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CURVES OF LOW DEGREES ON PROJECTIVE VARIETIES

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We work over the field of complex numbers. For $X \subset \mathbf{P}^N$ a smooth projective variety, we let $\mathscr{C}_d^g(X) \subset \mathrm{Hilb}(X)$ be the (quasi-projective) moduli space of smooth, genus-g, degree-d curves on X.

We want to explain how theses spaces can be used, in very specific cases, to study the geometry of X. One of the first striking example of this was the proof by Clemens and Griffiths of the non-rationality of smooth cubic 3-folds using the (smooth projective) surface parametrizing the lines that it contains.

We will start with this particular example, then move on to cubic 4-folds, then to Fano varieties of degree 10.

The general philosophy is that when the degree of a hypersurface X is very small with respect to its dimension, spaces of rational curves on X tend to become more and more "rational"; it is known that they have a rationally connected compactification, for example ([HS]). Going in another direction, I will describe how these spaces can help understand the geometry of the variety X in a few particular cases: cubic hypersurfaces of dimension 3 or 4, and Fano varieties of degree 10.

1. Curves on cubic 3-folds

Recall that the *maximal rationally connected fibration* (mrc fibration for short) of a smooth (complex) variety X is a rational dominant map $\rho : X \dashrightarrow R(X)$ such that for $z \in R(X)$ very general, any rational curve in X that meets $\rho^{-1}(z)$ is contained in $\rho^{-1}(z)$. The general fibers are proper and rationally connected, and the fibration ρ is unique up to birational equivalence.

Let $X \subset \mathbf{P}^4$ be a general (although some results are known for *any smooth* X) cubic hypersurface. The intermediate Jacobian $J(X) := H^{2,1}(X)^\vee / H_3(X, \mathbf{Z})$ is a 5-dimensional principally polarized abelian variety. We have:

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- for $g=0$ or $d\le 5$, $\mathscr{C}^g_d(X)$ is integral of dimension $2d$ ([HRS3], Theorem 1.1);
- for $d\le 5$, $\mathrm{aj}:\mathscr{C}^g_d(X)\to J(X)$ is the mrc fibration ([HRS2], Theorem 1.1);
- for $d\ge 4$, $\mathrm{aj}:\mathscr{C}^0_d(X)\to J(X)$ is dominant with irreducible general fibers.

It is natural to ask whether $\mathrm{aj}:\mathscr{C}^0_d(X)\to J(X)$ is the mrc fibration for all d (i.e., are the fibers rationally connected?).

1.1. **Lines.**

- $\mathscr{C}^0_1(X)$ is a smooth projective irreducible surface of general type;
- the image of $\mathrm{aj}:\mathscr{C}^0_1(X)\to J(X)$ is a surface S with minimal class $\theta^3/3!$ and $S-S$ is a theta divisor;
- aj induces an isomorphism $\mathrm{Alb}(\mathscr{C}^0_1(X))\simeq J(X)$ ([CG]).

The second item yields a proof of Torelli: *the period map $\mathscr{X}^3_3\to\mathscr{A}_5$ is injective* (the dimension of $\mathscr{X}^3_3$ is 10).

1.2. **Conics.**

- $\overline{\mathscr{C}}^0_2(X)$ is a smooth projective irreducible fourfold;
- the image of $\mathrm{aj}:\overline{\mathscr{C}}^0_2(X)\to J(X)$ is a $\mathbf{P}^2$-bundle over S (a conic is uniquely determined by a line in X and a 2-plane containing the line).

1.3. **Plane cubics.**

- $\overline{\mathscr{C}}^1_3(X)$ is isomorphic to $G(2,\mathbf{P}^4)$;
- the Abel-Jacobi map is constant.

1.4. **Twisted cubics.**

- $\mathrm{aj}:\mathscr{C}^0_3(X)\to J(X)$ is birational to a $\mathbf{P}^2$-bundle over a theta divisor ([HRS2], §4).

1.5. **Elliptic quartics.**

- $\mathrm{aj}:\mathscr{C}^1_4(X)\to J(X)$ is birational to a $\mathbf{P}^6$-bundle over S ([HRS2], §4.1).

1.6. **Normal rational quartics.**

- $\mathrm{aj}:\mathscr{C}^0_4(X)\to J(X)$ is dominant and the general fiber is birational to X ([IMa], Theorem 5.2), hence unirational.

1.7. **Normal elliptic quintics.**

- $\mathscr{C}_5^1(X)$ is an irreducible 10-fold;
- there is a factorization ([MT1], Theorem 5.6; [IMa], Theorem 3.2)
$$\mathrm{aj} : \overline{\mathscr{C}}_5^1(X) \xrightarrow{\alpha} M(2;0,2) \xrightarrow{\beta} J(X),$$
where
  - $M(2;0,2)$ is some component of the moduli space of rank-2 stable vector bundles on X with $c_1 = 0$ and $c_2 = 2$;
  - α is a $\mathbf{P}^5$-bundle over an open subset of $M(2;0,2)$;
  - β is birational (proved in [IMa] via ingenious geometrical constructions).

The map α is obtained via the Serre construction: to $C \in \mathscr{C}_5^1(X)$, one can associate by the Serre construction a stable rank-2 vector bundle E_C on X with Chern classes $c_1 = 0$ and $c_2 = 2$ such that C is the zero-locus of a section of $E_C(1)$.

The fibers of α are $\mathbf{P}(H^0(X, E_C(1))) \simeq \mathbf{P}^5$ hence the Abel-Jacobi map factors through α.

According to Murre, the Chow group of algebraic 1-cycles of fixed degree on X_3 modulo rational equivalence is canonically isomorphic to $J(X)$. The map β can then be defined directly as $E \mapsto c_2(E)$.

1.8. **Normal elliptic sextics.**

- $\mathscr{C}_6^1(X)$ is an irreducible 12-fold;
- $\mathrm{aj} : \mathscr{C}_6^1(X) \to J(X)$ is dominant and the general fiber is rationally connected ([V], Theorem 2.1).

1.9. **Fano 3-folds X_{14} of degree 14 and index 1.** There is a very interesting relationship with Fano 3-folds X_{14} of degree 14 and index 1 ([MT1], [IMa], [K]). They are obtained as linear sections of $G(2,6) \subset \mathbf{P}^{14}$ by a $\mathbf{P}^9$.

Let $C \in \mathscr{C}_5^1(X_3)$ and let $\pi : \widetilde{X} \to X$ be its blow-up, with exceptional divisor E. We have
$$-K_{\widetilde{X}_3} \underset{\mathrm{lin}}{\equiv} -\pi^* K_{X_3} - E \underset{\mathrm{lin}}{\equiv} 2\pi^* H - E.$$
This linear system induces a morphism $\widetilde{X} \to \mathbf{P}^4$ which is a small contraction φ onto (the normalization of) its image. Its non-trivial fibers are the strict transforms of the 25 lines bisecant to C: the divisor E is φ-ample hence there is a flop
$$\chi : \widetilde{X}_3 \xrightarrow{\varphi} \bar{X}_3 \xleftarrow{\varphi'} \widetilde{X}_3',$$
where $\widetilde{X}_3'$ is smooth projective and $\chi(E)$ is φ'-antiample.

We have $\rho(\widetilde{X}'_3) = 2$. Since the extremal ray generated by the class of curves contracted by φ' has $K_{\widetilde{X}'_3}$-degree 0 and $K_{\widetilde{X}'}$ is not nef ($-K_{\widetilde{X}'_3} = \varphi'^*\bar{H}$), the other extremal ray is $K_{\widetilde{X}'_3}$-negative and defines a contraction $\pi' : \widetilde{X}'_3 \to X'_3$ and one checks:

- X'_3 is a smooth Fano threefold X_{14} of index 1, with Picard group generated by $H' := -K_{X'_3}$;
- π' is the blow-up of an elliptic quintic curve $C' \subset X'_3$, with exceptional divisor $E' \equiv 5\varphi'^*\bar{H} - 3\chi_*(E)$ and $\chi^*\pi'^*H' \underset{\text{lin}}{\equiv} 7H - 4E$.

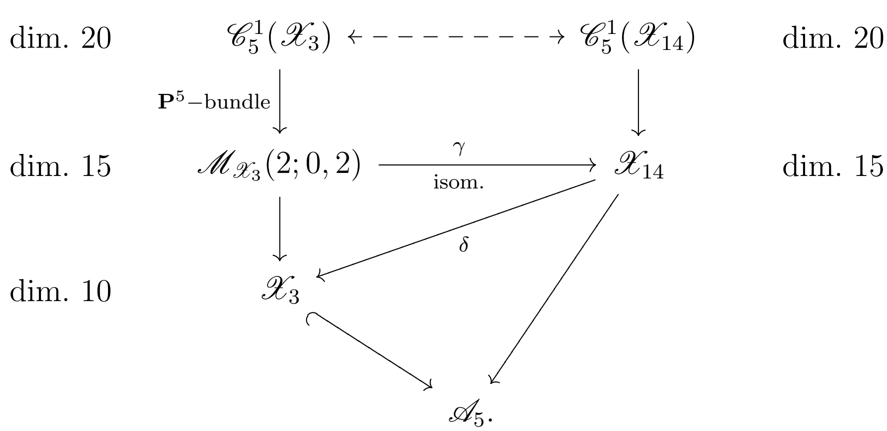
Conversely, given an elliptic quintic curve C' in a X_{14}, one can construct a quintic C in an X_3. In other words, we have a birational isomorphism

$$\mathscr{C}^1_5(\mathscr{X}_3) \dashrightarrow \mathscr{C}^1_5(\mathscr{X}_{14})$$

between 20-dimensional varieties. We have:

- The intermediate Jacobians of X_3 and X_{14} are isomorphic (because $J(X_3) \times J(C) \simeq J(X_{14}) \times J(C')$, $J(X_3)$ is not a product, and $J(X_3)$ and $J(X_{14})$ have same dimension). Since we have Torelli for X_3, the X_3 obtained from $C' \subset X_{14}$ is uniquely determined.
- The variety X_{14} only depends on E_C (Kuznetsov proves in [K] that $\mathbf{P}(\mathscr{S}|_{X_{14}})$ is obtained by a flop of $\mathbf{P}(E_C)$).

So we get a commutative diagram



The map γ is actually an isomorphism of stacks and the fiber of δ (between stacks) at $[X_3]$ is the (quasi-projective) moduli space of locally free rank-2 stable E such that $c_1(E) = 0$ and $H^1(X_3, E(-1)) = 0$ (instanton bundles), an open subset of $J(X_3)$ ([K], Theorem 2.9).

2. Curves on cubic 4-folds

Let $Y \subset \mathbf{P}^5$ be a general cubic hypersurface. Since we are in even dimension, there is no intermediate Jacobian.

The variety $\mathscr{C}_d^0(Y)$ is integral of dimension $3d+1$ ([HRS3], Theorem 1.1 again). To study these varieties, one could use, instead of the Abel-Jacobi map, their mrc fibration ([dJS]). Here is what is known, or conjectured:

- $\mathscr{C}_1^0(Y)$ is a symplectic 4-fold ([BD]) hence is its own reduction;
- the map $\mathscr{C}_2^0(Y) \dashrightarrow \mathscr{C}_1^0(Y)$ given by residuation is a $\mathbf{P}^3$-bundle so this is the mrc fibration;
- $\mathscr{C}_3^0(Y)$ is uniruled (a general cubic curve lies on a unique cubic surface and moves in a 2-dimensional linear system on it);
- there is a map $\mathscr{C}_4^0(Y) \dashrightarrow \mathscr{J}(Y)$ (where $\mathscr{J}(Y) \to (\mathbf{P}^5)^\vee$ is the relative intermediate Jacobian of smooth hyperplane sections of Y) whose general fibers are these cubics (§1.6) hence are unirational; moreover, $\mathscr{J}(Y)$ should be non-uniruled (see also Example 2.2 below), so this should be the mrc fibration;
- for $d \geq 5$ odd, there is ([dJS], Theorem 1.2; [KM]) a holomorphic 2-form on (a smooth nonsingular model of) $\mathscr{C}_d^0(Y)$ which is *non-degenerate at a general point.* In particular, $\mathscr{C}_d^0(Y)$ is not uniruled.

2.1. Constructing symplectic forms on moduli spaces.

Mukai proved in 1984 that any moduli space of simple sheaves on a K3 or abelian surface has a closed non-degenerate holomorphic 2-form : the tangent space to the moduli space $\mathscr{M}$ at a point $[\mathscr{F}]$ representing a simple sheaf $\mathscr{F}$ on a smooth projective variety Y is isomorphic to $\mathrm{Ext}^1(\mathscr{F},\mathscr{F})$. The Yoneda coupling

$$\mathrm{Ext}^1(\mathscr{F},\mathscr{F}) \times \mathrm{Ext}^1(\mathscr{F},\mathscr{F}) \longrightarrow \mathrm{Ext}^2(\mathscr{F},\mathscr{F})$$

is skew-symmetric whenever $[\mathscr{F}]$ is a *smooth* point of the moduli space. When Y is a symplectic surface S with a symplectic holomorphic form $\omega \in H^0(S,\Omega_S^2)$, Mukai composes the Yoneda coupling with the map

$$\mathrm{Ext}^2(\mathscr{F},\mathscr{F}) \xrightarrow{\mathrm{Tr}} H^2(S,\mathscr{O}_S) \xrightarrow{\cup\,\omega} H^2(S,\Omega_S^2) = \mathbf{C},$$

and this defines the symplectic structure on $\mathscr{M}_{\mathrm{sm}}$.

Over an n-dimensional variety Y such that $h^{q,q+2}(Y) \neq 0$, we

- pick a nonzero element $\omega \in H^{n-q-2}(Y,\Omega_Y^{n-q})$;

- use the exterior power $\mathrm{At}(\mathscr{F})^{\wedge q} \in \mathrm{Ext}^q(\mathscr{F}, \mathscr{F} \otimes \Omega_Y^q)$ of the Atiyah class[1] $\mathrm{At}(\mathscr{F}) \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{F} \otimes \Omega_Y^1)$;

to define

$$\mathrm{Ext}^2(\mathscr{F}, \mathscr{F}) \xrightarrow{\mathrm{At}(\mathscr{F})^{\wedge q}\circ\bullet} \mathrm{Ext}^{q+2}(\mathscr{F}, \mathscr{F} \otimes \Omega_Y^q) \xrightarrow{\mathrm{Tr}} H^{q+2}(Y, \Omega_Y^q) \xrightarrow{\cup\omega} H^n(Y, \Omega_Y^n) \simeq \mathbf{C}.$$

Composing the Yoneda coupling with this map provides a closed (possibly degenerate) 2-form α_ω on the moduli space $\mathscr{M}$.

Example 2.1. Since $h^{1,3}(Y) = 1$, the construction provides a (unique) 2-form on every moduli space of sheaves on Y. Kuznetsov & Markouchevitch use this construction in a round-about way to produce a symplectic structure on the (smooth) fourfold $\mathscr{C}_1^0(Y)$ of lines $L \subset Y$ (originally discovered by Beauville and Donagi by a deformation argument; the simple-minded idea to look at sheaves of the form $\mathscr{O}_L$ does not work).

Example 2.2. Let $\mathscr{N}_Y$ be the (quasi-projective) moduli space of sheaves on Y of the form i_*E, where $i : X \to Y$ is a non-singular hyperplane section of Y and $[E] \in M_X(2; 0, 2)$. By §1.7, $\mathscr{N}_Y$ is a torsor under the (symplectic) relative intermediate Jacobian $\mathscr{J}_Y$ of smooth hyperplane sections of Y. The Donagi-Markman symplectic structure ([DM], 8.5.2) on $\mathscr{J}_Y$ induces a symplectic structure on $\mathscr{N}_Y$ which should be the same as the Kuznetsov-Markouchevitch structure ([MT2]; [KM], Theorem 7.3 and Remark 7.5). Note that since we do not know whether the Donagi-Markman 2-form on $\mathscr{J}_Y$ extends to a smooth compactification, we cannot deduce that $\mathscr{J}_Y$ is not uniruled.

Example 2.3. Let X_{10}^4 be the smooth Fano fourfold obtained by intersecting $G(2, V_5)$ in its Plücker embedding by a general hyperplane and a general quadric. Then $h^{3,1}(X_{10}^4) = 1$. The Hilbert scheme $\overline{\mathscr{C}}_2^0(X_{10}^4)$ of (possibly degenerate) conics in X_{10}^4 is smooth, hence it is endowed, by the construction above, with a canonical global holomorphic 2-form. Since $\mathscr{C}_2^0(X_{10}^4)$ has dimension five, this form must be degenerate.

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[1] Let $\Delta : Y \to Y \times Y$ be the diagonal embedding and let $\Delta(Y)^{(2)} \subset Y \times Y$ be the closed subscheme defined by the sheaf of ideals $\mathscr{I}_{\Delta(Y)}^2$. Since $\mathscr{I}_{\Delta(Y)}/\mathscr{I}_{\Delta(Y)}^2 \sim \Omega_Y$, we have an exact sequence

$$0 \to \Delta_*\Omega_Y \to \mathscr{O}_{\Delta(Y)^{(2)}} \to \Delta_*\mathscr{O}_Y \to 0.$$

If $\mathscr{F}$ is a locally free sheaf on Y, we obtained an exact sequence

$$0 \to \mathscr{F} \otimes \Omega_Y \to p_{1*}(p_2^*(\mathscr{F} \otimes \mathscr{O}_{\Delta(Y)^{(2)}})) \to \mathscr{F} \to 0.$$

hence an extension class $\mathrm{At}_{\mathscr{F}} \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{F} \otimes \Omega_Y)$. The same construction can be extended to any coherent sheaf on Y by working in the derived category (Illusie).

There is a naturally defined rational map

$$\overline{\mathscr{C}}_2^0(X_{10}^4) \dashrightarrow \mathbf{P}(H^0(I_{X_{10}^4}(2)) \simeq \mathbf{P}^5$$

whose image a *sextic EPW hypersurface* $Z_{X_{10}^4} \subset \mathbf{P}^5$ (see [O1] for the definition of Eisenbud-Popescu-Walter (EPW for short) sextics). In the Stein factorization

$$\mathscr{C}_2^0(X_{10}^4) \dashrightarrow Y_{X_{10}^4} \to Z_{X_{10}^4},$$

the projective variety $Y_{X_{10}^4}$ is a smooth fourfold over which $\mathscr{C}_2^0(X_{10}^4)$ is (essentially) a smooth fibration in projective lines. Thus the 2-form on $\mathscr{C}_2^0(X_{10}^4)$ thus descends to $Y_{X_{10}^4}$ and this makes $Y_{X_{10}^4}$ into a holomorphic symplectic fourfold ([IM2]; originally [O1]) called a *double EPW sextic.*

2.2. **Periods for cubic 4-folds.** Since Y has even dimension, it has no intermediate Jacobian, but an interesting Hodge structure:

$$\begin{array}{ccccccc} H^4(Y,\mathbf{C})_{\text{prim}} & = & H^{1,3}(Y) & \oplus & H^{2,2}(Y)_{\text{prim}} & \oplus & H^{3,1}(Y) \\ \text{dimensions:} & & 1 & & 20 & & 1 \end{array}$$

with period domain

$$\mathscr{D}^{20} = \{[\omega] \in \mathbf{P}^{21} \mid Q(\omega,\omega) = 0,\ Q(\omega,\bar{\omega}) > 0\},$$

a bounded symmetric domain of type IV. We get a period map

$$\mathscr{X}_3^4 \to \mathscr{D}^{20}/\Gamma$$

where the discrete arithmetic group Γ can be explicitely described. Voisin proved that it is injective and its (open) image was explicitly described in [L].

2.3. **Periods for cubic 3-folds.** We can construct another period map for cubic threefolds: to such a cubic $X \subset \mathbf{P}^4$, associate the cyclic triple cover $Y_X \to \mathbf{P}^4$ branched along X. It is a cubic fourfold, hence this construction defines a map $\mathscr{X}_3^3 \to \mathscr{D}^{20}/\Gamma$.

However, because of the presence of an automorphism of Y_X of order 3, we can restrict the image and define a period map (Allcock-Carlson-Toledo)

$$\mathscr{X}_3^3 \to \mathscr{D}^{10}/\Gamma',$$

where

$$\mathscr{D}^{10} := \{\omega \in \mathbf{P}^{10} \mid Q(\omega,\bar{\omega}) < 0\} \simeq \mathbf{B}^{10}$$

and the discrete arithmetic group Γ' can be explicitely described. Again, it is an isomorphism onto an explictily described open subset of $\mathscr{D}^{10}/\Gamma'$.

3. Fano varieties of degree 10

We define, for $k \in \{3, 4, 5\}$, a degree-10 Fano k-fold by

$$X_{10}^k := G(2, V_5) \cap \mathbf{P}^{k+4} \cap \Omega \subset \mathbf{P}(\wedge^2 V_5),$$

where $\mathbf{P}^{k+4}$ is a general $(k+4)$-plane and Ω a general quadric. Let $\mathscr{X}_{10}^k$ be the moduli stack for smooth varieties of type X_{10}^k.

Enriques proved that all (smooth) X_{10}^3, X_{10}^4, and X_{10}^5 are unirational. A general X_{10}^3 is not rational, whereas all (smooth) X_{10}^5 are rational (Semple, 1930). Some smooth X_{10}^4 are rational (Prokhorov), but the rationality of a general X_{10}^4 is an open question.

We have

$$I_{X_{10}^k}(2) \simeq \mathbf{C}\Omega \oplus V_5,$$

where V_5 corresponds to the rank-6 *Plücker quadrics* $\Omega_v : \omega \mapsto \omega \wedge \omega \wedge v$. In the 5-plane $\mathbf{P}(I_{X_{10}^k}(2))$, the degree-$(k+5)$ hypersurface corresponding to singular quadrics decomposes as

$$(k-1)\mathbf{P}(V_5) + Z_{X_{10}^k}^\vee,$$

where $Z_{X_{10}^k}^\vee$ is an *EPW sextic* ([IM2], §2.2; this is indeed the (projective) dual of the sextic defined in Example 2.3 when $k = 4$).

If $\mathscr{EPW}$ is the 20-dimensional moduli space of EPW sextics, we get morphisms

$$\mathrm{epw}^k : \mathscr{X}_{10}^k \to \mathscr{EPW}$$

which are *dominant* ([IM2], Corollary 4.17). Note that

$$\begin{aligned} \dim(\mathscr{X}_{10}^5) &= 25, \\ \dim(\mathscr{X}_{10}^4) &= 24, \\ \dim(\mathscr{X}_{10}^3) &= 22. \end{aligned}$$

Proposition 3.1 (Debarre-Iliev-Manivel)**.** *For a general EPW sextic $Z \subset \mathbf{P}(V_5)$, with dual $Z^\vee \subset \mathbf{P}(V_5^\vee)$, the fiber* $(\mathrm{epw}^3)^{-1}([Z])$ *is isomorphic to the smooth surface* $\mathrm{Sing}(Z^\vee)$.

3.1. **Gushel degenerations.** The link between these Fano varieties of various dimensions can be made through a construction of Gushel analogous to what we did with cubics.

Let $CG \subset \mathbf{P}(\mathbf{C} \oplus \wedge^2 V_5)$ be the cone, with vertex $v = \mathbf{P}(\mathbf{C})$, over the Grassmannian $G(2, V_5)$. Intersect CG with a general quadric $\Omega \subset \mathbf{P}(\mathbf{C} \oplus \wedge^2 V_5)$ and a linear space $\mathbf{P}^{k+4}$ to get a Fano variety Z^k of dimension k. There are two cases:

- either $v \notin \mathbf{P}^{k+4}$, in which case Z^k is isomorphic to the intersection X_{10}^k of $G(2, V_5)$ with the projection of $\mathbf{P}^{k+4}$ to $\mathbf{P}(\wedge^2 V_5)$ and a quadric;

- or $v \in \mathbf{P}^{k+4}$, in which case $\mathbf{P}^{k+4}$ is a cone over a $\mathbf{P}^{k+3} \subset \mathbf{P}(\wedge^2 V_5)$ and Z^k is a double cover X_G^k of $G(2, V_5) \cap \mathbf{P}^{k+3}$ branched along its intersection X_{10}^{k-1} with a quadric.

The second case is a specialization of the first, and the EPW sextics Z_{X^k} from the first case degenerate to the sextics $Z_{X_G^k}$ from the second case. Moreover, in the second case, *the sextics* $Z_{X_G^k}$ *and* $Z_{X_{10}^{k-1}}$ *are the same.*

Let $\mathscr{X}_G^k$ be the moduli stack for smooth varieties of type X_{10}^k and their Gushel degenerations. The Gushel constructions yields morphisms $\mathscr{X}_{10}^k \to \mathscr{X}_G^{k+1}$ and such that the diagrams

$$\begin{array}{ccc} \mathscr{X}_{10}^k & \longrightarrow & \mathscr{X}_G^{k+1} \\ & {\scriptstyle \mathrm{epw}^k} \searrow & \downarrow {\scriptstyle \mathrm{epw}_G^{k+1}} \\ & & \mathscr{E}\mathscr{P}\mathscr{W} \end{array}$$

commute.

We can perform a "'double Gushel construction" as follows. Let $CCG \subset \mathbf{P}(\mathbf{C}^2 \oplus \wedge^2 V_5)$ be the cone, with vertex $L = \mathbf{P}(\mathbf{C}^2)$, over the Grassmannian $G(2, V_5)$. Intersect CCG with a general quadric $\Omega \subset \mathbf{P}(\mathbf{C}^2 \oplus \wedge^2 V_5)$ and a codimension-2 linear space $\mathbf{P}^9$ to get a Fano variety Z of dimension 6.

- If $L \cap \mathbf{P}^9 = \varnothing$, the variety Z is smooth of type $\mathscr{X}_{10}^5$.
- If $L \subset \mathbf{P}^9$, in which case $\mathbf{P}^9$ is a cone over a $\mathbf{P}^7 \subset \mathbf{P}(\wedge^2 V_5)$, the corresponding variety Z_0 meets L at two points p and q, and the projection from L induces a rational conic bundle $Z_0 \dashrightarrow W_5^4 := G(2, V_5) \cap \mathbf{P}^7$, undefined at p and q, whose discriminant locus is a threefold X_{10}^3. Blowing up these two points, we obtain a conic bundle $\widehat{Z}_0 \to W_5^4$ with two disjoint sections corresponding to the two exceptional divisors. These two sections trivialize the double étale cover $\widetilde{X}_{10}^3 \to X_{10}^3$ of the discriminant.

We call the singular variety Z_0 a double-Gushel degeneration. If we let $\overline{\mathscr{X}}_{GG}^5$ be the moduli stack for smooth varieties of type X_{10}^5 and their Gushel and double-Gushel degenerations, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathscr{X}_{10}^3 & \longrightarrow & \mathscr{X}_G^4 & \longrightarrow & \mathscr{X}_{GG}^5 \\ & {\scriptstyle \mathrm{epw}^3} \searrow & \downarrow {\scriptstyle \mathrm{epw}_G^4} & \swarrow {\scriptstyle \mathrm{epw}_{GG}^5} & \\ & & \mathscr{E}\mathscr{P}\mathscr{W}\,. & & \end{array}$$

Question 3.2. Is epw_{GG}^5 proper with general fiber isomorphic to $\mathbf{P}^5$?

Period maps. Both $H^3(X^3,\mathbf{Q})$ and $H^5(X^5,\mathbf{Q})$ are 20-dimensional and carry Hodge structures of weight 1. This gives rise to period maps

$$\wp^3 : \mathscr{X}^3 \to \mathscr{A}_{10} \quad \text{and} \quad \wp^5 : \mathscr{X}^5 \to \mathscr{A}_{10}.$$

By [DIM], the general fibers of $\wp^3$ are unions of proper surfaces that come in pairs:

- $\mathscr{F}_{X^3_{10}}$, isomorphic to $\mathscr{C}^0_2(X^3_{10})/\iota$ and to $\mathrm{Sing}(Z^\vee_{X^3_{10}})$;
- $\mathscr{F}^\star_{X^3_{10}}$, the analogous surface for any line-transform of X^3_{10}.

In particular, by Proposition 3.1, there is a commutative diagram

$$\begin{array}{ccccc} \mathscr{X}^3 & \xrightarrow{\ \mathrm{epw}^3\ } & \mathscr{E}\mathscr{P}\mathscr{W} & \longrightarrow & \mathscr{E}\mathscr{P}\mathscr{W}/\text{duality} \\ & {\scriptstyle \wp^3}\searrow & \downarrow{\scriptstyle \wp} & \swarrow{\scriptstyle \bar\wp} & \\ & & \mathscr{A}_{10}, & & \end{array}$$

where the map $\bar\wp$ is generically finite (presumably birational) onto its image. Since $J(X^3_{10})$ is isomorphic to the Albanese variety of the surface $\mathscr{C}^0_2(X^3_{10})$ ([Lo]), the map $\wp$ is defined by sending the class of an EPW sextic Z to the Albanese variety of the canonical double cover of its singular locus (the resulting principally polarized abelian varieties are isomorphic for Z and $Z^\vee$).

Theorem 3.3. *The image of $\wp^3$ and the image of $\wp^5$ have same closures.*

Proof. This is proved using a double Gushel degeneration (see §3.1): with the notation above, one proves that $J^5(\widehat{Z}_0)$ is still a 10-dimensional principally polarized abelian variety which is a limit of intermediate Jacobians of Fano fivefolds of type $\mathscr{X}^5_{10}$, hence belongs to the closure of $\mathrm{Im}(\wp^5)$.

Since, by Lefschetz theorem, we have $H^5(W_4,\mathbf{Q}) = H^3(W_4,\mathbf{Q}) = 0$, the following lemma implies that the intermediate Jacobians $J^5(\widehat{Z}_0)$ and $J^3(X^3_{10})$ are isomorphic. This proves already that the image of $\wp^3$ is contained in the closure of the image of $\wp^5$.

Lemma 3.4. *Let Z and W be smooth projective varieties satisfying $H^k(W,\mathbf{Q}) = H^{k-2}(W,\mathbf{Q}) = 0$. Let $\pi : Z \to W$ be a conic bundle with smooth irreducible discriminant divisor $X \subset W$. Assume further that $\pi^{-1}(X)$ is reducible. There is an isomorphism of Hodge structures*

$$H^k(Z,\mathbf{Z})/\mathrm{tors} \simeq H^{k-2}(X,\mathbf{Z})/\mathrm{tors}.$$

To finish the proof of the theorem, we prove, by computing the kernel of its differential, that the fibers of $\wp^5$ have dimension at least 5. $\square$

Questions 3.5. Is $\wp^5$ equal to $\wp \circ \mathrm{epw}^5$?

Question 3.6. Can one extend epw^3 to the family of nodal X^3_{10} with values in the compactifications studied in [O2]? Can one extend $\wp$ to these compactifications?

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