UNIRATIONALITY AND EXISTENCE OF INFINITELY TRANSITIVE MODELS

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Abstract. We study unirational algebraic varieties and the fields of rational functions on them. We show that after adding a finite number of variables some of these fields admit an *infinitely transitive model*. The latter is an algebraic variety with the given field of rational functions and an infinitely transitive regular action of an algebraic group generated by unipotent algebraic subgroups. We expect this property holds for all unirational varieties and in fact is a peculiar one for this class of algebraic varieties among those varieties which are rationally connected.

1. Introduction

This article aims to relate unirationality of a given algebraic variety with the property of being a homogeneous space with respect to unipotent algebraic group action. More precisely, let X be an algebraic variety defined over a field $\mathbf{k}$, and $\mathrm{Aut}(X)$ be the group of regular automorphisms of X. We assume for clarity that $\mathrm{Aut}(X)$ carries the structure of an (infinite dimensional or pro-) algebraic group in the sense of e.g. [17], [18]. Let also $\mathrm{SAut}(X) \subseteq \mathrm{Aut}(X)$ be the (closed normal) subgroup generated by algebraic groups isomorphic to the additive group $\mathbb{G}_a$.

Definition 1.1 (cf. [1])**.** We call X *infinitely transitive* if for any $k \in \mathbb{N}$ and any two collections of points $\{P_1, \ldots, P_k\}$ and $\{Q_1, \ldots, Q_k\}$ on X there exists an element $g \in \mathrm{SAut}(X)$ such that $g(P_i) = Q_i$ for all i. Similarly, we call X *stably infinitely transitive* if the variety $X \times \mathbf{k}^m$ is infinitely transitive for some m.

Recall that in Birational Geometry adding a number m of algebraically independent variables to the function field $\mathbf{k}(X)$ is referred to as *stabilization*. Geometrically this precisely corresponds to taking the product $X \times \mathbf{k}^m$ with the affine space. Note also that if X is infinitely transitive, then it is unirational, i.e., $\mathbf{k}(X) \subseteq \mathbf{k}(y_1, ...y_m)$ for some $\mathbf{k}(X)$-transcendental elements y_i (see [1, Proposition 5.1]). This suggests to regard (stable) infinite transitivity as a birational property of X (in particular, we will usually assume the test variety X to be smooth and projective):

Definition 1.2. We call X *stably b-infinitely transitive* if the field $\mathbf{k}(X)(y_1, ...y_m)$ admits an infinitely transitive model (not necessarily smooth or projective) for some m and $\mathbf{k}(X)$-transcendental elements y_i. If $m = 0$, we call X *b-infinitely transitive.*

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Example 1.3. The affine space $X := \mathbf{k}^{\dim X}$ is stably infinitely transitive (and infinitely transitive when $\dim X \geq 2$). More generally, any rational variety is stably b-infinitely transitive, and it is b-infinitely transitive if the dimension ≥ 2.

Example 1.3 suggests that being stably b-infinitely transitive does not give anything interesting for rational varieties. In the present article, we put forward the following:

Conjecture 1.4. *Any unirational variety X is stably b-infinitely transitive.*

Thus, Conjecture 1.4 together with the above unirationality result from [1, Proposition 5.1] provides a (potential) characterization of unirational varieties among all those which are rationally connected. Note that the class of rationally connected varieties contains all stably b-infinitely transitive varieties. At the same time, not every rationally connected variety is stably b-infinitely transitive, as one may expect from the case of (Fano) hypersurfaces of degree d in $\mathbb{P}^{n+1}$, $d > \frac{2}{3}(n+3)$ (see [11]).

Remark 1.5. Originally, the study of infinitely transitive varieties was initiated in the paper [9]. We also remark one application of these varieties to the Lüroth problem in [1], where a non-rational infinitely transitive variety was constructed. See [5], [4], [6], [10], [14], [15] and [16] for other results, properties and applications of infinitely transitive (and related) varieties.

We are going to verify Conjecture 1.4 for some particular cases of X (see Theorems 2.2, 2.3 and Propositions **??**, 3.6, 3.8 and 3.10 below). At this stage, one should note that it is not possible to lose the stabilization assumption in Conjecture 1.4:

Example 1.6. Any three-dimensional algebraic variety X with an infinitely transitive model is rational. Indeed, let us take a one-dimensional algebraic subgroup $G \subset \mathrm{SAut}(X)$ acting on X with a free orbit. Then X is birationally isomorphic to $G \times Y$ (see Remark 2.11 below), where Y is a rational surface (since X is unirational). On the other hand, if $X := X_3 \subset \mathbb{P}^4$ is a smooth cubic hypersurface, then it is unirational but not rational (see [3]). However, Conjecture 1.4 is true as stated for X_3, because X_3 is stably b-infinitely transitive (see Proposition 3.6 below). In this context, it would be also interesting to settle down the case of the quartic hypersurface X_4 in $\mathbb{P}^4$ (or, more generally, in $\mathbb{P}^n$ for arbitrary n), which relates our subject to the old problem of (non-)unirationality of (generic) X_4 (cf. **3.10** and Remark 3.11 below).

Notation and conventions. Throughout the paper we keep up with the following:

- $\mathbf{k}$ is an algebraically closed field of characteristic zero and $\mathbf{k}^*$ is the multiplicative group of $\mathbf{k}$;
- $X_1 \approx X_2$ denotes birational equivalence between two algebraic varieties X_1 and X_2;
- we abbreviate infinite transitivity (transitive, transitively, etc.) to inf. trans.

2. Varieties with many cancellations

2.1. The set-up. The goal of the present section is to prove the following:

Theorem 2.2. *Let $K := \boldsymbol{k}(X)$ for some (smooth projective) algebraic variety X of dimension n over $\boldsymbol{k}$. We assume there are n distinct presentations (we call them* cancellations *(of K or X)) $K = K'(x_i)$ for some K'-transcendental algebraically independent elements x_i. Then there exists an inf. trans. model of $K(y_1, \ldots, y_n)$ for some K-transcendental elements y_i.*

Let us put Theorem 2.2 into a geometric perspective. Namely, the presentation $K = K'(x_i)$ reads as there exists a (smooth projective) model of K, say X_i^n, with a surjective regular map $\pi_i : X_i^n \to Y_i^{n-1}$ and the general fiber $\simeq \mathbb{P}^1$ such that π_i admits a section over an open subset in Y_i^{n-1}. Moreover, by resolving indeterminacies, we may assume $X_i^n := X$ fixed for all i. Then, since K admits n cancellations, n vectors, each tangent to a fiber of some π_i, span the tangent space to X at the generic point.

Here is a geometric counterpart of Theorem 2.2:

Theorem 2.3. *Let X be a smooth projective variety of dimension n. Assume that there exist n flat morphisms $\pi_i : X \to Y_i$ satisfying the following:*

- **(1)** *Y_i is a (normal) projective variety such that π_i admits a section over an open subset in Y_i;*
- **(2)** *for general point $\zeta \in X$ and the fiber $F_i := \pi_i^{-1}(\pi_i(\zeta)) \simeq \mathbb{P}^1$, vector fields $T_{F_1,\zeta}, \ldots, T_{F_n,\zeta}$ span the tangent space $T_{X,\zeta}$.*

Then X is stably b-inf. trans.

Let X be as in Theorem 2.3. Fix an embedding $Y_i \subseteq \mathbb{P}^{N_i}$, $1 \le i \le n$, and consider the affine cone H_i^* minus the origin over Y_i. Note that $H_i^* \to Y_i$ is a principal toric bundle, and we define $L_i^* := \pi_i^*(H_i^*)$, a principal toric bundle over X. Put also $\mathfrak{T}_X := L_1^* \times_X \ldots \times_X L_n^*$. Note that $\mathfrak{T}_X \to X$ is a principal toric bundle which has a section (the diagonal) and all fibers isomorphic to $(\mathbf{k}^*)^n$. In particular, we have $\mathfrak{T}_X \approx X \times \mathbf{k}^n$, and Theorem 2.3 follows from

Proposition 2.4. *$\mathfrak{T}_X$ is stably b-inf. trans.*

2.5. Proof of Proposition 2.4: one-dimensional case. Let us illustrate the proof of Proposition 2.4 in the simplest case when $X = \mathbb{P}^1$. Consider a non-trivial principal toric bundle $\mathfrak{T}$ over X of rank ≥ 3. Then we have $\mathfrak{T} = \mathcal{O}(n_1)^* \times_{\mathbb{P}^1} \ldots \times_{\mathbb{P}^1} \mathcal{O}(n_r)^*$ for some ≥ 3 and $n_j \ne 0$ for at least one j.

Lemma 2.6. *The following holds:*

- *$\mathfrak{T} \simeq (\boldsymbol{k}^2 \setminus \{0\})/\mathbb{Z}_m \times T$ for some m and the torus $T = (\boldsymbol{k}^*)^{r-1}$;*
- *the natural projection $\pi : \mathfrak{T} \to T$ is $Aut(\mathfrak{T})$-equivariant and the $Aut(\mathfrak{T})$-action on T is transitive.*

Proof. Let $m\mathbb{Z} \subseteq \mathbb{Z} = \mathrm{Pic}(X)$ be the sublattice generated by $\mathcal{O}(n_i)$ for all i. Then we may assume that $m = n_1$ and $m|n_i$ for all i. Note that $C := \mathcal{O}_X(m)^*$ is the affine cone minus the origin over a rational normal curve of degree m. Let us lift all $\mathcal{O}(n_i)^*$ to C. Then we get

$$\mathfrak{T} \simeq C \times_C \mathcal{O}(n_2)^* \times_C \ldots \times_C \mathcal{O}(n_r)^* = (\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m \times T$$

because $C \simeq (\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m$ and $\mathcal{O}(n_i)^*$ are trivial over C for all i.

Further, the morphism $\pi : \mathfrak{T} \to T$ is given by invertible regular functions on $\mathfrak{T}$. Since there are no such (non-constant) functions on $(\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m$, π is $\mathrm{Aut}(\mathfrak{T})$-equivariant. The transitivity statement is evident. □

Let $\mathrm{Aut}_f(\mathfrak{T}) \subseteq \mathrm{Aut}(\mathfrak{T})$ be the subgroup which preserves the fibers of π from Lemma 2.6. Then $\mathrm{Aut}_f(\mathfrak{T})$ can be identified with the space of regular maps $T \to \mathrm{Aut}((\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m)$ (whose images contain the identity map $\mathrm{id} \in \mathrm{Aut}((\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m)$) and hence is a closed subgroup in $\mathrm{Aut}(\mathfrak{T})$. The same holds for the group

$$\mathrm{Aut}_f^u(\mathfrak{T}) := \mathrm{SAut}((\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m)(\mathbf{k}[T]) \subseteq \mathrm{Aut}_f(\mathfrak{T}) \cap \mathrm{SAut}(\mathfrak{T}).$$

Lemma 2.7. *In the notation of Lemma 2.6, the group $SAut(\mathfrak{T})$ acts fiberwise inf. trans. on $\pi : \mathfrak{T} \to T$. More precisely, let $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ be distinct points such that $\pi(P_i) = \pi(Q_i)$ for all i. Then there exists $g \in Aut_f^u(\mathfrak{T})$ such that $g(P_i) = Q_i$ for all i.*

Proof. The group $\mathrm{SAut}((\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m)$ acts inf. trans. on $(\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m$. Let us take $g_i \in \mathrm{SAut}((\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m)$ such that $g_i(P_i) = Q_i$ for all i.

Further, if $h : \mathbf{k}^N \to \mathrm{SAut}((\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m)$ is a regular map, then any algebraic subgroup $\mathbb{G}_a \subseteq \mathrm{SAut}((\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m)$ defines a regular map $\mathbf{k}^{N+1} \to \mathrm{SAut}((\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m)$ as follows:

$$(\mathbf{k}^N,\, g) \mapsto h(\mathbf{k}^N)g,\ \ g \in \mathbb{G}_a.$$

In particular, we may assume that the above $g_i \in h(\mathbf{k}^N)$ for all i and some N and h. Let us also pick up a regular map $p : T \to \mathbf{k}^N$ with $h \circ p(P_i) = g_i$ for all i. Then $h \circ f$ defines an element $g \in \mathrm{Aut}_f^u(\mathfrak{T})$ such that $g(P_i) = Q_i$ for all i. □

Lemmas 2.6, 2.7 provide $\mathfrak{T} = (\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m \times T$ with the inf. trans. model $(\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m \times \mathbf{k}^{r-1}$.

2.8. Proof of Proposition 2.4: general case. Let X be a smooth projective variety. We assume that there exists a flat morphism $\pi : X \to Y$ onto a (normal) projective variety Y such that the general fiber F of π is isomorphic to $\mathbb{P}^1$ and π admits a section over an open subset in Y. Let $L \in \mathrm{Pic}(X)$ be such that $L\big|_F = \mathcal{O}(m)$ for some $m \neq 0$. Take also a very ample line bundle $H \in \mathrm{Pic}(Y)$ and form a principal toric bundle $\hat{T} := L^* \times_X \pi^*(H^*)$ over X. Then we get the following relative version of Lemma 2.7:

Lemma 2.9. *For the natural projection $\pi_H : \hat{T} \to H^*$, the group $SAut(\hat{T})$ acts fiberwise inf. trans. on $\hat{T}$.*

Proof. Note that the general fiber of π_H equals $(\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m$ and H^* is quasi-affine (cf. the proof of Lemma 2.6). Also, the assumption on π implies that the fibration $\pi_H : \hat{T} \to H^*$ is birationally trivial over an open subset $U \subseteq H^*$. Then we get the exact situation of **2.5**, with T replaced by U, $\mathfrak{T}$ replaced by $(\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m \times U$, and $\pi : \mathfrak{T} \to T$ replaced by $\pi_H : (\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m \times U \to U$. The same argument as in the proof of Lemma 2.7 shows that the group $G := \mathrm{Aut}_f((\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m \times U)$ acts fiberwise inf. trans. on $(\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_m \times U$ (with respect to projection π_H). It remains to show that any $g \in \mathbb{G}_a \subseteq G$ extends to an element in $\mathrm{Aut}(\hat{T})$ (then it will be automatically from $\mathrm{SAut}(\hat{T})$).

Obviously, g induces a birational automorphism of $\hat{T}$, with possible poles contained in $S := H^* \setminus U$. However, since H^* is quasi-affine, there exists a regular function $f_S \in \mathcal{O}_{H^*}$ such that the vector field $f_S \partial/\partial t$ (for the vector field $\partial/\partial t$ generated by $\mathbb{G}_a \ni g$) has no poles on $\hat{T}$. Then, since $\partial/\partial t$ is locally nilpotent, we can choose f_S to be such that $\mathbb{G}_a = \exp(\mathbf{k}\partial/\partial t) \simeq \exp(f_S\mathbf{k}\partial/\partial t)$ (cf. [8, the proof of Lemma 8.2] for a similar argument). $\square$

Let $\mathfrak{T}_X$ be as in Proposition 2.4. Because of the condition **(1)** in Theorem 2.3, summing up the constructions in **2.5** and Lemma 2.9, we arrive at n (birationally trivial) projections $\pi_{H_i} : \mathfrak{T}_X \to H_i^*$, with generic fiber of π_{H_i} equal $(\mathbf{k}^2 \setminus \{0\})/\mathbb{Z}_{m_i} \times T_i$ and T_i being a torus, $\dim T_i = n-1$. Furthermore, for each i there are groups $\mathrm{Aut}^u_{f,i}(\mathfrak{T}_X) \subseteq \mathrm{SAut}(\mathfrak{T}_X)$, similar to $\mathrm{Aut}^u_f(\mathfrak{T})$ from **2.5**, acting fiberwise with respect to π_{H_i} ($\mathrm{Aut}^u_{f,i}(\mathfrak{T}_X)$-action is trivial on the T_i-factor of the general fiber of π_{H_i}). Let $\mathrm{Aut}_H(\mathfrak{T}_X) \subseteq \mathrm{SAut}(\mathfrak{T}_X)$ be the (closed) subgroup generated by $\mathrm{Aut}^u_{f,i}(\mathfrak{T}_X)$ for all i.

Now, pick up the point $\zeta \in X$ as in Theorem 2.3, identify ζ with a (generic) point on $\mathfrak{T}_X$, and put $O_\zeta := \mathrm{Aut}_H(\mathfrak{T}_X) \cdot \zeta$. Proposition 2.4 now follows from

Lemma 2.10. *The group $Aut_H(\mathfrak{T}_X)$ acts inf. trans. on O_ζ. Moreover, we have $\mathfrak{T}_X \approx O_\zeta \times T_s$, where T_s is a torus. In particular, $\mathfrak{T}_X$ is stably b-inf. trans.*

Proof. The group $\mathrm{Aut}_H(\mathfrak{T}_X)$ is irreducible as an (infinite dimensional) algebraic variety (see e.g. [18] for the general argument). Note also that for any $P \in O_\zeta$, those $g \in \mathrm{Aut}_H(\mathfrak{T}_X)$ which map P to a given open subset in $\mathfrak{T}_X$ form an open subgroup in $\mathrm{Aut}_H(\mathfrak{T}_X)$. In particular, every π_{H_i} determines an open subgroup $U_{i,P} \subseteq \mathrm{Aut}_H(\mathfrak{T}_X)$ which maps P to a generic fiber of π_{H_i}.

Let $\{P_1, \ldots, P_k\}$, $\{Q_1, \ldots, Q_k\}$ be two finite subsets in O_ζ. For $P := P_j$, Q_j in the preceding discussion, we obtain an open locus $\bigcap_j U \cdot P_j = \bigcap_j U \cdot Q_j$ in O_ζ, where $U := \bigcap_{i,j}(U_{i,P_j} \cap U_{i,Q_j})$ is open and non-empty due to the irreducibility of $\mathrm{Aut}_H(\mathfrak{T}_X)$. Let $z_1, \ldots, z_N$ be local analytic coordinates on U. We have $\partial/\partial z_i \in T_{\mathfrak{T}_X,\zeta}$ for all i due to the condition **(2)** in Theorem 2.3. We also have $\exp(\mathbf{k}\partial/\partial z_i) \subset \mathrm{Aut}_H(\mathfrak{T}_X)$ for all i (cf. the proof of Lemma 2.9). Then, similar to the proof of Lemma 2.7, we find $g := g_{i_1} \ldots g_{i_n}$ for some $g_{i_j} \in \mathrm{Aut}_{f,i_j}(\mathfrak{T}_X)$, such that $g(P_j) = Q_j$ for all j. Hence $\mathrm{Aut}_H(\mathfrak{T}_X)$ acts inf. trans. on O_ζ.

Further, complete $\partial/\partial z_i$ to a basis on $T_{\mathfrak{T}_X,\zeta}$, $\{\partial/\partial z_1, \ldots, \partial/\partial z_N, \partial/\partial z_{N+1}, \ldots, \partial/\partial z_{2n}\}$. Vectors $\partial/\partial z_i$, $i \geq N+1$, span the tangent space $T_{T_s,\zeta}$, where $T_s \ni \zeta$ is a torus whose descend to X determines a toric bundle, trivial on each fiber F_i (see Theorem 2.3, **(2)**). In particular, $T_s \subset T_i$ for all i, and T_s is transversal to O_ζ. The latter implies that $\mathfrak{T}_X \approx O_\zeta \times T_s$. $\square$

Remark 2.11. Conversely, in view of Theorem 2.3, given a b-inf. trans. variety X there exist $\dim X$ cancellations of X. Indeed, for general point $\zeta \in X$ we can find $\dim X$ tangent vectors spanning $T_{X,\zeta}$, such that each vector generates a copy of $\mathbb{G}_a =: G_i \subseteq \mathrm{SAut}(X)$, $1 \leq i \leq n$ (cf. the proof of Lemma 2.10). Let $\mathfrak{G} \subseteq \mathrm{SAut}(X)$ be the subgroup generated by the groups $G_2, \ldots, G_n$. Then we have $X \approx G_1 \times \mathfrak{G} \cdot \zeta$.

3. Examples

Here we collect several examples and properties of (stably) b-inf. trans. varieties.

3.1. Quotients. Let us start with the projective space $\mathbb{P}^n$, $n \geq 2$, and a finite group $G \subset PGL_{n+1}(\mathbf{k})$. Notice that the quotient $\mathbb{P}^n/G$ is stably b-inf. trans. Indeed, let us replace G by its central extension $\tilde{G}$ acting linearly on $V := \mathbf{k}^{n+1}$, so that $V/\tilde{G} \approx \mathbb{P}^n/G \times \mathbb{P}^1$. Further, form the product $V \times V$ with the diagonal $\tilde{G}$-action, and take the quotient $V' := (V \times V)/\tilde{G}$. Then, projecting on the first factor we get $V' \approx V \times V/\tilde{G}$, and similarly for the second factor. This implies that V' admits $2n+2$ cancellations (cf. Theorem 2.2). Hence V' is stably b-inf. trans. by Theorem 2.3. The argument just used can be summarized as follows:

Lemma 3.2. *Let $X \to S$ be a $\mathbb{P}^m$-fibration for some $m \in \mathbb{N}$. Then the product $X \times_S X \approx X \times \boldsymbol{k}^m$ admits $2m$ algebraically independent cancellations over S.*

Proof. Note that $X \times_S X$ has two projections (left and right) onto X, both having a section (the diagonal $\Delta_X \subset X \times_S X$), hence the corresponding $\mathbb{P}^m$-fibrations are birational (over S) to $X \times \mathbf{k}^m$. This gives $2m$ algebraically independent cancellations over S. $\square$

Corollary 3.3. *Assume that X carries a collection of distinct birational structures of $\mathbb{P}^{m_i}$-bundles, $\pi_i : X \to S_i$, with the condition that the tangent spaces of generic fibers of π_i span the tangent space of X at the generic point. Then X is stably b-inf. trans.*

Proof. Indeed, after multiplying by the maximum of m_i we may assume that all $\mathbb{P}^{m_i}$-bundles provide with at least $2m_i$ different cancellations (see Lemma 3.2). We can now apply Theorem 2.3. $\square$

Remark 3.4. It seems plausible that given an inf. trans. variety X and a finite group $G \subset \mathrm{Aut}(X)$, variety X/G is stably b-inf. trans. (though the proof of this fact requires a finer understanding of the group $\mathrm{SAut}(X)$). At this stage, note also that if G is cyclic, then there exists a G-fixed point on X. Indeed, since X is unirational (cf. Section 1), it has trivial algebraic fundamental group $\pi_1^{\mathrm{alg}}(X)$ (see [13]). Then, if the G-action is free on X, we get $G \subset \pi_1^{\mathrm{alg}}(X/G) = \{1\}$ for X/G smooth unirational, a contradiction. This fixed-point-non-freeness property of X relates X to homogeneous spaces, and it would be interesting to investigate whether this is indeed the fact, i.e., in particular, does X, after stabilization and passing to birational model, admit a uniformization which is a genuine (finite dimensional) algebraic group?[1)]

3.5. Cubic hypersurfaces. Let $X_3 \subset \mathbb{P}^{n+1}$, $n \geq 2$, be a smooth cubic. Then

Proposition 3.6. *X_3 is stably b-inf. trans.*

Proof. Let $L \subset X_3$ be a line and $\pi : X_3 \dashrightarrow \mathbb{P}^{n-1}$ the linear projection from L. Let us resolve the indeterminacies of π by blowing up X_3 at L. We arrive at a smooth variety X_L together with a morphism $\pi_L : X_L \to \mathbb{P}^{n-1}$ whose general fiber is $\mathbb{P}^1$ ($\simeq$ a conic in $\mathbb{P}^2$). Varying $L \subset X_3$, we then apply Lemma 3.2 and Corollary 3.3 to get that X_3 is stably b-inf. trans. $\square$

[1)]This question was suggested by J.-L. Colliot-Thélène in connection with Conjecture 1.4. However, there are reasons to doubt the positive answer, since, for example, it would imply that X is (stably) birationally isomorphic to G/H, where both G, H are (finite dimensional) reductive algebraic groups. Even more, up to stable birational equivalence we may assume that $X = G'/H'$, where H' is a finite group and G' is the product of a general linear group, Spin groups and exceptional Lie groups. The latter implies, among other things, that there are only countably many stable birational equivalence classes of unirational varieties, but we could not develop a rigorous argument to bring this to contradiction.

3.7. Quartic hypersurfaces. Let $X_4 \subset \mathbb{P}^n$, $n \geq 4$, be a quartic hypersurface with a line $L \subset X_4$ of double singularities. Then

Proposition 3.8. *X_4 is stably b-inf. trans.*

Proof. Consider the cone $\mathfrak{X}_4 \subset \mathbb{P}^{n+1}$ over X_4. Then $\mathfrak{X}_4$ contains a plane Π of double singularities. Pick up a (generic) line $L' \subset \Pi$ and consider the linear projection $\mathfrak{X}_4 \dashrightarrow \mathbb{P}^{n-1}$ from L'. This induces a conic bundle structure on $\mathfrak{X}_4$, similarly as in the proof of Proposition 3.6, and varying L' in Π as above we obtain that $\mathfrak{X}_4$ is stably b-inf. trans. Then, since $\mathfrak{X}_4 \approx X_4 \times \mathbf{k}$, Proposition 3.8 follows. $\square$

3.9. Complete intersections. Let $X_{2\cdot 2\cdot 2} \subset \mathbb{P}^6$ be the complete intersection of three quadrics. Then

Proposition 3.10. *$X_{2\cdot 2\cdot 2}$ is stably b-inf. trans.*

Proof. Let $L \subset X_{2\cdot 2\cdot 2}$ be a line and $X_L \to X_{2\cdot 2\cdot 2}$ the blowup of L. Then the threefold X_L carries the structure of a conic bundle (see [7, Ch. 10, Example 10.1.2, (ii)]). Now, varying L and applying the same arguments as in the proof of Proposition 3.6, we obtain that $X_{2\cdot 2\cdot 2}$ is stably b-inf. trans. $\square$

Remark 3.11. Fix $n, r \in \mathbb{N}$, $n \gg r$, and a sequence of integers $0 < d_1 \leq \ldots \leq d_m$, $m \geq 2$. Let us assume that $(n-r)(r+1) \geq \sum_{i=1}^{m} \binom{d_i + r}{r}$. Consider the complete intersection $X := H_1 \cap \ldots \cap H_m$ of hypersurfaces $H_i \subset \mathbb{P}^n$ of degree d_i. Then it follows from the arguments in [12] that X contains a positive dimensional family of linear subspaces $\simeq \mathbb{P}^r$. Moreover, X is unirational, provided X is generic. It would be interesting to adopt the arguments from the proofs of Propositions 3.6, 3.8 and 3.10 to this more general setting in order to prove that X is stably b-inf. trans.

Remark 3.12. Propositions 3.6, 3.8 and 3.10 (cf. Remark 3.11) provide an alternative method of proving unirationality of complete intersections (see [7, Ch. 10] for recollection of classical arguments). Note also that (generic) $X_{2\cdot 2\cdot 2}$ is non-rational (see for example [19]), and (non-)rationality of the most of other complete intersections considered above is not known. At the same time, verifying stable b-inf. trans. property of other (non-rational) Fano manifolds (cf. [7, Ch. 10, Examples 10.1.3, (ii), (iii), (iv)]) is out of reach for our techniques at the moment.

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