivial on $\pi_2(A) \subset G_2$.

To prove (b) note that G_{12}, being a toroidally trivial extension of G_1, is induced from an abelian extension $\tilde{G}_1^{\mathrm{ab}}$ of G_1^{ab}:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z_2 & \longrightarrow & G_{12} & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Z_2 & \longrightarrow & \tilde{G}_1^{\mathrm{ab}} & \longrightarrow & G_1^{\mathrm{ab}} & \longrightarrow & 1\,. \end{array}$$

Now $\tilde{G}_1^{\mathrm{ab}}$ splits as a direct product $A_2 + A_1$ of abelian groups, where A_1 is an abelian extension of G_1^{ab} of the same p-rank as G_1^{ab}, i.e. $\dim_{\mathbb{F}_p}(A_1/p) = \dim_{\mathbb{F}_p}(G_1^{\mathrm{ab}}/p)$, and $A_2 + A_1$ is then a trivial abelian extension of A_1. It follows that

$$G_{12} = A_2 \times G_1'$$

where G_1' is the preimage of A_1 in G_{12} under the map $G_{12} \to \tilde{G}_1^{\mathrm{ab}}$. There is a natural map $G_1' \to G_1$. We claim that the induced map

$$H^*_{\mathrm{s}}(G_1,\,\mathbb{Z}/p) \to H^*_{\mathrm{s}}(G_1', \mathbb{Z}/p)$$

is injective.

We prove that G_1 and G_1' have a faithful representation W with the same quotient. Indeed, let V be a faithful representation of G_1 and let $W = V \oplus \prod_j (\mathbb{C})_{\chi_j}$ where $\prod_j (\mathbb{C})_{\chi_j}$ gives an embedding of the abelian group A_1 into a torus via characters χ_j. This W is then a faithful representation of G_1 and G_1'. Let g_j be generators of A_1 dual to the characters χ_j, and $h_j = g_j^{p^{i_j}}$ generators of G_1^{ab}. Let V^L the open part of V where the action is free, then the map

$$V^L \times \prod_j (\mathbb{C}^*)_{\chi_j} \to V^L \times \prod_j (\mathbb{C}^*)_{\chi_j}$$

which is the identity on V^L and maps the coordinate z_j in $(\mathbb{C}^*)_{\chi_j}$ to $z_j^{p_{i_j}}$, induces an isomorphism

$$(V^L \times \prod_j (\mathbb{C}^*)_{\chi_j})/G_1' \to (V^L \times \prod_j (\mathbb{C}^*)_{\chi_j})/G_1 \,.$$

This finishes the proof of (b).

Let us now prove (a), i.e. we assume that $H^*_{\mathrm{s}}(G_1, \mathbb{Z}/p)$ is detected by abelian subgroups and we want to show the same for G_{12}. It suffices to treat the case where we are in the situation

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & G_{12} & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \Vert & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & \mathbb{Z}/p^{i+1} & \longrightarrow & \mathbb{Z}/p^i & \longrightarrow & 1\,. \end{array}$$

Let G_{12}^c be the $\mathbb{C}^*$-extension induced by $G_{12} \to G_1$ resp. $G_{12} \to G_2$. Then G_{12}^c is trivial: $G_{12}^c \simeq G_i \times \mathbb{C}^*$.
Consider a generically free linear representation V of G_1 and a linear representation $V \oplus \mathbb{C}^M$ of G_{12} where the quotient $\mathbb{Z}/p^{i+1}$ acts in $\mathbb{C}^M$ via multiplication by a p^{i+1}st root of unity. Let V^0 be the open part of V where the G_1-action is free. Note that likewise G_{12}^c acts naturally in $V \oplus \mathbb{C}^M$ (here $\mathbb{C}^*$ acts in $\mathbb{C}^M$ via homotheties). We have natural maps

$$\begin{array}{ccc} \mathcal{L}^{\times} := (V^0 \oplus (\mathbb{C}^M - \{0\}))/G_{12} & \xrightarrow{\ p\ } & \mathcal{P} := (V^0 \oplus (\mathbb{C}^M - \{0\}))/G_{12}^c \\ & & \downarrow q \\ & & \mathcal{B} := V^0/G_1\,. \end{array}$$

Here q makes $\mathcal{P}$ into a (trivial) projective bundle over $\mathcal{B}$, and $\mathcal{L}^{\times}$ is a line bundle over $\mathcal{P}$ with zero section removed. Note that if we replace V by a direct sum $V \oplus V \oplus \dots \oplus V$ with sufficiently many copies of V and make M large, we can assume that the group actions on the respective spaces are free in high codimension, hence the $\mathbb{Z}/p$-cohomology of $\mathcal{L}^{\times}$, $\mathcal{P}$ resp. $\mathcal{B}$ agrees with the group cohomology $H^*(G_{12}, \mathbb{Z}/p)$, $H^*(G_{12}^c, \mathbb{Z}/p)$ resp. $H^*(G_1, \mathbb{Z}/p)$ up to arbitrarily high degree. Now

$$H^*_{\mathrm{s}}(G_{12}^c, \mathbb{Z}/p) \simeq H^*_{\mathrm{s}}(G_1, \mathbb{Z}/p)$$

via the pull-back map q (the stable cohomology of the algebraic group $\mathbb{C}^*$ is trivial). Moreover, for the usual cohomology it clearly holds

$$H^*(\mathcal{P}, \mathbb{Z}/p) \simeq H^*(\mathcal{B}, \mathbb{Z}/p)[b]$$

where b is a generator of $H^2((\mathbb{C}^M - \{0\})/\mathbb{C}^*, \mathbb{Z}/p)$. Now we have

$$H^*_{\mathrm{s}}(\mathcal{L}^\times, \mathbb{Z}/p) \hookrightarrow H^*_{\mathrm{s}}(\mathcal{P}, \mathbb{Z}/p) \otimes H^*_{\mathrm{s}}(\mathbb{C}^*, \mathbb{Z}/p),$$

where $H^*_{\mathrm{s}}(\mathbb{C}^*, \mathbb{Z}/p)$ is the stable cohomology of the *algebraic variety* $\mathbb{C}^*$. To see this it is sufficient to consider the spectral sequence of the fibration p

$$H^i(\mathcal{P}, H^j(\mathbb{C}^*, \mathbb{Z}/p)) \implies H^{i+j}(\mathcal{L}^\times, \mathbb{Z}/p)$$

to show that every cohomology class $\alpha \in H^*(\mathcal{L}^\times, \mathbb{Z}/p)$ can be written as $\beta + \gamma \otimes t$ with $\beta, \gamma \in H^*(\mathcal{P}, \mathbb{Z}/p)$ and t a generator of $H^1(\mathbb{C}^*, \mathbb{Z}/p)$, and then pass to the stabilized cohomology. Put differently, we have maps

$$\begin{array}{ccc} \Gamma & \xrightarrow{r} & G_{12} \\ {\scriptstyle s}\downarrow & & \\ G_1 \times \mathbb{Z}/p & & \end{array}$$

where Γ is a profinite group, the decomposition group of the valuation corresponding to the zero section of $\mathcal{L}^\times$, which splits as $\Gamma' \oplus \hat{\mathbb{Z}}$, and r^* resp. s^* stabilize the cohomology of G_{12} resp. $G_1 \times \mathbb{Z}/p$. Note that there is some profinite abelian group $\hat{A}$ in Γ lying over every abelian subgroup in $G_1 \times \mathbb{Z}/p$ (one can take the valuation subgroup associated to the (composite) valuation given by a system of A-invariant coordinate hyperplanes). Then $\hat{A}$ stabilizes the cohomology of A. Now given a cohomology class in G_{12}, when pulled back to Γ, it is in the image of the cohomology of $G_1 \times \mathbb{Z}/p$, hence nontrivial on some such $\hat{A}$. Hence the original class will be nontrivial in the stable cohomology of the image of $\hat{A}$ under r in G_{12}. □

3. Comparison to the notion of isoclinism

In [Hall], P.Hall introduced the notion of *isoclinism* of finite groups which morally speaking means that the two groups have the same commutator function.

Definition 3.1. Two finite groups G_1 and G_2 with centers Z_1 and Z_2 are said to be *isoclinic* if there are isomorphisms

$$i : G_1/Z_1 \to G_2/Z_2, \quad j : [G_1, G_1] \to [G_2, G_2]$$

such that

$$\begin{array}{ccc} G_1/Z_1 \times G_1/Z_1 & \xrightarrow{(i,\,i)} & G_2/Z_2 \times G_2/Z_2 \\ {\scriptstyle [\cdot,\cdot]}\downarrow & & {\scriptstyle [\cdot,\cdot]}\downarrow \\ [G_1,\, G_1] & \xrightarrow{j} & [G_2,\, G_2] \end{array}$$

commutes.

As was remarked above, if G_1 and G_2 are toroidally related extensions of the group G, then G_{12} is a toroidally trivial extension of both G_1 and G_2; which means that it is an extension induced from an abelian extension of G_1^{ab} resp. G_2^{ab}. Hence G_1 and G_2 have the same commutator function (that of G_{12}) and are isoclinic.

Suppose conversely that G_1 and G_2 are isoclinic. Then naturally $G = G_1/Z_1 \simeq G_2/Z_2$ and we want to show that

$$e_1 \,:\, 1 \to Z_1 \to G_1 \to G \to 1\,,$$
$$e_2 \,:\, 1 \to Z_2 \to G_2 \to G \to 1$$

are toroidally related. We have to show that e_1 and e_2 map to the same element under $\iota_i \circ \alpha_i$ where α_i is the map in the sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_1(G,\,\mathbb{Z}),\, Z_i) \to H^2(G,\, Z_i) \xrightarrow{\alpha_i} \mathrm{Hom}(H_2(G,\,\mathbb{Z}),\, Z_i) \to 0\,.$$

and ι_i is the map

$$\iota_i \,:\, \mathrm{Hom}(H_2(G,\,\mathbb{Z}),\, Z_i) \to \mathrm{Hom}(H_2(G,\,\mathbb{Z}),\, (\mathbb{C}^*)^r)$$

induced by an appropriately chosen embedding $Z_i \hookrightarrow (\mathbb{C}^*)^r$. To do this we use an interpretation of the Schur multiplier $H_2(G,\,\mathbb{Z})$ from [Karp], section 2.6, in terms of commutator relations. Let $\langle G,\, G\rangle$ be the free group generated by all pairs $\langle x,\, y\rangle$ with $x,\, y \in G$ together with its natural map $c \,:\, \langle G,\, G\rangle \to [G,\, G]$ with $c(\langle x,\, y\rangle) = [x,\, y] = xyx^{-1}y^{-1}$. The kernel of p, denoted by $C(G)$, consists of relations among commutators in G. Moreover, there are the following universal commutator relations valid in any group G:

$$\langle x,\, x\rangle,\ \langle x,\, y\rangle\langle y,\, x\rangle,\ \langle y,\, z\rangle^x\langle x,\, z\rangle\langle xy,\, z\rangle^{-1}\,,$$
$$\langle y,\, z\rangle^x\langle y,\, z\rangle^{-1}\langle x,\, [y,\, z]\rangle^{-1}$$

where for $x,\, y,\, z \in G$, $\langle y,\, z\rangle^x = \langle xyx^{-1},\, xzx^{-1}\rangle$. The smallest normal subgroup in $C(G)$ containing all these universal relations is denoted by $B(G)$. Let $H(G) = C(G)/B(G)$ be the quotient. The universal relations are the ones that hold in a free group. We have now, by Theorem 2.6.6 of [Karp], that naturally $H(G) \simeq H_2(G,\,\mathbb{Z})$, the Schur multiplier, for a finite group G. In fact this is a consequence of Hopf's formula for $H_2(G, \mathbb{Z})$ which says that if $F \twoheadrightarrow G$ is a free presentation of G with subgroup of relations R, then $H_2(G, \mathbb{Z}) \simeq (R \cap [F, F])/[F, R]$. Now we want to reinterpret the maps

$$\iota_i \circ \alpha_i \,:\, H^2(G, Z_i) \to \mathrm{Hom}(H(G), Z_i) \hookrightarrow \mathrm{Hom}(H(G), (\mathbb{C}^*)^r)$$

(see also the proof of Theorem 2.6.6 in [Karp]): if A is one of Z_1 or Z_2, α one of α_i, then to a central extension $e \in H^2(G, A)$ given by

$$1 \to A \to \tilde{G} \to G \to 1$$

we first associate the homomorphism $\langle G, G\rangle \to \tilde{G}$ which maps $\langle x, y\rangle$ to $[\bar{x}, \bar{y}]$, $\bar{x}$, $\bar{y}$ lifts of x, y in $\tilde{G}$. This homomorphism maps $C(G)$ into A (in fact onto $A \cap [\tilde{G}, \tilde{G}]$), and it maps $B(G)$ to $\{1\}$, hence associated to e one gets a homomorphism

$$\psi_e \,:\, H(G) \to A \cap [\tilde{G}, \tilde{G}] \subset A\,.$$

Then $\alpha(e) = \psi_e$. It is obvious that if e_1 and e_2 have the same commutator function, then we can choose embeddings $Z_1 \hookrightarrow (\mathbb{C}^*)^r$, $Z_2 \hookrightarrow (\mathbb{C}^*)^r$ which agree on $Z_1 \cap [G_1, G_1] \simeq Z_2 \cap [G_2, G_2]$ and such that then $\psi_{e_1} = \psi_{e_2}$, viewed as maps into $(\mathbb{C}^*)^r$. Hence we have proven

Theorem 3.2. *The notions of being toroidally related and isoclinic coincide. In particular, if G_1 and G_2 are isoclinic, then generically free linear quotients for G_1 and G_2 are stably equivalent.*

This answers Question (1.11) of [HKK] in the affirmative; the partial result that G_1 and G_2 have isomorphic second unramified cohomology groups has been proven in [Mor].

We give an additional elementary argument for the implication "G_1 and G_2 isoclinic" $\implies$ "the extensions e_1 and e_2 are toroidally related". Suppose that G_1 and G_2 are isoclinic, and consider the fiber product over G as above:

$$\begin{array}{ccc} G_{12} = G_1 \times_G G_2 & \xrightarrow{\pi_1} & G_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow \\ G_2 & \longrightarrow & G\,. \end{array}$$

We want to show that then the extensions $1 \to Z_2 \to G_{12} \to G_1 \to 1$ and $1 \to Z_1 \to G_{12} \to G_2 \to 1$ are toroidally trivial. Note that the preimages of $[G_1, G_1] \subset G_1$ and $[G_2, G_2] \subset G_2$ coincide with $[G_{12}, G_{12}]$ and this group equals

$$[G_{12}, G_{12}] = \langle ([g_1, h_1], j([g_1, h_1]))\rangle \quad g_1, h_1 \in G_1$$

which is a "diagonal subgroup" of G_{12} which intersects both $Z_1 \simeq \{(z_1, 1) \,|\, z_1 \in Z_1\} \subset G_{12}$ and $Z_2 \simeq \{(1, z_2) \,|\, z_2 \in Z_2\}$ trivially. In other words, Z_1 maps isomorphically to $G_2/[G_2,\, G_2] = G_2^{\mathrm{ab}}$ and Z_2 isomorphically to $G_1/[G_1,\, G_1] = G_1^{\mathrm{ab}}$. Therefore, for example, the extension G_{12} of G_2 by Z_1 is induced by an abelian extension

$$1 \to Z_1 \to G_{12}/[G_{12},\, G_{12}] \to G_2/[G_2,\, G_2] \to 1\,.$$

The same holds for the extension G_{12} of G_1. Hence these two are toroidally trivial, hence e_1 and e_2 are toroidally related.

The following Remark is sometimes useful and summarizes some compatibilities of isoclinism with passing to subgroups or quotients. It can be found already in [Hall].

Remark 3.3. An isoclinism between G_1 and G_2 sets up a bijective correspondence between subgroups of G_1 containing Z_1 and subgroups of G_2 containing Z_2, and corresponding subgroups are isoclinic.
In particular, a centralizer in the group G_2 is isoclinic to a centralizer in G_1.
Moreover, an isoclinism also gives a bijective correspondence between quotient groups G_1/K_1 and G_2/K_2, where $K_1 \subset [G_1,\, G_1]$ and $K_2 \subset [G_2,\, G_2]$, and corresponding quotient groups are isoclinic.

4. Stable cohomology of wreath products

Now we want to use Proposition 2.2 to compute the stable cohomology of the iterated wreath product $G_n = \mathbb{Z}/p \wr \mathbb{Z}/p \wr \dots \wr \mathbb{Z}/p$ of groups $\mathbb{Z}/p$ (there are n-factors $\mathbb{Z}/p$). We first define a class of groups which will turn out to be stable under taking iterated centralizers, provided one identifies groups which are isoclinic/toroidally related, and contains G_n.

Definition 4.1. Let $\mathcal{D}$ be a class of groups with the property that (1) every group in $\mathcal{D}$ has stably rational generically free linear quotients, and (2) the centralizer of any element in a group in $\mathcal{D}$ again belongs to $\mathcal{D}$.
Then we define a group to belong to the class $\mathcal{C}_p(\mathcal{D})$ if it can be reached starting from a group in $\mathcal{D}$ by a finite number of the following operations, which successively enlarge the set of groups in $\mathcal{C}_p(\mathcal{D})$ already constructed:

(a) taking a wreath product with a group $\mathbb{Z}/p$, i.e. passing from H to $H \wr \mathbb{Z}/p$,
(b) taking a finite direct product,
(c) passing from H to an isoclinic group H'.

It follows that all the groups in $\mathcal{C}_p(\mathcal{D})$ have trivial higher unramified cohomology, in fact they all have stably rational quotients. Moreover, clearly G_n belongs to $\mathcal{C}_p$ if we take $\mathcal{D}$ to consist only of the group $\mathbb{Z}/p$.

The key result will be Proposition 4.3 below. We precede it with a Lemma on isoclinism types of centralizers in a wreath product $H = H' \wr \mathbb{Z}/p$ which will be used in the proof of Proposition 4.3.

Lemma 4.2. *Let H' be a finite group, and let $H = H' \wr \mathbb{Z}/p$ be its wreath product with $\mathbb{Z}/p$. Let $x \in H$ be some element, and $Z_H(x)$ be its centralizer. Then one of the following is true:*

(a) *The element x is contained in $(H')^p \subset H$, $Z_H(x)$ does* not *surject onto the quotient $\mathbb{Z}/p$ under the natural projection $H \twoheadrightarrow \mathbb{Z}/p$, and the centralizer $Z_H(x)$ is a product*
$$Z_{H'}(x_1) \times \cdots \times Z_{H'}(x_p)$$
of the centralizers $Z_{H'}(x_i)$ of the components x_i of x with respect to the product $H' \times ... \times H'$ (p-factors) in H.

(b) *The cyclic subgroup $\langle x \rangle$ generated by x in H surjects onto the quotient $\mathbb{Z}/p$ under the natural projection $H \twoheadrightarrow \mathbb{Z}/p$, and if $x^p = ((a, a, \dots, a), \mathrm{id})$, then $Z_H(x)$ is* isoclinic *to*
$$Z_{H'}(a) \times \mathbb{Z}/p\,.$$

(c) *The element x is contained in $(H')^p$ and $Z_H(x)$ surjects onto $\mathbb{Z}/p$. Then $x = (x', \dots, x') \in (H')^p$ and $Z_H(x)$ is isomorphic to*
$$Z_{H'}(x) \wr \mathbb{Z}/p\,.$$

Proof. The cases enumerated in (a), (b), (c) obviously cover all the possibilities and are mutually exclusive. We deal with them one by one.

Case (a) The centralizer $Z_H(x)$ is contained in $(H')^p$. Then $Z_H(x)$ is obviously the product of the centralizers of components.

Case (b) Suppose that $\langle x \rangle$ surjects onto the quotient $\mathbb{Z}/p$ of $H = H' \wr \mathbb{Z}/p$. After conjugating by an element in $(H')^p$ we may assume $x = ((a, \mathrm{id}, \dots, \mathrm{id}), \sigma)$ for some $a \in H'$ and $\sigma \in \mathbb{Z}/p$ a generator. Indeed, if a priori $x = ((x_1, \dots, x_p), \sigma)$, it is sufficient for this to solve the equations
$$c_i x_i c_{\sigma^{-1}(i)}^{-1} = 1 \text{ for } i \neq 1$$
in elements c_i of H', which is always possible successively, and conjugate by $c = ((c_1, \dots, c_p), \mathrm{id})$. To see that it is indeed possible successively,

put e.g. $c_{\sigma^{-1}(1)} = 1$, $c_{\sigma^{-2}(1)} = x_{\sigma^{-1}(1)}$, $c_{\sigma^{-3}(1)} = x_{\sigma^{-1}(1)}x_{\sigma^{-2}(1)}$, $\dots$, $c_{\sigma^{-p}(1)} = c_1 = x_{\sigma^{-1}(1)} \dots x_{\sigma^{-(p-1)}(1)}$, $a = x_{\sigma^{-1}(1)} \dots x_{\sigma^{-(p-1)}(1)} x_1 = \prod_i x_i$. We will assume therefore now that $x = ((a, \mathrm{id}, \dots, \mathrm{id}), \sigma)$ with σ a generator of $\mathbb{Z}/p$ for simplicity.

Clearly, $Z_H(x)$ is generated by $\langle x \rangle$ and those $y = ((y_1, \dots, y_p), \mathrm{id})$ which commute with x. These elements y will also commute with all powers of x, hence with $((a, a, \dots, a), \mathrm{id})$. That is, it is necessary that a commutes with every y_j. The condition

$$((y_1, \dots, y_p), 1)((a, 1, \dots, 1), \sigma) = ((a, 1, \dots, 1), \sigma)((y_1, \dots, y_p), 1)$$

reads

$$\begin{aligned} y_1 a &= a y_{\sigma^{-1}(1)} \\ y_2 &= y_{\sigma^{-1}(2)}\,, \\ &\vdots \\ y_p &= y_{\sigma^{-1}(p)}\,. \end{aligned}$$

This implies $y_{\sigma^{-1}(1)} = y_{\sigma^{-2}(1)} = \dots = y_{\sigma^{-(p-1)}(1)} = y_1$, so the elements y which commute with x are precisely those such that $y_1 = \dots = y_p$ and all of them commute with a.
In this case it follows that $Z_H(x)$ is an extension

$$0 \to Z_{H'}(a) \to Z_H(x) \to \mathbb{Z}/p \to 0$$

which will in general be nontrivial; however, we claim that $Z_H(x)$ is toroidally related to the product $Z_{H'}(a) \times \mathbb{Z}/p$. More precisely, $Z_H(x)$ is a central extension

$$1 \to \langle a \rangle \to Z_H(x) \to Z_H(x)/\langle a \rangle \simeq Z_{H'}(a)/\langle a \rangle \times \mathbb{Z}/p \to 1$$

of $Z_{H'}(a)/\langle a \rangle \times \mathbb{Z}/p$ by the cyclic group generated by a. We claim that this extension is toroidally related to the extension

$$1 \to \langle a \rangle \to Z_{H'}(a) \times \mathbb{Z}/p \to Z_{H'}(a)/\langle a \rangle \times \mathbb{Z}/p \to 1\,.$$

Indeed, look at the extension $\widetilde{Z_H(x)}$ of $Z_{H'}(a)/\langle a \rangle \times \mathbb{Z}/p$ by $\mathbb{Q}/\mathbb{Z}$ induced by $Z_H(x)$:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & Z_H(x) & & & & \\
 & & & & \downarrow & & & & \\
1 & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & \widetilde{Z_H(x)} & \longrightarrow & Z_{H'}(a)/\langle a\rangle \times \mathbb{Z}/p & \longrightarrow & 1 \\
 & & & \searrow^{\cdot N} & \downarrow & & & & \\
 & & & & \mathbb{Q}/\mathbb{Z} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array}
$$

As $\mathbb{Q}/\mathbb{Z}$ is divisible, the element $a \in Z_{H'}(a) \subset \widetilde{Z_H(x)}$ is a p-th power of a central element A in $\widetilde{Z_H(x)}$. In particular, $\widetilde{Z_H(x)}$ contains also $Z_{H'}(a) \times \mathbb{Z}/p$; we can map a generator of $\mathbb{Z}/p$ to $((a,\, 1, \ldots,\, 1),\, \sigma)A^{-1}$ in $Z_H(x)$.

Case (c) We have $x \in (H')^p$, but there is an element $g \in Z_H(x)$ such that the subgroup generated by it surjects onto $\mathbb{Z}/p$. The element g is then conjugate to $((a,\, \mathrm{id}, \ldots,\, \mathrm{id}),\, \sigma)$ for some $a \in H'$ and $\sigma \in \mathbb{Z}/p$ a generator. We can assume then that $g = ((a,\, \mathrm{id}, \ldots,\, \mathrm{id}),\, \sigma)$. Now $g^p = ((a,\, a, \ldots,\, a),\, \mathrm{id}) \in Z_{(H')^p}(x)$, and as this is a product of the centralizers of the components of x, also $b = ((a,\, \mathrm{id}, \ldots,\, \mathrm{id}),\, \mathrm{id}) \in Z_H(x)$. So $b^{-1} \cdot g = ((\mathrm{id}, \ldots,\, \mathrm{id}),\, \sigma)$ is also always in $Z_H(x)$ in this case. Hence $x = (x', \ldots,\, x')$ here, and $Z_H(x) = Z_{H'}(x') \wr \mathbb{Z}/p$. ☐

Proposition 4.3. *Suppose G is a group in $\mathcal{C}_p(\mathcal{D})$ and let $h \in G$ be some element. Then the centralizer $Z_G(h)$ is again a group in $\mathcal{C}_p(\mathcal{D})$.*

Proof. If we denote $\Longrightarrow$ one of the construction steps in Definition 4.1, and by $G_0 \Longrightarrow G_1$, $G_0, G_1 \in \mathcal{C}_p(\mathcal{D})$, the fact that G_1 is gotten from G_0 applying one construction step, then we call

$$G_0 \Longrightarrow G_1 \Longrightarrow \ldots \Longrightarrow G_s$$

with G_0 in $\mathcal{D}$ a *chain of length* s and G_s the *end* of the chain. Let $\mathcal{C}_p(\mathcal{D})^{\leq s}$ be the subclass of all groups in $\mathcal{C}_p(\mathcal{D})$ which are ends of chains of length $\leq s$. We prove by induction on s, starting from $s = 0$, that every centralizer of an element in a group in $\mathcal{C}_p(\mathcal{D})^{\leq s}$ again belongs to $\mathcal{C}_p(\mathcal{D})$.

The beginning of the induction is trivial because the assertion of Proposition 4.3 holds by assumption for all the groups in $\mathcal{D}$. Now suppose it holds for all groups in $\mathcal{C}_p(\mathcal{D})^{\le s}$. Suppose then H is a group in $\mathcal{C}_p(\mathcal{D})^{\le s+1}$ constructed out of $\mathcal{C}_p(\mathcal{D})^{\le s}$ according to the rules in Definition 4.1. We have the following possibilities.

(1) The group H is a finite product
$$H = H_1' \times \cdots \times H_N'$$
of groups H_i' in $\mathcal{C}_p(\mathcal{D})^{\le s}$. Then the centralizer $Z_H(x)$ of an element $x \in H$ is the product of the centralizers of the components x_i of x. Each of the $Z_{H_i}(x_i)$ belongs to $\mathcal{C}_p(\mathcal{D})$ by the induction hypothesis, hence so does the product as $\mathcal{C}_p(\mathcal{D})$ is closed under taking finite products by definition.

(2) The group H is isoclinic to a group H' in $\mathcal{C}_p(\mathcal{D})^{\le s}$. By Remark 3.3, centralizers of elements in H are isoclinic to centralizers of elements in H'. The latter however belong to $\mathcal{C}_p(\mathcal{D})$ by induction. As $\mathcal{C}_p(\mathcal{D})$ is closed under passage to isoclinic groups, H belongs to $\mathcal{C}_p(\mathcal{D})$, too, in this case.

(3) The group H is a wreath product
$$H = H' \wr \mathbb{Z}/p$$
where H' belongs to $\mathcal{C}_p(\mathcal{D})^{\le s}$. According to Lemma 4.2 above, we see, using the induction hypothesis and the definition of the class $\mathcal{C}_p(\mathcal{D})$, that H also belongs to $\mathcal{C}_p(\mathcal{D})$.

This concludes the proof. □

Theorem 4.4. *The stable cohomology $H^*_{\mathrm{s}}(G, \mathbb{Z}/p)$ is detected by abelian subgroups for any group G in $\mathcal{C}_p(\mathcal{D})$.*

Proof. Here we use Lemma 1.5 of [B-B11] inductively. Then everything follows from Proposition 4.3, saying that $\mathcal{C}_p(\mathcal{D})$ is closed under taking centralizers, and induction over the cohomological degree: note that all groups in $\mathcal{C}_p(\mathcal{D})$ have trivial higher unramified cohomology and that the stable cohomology $H^1_{\mathrm{s}}(G, \mathbb{Z}/p)$ of any finite group G is detected by abelian subgroups: this is so because any nontrivial character $\chi : G \to \mathbb{Z}/p$ is nontrivial on a cyclic subgroup in G. □

Corollary 4.5. *The stable cohomology $H^*_{\mathrm{s}}(G_n, \mathbb{Z}/p)$ is detected by abelian subgroups.*

Proof. Take $\mathcal{D} = \{\mathbb{Z}/p\}$ and apply Theorem 4.4. □

The following result allows us to determine $H^*_{\mathrm{s}}(G_n, \mathbb{Z}/p)$ rather precisely. It follows from Corollary 4.5, but requires some additional work.

Theorem 4.6. *The stable cohomology $H^*_{\mathrm{s}}(G_n,\, \mathbb{Z}/p)$ is detected by elementary abelian subgroups.*

Proof. It will be sufficient to prove:

> Every nontrivial class $\alpha \in H^k_{\mathrm{s}}(G_n,\, \mathbb{Z}/p)$, $k > 1$, is nontrivial on the subgroup $G^p_{n-1} \subset G_{n-1} \wr \mathbb{Z}/p = G_n$.

Then the assertion of Theorem 4.6 will follow by induction and the fact that $H^1_{\mathrm{s}}(G_n,\, \mathbb{Z}/p)$ is always detected by elementary abelian subgroups: in fact, every nontrivial character $\chi : G_n \to \mathbb{Z}/p$ is nontrivial on G^p_{n-1} or else nontrivial on the quotient $\mathbb{Z}/p$.

By Theorem 4.5 a class α as above which is trivial on G^p_{n-1} must be nontrivial on some abelian subgroup A which surjects onto $\mathbb{Z}/p$ under the composite map $A \hookrightarrow G_n \twoheadrightarrow \mathbb{Z}/p$. Then A is contained in a subgroup $B \wr \mathbb{Z}/p \subset G_n$ where $B^p \subset G^p_{n-1}$ is abelian: we can take for B the image of $A \cap G^p_{n-1}$ in G_{n-1} under any of the coordinate projections $G^p_{n-1} \to G_{n-1}$. Note that if $x = ((a,\, 1,\, 1, \ldots,\, 1),\, \sigma)$ is an element in A such that $\langle x \rangle$ surjects onto $\mathbb{Z}/p$ and if $b = ((b_1, \ldots,\, b_p),\, 1) \in A \cap G^p_{n-1}$, then the equation $xbx^{-1} = b$ together with the fact that a commutes with every b_i implies that $b_1 = \cdots = b_p$, so all coordinate projections are the same.

Thus we get a nontrivial class in $H^k_{\mathrm{s}}(B \wr \mathbb{Z}/p,\, \mathbb{Z}/p)$ which with slight abuse of notation we denote again by α. We have to recall some results about the structure of the cohomology of $B \wr \mathbb{Z}/p$: by Nakaoka's Theorem (see [Evens]) one has an isomorphism

$$H^*(B \wr \mathbb{Z}/p,\, \mathbb{Z}/p) \simeq H^*(\mathbb{Z}/p,\, H^*(B,\, \mathbb{Z}/p)^{\otimes p})$$

where we consider the cohomology $H^*(B, \mathbb{Z}/p)^{\otimes p}$ as a $\mathbb{Z}/p[\mathbb{Z}/p]$-module (with nontrivial action). However, $H^*(B, \mathbb{Z}/p)^{\otimes p}$ is a direct sum of trivial $\mathbb{Z}/p[\mathbb{Z}/p]$-modules and free $\mathbb{Z}/p[\mathbb{Z}/p]$-modules (see [A-M], p. 117). The trivial modules are generated by norm elements $x \otimes \cdots \otimes x \in H^*(B,\, \mathbb{Z}/p)^{\otimes p}$, $x \in H^*(B,\, \mathbb{Z}/p)$. The free modules do not contribute to the cohomology. Let $b_1,\, b_2, \ldots$ be a basis for $H^*(B,\, \mathbb{Z}/p)$. Hence there is a natural splitting

$$\begin{aligned} H^*(B \wr \mathbb{Z}/p,\, \mathbb{Z}/p) &= H^0(\mathbb{Z}/p,\, H^*(B,\, \mathbb{Z}/p)^{\otimes p}) \oplus \bigoplus_{k>0,\, i} H^k(\mathbb{Z}/p,\, T_i) \\ &\simeq H^*(B^p,\, \mathbb{Z}/p)^{\mathbb{Z}/p} \oplus \bigoplus_{k>0,\, i} H^k(\mathbb{Z}/p,\, T_i)\,, \end{aligned}$$

the direct sum running over all the trivial modules T_i, generated by $b_i \otimes \cdots \otimes b_i$, which occur.

Consider now a faithful toric representation R_B for B that stabilizes the cohomology of B with open part R^o_B where the B action is free. We construct the faithful $B \wr \mathbb{Z}/p$ representation $R_B^p \oplus \mathbb{C}$ where $\mathbb{Z}/p$ acts via a p-th root of unity in $\mathbb{C}$ and rotates the copies of R_B. This has an open toric free part $(R_B^o)^p \times \mathbb{C}^*$ and the quotient $Q := ((R_B^o)^p \times \mathbb{C}^*)/(B \wr \mathbb{Z}/p)$ has the structure of a torus fibration with fibre $(R_B^o)^p/B^p$ over $\mathbb{C}^* \simeq (\mathbb{C}^*)/(\mathbb{Z}/p)$. The fundamental group $\pi_1(Q)$ yields a partial stabilization for the cohomology of $B \wr \mathbb{Z}/p$ and is of the form $B^s \rtimes \mathbb{Z}$ where $B^s = \pi_1(R_B^o/B)$ and stabilizes the cohomology of B. We consider the image of the cohomology of $B \wr \mathbb{Z}/p$ in the cohomology of $B^s \wr \mathbb{Z}/p$. That map can be factored

$$H^*(B \wr \mathbb{Z}/p,\, \mathbb{Z}/p) \xrightarrow{f_1} H^*(B^s \wr \mathbb{Z}/p,\, \mathbb{Z}/p) \xrightarrow{f_2} H^*(B^s \rtimes \mathbb{Z},\, \mathbb{Z}/p)$$

and the cohomologies of $B^s \wr \mathbb{Z}/p$ and also of $B^s \rtimes \mathbb{Z}$ can be described analogously to what was said above: first, clearly,

$$H^*(B^s \wr \mathbb{Z}/p,\, \mathbb{Z}/p) \simeq H^*(\mathbb{Z}/p,\, H^*(B^s,\, \mathbb{Z}/p)^{\otimes p})$$

and the description is entirely the same as before. Now for $B^s \rtimes \mathbb{Z}$ we also have, see [Evens], discussion on page 19 and proof of Theorem 5.3.1, that

$$H^*(B^s \wr \mathbb{Z},\, \mathbb{Z}/p) \simeq H^*(\mathbb{Z},\, H^*(B^s,\, \mathbb{Z}/p)^{\otimes p})$$

where now we consider $H^*(B^s, \mathbb{Z}/p)^{\otimes p}$ as a $\mathbb{Z}/p[\mathbb{Z}]$ module via the quotient map $\mathbb{Z} \to \mathbb{Z}/p$. In particular, the free $\mathbb{Z}/p[\mathbb{Z}/p]$-submodules of $H^*(B^s, \mathbb{Z}/p)^{\otimes p}$ may contribute to the cohomology $H^*(\mathbb{Z}, H^*(B^s, \mathbb{Z}/p)^{\otimes p})$ now (but those classes do not come from $H^*(B \wr \mathbb{Z}/p,\, \mathbb{Z}/p)$). The maps f_1 and f_2 have a natural description using the previous isomorphisms: the map f_1 is just induced by the map of coefficients $H^*(B,\, \mathbb{Z}/p)^{\otimes p} \to H^*(B^s,\, \mathbb{Z}/p)^{\otimes p}$ (which is a stabilization map for the cohomology of B^p), and f_2 is induced by the natural surjection of groups $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p$.

From this description we see that if a class α is in $H^*(B^p,\, \mathbb{Z}/p)^{\mathbb{Z}/p}$ and stable, then it is detected already on B^p. Moreover, the classes in $H^k(\mathbb{Z}/p, T_i)$ can only be stable if $k = 1$ and T_i is generated by $\beta \otimes \cdots \otimes \beta$ with β stable (and part of the chosen basis for $H^*(B,\, \mathbb{Z}/p)$). Let $\deg(\beta) =: b$, and let us show that in fact all classes $\alpha = \tau \cup (\beta \otimes \cdots \otimes \beta)$ with $b > 0$ and τ some generator of $H^1(\mathbb{Z}/p,\, \mathbb{Z}/p)$ are unstable in $H^*(B \wr \mathbb{Z}/p,\, \mathbb{Z}/p)$, which will prove Theorem 4.6. The degree of α is $1 + pb$. However, α is then induced from a class α' in $E \wr \mathbb{Z}/p$, where $E \simeq (\mathbb{Z}/p)^b$ is elementary abelian, via some surjection $B \twoheadrightarrow E$: in fact, we may assume β is a monomial in $e_1, \dots, e_r$, the latter being some basis of $H^1(B,\, \mathbb{Z}/p)$ and then the surjection is just a coordinate projection followed by reduction to $\mathbb{Z}/p$. If α were stable, then α' would be stable. This is however clearly not so if $b > 0$ for in that case $E \wr \mathbb{Z}/p$

has a faithful representation of dimension pb (let each standard copy of $\mathbb{Z}/p$ in $E \simeq \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$ act on $\mathbb{C}$ via a nontrivial character and let $\mathbb{Z}/p$ rotate those copies). Hence all classes in the cohomology of $E \wr \mathbb{Z}/p$ of degrees $> pb$ are killed under stabilization for dimension reasons. □

We have already proven a lot more, but let us record the easy

Corollary 4.7. *The stable cohomology $H^*_{\mathrm{s}}(G_n,\, \mathbb{Z}/p)$ is detected by the two subgroups G^p_{n-1} and $G_{n-1} \times \mathbb{Z}/p$.*

Proof. It is known ([Mui], p.349) that every maximal elementary abelian p-subgroup of G_n is contained in G^p_{n-1} or $G_{n-1} \times \mathbb{Z}/p$. □

We can now also say very precisely how the cohomology ring $H^*_{\mathrm{s}}(G_n \mathbb{Z}/p)$ is structured.

Theorem 4.8. *The stable cohomology $H^*_{\mathrm{s}}(G_n, \mathbb{Z}/p)$ is determined inductively as follows: there is an isomorphism*

$$i \,:\, H^*_{\mathrm{s}}(G_n,\, \mathbb{Z}/p) \simeq H^*(G^p_{n-1},\, \mathbb{Z}/p)^{\mathbb{Z}/p} \oplus H^1(\mathbb{Z}/p,\, \mathbb{Z}/p)$$

where $i = (i_1,\, i_2)$ and i_1 is the restriction map $\mathrm{res}^{G_n}_{G^p_{n-1}}$ and i_2 is the restriction map to the subgroup $\mathbb{Z}/p$ in $G_n = G_{n-1} \wr \mathbb{Z}/p$.

Proof. This follows rather easily from the proof of Theorem 4.6, but let us give another proof using Steenrod operations and the Bloch-Kato conjecture to show how everything falls into place.

By Steenrod's description of the cohomology of wreath products (see e.g. [A-M], IV.4 and IV.7) one has

$$H^*(G_{n-1} \wr \mathbb{Z}/p) = H^*(G^p_{n-1},\, \mathbb{Z}/p)^{\mathbb{Z}/p} + \{\Gamma(\alpha) \cup \theta_i\}$$

where $H^*(G^p_{n-1}, \mathbb{Z}/p)^{\mathbb{Z}/p}$ is the image of the restriction map to G^p_{n-1} and $\alpha \in H^k(G_{n-1},\, \mathbb{Z}/p)$, $\Gamma(\alpha) \in H^{kp}(G_{n-1} \wr \mathbb{Z}/p,\, \mathbb{Z}/p)$ is the total Steenrod power Γ applied to α, and θ_i are some classes in $H^i(\mathbb{Z}/p, \mathbb{Z}/p)$ with $i \geq 1$. A little more conceptionally, one can say that by Nakaoka's Theorem, as above, one has an isomorphism

$$H^*(G_{n-1} \wr \mathbb{Z}/p,\, \mathbb{Z}/p) \simeq H^*(\mathbb{Z}/p,\, H^*(G_{n-1},\, \mathbb{Z}/p)^{\otimes p})$$

and there is a natural splitting

$$\begin{aligned} H^*(G_{n-1} \wr \mathbb{Z}/p,\, \mathbb{Z}/p) &= H^0(\mathbb{Z}/p,\, H) \oplus \bigoplus_{k>0,\, T_x} H^k(\mathbb{Z}/p,\, T_x) \\ &\simeq H^*(G^p_{n-1},\, \mathbb{Z}/p)^{\mathbb{Z}/p} \oplus \bigoplus_{k>0,\, T_x} H^k(\mathbb{Z}/p,\, T_x)\,, \end{aligned}$$

the direct sum running over all the trivial modules T_x, generated by $x\otimes\cdots\otimes x$, which occur. The point of the total Steenrod power Γ is then that it realizes the generator of $H^k(\mathbb{Z}/p, T_x)$ as an explicit class $\theta_i\cup\Gamma(x)$ in the cohomology of G_n (here $\theta_i \in H^k(\mathbb{Z}/p,\, \mathbb{Z}/p)$) in a way that is compatible with cup products and functorial for group homomorphisms. We still need to remark that $\Gamma(x)$ restricts to $H^*(G_{n-1}^p,\, \mathbb{Z}/p)$ as $x\otimes\cdots\otimes x$ and its restriction to $G_{n-1}\times\mathbb{Z}/p$ is of the form

$$\sum_j D^j(x)\cup g_j$$

where, keeping in mind $H^*(G_{n-1}\times\mathbb{Z}/p,\, \mathbb{Z}/p) = H^*(G_{n-1},\, \mathbb{Z}/p)\otimes H^*(\mathbb{Z}/p, \mathbb{Z}/p)$, the element g_j is a generator of $H^j(\mathbb{Z}/p, \mathbb{Z}/p)$ and $D^j(x)$ is a certain class in $H^{\deg(x)p-j}(G_{n-1},\, \mathbb{Z}/p)$. Here $D^j(x)$ is, if nontrivial, equal -up to a sign- to a Steenrod power $P^s(x)$ or $\beta P^s(x)$, where β is the Bockstein operator. We do not need the exact formula (which is in [A-M], p. 184, (1.12)), but just that, due to the fact that the Steenrod operations P^s and the Bockstein are zero in stable cohomology, $D^j(x)\cup g_j$ can just be nontrivial in stable cohomology if D^j is the identity up to a sign and $j\leq 1$. Which means $\deg(x)p-j=\deg(x)$, and this can happen for odd primes p only when x is in degree 0. Hence we see that elements of the form $\Gamma(\alpha)\cup\theta_i$ give nontrivial classes in stable cohomology (in view of Corollary 4.7, which of course we use all the time now) if and only if this is in fact just a class θ_i coming from $H^1(\mathbb{Z}/p,\, \mathbb{Z}/p)$.

Moreover the elements in $H^*(G_{n-1}^p,\, \mathbb{Z}/p)^{\mathbb{Z}/p}$ we are getting are exactly of two types: norms and traces; norms being elements of the form $x\otimes\cdots\otimes x$, $x\in H^*(G_{n-1},\, \mathbb{Z}/p)$ (these are stable if and only if x is stable) and traces being of the form

$$\sum_{i=0}^{p}\sigma^i(x_1\otimes\cdots\otimes x_p)$$

$x_1\otimes\cdots\otimes x_p\in H^*(G_{n-1}^p,\, \mathbb{Z}/p)$ and σ is a cyclic shift. These are stable if and only if all the x_i are stable. □

5. Applications to finite groups of Lie type

We would like to give some simple applications of the fore-going material to the computation of stable cohomology of some finite groups of Lie type. Consider the group $\mathrm{GL}_n(\mathbb{F}_q)$ of automorphisms of $\mathbb{F}_q^n$ where $\mathbb{F}_q$ is the finite field with $q=p^\varrho$ elements.

We assume that p and l are prime numbers with $(p,\, l) = 1$ and $l \neq 2$. Then by [A-M], Theorem VII.4, 4.1 and Corollary 4.3, the l-Sylow subgroup $\mathrm{Syl}_l(\mathrm{GL}_n(\mathbb{F}_q)$ is a product of groups of the form:

$$\mathbb{Z}/l^r \wr \mathbb{Z}/l \cdots \wr \mathbb{Z}/l\,.$$

Hence Theorem 4.4 shows

Theorem 5.1. *Let p and l be two primes with $(p, q) = 1$, $l \neq 2$, $q = p^{\varrho}$, ϱ, $n \geq 1$ integers. Then the stable cohomology of $H^*_{\mathrm{s}}(\mathrm{GL}_n(\mathbb{F}_q),\, \mathbb{Z}/l)$ is detected by abelian subgroups (over any base field k). In particular, the unramified cohomology $H^*_{\mathrm{nr}}(\mathrm{GL}_n(\mathbb{F}_q),\, \mathbb{Z}/l)$ is trivial. The same statements hold for $\mathrm{SL}_n(\mathbb{F}_q)$ (the l-Sylow is the same as that of $\mathrm{GL}_n(\mathbb{F}_q)$).*

One should compare this to the results obtained in [BPT]. We hope to give a more exhaustive treatment of other classes of finite groups of Lie type elsewhere.

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