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Algebraic Surfaces in Positive Characteristic

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Summary. These notes are an introduction to and an overview over the theory of algebraic surfaces over algebraically closed fields of positive characteristic. After a little bit of background in characteristic-p-geometry, we sketch the Kodaira–Enriques classification. Next, we turn to more special characteristic-p topics, and end with lifting results, as well as applications to geometry in characteristic zero. We assume that the reader has a background in complex geometry and has seen the Kodaira–Enriques classification of complex surfaces before.

Key words: Algebraic surfaces, arithmetic, positive characteristic.

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1 Introduction

These notes grew out of a series of lectures given at Sogang University, Seoul in October 2009. They were meant for complex geometers, who are not familiar with characteristic-p-geometry, but who would like to see similarities, as well as differences to complex geometry. More precisely, these notes are on algebraic surfaces in positive characteristic, and assume familiarity with the complex side of this theory, say, on the level of Beauville's book [Be96].

Roughly speaking, the theory of curves in characteristic zero and $p > 0$ look very similar, and many fundamental results were already classically known to hold in arbitrary characteristic. Also, curves lift from characteristic p over the Witt ring to characteristic zero, which implies that many "characteristic-p-pathologies" cannot happen. Abelian varieties admit at least formal lifts over the Witt ring, and we refer to Section 11 for details and implications.

However, from dimension two on, geometry in positive characteristic displays more and more differences to classical complex geometry. In fact, this geometry has long been considered as "pathological" and "exotic", as even reflected in the titles of a series of articles by Mumford (the first one being

[Mu61]). There, the emphasis was more on finding and exploring differences to the classical theory. For a short overview over the main new phenomena for surfaces in positive characteristic, we refer to [I-S96, Section 15].

However, in their three fundamental articles [Mu69a], [B-M2] and [B-M3], Bombieri and Mumford established the Kodaira–Enriques classification of algebraic surfaces in positive characteristic. Together with Artin's results [Ar62] and [Ar66] on surface singularities, especially rational and Du Val singularities, as well as work of Ekedahl [Ek88] on pluricanonical systems of surfaces of general type (extending Bombieri's results to characteristic p), this sets the scene in positive characteristic. It turns out that surface theory in positive characteristic is in many respects not so different from characteristic zero, at least, if one takes the right angle.

As over the complex numbers, there is a vast number of examples, counter-examples, and (partial) classification results for special classes of surfaces in positive characteristic. Unfortunately, it was impossible for me to mention all of them in these introductory notes – for example, I could have written much more on K3 surfaces, elliptic surfaces, and (birational) automorphisms of surfaces.

These notes are organized as follows:

Preparatory Material

Section 2 We introduce the various Frobenius morphisms, and proceed to basic results on algebraic curves. Finally, we discuss finite, constant, and infinitesimal group schemes, as well as the three group schemes of length p.

Section 3 We recall Hodge-, étale and deRham (hyper-) cohomology. Next, we discuss Albanese and Picard schemes, non-closed differential forms, and their relation to (non-)degeneracy of the Frölicher spectral sequence from Hodge- to deRham-cohomology. Finally, we sketch how crystalline cohomology links all the above mentioned cohomology theories.

Classification of Algebraic Surfaces

Section 4 We discuss blow-ups and Castelnuovo's contraction theorem, introduce minimal models, and describe the structure of rational and birational maps of surfaces. We classify birationally ruled surfaces, and state the rationality theorem of Castelnuovo–Zariski.

Section 5 We recall elliptic fibrations, and discuss the phenomenona of quasi-elliptic fibrations and wild fibers. Then, we state the canonical bundle formula and give the possible degeneration types of fibers in (quasi-)elliptic fibrations.

Section 6 We sketch the Kodaira–Enriques classification of algebraic surfaces according to their Kodaira dimension.

Section 7 We discuss the four classes of minimal surfaces of Kodaira dimension zero in greater detail. We put an emphasis on the non-classical classes of Enriques surfaces in characteristic 2, as well as the new classes of quasi-hyperelliptic surfaces in characteristics 2 and 3.

Section 8 We start with Ekedahl's work on pluricanonical maps of surfaces of general type. Then, we continue with what is known about various inequalities

(Noether, Castelnuovo, Bogomolov–Miyaoka–Yau) in positive characteristic, and end with a couple of results on surfaces of general type with small invariants.

Special Topics in Positive Characteristic

Section 9 We study uniruled surfaces that are not birationally ruled, and introduce two notions of supersingularity, due to Artin and Shioda. Then, we discuss these notions for K3 surfaces. Next, we turn to surfaces over finite fields, zeta functions, and the Tate conjecture.

Section 10 We explain Jacobson's correspondence for purely inseparable field extensions. On the geometric level, this corresponds to p-closed foliations. We give applications to global vector fields on K3 surfaces and end by discussing quotients by infinitesimal group schemes.

From Positive Characteristic to Characteristic Zero

Section 11 We recall the ring of Witt vectors, discuss various notions of what it means to "lift to characteristic zero", and discuss, what each type of liftings implies. We end by giving examples and counter-examples.

Section 12 As an application of characteristic-p-geometry, we establish infinitely many rational curves on complex projective K3 surfaces of odd Picard rank using reduction modulo p and special characteristic-p features.

Finally, we advise the reader who is interested in learning surface theory over algebraically closed ground fields of arbitrary characteristic from scratch (including proofs) to have a look at Badescu's excellent text book [Ba01]. From there, the reader can proceed to more advanced topics, including the original articles by Bombieri and Mumford mentioned above, as well as the literature given in these notes.

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2 Frobenius, curves and group schemes

Before dealing with surfaces, we first shortly review a little bit of background material. Of course, the omnipresent Frobenius morphism has to be mentioned first – in many cases, when a characteristic zero argument breaks down in positive characteristic, inseparable morphisms and inseparable field extensions are responsible. The prototype of an inseparable morphism is the *Frobenius morphism*, and in many situations it also provides the solution to a problem. Next, we give a very short overview over curves and group scheme actions. We have chosen our material in view of what we need for the classification and description of surfaces later on.

2.1 Frobenius

Let us recall that a field k of positive characteristic p is called *perfect*, if its Frobenius morphism $x \mapsto x^p$ is surjective, i.e., if every element in k has a p.th root in k. For example, finite fields and algebraically closed fields are perfect. On the other hand, function fields of varieties in positive characteristic are almost never perfect.

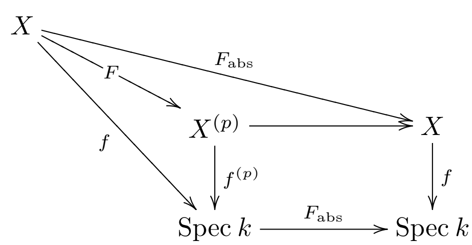
Let X be an n-dimensional variety over a field k with structure morphism $f : X \to \operatorname{Spec} k$. Then, the morphism that is the identity on points of X and is $x \mapsto x^p$ on the structure sheaf $\mathcal{O}_X$ is called the *absolute Frobenius morphism* F_{abs} of X.

However, the absolute Frobenius morphism is not "geometric": namely, it acts as $x \mapsto x^p$ on the ground field k, which is non-trivial except for $k = \mathbb{F}_p$. To obtain a morphism over k, we first form the pull-back

$$X^{(p)} \;:=\; X \times_{\operatorname{Spec} k} \operatorname{Spec} k \xrightarrow{\mathrm{pr}_2} \operatorname{Spec} k$$

with respect to the structure map $f : X \to \operatorname{Spec} k$ and with respect to the absolute Frobenius $F_{\mathrm{abs}} : \operatorname{Spec} k \to \operatorname{Spec} k$. This *Frobenius pullback* $f^{(p)} := \mathrm{pr}_2 : X^{(p)} \to \operatorname{Spec} k$ is a new variety over k. If k is perfect then X and $X^{(p)}$ are abstractly isomorphic as schemes, but not as varieties over k, see below.

Now, by the universal property of pull-backs, we obtain a morphism of varieties over k, the *k-linear Frobenius morphism* $F : X \to X^{(p)}$



In more down to earth terms and for affine space this simply means

$$\begin{array}{lcccccl} & & k[x_1,...,x_n] & \to & k[x_1,...,x_n] & & \\ \text{absolute Frobenius} & : & f(x_1,...,x_n) & \mapsto & (f(x_1,...,x_n))^p & & \\ k\text{-linear Frobenius} & : & c & \mapsto & c & & \text{if } c \in k \\ & & x_i & \mapsto & x_i^p & & \end{array}$$

When dealing with varieties over finite fields there are even more Frobenius morphisms: over the field $\mathbb{F}_q$ with $q = p^n$ elements one has a Frobenius morphism $F_q : x \mapsto x^q$, and for technical reasons sometimes its inverse has to be considered, see, for example [Har77, Appendix C.4]. Depending on author and context all these morphisms and various base-changes are called "Frobenius" and so, a little care is needed.

Next, if X is n-dimensional over k, then the k-linear Frobenius $F : X \to X^{(p)}$ is a finite morphism of degree p^n. Moreover, if k is perfect then, on the level of function fields, this morphism corresponds to the inclusion

$$k(X^{(p)}) \,=\, k(X)^p \,\subseteq\, k(X)\,.$$

Note that $k(X)^p$, the set of p.th powers of $k(X)$, is in fact a field: it is not only closed under multiplication, but also under addition since $x^p + y^p = (x+y)^p$ in characteristic p. Let us also fix an algebraic closure $\overline{K}$ of $K = k(X)$. For every integer $i \geq 0$ we define

$$K^{p^{-i}} \,:=\, \{\, x \in \overline{K} \,|\, x^{p^i} \in K \,\}$$

and note that these sets are in fact fields. The field $K^{p^{-i}}$ is a finite and purely inseparable extension of K of degree p^{ni}. Their union $K^{p^{-\infty}}$ as i tends to infinity is called the *perfect closure* of K in $\overline{K}$, as it is the smallest subfield of $\overline{K}$ that is perfect and contains K.

Definition 2.1 *Let L be a finite and purely inseparable extension of K. The* height *of L over K is defined to be the minimal i such that $K \subseteq L \subseteq K^{p^{-i}}$.*

Similarly, if $\varphi : Y \to X$ is a purely inseparable and generically finite morphism of varieties over a perfect field k, then the *height of* φ is defined to be the height of the extension of function fields $k(Y)/k(X)$. For example, the k-linear Frobenius morphism is of height one.

For more on inseparable morphisms, we refer to Section 10.

2.2 Curves

Most of the results of this section are classical, and we refer to [Har77, Chapter IV] or [Liu02] for details, specialized topics and further references. Let C be a smooth projective curve over an algebraically closed field of characteristic $p \geq 0$. Then its *geometric genus* is defined to be

$$g(C) \,:=\, h^0(C, \omega_C) \,=\, h^1(C, \mathcal{O}_C),$$

where ω_C denotes the dualizing sheaf. The second equality follows from Serre duality. Since C is smooth over k, the sheaf ω_C is isomorphic to the sheaf of Kähler differentials $\Omega_{C/k}$.

Let $\varphi : C \to D$ be a finite morphism between smooth curves. Let us also assume that φ is separable, i.e., the induced field extension $k(D) \subset k(C)$ is separable. Then, the *Riemann–Hurwitz formula* states that there is a linear equivalence of divisors on C

$$K_C \sim \varphi^*(K_D) + \sum_{P\in C} \mathrm{length}(\Omega_{C/D})_P \cdot P\,.$$

Here, $\Omega_{C/D}$ is the sheaf of relative Kähler differentials. Since φ is separable, it is generically étale, and thus, $\Omega_{C/D}$ is a torsion sheaf supported in finitely many points. By definition, the points in the support of this sheaf are called *ramification points*. In case φ is inseparable, $\Omega_{C/D}$ is non-trivial in every point, and every point would count as ramification point.

For a point $P \in C$ with image $Q = \varphi(P)$, and still assuming φ to be separable, we choose a local parameter $t \in \mathcal{O}_{D,Q}$ and define the *ramification index* e_P of φ at P to be the valuation of $\varphi^{\#}(t)$ in $\mathcal{O}_{C,P}$, see [Har77, Section IV.2]. Then, the ramification at a ramification point P is called *tame*, if e_P is not divisible by $p = \mathrm{char}(k)$, and it is called *wild* otherwise. We have

$$\mathrm{length}(\Omega_{C/D})_P \begin{cases} = e_P - 1 & \text{if } P \text{ is tame} \\ > e_P - 1 & \text{if } P \text{ is wild} \end{cases}.$$

In general, it is very difficult to bound e_P whenever the ramification is wild, see the example below. An important case where one can say more about wild ramification is in case φ is a Galois morphism: then, one can define for every wild ramification point P certain subgroups of the Galois group, the so-called *higher ramification groups*, that control the length of $\Omega_{C/D}$ at P, cf. [Se68, Chapitre IV.1].

Example 2.2 *Galois covers with Galois group* $\mathbb{Z}/p\mathbb{Z}$ *are called* Artin–Schreier covers. *An example is* $\varphi : C \to \mathbb{P}^1$ *given by the projective closure and normalization of the affine equation*

$$z^p - z = t^{hp-1}\,.$$

This cover is branched only over $t = \infty$, *and the ramification is wild of index* $e_\infty = p(p-1)h$. *Thus,* C *is a curve of genus* $1 - p + \frac{1}{2}p(p-1)h$ *and there are* $(p-1)h$ *non-trivial higher ramification groups.*

In particular, φ *defines a non-trivial étale cover of* $\mathbb{A}^1$, *which implies that the affine line* $\mathbb{A}^1$ *is* not *algebraically simply connected. In fact, by Raynaud's theorem (formerly Abhyankar's conjecture), every finite group that is generated by its p-Sylow subgroups occurs as quotient of* $\pi_1^{\text{ét}}(\mathbb{A}^1)$. *We refer to [B-L-R00] for an overview and references. However, it is still true that every étale cover of* $\mathbb{P}^1$ *is trivial, i.e.,* $\mathbb{P}^1$ *is algebraically simply connected [Gr60, Chapter XI.1].*

If φ is purely inseparable, there still exists a sort of Riemann–Hurwitz formula. We refer to [Ek87] or [Miy97, Lecture III] for more information on $\Omega_{C/D}$ in this case. In this case, the “ramification divisor” is defined only up to linear equivalence. On the other hand, the structure of purely inseparable morphisms between curves is simple: namely, every such morphism is just the composite of k-linear Frobenius morphisms (Proposition 10.3). However, from dimension two on, inseparable morphisms become more complicated. We will come back to this in Section 10.

Let us now give a rough classification of curves: If a smooth projective curve over a field k has genus zero, then $\omega_C^\vee$ is very ample and embeds C as a quadric in $\mathbb{P}_k^2$. Moreover, a quadric with a k-rational point is isomorphic to $\mathbb{P}_k^1$ over any field. Moreover, the Riemann–Hurwitz formula implies that every curve that is dominated by a curve of genus zero also has genus zero (Lüroth's theorem). Thus, since we assumed k to be algebraically closed, we find

Theorem 2.3 *If $g(C) = 0$ then $C \cong \mathbb{P}_k^1$, i.e., C is rational. Moreover, every unirational curve, i.e., a curve that is dominated by $\mathbb{P}_k^1$, is rational.*

Although unirational surfaces are rational in characteristic zero by Castelnuovo's theorem, this is false in positive characteristic, see Theorem 9.3.

For curves of genus one, we refer to [Har77, Chapter IV.4] or [Sil86]. Their classification is as follows:

Theorem 2.4 *Let C be a smooth projective curve of genus $g(C) = 1$ over an algebraically closed field k of characteristic $p \geq 0$. Then*

1. *after choosing a point $O \in C$ there exists the structure of an Abelian group on the points of C, i.e., C is an Abelian variety of dimension one, an* elliptic curve.
2. *The linear system $|2O|$ defines a finite morphism of degree 2*
$$\varphi \,:\, C \,\to\, \mathbb{P}_k^1\,.$$
3. *There exists a j-*invariant *$j(C) \in k$ such that two genus one curves are isomorphic if and only if they have the same j-invariant.*
4. *If $p \neq 2$ then φ is branched over four points and j can be computed from the cross ratio of these four points.*
5. *The linear system $|3O|$ embeds C as a cubic curve into $\mathbb{P}_k^2$. Moreover, if $p \neq 2, 3$ then C can be given by an affine equation (Weierstraß equation)*
$$y^2 \,=\, x^3 \,+\, ax \,+\, b$$
for some $a, b \in k$.

We note that the description of complex elliptic curves as quotients of $\mathbb{C}$ by lattices also has an analog in positive characteristic. This leads to the theory of *Drinfel'd modules*, which is parallel to the theory of elliptic curves but has not so much to do with the theory of curves, see [Go96, Chapter 4].

A curve C of genus $g \geq 2$ is called *hyperelliptic* if there exists a separable morphism $\varphi : C \to \mathbb{P}_k^1$ of degree 2. If $p = \mathrm{char}(k) \neq 2$ then φ is branched over

$2g+2$ points. (We also note that a morphism of degree 2 in characteristic $\neq 2$ is automatically separable.) On the other hand, if $p = 2$ then every ramification point of φ is wildly ramified and thus, there are at most $g+1$ branch points. In any characteristic, curves of genus $g = 2$ are hyperelliptic and the generic curve of genus $g \geq 3$ is not hyperelliptic.

Theorem 2.5 *If $g \geq 2$ then $\omega_C^{\otimes 2}$ is very ample if and only if C is not hyperelliptic. In any case, $\omega_C^{\otimes 3}$ is very ample for all curves with $g \geq 3$ and $\omega_C^{\otimes 4}$ is very ample for all curves of genus $g \geq 2$.*

Thus, curves of genus $g \geq 2$ embed into some fixed projective space that depends on g only. This is the first step towards constructing their moduli spaces. More precisely, Deligne and Mumford [D-M69] showed the existence of a Deligne–Mumford stack, flat and of dimension $3g-3$ over $\operatorname{Spec} \mathbb{Z}$ that parametrizes curves of genus g. Thus, the moduli space of curves in positive characteristics arises by reducing the one over $\operatorname{Spec} \mathbb{Z}$ modulo p.

Let us finally mention a couple of facts concerning automorphism groups:

1. If $p \neq 2, 3$ then the automorphism group of an elliptic curve, i.e., automorphisms fixing the neutral element O, has order 2, 4 or 6.
2. However, the elliptic curve with $j = 0$ has 12 automorphisms if $p = 3$ and even 24 automorphisms if $p = 2$, see [Sil86, Theorem III.10.1].
3. The automorphism group of a curve of genus $g \geq 2$ is finite. However, the Hurwitz bound $84(g-1)$ on its in characteristic zero can be violated. We refer to [Har77, Chapter IV.2, Exercise 2.5] for details and further references

Let us note that some classes of surfaces arise as quotients $(C_1 \times C_2)/G$, where C_1, C_2 are curves with G-actions. Now, in positive characteristic larger automorphism groups may show up and thus, new possibilities have to be considered. For example, we will see in Section 7.4 that hyperelliptic surfaces arise as quotients of products of elliptic curves in any characteristic. It is remarkable that *no* new classes arise in characteristic 2 and 3 from larger automorphism groups of elliptic curves with $j = 0$.

2.3 Group schemes

Constructions with groups are ubiquitous in geometry. Instead of finite groups we will consider *finite and flat group schemes* G over a ground field k, which we assume to be algebraically closed of characteristic $p \geq 0$. We refer to [Wa79] or [Ta97] for overview, details and references.

Thus, $G = \operatorname{Spec} A$ for some finite-dimensional k-algebra A, and there exist morphisms

$$O : \operatorname{Spec} k \to G \quad \text{and} \quad m : G \times G \to G$$

where m stands for multiplication, and O for the neutral element. These morphisms have to fulfill certain axioms that encode that G is a group object in the category of schemes. We refer to [Wa79, Chapter I] for the precise definition and note that it amounts to saying that A is a *Hopf algebra*. The dimension $\dim_k A$ is called the *length*, or *order*, of the group scheme G.

The following construction associates to every finite group a finite flat group scheme: for a finite group or order n with elements $g_1, ..., g_n$ we take a disjoint union of n copies of Spec k, one representing each g_i, and define m via the multiplication in the group we started with. This defines the *constant group scheme* associated to a finite group. Conversely, we have

Theorem 2.6 *A finite flat group scheme G of length prime to p over an algebraically closed field is a constant group scheme.*

In particular, over an algebraically closed field of characteristic zero, we obtain an equivalence between the categories of finite groups and finite flat group schemes.

One feature of constant group schemes is that the structure morphism $G \to \mathrm{Spec} k$ is étale, i.e., $A = H^0(G, \mathcal{O}_G)$ is a separable k-algebra. For example, consider the constant group scheme $\mathbb{Z}/p\mathbb{Z}$, which is of length p. As an algebra, A is isomorphic to k^p with componentwise addition and multiplication, and thus étale over k. On the other hand, we will see below that there are two different structures of group schemes on Spec $k[x]/(x^p)$, which is not reduced – these are examples of *infinitesimal group schemes*. Let us first note that infinitesimal group schemes are a particular characteristic p phenomenon:

Theorem 2.7 (Cartier) *Group schemes over fields of characteristic zero are smooth and thus, reduced.*

To give examples of infinitesimal group schemes, we consider $\mathbb{G}_a$ and $\mathbb{G}_m$. Here, $\mathbb{G}_a$ denotes the group scheme corresponding to the additive group, i.e., $(\mathbb{G}_a(k), \circ) = (k, +)$. Similarly, $\mathbb{G}_m$ denotes the group scheme corresponding to the multiplicative group of k, i.e., $(\mathbb{G}_m(k), \circ) = (k^\times, \cdot)$, see [Wa79, Chapter I.2] - these group schemes are affine but not finite over k. Then, the first example of an infinitesimal group scheme is μ_p, the group scheme of *p.th roots of unity*. Namely, there exists a short exact sequence of group schemes (in the flat topology)

$$0 \to \mu_p \to \mathbb{G}_m \xrightarrow{x \mapsto x^p} \mathbb{G}_m \to 0\,.$$

We note that the kernel μ_p is infinitesimal because of the equality $x^p - 1 = (x-1)^p$ in characteristic p. The second example is α_p, the kernel of Frobenius on $\mathbb{G}_a$, i.e., we have a short exact sequence

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0\,.$$

Both group schemes, α_p and μ_p, are isomorphic to Spec $k[x]/(x^p)$ as schemes, and are thus infinitesimal (non-reduced), but have different multiplication maps. Together with $\mathbb{Z}/p\mathbb{Z}$ these are all group schemes of length p:

Theorem 2.8 (Oort–Tate) *A finite flat group scheme of length p over an algebraically closed field of characteristic p is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, μ_p or α_p.*

For more general results we refer to [O-T70] and [Oo66]. Let us also mention that there exist non-Abelian group schemes of order p^2. Thus, in positive characteristic, the theory of finite flat group schemes is richer than the theory of finite groups, already over an algebraically closed field.

For example, if E is an elliptic curve in characteristic p, then multiplication by p induces a morphism $E \to E$, whose kernel $E[p]$ is a finite flat group scheme of length p^2 (as expected from characteristic zero). More precisely, and still assuming k to be algebraically closed,

$$E[p] \cong \begin{cases} \text{either} & \mu_p \oplus (\mathbb{Z}/p\mathbb{Z}) & \text{and } E \text{ is called } \textit{ordinary}, \\ \text{or} & M_2 & \text{a non-split extension of } \alpha_p \text{ by itself} \\ & & \text{and } E \text{ is called } \textit{supersingular}. \end{cases}$$

Looking at k-rational points, we find $E[p](k) = \mathbb{Z}/p\mathbb{Z}$ if E is ordinary, and $E[p](k) = 0$ if E is supersingular. Thus, k-rational points do not suffice to see the full p-torsion, and the theory of finite flat group schemes is really needed. As the name suggests, the generic elliptic curve is ordinary. More precisely, a theorem of Deuring states that there exist approximately $p/12$ supersingular elliptic curves in characteristic p, see [Sil86, Theorem V.4.1].

In classical algebraic geometry, one often constructs interesting and new varieties as Galois-covers or quotients by finite groups of "well-understood" varieties. In positive characteristic, one very successful way to construct a "pathological characteristic-p" example is via purely inseparable covers, or, via quotients by infinitesimal group schemes. The role of Galois covers is often played by torsors under α_p and μ_p. We come back to this in Section 10.5.

3 Cohomological tools and invariants

This section circles around algebraic versions of Betti and Hodge numbers, and deRham-cohomology. Especially towards the end, the subjects get deeper, our exposition becomes sketchier and we advise the reader interested in surface theory only, to skip all but the first three paragraphs.

In this section, X will be a smooth and projective variety of arbitrary dimension over an algebraically closed field k of characteristic $p \geq 0$.

3.1 Hodge numbers

As usual, we define the *Hodge numbers* of X to be

$$h^{i,j}(X) := \dim_k H^j(X, \Omega_X^i).$$

We note that Serre duality holds for projective Cohen–Macaulay schemes over any field [Har77, Chapter III.7], and in particular we find

$$h^{i,j}(X) \,=\, h^{n-i,n-j}(X), \qquad \text{where} \qquad n = \dim(X)\,.$$

Over the complex numbers, complex conjugation induces the Hodge symmetry $h^{i,j} = h^{j,i}$, see, for example [G-H78, Chapter 0.7]. However, even for a smooth projective surface in positive characteristic, the numbers $h^{0,1}$ and $h^{1,0}$ may be different. For example, in [Li08a, Theorem 8.3] we constructed a sequence $\{X_i\}_{i\in\mathbb{N}}$ of surfaces with fixed $\mathrm{Pic}^0_{\mathrm{red}}$ in characteristic 2, where $h^{1,0}(X_i) - h^{0,1}(X_i)$ tends to infinity.

3.2 Betti numbers

An algebraic replacement for singular cohomology is *ℓ-adic cohomology*, whose construction is due to Grothendieck. We refer to [Har77, Appendix C] for motivation, as well as to [Mil80] for a complete treatment. Let us here only describe its basic properties: let ℓ be a prime number different from p and let $\mathbb{Q}_\ell$ be the field of ℓ-adic numbers, i.e., the completion of $\mathbb{Q}$ with respect to the ℓ-adic valuation, see also Section 11.1. Then,

1. the ℓ-adic cohomology groups $H^i_{\text{ét}}(X, \mathbb{Q}_l)$ are finite-dimensional $\mathbb{Q}_\ell$-vector spaces,
2. they are zero for $i < 0$ and $i > 2\dim(X)$,
3. the dimension of $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ is independent of ℓ (here, $\ell \neq p$ is crucial), and we denote it by $b_i(X)$, the i.th *Betti number*,
4. $H^*_{\text{ét}}(X, \mathbb{Q}_\ell)$ satisfies Poincaré duality.

If $k = \mathbb{C}$, then so-called *comparison theorems* show that these Betti numbers coincide with the topological ones. Let us also mention the following feature: if k is not algebraically closed, then the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ acts on the ℓ-adic cohomology groups of $X_{\overline{k}}$, which gives rise to interesting representations of $\mathrm{Gal}(\overline{k}/k)$.

For the following two classes of varieties, ℓ-adic cohomology and Hodge invariants are precisely as one would expect them from complex geometry:

1. If C is a smooth and projective curve over k, then $b_0 = b_2 = 1$, $b_1 = 2g$, and $h^{1,0} = h^{0,1} = g$.
2. If A is an Abelian variety of dimension g over k then $b_0 = b_{2g} = 1$, $b_1 = 2g$, and $h^{0,1} = h^{1,0} = g$. Moreover, there exists an isomorphism
$$\Lambda^i H^1_{\text{ét}}(A, \mathbb{Q}_l) \,\cong\, H^i_{\text{ét}}(A, \mathbb{Q}_\ell)$$
giving – among many other things – the expected Betti numbers.

However, for more general classes of smooth and projective varieties, the relations between Betti numbers, Hodge invariants, deRham-cohomology and the Frölicher spectral sequence in positive characteristic are more subtle than over the complex numbers, as we shall see below.

Let us first discuss $h^{1,0}$, $h^{0,1}$ and b_1 in more detail, since this is important for the classification of surfaces. Also, these numbers can be treated fairly elementary.

3.3 Picard scheme and Albanese variety

If X is smooth and proper over a field k, then there exists an Abelian variety Alb(X) over k, the *Albanese variety* of X, and an *Albanese morphism*

$$\mathrm{alb}_X : X \to \mathrm{Alb}(X).$$

The pair $(\mathrm{Alb}(X), \mathrm{alb}_X)$ is characterized by the universal property that every morphism from X to an Abelian variety factors over alb_X. For a purely algebraic construction, we refer to [Se58b].

Next, the Picard functor, which classifies invertible sheaves on X, is representable by a group scheme, the *Picard scheme* $\mathrm{Pic}(X)$, whose neutral element is $[\mathcal{O}_X]$, see [Gr61]. We denote by $\mathrm{Pic}^0(X)$ the identity component of $\mathrm{Pic}(X)$. Deformation theory provides us with a natural isomorphism

$$T\mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X),$$

where $T\mathrm{Pic}^0(X)$ denotes the Zariski-tangent space at $[\mathcal{O}_X]$.

Now, group schemes over fields of positive characteristic may be non-reduced (the group schemes μ_p and α_p from Section 2.3 are examples), but the reduction of $\mathrm{Pic}^0(X)$ is still an Abelian variety, which is the dual Abelian variety of $\mathrm{Alb}(X)$, see [Ba01, Chapter 5]. Also, the first Betti number b_1 is twice the dimension of $\mathrm{Alb}(X)$. Thus, we get

$$\frac{1}{2} b_1(X) = \dim \mathrm{Alb}(X) = \dim \mathrm{Pic}^0(X).$$

Since, the dimension of the Zariski tangent space at $[\mathcal{O}_X] \in \mathrm{Pic}^0(X)$ is at least equal to the dimension of $\mathrm{Pic}^0(X)$, we find

$$h^{0,1}(X) = h^1(X, \mathcal{O}_X) \geq \frac{1}{2} b_1(X),$$

with equality if and only if $\mathrm{Pic}^0(X)$ is a reduced group scheme, i.e., an Abelian variety. By Cartier's theorem (Theorem 2.7), group schemes over a field of characteristic zero are reduced. As a corollary, we obtain a purely algebraic proof of the following fact

Proposition 3.1 *A smooth and proper variety over a field of characteristic zero satisfies* $b_1(X) = 2h^{0,1}(X)$.

For curves and Abelian varieties over arbitrary fields, b_1, $h^{1,0}$ and $h^{0,1}$ are precisely as over the complex numbers. On the other hand, over fields of positive characteristic,

1. there do exist surfaces with $h^{0,1} > b_1/2$, i.e., with non-reduced Picard schemes, see [Ig55b] and [Se58a].

In [Mu66, Lecture 27], the non-reducedness of $\mathrm{Pic}^0(X)$ is related to non-trivial Bockstein operations β_n from subspaces of $H^1(X, \mathcal{O}_X)$ to quotients of $H^2(X, \mathcal{O}_X)$. In particular, a smooth projective variety with $h^2(X, \mathcal{O}_X) = 0$ has a reduced $\mathrm{Pic}^0(X)$, which applies, for example, to curves. In the case of surfaces, a quantitative analysis of which classes can have non-reduced Picard schemes has been carried out in [Li09a].

3.4 Differential one-forms

We shall see in Section 9.1 that in positive characteristic, the pull-back of a non-zero differential form under a morphism may become zero. However, by a fundamental theorem of Igusa [Ig55a], every non-trivial global 1-form on $\mathrm{Alb}(X)$ pulls back, via alb_X, to a non-zero global 1-form on X. This implies the estimate

$$h^{1,0}(X) \,=\, h^0(X, \Omega_X^1) \,\geq\, \frac{1}{2} b_1(X)\,.$$

Moreover, all global 1-forms arising as pull-back from $\mathrm{Alb}(X)$ are d-closed, i.e., closed under the exterior derivative.

We have $h^{1,0} = b_1/2$ for curves and Abelian varieties over arbitrary fields and their global 1-forms are d-closed. On the other hand, over fields of positive characteristic,

1. there do exist surfaces with $h^{1,0} > b_1/2$, i.e., with "too many" global 1-forms, see [Ig55b], and
2. there do exist surfaces with global 1-forms that are not d-closed, see [Mu61] and [Fo81]. These forms give rise to a non-zero differential in their Frölicher spectral sequences, which thus do *not* degenerate at E_1.

We refer to [Ill79, Proposition II.5.16] for more results and to [Ill79, Section II.6.9] for the connection to Oda's subspace in first deRham cohomology.

3.5 Igusa's inequality

We denote by ρ the rank of the Néron–Severi group $\mathrm{NS}(X)$, which is always finite. More precisely, Igusa's theorem [Ig60] states

$$\rho(X) \,\leq\, b_2(X)\,.$$

This follows from the existence of a Chern map from $\mathrm{NS}(X)$ to second ℓ-adic or crystalline cohomology. On the other hand, $d\log$ induces a "naive" cycle map

$$d\log \,:\, \mathrm{NS}(X) \otimes_{\mathbb{Z}} k \,\to\, H^1(X, \Omega_X^1)\,,$$

which is injective in characteristic zero, and which then implies the inequality $\rho \leq h^{1,1}$, see, for example, [Ba01, Exercise 5.5]. However, this map may fail to be injective in positive characteristic, as the example of supersingular Fermat surfaces [Sh74] shows. More precisely, these surfaces satisfy $b_2 = \rho > h^{1,1}$, see also Section 9.4.

3.6 Kodaira vanishing

Raynaud [Ra78] gave the first counter-examples to the Kodaira vanishing theorem in positive characteristic. However, we mention the following results that tell us that the situation is not too bad:

1. If $\mathcal{L}$ is an ample line bundle then $\mathcal{L}^{\otimes\nu}$, $\nu \gg 0$ fulfills Kodaira vanishing (in fact, this is just Serre vanishing) [Har77, Theorem III.7.6].
2. If a smooth projective variety of dimension $< p$ lifts over $W_2(k)$ then Kodaira vanishing holds, see [Ill02, Theorem 5.8], [D-I87, Corollaire 2.11], and Section 11.2. Under stronger lifting assumptions, also Kawamata–Viehweg vanishing holds [Xi10].
3. Kodaira vanishing, and even stronger vanishing results hold for the (admittedly rather special) class of *Frobenius-split* varieties [B-K05, Theorem 1.2.9].
4. In [Ek88, Section II], Ekedahl develops tools to handle possible failures of Kodaira vanishing, see also Section 8.1.
5. In [Xi11], Xie shows that all surfaces violating Kodaira-Ramanujam vanishing arise as in [Ra78].

Here, $W(k)$ denotes the ring of Witt vectors and $W_2(k)$ the ring of Witt vectors of length 2, see Section 11.1. Let us just mention that if k is a perfect field then $W(k)$ is a complete discrete valuation ring of characteristic zero with residue field k, and that this ring is in a certain sense minimal and universal.

3.7 Frölicher spectral sequence

Let Ω_X^i be the sheaf of (algebraic) differential i-forms. These sheaves, together with the exterior derivative d form a complex, the *(algebraic) deRham-complex* (Ω_X^*, d). Now, the Zariski topology is too coarse to have a Poincaré lemma. Thus, we define *(algebraic) deRham-cohomology* $H^*_{\mathrm{dR}}(X/k)$ to be the hypercohomology of this complex. In particular, there always exists a spectral sequence

$$E_1^{i,j} \;=\; H^j(X, \Omega_X^i) \;\Rightarrow\; H^{i+j}_{\mathrm{dR}}(X/k),$$

the *Frölicher spectral sequence*, from Hodge- to deRham-cohomology. If $k = \mathbb{C}$, and X is proper over k, then these cohomology groups and the spectral sequence coincide with the analytic ones, see [Gr66]. Already the existence of the Frölicher spectral sequence implies for all $m \geq 0$ the inequality

$$\sum_{i+j=m} h^j(X, \Omega_X^i) \;\geq\; h^m_{\mathrm{dR}}(X/k)\,.$$

Equality for all m is equivalent to the degeneration of this spectral sequence at E_1. Over the complex numbers, degeneration at E_1 is true - however, the classical proof uses methods from differential geometry, functional analysis and partial differential equations, see [G-H78, Chapter 0.7]. On the other hand, if a variety of positive characteristic admits a lift over $W_2(k)$, then we have the following result from [D-I87] (but see [Ill02] for an expanded version):

Theorem 3.2 (Deligne–Illusie) *Let X be a smooth and projective variety in characteristic $p \geq \dim X$ and assume that X admits a lift over $W_2(k)$. Then the Frölicher spectral sequence of X degenerates at E_1.*

The assumptions are fulfilled for curves and Abelian varieties, see Section 11. Moreover, if a smooth projective variety X in characteristic zero admits a model over $W(k)$ for some perfect field of characteristic $p \geq \dim X$ it follows from semi-continuity that the Frölicher spectral sequence of X degenerates at E_1 in characteristic zero. From this one obtains purely algebraic proofs of the following

Theorem 3.3 *Degeneration at E_1 holds for*

1. *smooth projective curves and Abelian varieties over arbitrary fields, and*
2. *smooth projective varieties over fields of characteristic zero.*

We already mentioned above that varieties with global 1-forms that are not d-closed, such as Mumford's surfaces [Mu61], provide examples where degeneration at E_1 does *not* hold.

3.8 Crystalline cohomology

To link deRham-, Betti- and Hodge-cohomology, we use *crystalline cohomology*. Its construction, due to Berthelot and Grothendieck, is quite involved [B-O78]. This cohomology theory takes values in the Witt ring $W = W(k)$, which is a discrete valuation ring if k is perfect, see Section 11. In case a smooth projective variety lifts to some $\mathcal{X}/W(k)$, crystalline cohomology is the deRham cohomology $H^*_{\mathrm{dR}}(\mathcal{X}/W(k))$. It was Grothendieck's insight, and the starting point of crystalline cohomology, that this deRham cohomology does not depend on the choice of lift $\mathcal{X}$. One of the main technical difficulties to overcome defining crystalline cohomology for arbitrary smooth and proper varieties is that they usually do *not* lift over $W(k)$.

If X is a smooth projective variety over a perfect field k then

1. the groups $H^i_{\mathrm{cris}}(X/W)$ are finitely generated W-modules,
2. they are zero for $i < 0$ and $i > 2\dim(X)$,
3. there are actions of Frobenius and Verschiebung on $H^i_{\mathrm{cris}}(X/W)$,
4. $H^*_{\mathrm{cris}}(X/W) \otimes_W K$ satisfies Poincaré duality, where K denotes the field of fractions of W, and
5. if X lifts over $W(k)$ then crystalline cohomology is isomorphic to the deRham cohomology of a lift.

We remind the reader that in order to get the "right" Betti numbers from the ℓ-adic cohomology groups, we had to assume $\ell \neq p$. Crystalline cohomology takes values in $W(k)$ (recall $W(\mathbb{F}_p) \cong \mathbb{Z}_p$ with field of fractions $\mathbb{Q}_p$), and this is the "right" cohomology theory for $\ell = p$. In fact,

$$b_i(X) \;=\; \dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(X, \mathbb{Q}_\ell) \;=\; \text{rank}_W\, H^i_{\text{cris}}(X/W) \quad \text{for all} \quad \ell \neq p,$$

i.e., the Betti numbers of X are encoded in the rank of crystalline cohomology. However, since the $H^*_{\text{cris}}(X/W)$ are W-modules, there may be non-trivial torsion - and this is precisely the explanation for the differences between Hodge- and Betti-numbers. More precisely, there is a universal coefficient formula, and for all $m \geq 0$ there are short exact sequences

$$0 \to H^m_{\text{cris}}(X/W) \otimes_W k \to H^m_{\text{dR}}(X/k) \to \text{Tor}_1^W(H^{m+1}_{\text{cris}}(X/W), k) \to 0\,.$$

(In view of what we already said in case X admits a lift over W, it should be plausible that there is a connection between crystalline and deRham cohomology.) In particular, Betti- and deRham-numbers coincide if and only if all crystalline cohomology groups are torsion-free W-modules.

3.9 Hodge–Witt cohomology

In [Ill79], Illusie constructed the *deRham-Witt complex* $W\Omega^*_X$ and studied its cohomology groups $H^j(X, W\Omega^i_X)$, the *Hodge–Witt cohomology groups*. For $i = 0$, these coincide with Serre's Witt vector cohomology groups introduced in [Se58a]. The Hodge–Witt cohomology groups are W-modules, whose torsion may not be finitely generated. In any case, there exists a spectral sequence, the *slope spectral sequence*

$$E_1^{i,j} \;=\; H^j(X, W\Omega^i_X) \;\Rightarrow\; H^{i+j}_{\text{cris}}(X/W)\,,$$

which degenerates at E_1 modulo torsion. We refer to [Ill79, Section II.7] for computations and further results.

Finally, using Hodge–Witt cohomology and slopes on crystalline cohomology, Ekedahl [Ek86, page 85] (but see also [Ill83]) proposed new invariants of smooth projective varieties: slope numbers, dominoes and Hodge–Witt numbers. It is not yet clear, what role they will eventually play in characteristic-p geometry. We refer to [Jo] for some results.

4 Birational geometry of surfaces

From this section on, we study smooth surfaces. To start with, we discuss their birational geometry, which turns out to be "basically the same" as over the complex numbers. Unless otherwise stated, results and proofs can be found in [Ba01], and we refer to [B-H75] for an overview different from ours.

4.1 Riemann–Roch

Let S be a smooth projective surface over an algebraically closed field k of characteristic $p \geq 0$. Actually, asking for properness would be enough: by a theorem of Zariski and Goodman, a surface that is smooth and proper over a field is automatically projective, see [Ba01, Theorem 1.28].

For every locally free sheaf $\mathcal{E}$, Grothendieck constructed Chern-classes $c_i(\mathcal{E})$ that take values in Chow-groups, ℓ-adic, or crystalline cohomology. As usual, for a smooth variety X with tangent sheaf Θ_X we set $c_i(X) := c_i(\Theta_X)$.

We have *Noether's formula*

$$\chi(\mathcal{O}_S) \;=\; \frac{1}{12}\left(c_1^2(S) \,+\, c_2(S)\right)\,.$$

Moreover, if $\mathcal{L}$ is an invertible sheaf on S, we have the *Riemann–Roch formula*

$$\chi(\mathcal{L}) \;=\; \chi(\mathcal{O}_S) \,+\, \frac{1}{2}\mathcal{L}\cdot(\mathcal{L}\otimes\omega_S^\vee)\,.$$

We note that Serre duality holds for Cohen–Macaulay schemes that are of finite type over a field. Thus, we have an equality $h^i(S,\mathcal{L}) = h^{2-i}(S,\omega_S\otimes\mathcal{L}^\vee)$ for surfaces. However, we have seen in Section 3.6 that Kodaira vanishing may not hold. Finally, if D is an effective divisor on S, then D is a Gorenstein curve and the *adjunction formula* yields

$$\omega_D \;\cong\; (\omega_S\otimes\mathcal{O}_S(D))\,|_D\,,$$

where ω_D and ω_S denote the respective dualizing sheaves. In particular, if D is reduced and irreducible, we obtain

$$2p_a(D) \,-\, 2 \;=\; D^2 \,+\, K_S\cdot D\,,$$

where p_a denotes the arithmetic genus of D. We refer to [Har77, Chapter V.1], [Har77, Appendix A], [Ba01, Chapter 5], [Fu98], and [Mil80] for details and further references.

4.2 Blowing up and down

First of all, blowing up a point on a smooth surface over an algebraically closed field has the same effect as over the complex numbers.

Proposition 4.1 *Let $f:\widetilde{S}\to S$ be the blow-up in a closed point and denote by E the exceptional divisor on $\widetilde{S}$. Then*

$$E\;\cong\;\mathbb{P}^1_k,\qquad E^2=-1,\qquad \textit{and}\qquad K_{\widetilde{S}}\cdot E=-1\,.$$

Moreover, the equalities

$$b_2(\widetilde{S}) \;=\; b_2(S)\,+\,1\qquad \textit{and}\qquad \rho(\widetilde{S}) \;=\; \rho(S)\,+\,1$$

hold true.

As in the complex case, we call such a curve E with $E^2 = -1$ and $E \cong \mathbb{P}^1$ an *exceptional* (-1)*-curve*. A surface that does not contain exceptional (-1)-curves is called *minimal*.

Conversely, exceptional (-1)-curves can be contracted and the proof (modifying a suitable hyperplane section) is basically the same as in characteristic zero, cf. [Ba01, Theorem 3.30] or [Har77, Theorem V.5.7]

Theorem 4.2 (Castelnuovo) *Let E be an exceptional (-1)-curve on a smooth surface S. Then, there exists a smooth surface S' and a morphism $f : S \to S'$, such that f is the blow-up of S' in a closed point with exceptional divisor E.*

Since b_2 drops every time one contracts an exceptional (-1)-curve, Castelnuovo's theorem implies that for every surface S there exists a sequence of blow-downs $S \to S'$ onto a minimal surface S'. In this case, S' is called a *minimal model* of S.

4.3 Resolution of indeterminacy

As in characteristic zero, a rational map from a surface extends to a morphism after a finite number of blow-ups in closed points, which gives *resolution of indeterminacy* of a rational map. Moreover, every birational (rational) map can be factored as a sequence of blow-ups and Castelnuovo blow-downs, see, e.g., [Har77, Chapter V.5].

4.4 Kodaira dimension

As over the complex numbers, the following two notions are crucial in the Kodaira–Enriques classification of surfaces: first, the n.th *plurigenus* $P_n(X)$ of a smooth projective variety X is defined to be

$$P_n(X) \;:=\; h^0(X, \omega_X^{\otimes n}).$$

Second, the *Kodaira dimension* $\kappa(X)$ is defined to be $\kappa(X) = -\infty$ if $P_n(X) = 0$ for all $n \geq 1$, or else

$$\kappa(X) \;:=\; \max_{n \in \mathbb{N}} \{\, \dim \phi_n(X) \,\}$$

where $\phi_n : X \dashrightarrow \mathbb{P}_k^{P_n(X)-1}$ denotes the n.th pluri-canonical (possibly only rational) map. This recalled, we have the following important result, cf. [Ba01, Corollary 10.22], which is already non-trivial in characteristic zero:

Theorem 4.3 *Let S be a smooth projective surface with $\kappa(S) \geq 0$. Then, S possesses a unique minimal model.*

4.5 Birationally ruled surfaces

We recall that a surface S is called *birationally ruled* if it is birational to $\mathbb{P}^1 \times C$ for some smooth curve C. Such surfaces are easily seen to satisfy $P_n(S) = 0$ for all $n \geq 1$, i.e., they are of Kodaira dimension $\kappa(S) = -\infty$. Conversely, one can show (see, e.g., [Ba01, Theorem 13.2]) that such surfaces with $\kappa(S) = -\infty$ possess a smooth rational curve that moves:

Theorem 4.4 *If S is birationally ruled, then $\kappa(S) = -\infty$. Conversely, if $\kappa(S) = -\infty$ then S is birationally ruled, i.e., birational to $\mathbb{P}^1 \times C$, and*

$$h^1(S, \mathcal{O}_S) \;=\; \frac{1}{2} b_1(S) \;=\; g(C)\,,$$

where $g(C)$ denotes the genus of C.

As in the complex case, minimal models for surfaces with $\kappa(S) = -\infty$ are not unique. More precisely, we have Nagata's result

Theorem 4.5 *Let S be a minimal surface with $\kappa(S) = -\infty$.*

1. *If $h^1(S, \mathcal{O}_S) \geq 1$, then the image C of the Albanese map is a smooth curve. Moreover, there exists a rank two vector bundle $\mathcal{E}$ on C such that $\mathrm{alb}_S : S \to C$ is isomorphic to $\mathbb{P}(\mathcal{E}) \to C$.*
2. *If $h^1(S, \mathcal{O}_S) = 0$, then S is isomorphic to $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \to \mathbb{P}^1$ with $d \neq 1$.*

Also, Castelnuovo's cohomological characterization of rational surfaces holds true. The proof in positive characteristic is due to Zariski [Za58], but see also the discussion in [B-H75, Part 4]:

Theorem 4.6 (Castelnuovo–Zariski) *For a smooth projective surface S, the following are equivalent*

1. *S is rational, i.e., birational to $\mathbb{P}^2$,*
2. *$h^1(S, \mathcal{O}_S) = P_2(S) = 0$*
3. *$b_1(S) = P_2(S) = 0$.*

So far, things look pretty much the same as over the complex numbers. However, one has to be a little bit careful with the notion of uniruledness: we will see in Section 9 below that unirationality (resp. uniruledness) does *not* imply rationality (resp. ruledness).

4.6 Del Pezzo surfaces

A surface S is called *del Pezzo*, or, *Fano*, if $\omega_S^\vee$ is ample. In every characteristic, these surfaces are rational. More precisely, they are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, or $\mathbb{P}^2$ blown-up in at most 8 points in general position. We refer to Várilly-Alvarado's lecture notes in this volume for details.

5 (Quasi-)elliptic fibrations

For the classification of surfaces of special type in characteristic zero, elliptic fibrations play an important role. In positive characteristic, *wild fibers* and *quasi-elliptic fibrations* are new features that show up. We refer to [Ba01, Section 7] for an introduction, [B-M2], [B-M3] for more details, to [S-S10] for an overview article with many examples, and to [C-D89, Chapter V] for more advanced topics.

5.1 Quasi-elliptic fibrations

Given a dominant morphism from a smooth surface S onto a curve in any characteristic, we may pass to its Stein factorization and obtain a fibration $S \to B$, cf. [Har77, Corollary III.11.5]. Then, its generic fiber S_η is an integral curve, i.e., reduced and irreducible [Ba01, Theorem 7.1]. Moreover, in characteristic zero Bertini's theorem implies that S_η is in fact smooth over the function field $k(B)$. Now, if $\mathrm{char}(k) = p > 0$, then it is still true that the generic fiber is a regular curve, i.e., all local rings are regular local rings. However, this does *not* necessarily imply that S_η is smooth over $k(B)$. Note that since S_η is one-dimensional, regularity is the same as normality. We refer to [Mat80, Chapter 11.28] for a discussion of smoothness versus regularity.

Suppose S_η is not smooth over $k(B)$ and denote by $\overline{k(B)}$ the algebraic closure of $k(B)$. Then $S_{\overline{\eta}} := S_\eta \times_{\mathrm{Spec}\, k(B)} \mathrm{Spec}\, \overline{k(B)}$ is still reduced and irreducible [Ba01, Theorem 7.1] but no longer regular and we denote by $S'_{\overline{\eta}} \to S_{\overline{\eta}}$ its normalization. Then, Tate's theorem on genus change in inseparable extensions [Ta52] (see [Sch09] for a modern treatment) states

Theorem 5.1 (Tate) *Under the previous assumptions, the normalization map $S'_{\overline{\eta}} \to S_{\overline{\eta}}$ is a homeomorphism, i.e., $S_{\overline{\eta}}$ has only unibranch singularities ("cusps"). Moreover, if $p \geq 3$, then the arithmetic genus of every cusp of $S_{\overline{\eta}}$ is divisible by $(p-1)/2$.*

If the generic fiber S_η has arithmetic genus one, and the fiber is not smooth over $k(B)$, then the normalization of $S_{\overline{\eta}}$ is $\mathbb{P}^1$. Also, $S_{\overline{\eta}}$ can have only one singularity, which, by Tate's theorem, must be a cusp of arithmetic genus one. Since $(p-1)/2$ divides this genus if $p \geq 3$, we find $p = 3$ as only solution. Thus:

Corollary 5.2 *Let $f : S \to B$ be a fibration from a smooth surface whose generic fiber S_η is a curve of arithmetic genus one, i.e., $h^1(S_\eta, \mathcal{O}_{S_\eta}) = 1$. Then*

1. *either S_η is smooth over $k(B)$,*
2. *or $S_{\overline{\eta}}$ is a singular rational curve with one cusp.*

The second case can happen in characteristic 2 and 3 only.

Definition 5.3 *If the generic fiber of a fibration $S \to B$ is a smooth curve of genus one, the fibration is called* elliptic*. If the generic fiber is a curve that is not smooth over $k(B)$, the fibration is called* quasi-elliptic*, which can exist in characteristic* 2 *and* 3 *only.*

Some authors require elliptic fibrations to have a section, which we do not. The literature is not consistent.

We refer to [B-M2] and [Ba01, Exercises 7.5 and 7.6] for examples of quasi-elliptic fibrations and to [B-M3] for more on the geometry of quasi-elliptic fibrations. For results on quasi-elliptic fibrations in characteristic 3, see [La79].

We note that quasi-elliptic surfaces are always uniruled, but may not be birationally ruled and refer to Section 9.3, where we discuss this in greater detail.

We also note that the situation gets more complicated in higher dimensions: Mori and Saito [M-S03] constructed Fano 3-folds X in characteristic 2 together with fibrations $X \to S$, whose generic fibers are conics in $\mathbb{P}^2_{k(S)}$ that become non-reduced over $\overline{k(S)}$. Such fibrations are called *wild conic bundles.*

5.2 Canonical bundle formula

Let S be a smooth surface and $f : S \to B$ be an elliptic or quasi-elliptic fibration. Since B is smooth, we obtain a decomposition

$$R^1 f_* \mathcal{O}_S \;=\; \mathcal{L} \oplus \mathcal{T}\,,$$

where $\mathcal{L}$ is an invertible sheaf and $\mathcal{T}$ is a torsion sheaf on B. In characteristic zero, the torsion sheaf $\mathcal{T}$ is always trivial.

Definition 5.4 *Let $b \in B$ a point of the support of $\mathcal{T}$. Then, the fiber of f above b is called a* wild fiber *or, an* exceptional fiber*.*

Proposition 5.5 *Let $f : S \to B$ be a (quasi-)elliptic fibration, $b \in B$ and F_b the fiber above b. Then the following are equivalent:*

1. $b \in \operatorname{Supp}(\mathcal{T})$, *i.e., F_b is a wild fiber,*
2. $h^1(F_b, \mathcal{O}_{F_b}) \geq 2$,
3. $h^0(F_b, \mathcal{O}_{F_b}) \geq 2$.

In particular, wild fibers are multiple fibers. Moreover, if F_b is a wild fiber, then its multiplicity is divisible by p, and we have $h^1(S, \mathcal{O}_S) \geq 1$.

The canonical bundle formula for relatively minimal (quasi-)elliptic fibrations has been proved in [B-M2] - as usual, *relatively minimal* means that there are no exceptional (-1)-curves in the fibers of the fibration.

Theorem 5.6 (Canonical bundle formula) *Let $f : S \to B$ be a relatively minimal (quasi-)elliptic fibration from a smooth surface. Then,*

$$\omega_S \cong f^*(\omega_B \otimes \mathcal{L}^\vee) \otimes \mathcal{O}_S\left(\sum_i a_i P_i\right),$$

where

1. *$m_i P_i = F_i$ are the multiple fibers of f,*
2. *$0 \leq a_i < m_i$,*
3. *$a_i = m_i - 1$ if F_i is not a wild fiber, and*
4. *$\deg(\omega_S \otimes \mathcal{L}^\vee) = 2g(B) - 2 + \chi(\mathcal{O}_S) + \mathrm{length}(\mathcal{T})$.*

For more results on the a_i's we refer to [C-D89, Proposition V.5.1.5], as well as to [K-U85] for more details on wild fibers.

5.3 Degenerate fibers of (quasi-)elliptic fibrations

Usually, an elliptic fibration has fibers that are not smooth and the possible cases have been classified by Kodaira and Néron. The list in positive characteristic is the same as in characteristic zero, cf. [C-D89, Chapter V, §1] and [Sil94, Theorem IV.8.2]. This is not such a surprise, as the classification of degenerate fibers rests on the adjunction formula and on matrices of intersection numbers, and these numerics do not depend on the characteristic of the ground field.

Let us recall that the possible singular fibers together with their Kodaira symbols are as follows (after reduction):

1. An irreducible rational curve with a node as singularity (I_1).
2. A cycle of $n \geq 2$ rational curves (I_n).
3. An irreducible rational curve with a cusp as singularity (II).
4. A configuration of rational curves forming a root system of type A_2^* (III), $\overline{A}_3$ (IV), $\widetilde{E}_6$ (IV^*), $\widetilde{E}_7$ (III^*), $\widetilde{E}_8$ (II^*) or $\widetilde{D}_n$ (I_{n-4}^*).

In the first two cases, the reduction is called *multiplicative* or *semi-stable*, whereas in the last two cases, it is called *additive* or *unstable*. The latter names come from the theory of Néron models, see [Sil94, Chapter IV] or [B-L-R90]. The former names are explained by the fact that semi-stable reduction remains semi-stable after pull-back, whereas unstable reduction may become semi-stable after pull-back. In fact, for every fiber with unstable reduction there exists a pull-back, whose reduction is semi-stable [Sil94, Proposition IV.10.3].

For an elliptic fibration $S \to B$ from a smooth surface, the second Chern class (Euler number), can be expressed in terms of the singular fibers by *Ogg's formula*

$$c_2(S) = \sum_i \nu(\Delta_i),$$

where i runs through the singular fibers, Δ_i is the *minimal discriminant* of the singular fiber and ν denotes its valuation. If a fiber has n irreducible components, then this minimal discriminant is as follows

$$\nu(\Delta) = \begin{cases} 1 + (n-1) & \text{if the reduction is multiplicative, i.e., of type } \mathrm{I}_n, \\ 2 + (n-1) + \delta & \text{if the reduction is additive.} \end{cases}$$

Here, δ is the *Swan conductor*, or, *wild part of the conductor* of the fiber, which is zero if $p \neq 2, 3$. We refer to [Sil94, Chapter IV, §10] for details and to [C-D89, Proposition 5.1.6] for a version for quasi-elliptic fibrations.

In a quasi-elliptic fibration, all fibers are additive and the geometric generic fiber is of type II, i.e., an irreducible rational curve with one cusp. After an inseparable base change of the base curve $B' \to B$, the normalized pull-back yields a fibration whose generic fiber is of genus zero, see also Theorem 9.4. All fibers are reduced or have multiplicity equal to the characterstic $p = 2, 3$. The list of possible geometric fibers is as follows [C-D89, Corollary 5.2.4]:

$$\begin{aligned} p = 3 &: \mathrm{II},\ \mathrm{IV},\ \mathrm{IV}^* \text{ and } \mathrm{II}^*, \\ p = 2 &: \mathrm{II},\ \mathrm{III},\ \mathrm{III}^*,\ \mathrm{II}^* \text{ and } \mathrm{I}_n^*. \end{aligned}$$

Finally, we mention that if a (quasi-)elliptic fibration from a surface has a section, then there exists a *Weierstraß model* [C-D89, Chapter 5, §5], which is more involved in characteristic $2, 3$ than in the other characteristics.

6 Enriques–Kodaira classification

We now come to the Kodaira–Enriques classification of surfaces. In positive characteristic, it is due to Bombieri and Mumford, see [Mu69a], [B-M2] and [B-M3]. Let S be a smooth projective surface of Kodaira dimension $\kappa(S)$.

6.1 Negative Kodaira dimension

First, let us recall and repeat Theorem 4.4:

Theorem 6.1 *If $\kappa(S) = -\infty$, then S is birationally ruled.*

In fact, $\kappa(S) = -\infty$ is equivalent to $p_{12}(S) = 0$, where p_{12} is the 12.th plurigenus [Ba01, Theorem 9.8]. Moreover, although their minimal models are not unique, they have the same structure as in characteristic zero by Theorem 4.5. In Section 9, we shall see that uniruled surfaces in positive characteristic may *not* fulfill $\kappa = -\infty$.

6.2 Positive Kodaira dimension

We recall that the *canonical ring* of a smooth and proper variety X is defined to be

$$R_{\mathrm{can}}(X) := \bigoplus_{n \geq 0} H^0\left(X, \omega_X^{\otimes n}\right).$$

This said, we have the following fundamental result

Theorem 6.2 (Zariski–Mumford) *The canonical ring $R_{\mathrm{can}}(S)$ of a smooth projective surface is a finitely generated k-algebra. If $\kappa(S) \geq 0$, then $R_{\mathrm{can}}(S)$ has transcendence degree $1 + \kappa(S)$ over k.*

We refer to [B-M2] and [Ba01, Corollary 9.10]. More generally, we refer to [Ba01, Chapter 14] for a discussion of Zariski decompositions and finite generation of the more general rings $R(S, D)$ for a $\mathbb{Q}$-divisor D on S.

For a surface with $\kappa(S) \geq 1$ one studies the *Iitaka-fibration*

$$S \dashrightarrow \operatorname{Proj} R_{\mathrm{can}}(S)\,.$$

By the theorem of Zariski–Mumford just mentioned, the right hand side is a projective variety of dimension $\kappa(S)$.

Theorem 6.3 *Let S be a minimal surface with $\kappa(S) = 1$. Then, (the Stein factorization of) the Iitaka fibration is a morphism, which is a relatively minimal elliptic or quasi-elliptic fibration.*

If $\kappa(S) = 1$ and $p \neq 2, 3$, then the fibration is elliptic and unique, $|mK_S|$ for $m \geq 14$ defines this fibration, and 14 is the optimal bound [K-U85]. The main difficulties dealt with by Katsura and Ueno [K-U85] are related to wild fibers. We note that their bound $m \geq 14$ is better than Iitaka's bound $m \geq 86$ over the complex numbers, since over the complex numbers also analytic surfaces that are not algebraic are taken into account.

Although we will discuss surfaces with $\kappa(S) = 2$, i.e., surfaces of general type, in Section 8, let us already anticipate Theorem 8.1:

Theorem 6.4 *Let S be a minimal surface with $\kappa(S) = 2$. Then, the Iitaka fibration is a birational morphism that contracts all rational (-2)-curves and nothing more.*

7 Kodaira dimension zero

As in the complex case, surfaces in positive characteristic that are of Kodaira dimension zero fall into four classes. However, there are new subclasses of Enriques surfaces in characteristic 2, and new subclasses of hyperelliptic surfaces, so-called *quasi-hyperelliptic surfaces*, in characteristic 2 and 3. In particular, there are no fundamentally new classes in characteristic $p \geq 5$.

We start with a result that follows from the classification, especially from the explicit classification of (quasi-)hyperelliptic surfaces:

Theorem 7.1 *Let S be a minimal surface with $\kappa(S) = 0$. Then, $\omega_S^{\otimes 12} \cong \mathcal{O}_S$, and in particular, $p_{12}(S) = 1$.*

The key to the classification of minimal surfaces with $\kappa(S) = 0$ is to use $K_S^2 = 0$ to rewrite Noether's formula (see Section 4.1) as follows:

$$10 + 12p_g = 8h^{0,1} + 2\Delta + b_2,$$

where $\Delta := 2h^{0,1} - b_1$ measures the defect of smoothness of $\mathrm{Pic}^0(S)$. We have $0 \leq \Delta \leq 2p_g$ in general, and $p_g \leq 1$, since $\kappa(S) = 0$. Also, all terms in the above formula are non-negative, which gives a finite list of possibilities, leading eventually to four classes, see the introduction of [B-M2] or [Ba01, Chapter 10]. Let us now discuss these classes in detail:

7.1 Abelian surfaces

These are two-dimensional Abelian varieties. Their main invariants are as in characteristic zero:

$$\begin{array}{cccc} \omega_S \cong \mathcal{O}_S & p_g = 1 & h^{0,1} = 2 & h^{1,0} = 2 \\ \chi(\mathcal{O}_S) = 1 & c_2 = 0 & b_1 = 4 & b_2 = 2 \end{array}$$

Abelian surfaces are usually studied within the framework of Abelian varieties of arbitrary dimension. There exists a huge amount of literature on Abelian varieties and their moduli spaces, both in characteristic zero and in positive characteristic, see, e.g., [Mu70].

By an (unpublished) result of Grothendieck [Ill05, Theorem 5.23], Abelian varieties lift formally to characteristic zero, see also Section 11.3.

For an Abelian variety A of dimension g, multiplication by p is a finite morphism. Its kernel $A[p]$ is a finite and flat group scheme of length p^{2g}, and refer to Section 2.2, where we already discussed elliptic curves ($g = 1$). The identity component $A[p]^0$ is infinitesimal of length at least g. The quotient $A[p]/A[p]^0$ is an étale group scheme isomorphic to $(\mathbb{Z}/p\mathbb{Z})^r$, for some $0 \leq r \leq g$. This quantity r is called the *p-rank* of A. For Abelian varieties of dimension at most two, the p-rank can be detected by the Frobenius-action $F : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$.

Definition 7.2 *An Abelian surface A is called*

1. ordinary *if $r = 2$. Equivalently, F acts bijectively on $H^1(A, \mathcal{O}_A)$.*
2. supersingular *if $r = 0$. Equivalently, F is zero on $H^1(A, \mathcal{O}_A)$.*

We remark that the image of the Albanese morphism of a uniruled surface is at most one-dimensional. Thus, an Abelian surface cannot be uniruled. We note this in view of Shioda's notion of supersingularity and its connection with unirationality discussed in Section 9.7.

7.2 K3 surfaces

These surfaces have the following invariants:

$$\begin{array}{cccc} \omega_S \cong \mathcal{O}_S & p_g = 1 & h^{0,1} = 0 & h^{1,0} = 0 \\ \chi(\mathcal{O}_S) = 2 & c_2 = 24 & b_1 = 0 & b_2 = 22 \end{array}$$

Their formal deformation spaces are smooth $W(k)$-algebras in any characteristic, i.e., the Bogomolov–Tian–Todorov unobstructedness theorem for K3 surfaces in positive characteristic is true [R-S76]:

Theorem 7.3 (Rudakov–Shafarevich) *A K3 surface has no global vector fields. Thus,*

$$H^2(S, \Theta_S) \cong H^2(S, \Omega_S^1 \otimes \omega_S) \overset{\mathrm{SD}}{\cong} H^0(S, \Theta_S)^\vee = 0$$

where SD denotes Serre duality. In particular, deformations of K3 surfaces are unobstructed.

For K3 surfaces over arbitrary fields, we have $h^2(\Theta_S) = h^{1,2} = h^{1,0}$ by Serre duality and $h^{0,1} = 0$ by our list of invariants. Over the complex numbers, vanishing of the former then follows easily from the Hodge symmetry $h^{1,0} = h^{0,1}$, which is induced by complex conjugation and thus, may not hold over arbitrary ground fields. The proof in positive characteristic of [R-S76] makes heavy use of purely characteristic-p-techniques, see Section 10.4. We note that over fields of positive characteristic and in dimension three the Bogomolov–Tian–Todorov unobstructedness theorem for Calabi–Yau varieties may fail, cf. [Hir99a] and [Sch04].

The vanishing $H^2(S, \Theta_S) = 0$ implies that K3 surfaces possess formal lifts over the Witt ring. Deligne [Del81a] showed in fact (see Section 11 for more on lifts):

Theorem 7.4 (Deligne) *K3 surfaces lift projectively to characteristic zero.*

The moduli space of polarized K3 surfaces in positive and mixed characteristic exists by a result of Rizov [Ri06]. However, it is still open, whether these moduli spaces are irreducible. What makes moduli spaces of (polarized) K3 surfaces so difficult to come by, is that no local or global Torelli theorems are known (except for supersingular K3 surfaces, see Section 9.8).

We come back to K3 surfaces in Section 9, where we discuss arithmetic conjectures and conjectural characterizations of unirational K3 surfaces.

7.3 Enriques surfaces

In characteristic $p \neq 2$ these surfaces have the following invariants:

$$\begin{array}{cccc} \omega_S \not\cong \mathcal{O}_S & \omega_S^{\otimes 2} \cong \mathcal{O}_S & p_g = 0 & h^{0,1} = 0 \\ \chi(\mathcal{O}_S) = 1 & c_2 = 12 & b_1 = 0 & b_2 = 10 \end{array}$$

Moreover, the canonical sheaf ω_S defines an étale double cover $\widetilde{S} \to S$, where $\widetilde{S}$ is a K3 surface. Also, there always exist elliptic or quasi-elliptic fibrations. Every such fibration has precisely two multiple fibers, both of which are not wild.

The most challenging case is characteristic 2, where Enriques surfaces are characterized by (here, $\equiv$ denotes numerical equivalence)

$$\omega_S \equiv \mathcal{O}_S \quad \chi(\mathcal{O}_S) = 1 \quad c_2 = 12 \quad b_1 = 0 \quad b_2 = 10$$

It turns out that $p_g = h^{0,1} \leq 1$, see [B-M3]. Since $b_1 = 0$, we conclude that the Picard scheme of an Enriques surface with $h^{0,1} = 1$ is not smooth. In this case, Frobenius induces a map $F : H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S)$, which is either zero or a bijection. We thus obtain three possibilities:

Definition 7.5 *An Enriques surface (in characteristic* 2*) is called*

1. classical *if* $h^{0,1} = p_g = 0$*, and*
2. non-classical *if* $h^{0,1} = p_g = 1$*. Such a surface is called*
   a) ordinary *if Frobenius acts bijectively on* $H^1(S, \mathcal{O}_S)$*, and*
   b) supersingular *if Frobenius is zero on* $H^1(S, \mathcal{O}_S)$*.*

All three types exist [B-M3]. We note that the terminology is inspired by Abelian surfaces, see Definition 7.2.

In any characteristic, every Enriques surface possesses elliptic or quasi-elliptic fibrations. Such a fibration always has multiple fibers. Moreover, if S is classical, then every (quasi-)elliptic fibration has precisely two multiple fibers, both of multiplicity two and neither of them is wild. If S non-classical, then there is only one multiple fiber, which is wild with multiplicity two. Finally, if S is non-classical and ordinary it does not possess quasi-elliptic fibrations. We refer to [C-D89, Chapter V.7] for details.

As explained in [B-M3, §3], every Enriques surface possesses a finite and flat morphism of degree two

$$\varphi : \widetilde{S} \to S$$

such that $\omega_{\widetilde{S}} \cong \mathcal{O}_{\widetilde{S}}$, $h^0(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = h^2(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = 1$, and $h^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = 0$, i.e., $\widetilde{S}$ is "K3-like". More precisely, in characteristic $\neq 2$, or, if S is non-classical and ordinary, then φ is étale of degree two and $\widetilde{S}$ is in fact a K3 surface. However, in the remaining cases in characteristic 2, $\widetilde{S}$ is only an integral Gorenstein surface that need not even be normal, since φ is a torsor under an infinitesimal group scheme. In any case and any characteristic, $\widetilde{S}$ is birational to the complete intersection of three quadrics in $\mathbb{P}^5$, see [Li10b], which generalizes results of Cossec [Co83] and Verra [Ve84].

Moreover, the moduli space of Enriques surfaces in characteristic $p \neq 2$ is irreducible, unirational, smooth and 10-dimensional. In characteristic 2, it consists of two irreducible, unirational, and 10-dimensional components, whose intersection is 9-dimensional. This intersection corresponds to non-classical supersingular surfaces, and outside their intersection the 10-dimensional components parametrize non-classical ordinary, and classical Enriques surfaces, respectively. We refer to [Li10b] for details, as well as [E-SB] for a complementary approach.

We refer to [B-M3], [La83b], [La88], [Li10b], and, of course, to [C-D89] for more details and partial classification results.

7.4 (Quasi-)hyperelliptic surfaces

In characteristic $p \neq 2, 3$, these surfaces have the following invariants:

$$\begin{array}{cccc} \omega_S \not\cong \mathcal{O}_S & \omega_S^{\otimes 12} \cong \mathcal{O}_S & p_g = 0 & h^{0,1} = 0 \\ \chi(\mathcal{O}_S) = 0 & c_2 = 0 & b_1 = 2 & b_2 = 2 \end{array}$$

Moreover, these surfaces are equipped with two elliptic fibrations: one is the Albanese fibration $S \to E$, where E is an elliptic curve, and then, there exists a second fibration $S \to \mathbb{P}^1$. It turns out that all these surfaces arise as quotients

$$S \,=\, (E \times F)/G\,,$$

where E and F are elliptic curves, and G is a group acting faithfully on both, E and F. The quotient yielding S is via the diagonal action. In particular, the classical list of Bagnera–DeFranchis (see [Ba01, List 10.27]) gives all classes.

The more complicated classes arise in characteristic 2 and 3. First, these surfaces have invariants (again, $\equiv$ denotes numerical equivalence)

$$\omega_S \equiv \mathcal{O}_S \quad \chi(\mathcal{O}_S) = 0 \quad c_2 = 0 \quad b_1 = 2 \quad b_2 = 2$$

It turns out that $1 \leq p_g + 1 = h^{0,1} \leq 2$, and that surfaces with $h^{0,1} = 2$ are precisely those with non-smooth Picard schemes [B-M3].

In any case, the Albanese morphism $S \to \mathrm{Alb}(S)$ is onto an elliptic curve, and its generic fiber is a curve of genus one. This motivates the following

Definition 7.6 *The surface is called* hyperelliptic*, if $S \to \mathrm{Alb}(S)$ is an elliptic fibration, and* quasi-hyperelliptic *if this fibration is quasi-elliptic.*

In both cases, there exists a second fibration $S \to \mathbb{P}^1$, which is always elliptic. Finally, for every (quasi-)hyperelliptic surface S, there exists

1. an elliptic curve E,
2. a curve C of arithmetic genus one, which is smooth if S is hyperelliptic, or rational with a cusp if S is quasi-hyperelliptic, and
3. a finite and flat group scheme G (possibly non-reduced), together with embeddings $G \to \mathrm{Aut}(C)$ and $G \to \mathrm{Aut}(E)$, where G acts by translations on E,

such that S is isomorphic to

$$S \,\cong\, (E \times C)/G\,.$$

The Albanese map arises as projection onto E/G with fiber C, and the other fibration onto $C/G \cong \mathbb{P}^1$ is elliptic with fiber E.

It turns out that G may contain infinitesimal subgroups, which gives rise to new cases even for hyperelliptic surfaces. In any case, it turns out that G is Abelian. This implies that the especially large and non-Abelian automorphism groups of elliptic curves with $j = 0$ in characteristic 2 and 3 (see Section

2.2) do *not* give rise to new classes. We refer to [B-M2] for the complete classification of hyperelliptic surfaces and to [B-M3] for the classification of quasi-hyperelliptic surfaces.

An interesting feature in characteristic 2 and 3 is the possibility that G acts trivially on $\omega_{E\times C}$, and thus, the canonical sheaf on S is trivial. In this case, we find $p_g = 1$, $h^{0,1} = 2$, and the Picard scheme of S is not reduced.

8 General type

In this section we discuss surfaces of general type, and refer to [BHPV, Chapter VII] and the references given there for the corresponding results over the complex numbers.

8.1 Pluricanonical maps

Let S be a minimal surface of general type. Clearly,

$$K_S^2 \,>\, 0$$

since some pluricanonical map has a two-dimensional image. However, we shall see below that Castelnuovo's inequality $c_2 > 0$ may fail. Let us recall that a *rational* (-2)*-curve* is a curve C on a surface with $C \cong \mathbb{P}^1_k$ and $C^2 = -2$.

Theorem 8.1 *Let S be a minimal surface of general type. Then, the (a priori) rational Iitaka fibration to the canonical model*

$$S \dashrightarrow S_{\mathrm{can}} := \mathrm{Proj}\, R_{\mathrm{can}}(S) \,=\, \mathrm{Proj}\, \bigoplus_{n\geq 0} H^0(S,\, \omega_S^{\otimes n})$$

is a birational morphism that contracts all rational (-2)-curves and nothing more.

Rational (-2)-curves form configurations, whose intersection matrices are negative definite. These matrices are Cartan matrices and correspond to Dynkin diagrams of type A, D and E. In particular, the morphism $S \to S_{\mathrm{can}}$ contracts these curves to DuVal singularities (also known as canonical singularities, or rational double points), see [Ba01, Chapter 3], as well as Artin's original papers [Ar62] and [Ar66].

Bombieri's results on pluricanonical systems were extended to positive characteristic in [Ek88], and refined in [SB91a] and [CFHR], and we refer to these articles for more results. We give a hint of how to modify the classical proofs below. Also, the reader who is puzzled by the possibility of purely inseparably uniruled surfaces of general type in the statements below might want to look at Section 9 first.

Theorem 8.2 (Ekedahl, Shepherd-Barron) *Let S be a minimal surface of general type and consider the linear system $|mK_S|$ on the canonical model S_{can}:*

1. *it is ample for $m \geq 5$ or if $m = 4$ and $K_S^2 \geq 2$ or $m = 3$ and $K_S^2 \geq 3$,*
2. *it is base-point free for $m \geq 4$ or if $m = 3$ and $K_S^2 \geq 2$,*
3. *it is base-point free for $m = 2$ if $K_S^2 \geq 5$ and $p \geq 11$ or $p \geq 3$ and S is not uniruled, and*
4. *it defines a birational morphism for $m = 2$ if $K_S^2 \geq 10$, S has no pencil of genus 2 curves and $p \geq 11$ or $p \geq 5$ and S is not uniruled.*

Next, we have the following version of Ramanujam-vanishing (see [Ek88, Theorem II.1.6] for the complete statement):

Theorem 8.3 (Ekedahl) *Let S be a minimal surface of general type and let $\mathcal{L}$ be an invertible sheaf that is numerically equivalent to $\omega_S^{\otimes i}$ for some $i \geq 1$. Then, $H^1(S, \mathcal{L}^\vee) = 0$ except possibly for certain surfaces in characteristic 2 with $\chi(\mathcal{O}_S) \leq 1$.*

On the other hand, minimal surfaces of general type with $H^1(S, \omega_S^\vee) \neq 0$ in characteristic 2 do exist [Ek88, Proposition I.2.14].

Bombieri's proof of the above results over the complex numbers is based on vanishing theorems $H^1(S, \mathcal{L}) = 0$ for certain more or less negative invertible sheaves. However, these vanishing results may fail in positive characteristic, see [Ra78] or Section 3.6. Ekedahl [Ek88] overcomes this problem as follows: he considers an invertible sheaf $\mathcal{L}$ and its Frobenius-pullback $F^*(\mathcal{L}) \cong \mathcal{L}^{\otimes p}$ as group schemes over S. Then, Frobenius induces a short exact sequence of group schemes (for the flat topology on S)

$$0 \to \alpha_{\mathcal{L}} \to \mathcal{L} \stackrel{F}{\to} F^*(\mathcal{L}) \to 0\,. \tag{1}$$

By definition, $\alpha_{\mathcal{L}}$ is the kernel of F, see also the definition of the group scheme α_p in Section 2.3. This $\alpha_{\mathcal{L}}$ is an infinitesimal group scheme over S and can be thought of as a possibly non-trivial family of α_p's over S.

Now, if $\mathcal{L}^\vee$ is ample, then $H^1(S, \mathcal{L}^{\otimes\nu}) = 0$ for $\nu \gg 0$ (Serre vanishing, see [Har77, Theorem III.7.6]). In order to get vanishing of $H^1(S, \mathcal{L})$, we assume that this is not the case and replace $\mathcal{L}$ by some $\mathcal{L}^{\otimes\nu}$ such that $H^1(S, \mathcal{L}) \neq 0$ and $H^1(S, \mathcal{L}^{\otimes p}) = 0$. Then, the long exact sequence in cohomology for (1) yields

$$H^1_{\mathrm{fl}}(S, \alpha_{\mathcal{L}}) \neq 0.$$

Such a cohomology class corresponds to an $\alpha_{\mathcal{L}}$-torsor, which implies that there exists a purely inseparable morphism of degree p

$$Y \stackrel{\pi}{\to} S,$$

where Y is an integral Gorenstein surface, whose dualizing sheaf satisfies $\omega_Y \cong \pi^*(\omega_S \otimes \mathcal{L}^{p-1})$. (The subscript fl in the cohomology group above denotes the flat topology, which is needed since $\alpha_{\mathcal{L}}$-torsors are usually only locally trivial with respect to the flat topology.)

For example, suppose S is of general type and $\mathcal{L} = \omega_S^{\otimes(-m)}$ for some $m \geq 1$. Then *either* $H^1(S, \mathcal{L}) = 0$ and one proceeds as in the classical case *or* there exists an inseparable cover $Y \to S$, where $\omega_Y^\vee$ is big and nef. The second alternative implies that S is inseparably dominated by a surface of special type, namely Y, and a further analysis of the situation leads *either* to a contradiction (establishing the desired vanishing result) *or* an explicit counter-example to a vanishing result. For example, Theorem 8.3 is proved this way.

8.2 Castelnuovo's inequality

Over the complex numbers, surfaces of general type satisfy *Castelnuovo's inequality* $c_2 > 0$. In [Ra78], Raynaud constructs minimal surfaces of general type with $c_2 < 0$ in every characteristic $p \geq 5$, i.e., this inequality fails. On the other, we have the following structure result:

Theorem 8.4 (Shepherd-Barron [SB91b]) *Let S be a minimal surface of general type.*

1. *If $c_2(S) = 0$, then S is inseparably dominated by a surface of special type.*
2. *If $c_2(S) < 0$, then the Albanese map $S \to \mathrm{Alb}(S)$ has one-dimensional image, whose generic fiber is a singular rational curve. In particular, S is uniruled.*

In characteristic $p \geq 11$, surfaces of general type satisfy $\chi(\mathcal{O}_S) > 0$.

We refer to [SB91b] for more detailed statements. There do exist surfaces of general type with $c_2 < 0$, but in view of Noether's formula $12\chi = c_1^2 + c_2$, one might ask whether the stronger inequality $\chi > 0$ still holds for surfaces of general type in positive characteristic. At least, we have the following analog of a theorem of Castelnuovo and DeFranchis:

Proposition 8.5 *Let S be a surface with $\chi(\mathcal{O}_S) < 0$. Then,*

1. *S is birationally ruled over a curve of genus $1 - \chi(\mathcal{O}_S)$, or*
2. *S is quasi-elliptic of Kodaira dimension $\kappa = 1$ and $p \leq 3$, or*
3. *S is a surface of general type and $p \leq 7$.*

Proof. If $\kappa = -\infty$, then S is birationally ruled over a curve of genus $1 - \chi(\mathcal{O}_S)$ and we get the first case. Also, by the explicit classification, there are no surfaces with $\kappa = 0$ and $\chi(\mathcal{O}_S) < 0$. For $\kappa = 2$, this is [SB91b, Theorem 8].

If $\kappa = 1$, then S admits a (quasi-)elliptic fibration $S \to B$, say with generic fiber F. Also, we may assume that S is minimal. In case F is smooth then [Dol72] yields $c_2(S) \geq e(F) \cdot e(B) = 0$, where e denotes the Euler number. Since $c_1^2(S) = 0$ for a relatively minimal (quasi-)elliptic fibration, Noether's formula yields $\chi(\mathcal{O}_S) = 0$. Thus, if $\chi(\mathcal{O}_S) < 0$, then the fibration must be quasi-elliptic and such surfaces exist for $p \leq 3$ only. □

Quasi-elliptic surfaces with $\chi(\mathcal{O}_S) < 0$ in characteristic $p \leq 3$ can be found in [Ra78], i.e., the first two cases of the previous proposition do exist. On the other hand, it is still unknown whether there do exist surfaces of general type with $\chi(\mathcal{O}_S) \leq 0$.

8.3 Noether's inequality

Every minimal surface of general type fulfills

$$K_S^2 \geq 2p_g(S) - 4 \qquad \textit{(Noether's inequality).}$$

Moreover, if the canonical map is composed with a pencil, then

$$K_S^2 \geq 3p_g(S) - 6 \qquad \textit{(Beauville's inequality)}$$

holds true. If the canonical map is birational onto its image, then

$$K_S^2 \geq 3p_g(S) - 7 \qquad \textit{(Castelnuovo's inequality)}$$

holds true, see [Li08b] and [Li10a]. In particular, this area of geography of surfaces of general type behaves as over the complex numbers.

We recall that surfaces that are extremal with respect to Noether's inequality are called *Horikawa surfaces*. More precisely, an *even Horikawa surface* is a minimal surface of general type with $K^2 = 2p_g - 4$, whereas an *odd Horikawa surface* satisfies $K^2 = 2p_g - 3$. These surfaces are classified in arbitrary characteristic in [Li08b] and [Li10a]. Basically, the same structure results as over the complex numbers hold for them: most of them arise as double covers of rational surfaces via their canonical maps. In characteristic 2, the canonical map may become purely inseparable, and then, the corresponding Horikawa surfaces are unirational, see also Section 9. We refer to the aforementioned articles for precise classification results, description of the moduli spaces, as well as the description of the subsets in these moduli spaces corresponding to surfaces with inseparable canonical maps. Finally, unirational Horikawa surfaces in characteristic $p \geq 3$ were systematically constructed in [L-S09].

Also, Beauville's result that minimal surfaces of general type with $K^2 < 3p_g - 7$ are double covers of rational surfaces via their canonical maps still holds in positive characteristic [Li10a].

8.4 Bogomolov–Miyaoka–Yau inequality

A minimal surface of general type over the complex numbers fulfills $K_S^2 \leq 9\chi(\mathcal{O}_S)$ or, equivalently, $K_S^2 \leq 3c_2(S)$. This is proved using analytic methods from differential geometry. Moreover, by a theorem of Yau, surfaces with $c_1^2 =$

$3c_2$ are uniformized by the complex 2-ball and thus, these surfaces are rigid by a theorem of Siu.

Minimal surfaces of general type with $c_2 \leq 0$ (counter-examples to Castelnuovo's inequality) provide counter-examples to the Bogomolov–Miyaoka–Yau inequality. But even if c_2 is positive, it may fail, as shown by Parshin [Pa72] and Szpiro [Sz79, Section 3.4.1]. More precisely, they construct series of examples, where c_2 is bounded and c_1^2 tends to infinity. Let us also mention the counter-examples of [BHH87, Kapitel 3.5.J], where covers of $\mathbb{P}^2$ ramified over special line configurations that only exist in positive characteristic are used. Similar constructions appeared in [Ea08].

Since Parshin's counter-examples have highly non-reduced Picard schemes, he asked in [Pa91], whether surfaces of general type with reduced Picard schemes satisfy the Bogomolov–Miyaoka–Yau inequality. Also this turns out to be wrong by the examples of Jang [Ja10].

In [Ek88, Remark (i) to Proposition 2.14], a 10-dimensional family of surfaces with $K^2 = 9$ and $\chi(\mathcal{O}_S) = 1$ in characteristic 2 is constructed, i.e., rigidity on the Bogomolov–Miyaoka–Yau line fails.

On the other hand, there is the following positive result [SB91a]

Theorem 8.6 (Shepherd-Barron) *If S is a minimal surface of general type in characteristic 2 that lifts over $W_2(k)$ then $c_1^2(S) \leq 4c_2(S)$ holds true.*

We refer to [SB91a] for results circling around Bogomolov's inequality $c_1^2(\mathcal{E}) \leq 4c_2(\mathcal{E})$ for stable rank 2 vector bundles.

8.5 Global vector fields

The tangent space to the automorphism group scheme of a smooth variety is isomorphic to the space of global vector fields. Since a surface of general type has only finitely many automorphisms, this implies that there are no global vector fields on a surface of general type in characteristic zero. However, in positive characteristic, the automorphism group scheme of a surface of general type has still finite length, but may contain infinitesimal subgroup schemes, which have non-trivial tangent spaces. Thus, infinitesimal automorphism group schemes of surfaces of general type in positive characteristic give rise to non-trivial global vector fields. For examples, we refer to Lang's article [La83a].

8.6 Non-classical Godeaux surfaces

Since $K_S^2 > 0$ for a minimal surface of general type, it is natural to classify surfaces with $K_S^2 = 1$. It turns out that these fulfill $1 \leq \chi(\mathcal{O}_S) \leq 3$ and thus, the lowest invariants possible are as follows:

Definition 8.7 *A* numerical Godeaux surface *is a minimal surface of general type with $\chi(\mathcal{O}_S) = K_S^2 = 1$. Such a surface is called* classical *if $p_g = h^{0,1} = 0$ and otherwise* non-classical.

In characteristic zero or in characteristic $p \geq 7$, numerical Godeaux surfaces are classical [Li09b]. Moreover, quotients of a quintic surface in $\mathbb{P}^3$ by a free $\mathbb{Z}/5\mathbb{Z}$-action (this construction is due to Godeaux) provide examples of classical Godeaux surfaces in characteristic $p \neq 5$. Classical and non-classical Godeaux surfaces in characteristic $p = 5$ have been constructed by Lang [La81] and Miranda [Mir84]. Non-classical Godeaux surfaces in characteristic $p = 5$ have been completely classified in [Li09b] - it turns out that all of them arise as quotients of (possibly highly singular) quintic surfaces in $\mathbb{P}^3$ by $\mathbb{Z}/5\mathbb{Z}$ or α_5. We finally note that non-classical Godeaux surfaces are precisely those numerical Godeaux surfaces that have non-reduced Picard schemes.

Quite generally, for every n there exists an integer $P(n)$ such that minimal surfaces of general type with $K^2 \leq n$ in characteristic $p \geq P(n)$ have a reduced Picard scheme [Li09a]. Thus, $P(1) = 7$, but $P(n)$ is unbounded as a function of n.

8.7 Surfaces with $\mathbf{p_g = 0}$

For a minimal surface of general type with $p_g = 0$ over the complex numbers, the inequality $\chi(\mathcal{O}_S) > 0$ forces $h^{0,1} = 0$, thus $\chi(\mathcal{O}_S) = 1$, and then, the Bogomolov–Miyaoka–Yau inequality implies $1 \leq K^2 \leq 9$. Interestingly, these (in-)equalities hold over any field:

Proposition 8.8 *Let S be a minimal surface of general type with $p_g(S) = 0$. Then, the equalities and inequalities*

$$h^{0,1}(S) = 0, \qquad \chi(\mathcal{O}_S) = 1, \qquad \textit{and} \qquad 1 \leq K_S^2 \leq 9$$

hold true.

PROOF. From $p_g = 0$ we get $\chi(\mathcal{O}_S) \leq 1$. Suppose first, that $\chi(\mathcal{O}_S) = 1$ holds. Then $h^{0,1} = 0$ and we find $b_1 = 0$, which yields $c_2 = 2 - 2b_1 + b_2 \geq 3$. But then, Noether's formula yields

$$K_S^2 \;=\; 12\chi(\mathcal{O}_S) - c_2(S) \;\leq\; 12 - 3 \;=\; 9,$$

which gives the desired (in-)equalities.

If $\chi(\mathcal{O}_S) \leq 0$, then Noether's formula implies $c_2(S) < 0$. By Theorem 8.4, the Albanese morphism $S \to \mathrm{Alb}(S)$ is a fibration over a curve. Thus, $b_2 \geq \rho(S) \geq 2$ using Igusa's inequality. Next, we have $h^{0,1} = 1 - \chi(\mathcal{O}_S)$ and in particular, $b_1 \leq 2(1 - \chi(\mathcal{O}_S))$. Thus,

$$c_2(S) \;=\; 2 - 2b_1 + b_2 \;\geq\; 2 \;-\; 4(1 - \chi(\mathcal{O}_S)) + 2 \;=\; 4\chi(\mathcal{O}_S).$$

But then, Noether's formula implies

$$12\chi(\mathcal{O}_S) \;=\; K_S^2 \;+\; c_2(S) \;\geq\; 4\chi(\mathcal{O}_S) \;+\; K_S^2$$

and we obtain

$$0 \geq 8\chi(\mathcal{O}_S) \geq K_S^2,$$

a contradiction. Thus, the $\chi(\mathcal{O}_S) \leq 0$-case cannot happen and we are done.

□

The first examples of algebraically simply connected surfaces of general type with $p_g = 0$ were constructed by Lee and Nakayama [L-N11], by adapting $\mathbb{Q}$-Gorenstein smoothing techniques to positive characteristic.

Theorem 8.9 (Lee–Nakayama) *There do exist algebraically simply connected surfaces of general type with $p_g = 0$, all values $1 \leq K^2 \leq 4$, and in all characteristics $p \geq 3$.*

9 Unirationality, supersingularity, finite fields, and arithmetic

In this and the following section we discuss more specialized characteristic-p topics. In this section, we circle around rationality, unirationality, their effect on Néron–Severi groups, and the formal Brauer group. We discuss these for K3 surfaces, and surfaces over finite fields. Finally, we discuss zeta functions and the Tate conjecture.

9.1 An instructive computation

To start with, let $\varphi : X \dashrightarrow Y$ be a dominant and generically finite morphism in characteristic zero. Then, the pull-back of a non-zero pluricanonical form is again a non-zero pluricanonical form. Thus, if $\kappa(X) = -\infty$, also $\kappa(Y) = -\infty$ holds true. However, over a field of positive characteristic p, the example

$$\varphi : t \mapsto t^p, \quad \text{and then} \quad \varphi^*(dt) = dt^p = pt^{p-1}dt = 0$$

shows that the pull-back of a non-zero pluricanonical form may become zero after pullback. In particular, the previous characteristic zero argument, which shows that the Kodaira dimension cannot increase under generically finite morphisms, breaks down.

However, if S is separably uniruled, i.e., if there exists a dominant rational map $\mathbb{P}^1 \times C \dashrightarrow S$ such that the finite field extension $k(S) \subset k(\mathbb{P}^1 \times C)$ is separable, then this phenomenon does not occur, we find $\kappa(S) = -\infty$ and applying Theorem 4.4, we conclude

Theorem 9.1 *A separably uniruled surface is birationally ruled.*

In particular, if a surface S is separably unirational, then $p_2(S) = 0$. But being dominated by a rational surface, its Albanese map is trivial and so $b_1(S) = 0$. Thus, S is rational by Theorem 4.6, and we have shown:

Theorem 9.2 *A separably unirational surface is unirational.*

9.2 Zariski surfaces

On the other hand, Zariski [Za58] gave the first examples of (inseparably) unirational surfaces in positive characteristic that are not rational: for a generic choice of a polynomial $f(x,y) \in k[x,y]$ of sufficiently large degree,

$$z^p - f(x,y) = 0 \qquad (2)$$

extends to an inseparable cover $X \to \mathbb{P}^2$, where X has "mild" singularities and where usually $\kappa(\widetilde{X}) \geq 0$ for some resolution of singularities $\widetilde{X} \to X$. By construction, we have an inclusion of function fields

$$k(x,y) \subset k(\widetilde{X}) = k(x,y)[\sqrt[p]{f(x,y)}] \subset k(\sqrt[p]{x}, \sqrt[p]{y})$$

i.e., $\widetilde{X}$ is unirational. Surfaces that arise as desingularizations of covers of the form (2) are called *Zariski surfaces*.

Theorem 9.3 (Zariski) *In every positive characteristic there do exist unirational surfaces that are not rational.*

However, we have seen in Theorem 4.6 that rational surfaces are still characterized as those surfaces that satisfy $h^{0,1} = p_2 = 0$.

9.3 Quasi-elliptic surfaces

If $S \to B$ is a quasi-elliptic fibration from a surface S with generic fiber F, then there exists a purely inseparable extension $L/k(B)$ of degree $p = \mathrm{char}(k)$, such that $F_L := F \times_{\mathrm{Spec}\, k(B)} \mathrm{Spec}\, L$ is not normal, i.e., the cusp "appears" over L, see [B-M3]. Thus, the normalization of F_L is isomorphic to $\mathbb{P}^1_L$, and we get the following result

Theorem 9.4 *Let S be a surface and $S \to B$ be a quasi-elliptic fibration. Then, there exists a purely inseparable and dominant rational map $B \times \mathbb{P}^1 \dashrightarrow S$, i.e., S is (purely inseparably) uniruled.*

In particular, if $S \to \mathbb{P}^1$ is a quasi-elliptic fibration, then S is a Zariski surface, and thus, unirational.

9.4 Fermat surfaces

If the characteristic $p = \mathrm{char}(k)$ does not divide n, then the *Fermat surface* S_n, i.e., the hypersurface

$$S_n := \{ x_0^n + x_1^n + x_2^n + x_3^n = 0 \} \subset \mathbb{P}^3_k$$

is smooth over k. For $n \leq 3$ it is rational, for $n = 4$ it is K3, and for $n \geq 5$ it is of general type. Shioda and Katsura have shown in [Sh74] and [K-S79] that

Theorem 9.5 (Katsura–Shioda) *For $n \geq 4$ and $p \nmid n$, the Fermat surface S_n in characteristic p is unirational if and only if there exists a $\nu \in \mathbb{N}$ such that $p^\nu \equiv -1 \mod n$.*

Shioda [Sh86] generalized this result to *Delsarte surfaces*. The example of Fermat surfaces shows that being unirational is quite subtle. Namely, one can show that the generic hypersurface of degree $n \geq 4$ in $\mathbb{P}^3_k$ is *not* unirational, and thus, being unirational is not a deformation invariant.

From the point of view of Mori theory it is interesting to note that unirational surfaces that are not rational are covered by *singular* rational curves. However, (unlike in characteristic zero) it is not possible to smoothen these families – after all, possessing a pencil of smooth rational curves implies that the surface in question is rational.

9.5 Fundamental group

There do exist geometric obstructions to unirationality: being dominated by a rational surface, the Albanese morphism of a unirational surface is trivial, and we conclude $b_1 = 0$. Moreover, Serre [Se59] showed that the fundamental group of a unirational surface is finite, and Crew [Cr84] that it does *not* contain p-torsion in characteristic p. A subtle invariant is the formal Brauer group (see Section 9.8 below), whose height can prevent a surface from being unirational (and that may actually be the only obstruction to unirationality for K3 surfaces).

9.6 Horikawa surfaces

Let us recall from Section 8.3 that a minimal surface of general type is called an *even Horikawa surface* if it satisfies $K^2 = 2p_g - 4$. This unbounded class is particularly easy to handle, since all such surfaces arise as double covers of rational surfaces. In view of the previous paragraph, let us also mention that they are algebraically simply connected. In [L-S09], we constructed unirational Horikawa surfaces in arbitrarily large characteristics and for arbitrarily large p_g. Thus, although the generic Horikawa surface is not unirational, being unirational is nevertheless a common phenomenon.

9.7 K3 surfaces and Shioda-supersingularity

We recall that the *Kummer surface* $\mathrm{Km}(A)$ of an Abelian surface A is the minimal desingularization of the quotient of A by the sign involution. In characteristic $p \neq 2$ the Kummer surface is always a K3 surface. Shioda [Sh77] determined when such surfaces are unirational - in particular, his result establishes the existence of unirational K3 surfaces in every characteristic $p \geq 3$:

Theorem 9.6 (Shioda) *Let A be an Abelian surface in characteristic $p \geq 3$. Then, the Kummer surface* $\mathrm{Km}(A)$ *is unirational if and only if A is a supersingular Abelian variety.*

We recall from Definition 7.2 that an Abelian variety is called supersingular if its p-torsion subgroup scheme $A[p]$ is infinitesimal.

To explain the notion of supersingularity introduced by Shioda [Sh74] let us recall from Section 3.5 that Igusa's inequality states $\rho \leq b_2$, where ρ denotes the rank of the Néron–Severi group and b_2 is the second Betti number.

Definition 9.7 *A surface S is called* supersingular in the sense of Shioda *if $\rho(S) = b_2(S)$ holds true.*

This notion is motivated by the following result, also from [Sh74]

Theorem 9.8 (Shioda) *Uniruled surfaces are Shioda-supersingular.*

The unirationality results on Kummer and Fermat surfaces show that these classes of surfaces are unirational if and only if they are supersingular in the sense of Shioda. This leads to the following

Conjecture 9.9 (Shioda) *A K3 surface is unirational if and only if it is Shioda-supersingular.*

Apart from Kummer surfaces (basically Theorem 9.6), this conjecture is known to be true in characteristic 2: the Néron–Severi lattices of Shioda-supersingular K3 surfaces have been classified in [R-S78] and using these results they show

Theorem 9.10 (Rudakov–Shafarevich) *Every Shioda-supersingular K3 surface in characteristic* 2 *possesses a quasi-elliptic fibration. In particular, these surfaces are Zariski surfaces and unirational.*

It is also is known in the following cases: for supersingular K3 surfaces with Artin invariant $\sigma_0 \leq 6$ (see Definition 9.13) and $p = 3$ [R-S78], for $\sigma_0 \leq 3$ and $p = 5$ [P-S06], and for elliptic K3 surfaces with p^n-torsion section [I-L10].

9.8 K3 surfaces and Artin-supersingularity

There exists yet another notion of supersingularity, apart from those of Definition 7.2 and Definition 9.7: for a K3 surface S over k, Artin [Ar74a] considers the functor that associates to every Artin-algebra A over k the Abelian group

$$\mathrm{Br} \,:\, A \mapsto \ker\left(H^2(S \times A, \mathcal{O}_{S\times A}^\times) \,\to\, H^2(S, \mathcal{O}_S^\times)\right) \,.$$

This functor is pro-representable by a one-dimensional formal group law, the so-called *formal Brauer group* $\widehat{\mathrm{Br}}(S)$ of S. (Of course, this functor can be studied for arbitrary varieties, not just K3 surfaces. Under suitable conditions, which are satisfied for K3 surfaces, it it pro-representable by a formal group law of dimension $h^{0,2}$, see [A-M77].)

Over a field of characteristic zero, there exists for every one-dimensional formal group law an isomorphism (the logarithm) to the additive formal group law $\widehat{\mathbb{G}}_a$. In positive characteristic, this need no longer be the case, and every formal group law has a discrete invariant, called the *height*. The height h is a strictly positive integer or infinity, and measures the complexity of multiplication by p in the group law. For example, $h = \infty$ means that multiplication by p is equal to zero. By a result of Lazard, h determines the formal group law if the ground field is algebraically closed. Over an algebraically closed field, a one-dimensional group law of height $h = 1$ is isomorphic to the multiplicative group law $\widehat{\mathbb{G}}_m$, whereas height $h = \infty$ corresponds to the additive group law $\widehat{\mathbb{G}}_a$. We refer to [Ha78] for more on formal group laws and to Artin's and Mazur's original article [A-M77] for applications to geometry.

For K3 surfaces, the height h of the formal Brauer group satisfies $1 \leq h \leq 10$ or $h = \infty$. This follows from the fact that $b_2 = 22$ together with the formula

$$\rho(S) \leq b_2(S) - 2h(S), \tag{3}$$

which holds if $h \neq \infty$, see [Ar74a]. Moreover, the height h stratifies the moduli space of K3 surfaces: K3 surfaces with $h = 1$ – these are called *ordinary* – are open in families, and surfaces with $h \geq h_0 + 1$ form a closed subset inside families of surfaces with $h \geq h_0$. We refer to [Ar74a], [Og01], and [G-K00] for more on the geometry of the height stratification of the moduli space.

Definition 9.11 *A K3 surface is called* supersingular in the sense of Artin*, if its formal Brauer group has infinite height.*

Shioda-supersingular K3 surfaces are Artin-supersingular, which follows from Formula (3). To prove the converse direction, one first reduces to the case of finite fields, where it follows from the Tate conjecture for K3 surfaces with $h = \infty$ (see the discussion below). For elliptic K3 surfaces with $h = \infty$, this latter conjecture was established by Artin [Ar74a], and for K3 surfaces possessing a degree 2 polarization by Rudakov, Shafarevich and Zink [RSZ82]. Finally, it was established by Charles [Ch12] and Maulik [Ma12] for every characteristic $p \geq 5$.

Theorem 9.12 *Let X be a K3 surface in characteristic $p > 0$. Assume that $p \geq 5$, or that X is elliptic, or that it possesses a degree 2 polarization. Then, X is Artin-supersingular if and only if it is Shioda-supersingular.*

To stratify the moduli space of Artin-supersingular K3 surfaces, we consider their Néron–Severi groups. The discriminant of the intersection form on $\mathrm{NS}(S)$ of an Artin-supersingular K3 surface S is equal to

$$\operatorname{disc} \mathrm{NS}(S) = \pm p^{2\sigma_0}$$

for some integer $1 \leq \sigma_0 \leq 10$ by [Ar74a].

Definition 9.13 *The integer* σ_0 *is called the* Artin invariant *of the Artin-supersingular K3 surface.*

In characteristic $p \geq 3$, Shioda-supersingular K3 surfaces with $\sigma_0 \leq 2$ are Kummer surfaces of supersingular Abelian surfaces. Their moduli space is one-dimensional but non-separated. Moreover, there is precisely one such surface with $\sigma_0 = 1$, and it arises as $\mathrm{Km}(E \times E)$, where E is a supersingular elliptic curve. We refer to [Sh79] and [Og79] for details, and to [Sch05] for the description in characteristic 2. From Theorem 9.6, it follows that Shioda-supersingular K3 surfaces with $\sigma_0 \leq 2$ are unirational (see also the discussion at the end of Section 9.7).

Ogus [Og79] established a Torelli theorem for Shioda-supersingular K3 surfaces with marked Picard lattices in terms of crystalline cohomology. Finally, we refer to [R-S81] for further results on K3 surfaces in positive characteristic.

9.9 Zeta functions and Weil Conjectures

If X is a smooth and projective variety of dimension d over a finite field $\mathbb{F}_q$, then we can count the number $\#X(\mathbb{F}_{q^n})$ of its $\mathbb{F}_{q^n}$-rational points and form its *zeta function* :

$$Z(X,t) \,:=\, \exp\left(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}\right).$$

Weil conjectured many properties of $Z(X,t)$, and it was Grothendieck's insight that many of these properties would follow from the existence of a suitable cohomology theory, namely ℓ-adic cohomology. These conjectures are now known to hold by work of Deligne, Dwork, Grothendieck, Weil and others - we refer to [Har77, Appendix C] for an overview and to [Mil80] for details. In particular, the zeta function is a rational function of the form

$$Z(X,t) \,=\, \frac{P_1(S,t)\,\cdot\,P_3(S,t)\,\cdot\,...\,\cdot\,P_{2d-1}(X,t)}{P_0(X,t)\,\cdot P_2(X,t)\,\cdot\,...\,\cdot\,P_{2d}(X,t)}\,,$$

where each $P_i(X,t)$ is a polynomial with integral coefficients, with constant term 1, and of degree equal to the i.th Betti number $b_i(X)$. In the extremal cases we have $P_0(X,t) = 1-t$ and $P_{2d}(X,t) = 1-q^d t$. Moreover, over the complex numbers, these polynomials factor as

$$P_i(X,t) \,=\, \prod_{j=1}^{b_i(X)} (1\,-\,\alpha_{ij}t)\,,$$

where the α_{ij} are complex numbers (in fact, algebraic integers) of absolute value $q^{i/2}$. Finally, there is a functional equation

$$Z(X,\frac{1}{q^d\,t}) \,=\, \pm\, q^{dE/2}\,t^E\,\cdot\,Z(X,t),$$

where E is the Euler number $c_d(X) = c_d(\Theta_X)$.

The Frobenius morphism $F_q : x \mapsto x^q$ acts trivially on $\mathbb{F}_q$, and topologically generates the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. It also induces an $\mathbb{F}_q$-linear morphism $F_q : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$. Now, an important characterization of $P_i(X,t)$ is that it is equal to the characteristic polynomial $\det(1 - t \cdot F_q^*)$ of the linear map F_q^* induced by F_q on $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$. In fact, taking this as definition for the $P_i(X,t)$'s, and noting that $\mathbb{F}_{q^n}$-rational points of X correspond to fixed-points under F_q^n, the rationality of the zeta-function and the specific form of its factors as given above follow from Lefschetz fixed-point formulae for powers of F_q^* on ℓ-adic cohomology [Har77, Appendix C.4].

There is an injective Chern map $c_1 : \mathrm{NS}(X_{\overline{\mathbb{F}}_q}) \otimes \mathbb{Q}_\ell \to H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, which is equivariant with respect to the Galois-actions of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on both sides. In particular, a non-torsion invertible sheaf $\mathcal{L} \in \mathrm{NS}(X)$ is Galois-invariant, and thus, $c_1(\mathcal{L})$ is a non-trivial and Galois-invariant class in $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$. Since Frobenius topologically generates $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, we see that $c_1(\mathcal{L})$ is an eigenvector of F_q^* for the eigenvalue q. In particular, q^{-1} is a root of $P_2(X,t)$ and thus, $(1 - qt)$ divides $P_2(X,t)$. Applying this argument to the whole of $\mathrm{NS}(X)$, we find that

$$(1 - qt)^{\rho(X)} \quad \text{divides} \quad P_2(X,t),$$

where $\rho(X)$ denotes the rank of $\mathrm{NS}(X)$. In [Ta65], Tate conjectured that the image $c_1(\mathrm{NS}(X) \otimes \mathbb{Q}_\ell)$ is not only a subspace, but is in fact equal to the whole eigenspace of F_q^* to the eigenvalue q. We shall now discuss this conjecture in greater detail:

9.10 Tate Conjecture

Let us now specialize to the case where $S := X$ is a smooth and projective surface over $\mathbb{F}_q$. For the factorization of $Z(S,t)$, we have $P_0(S,t) = 1 - t$, and $P_4(S,t) = 1 - q^2 t$. Since $H^1_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ is Galois-equivariantly isomorphic to $H^1_{\text{ét}}(\mathrm{Alb}(X)_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ via the Albanese morphism, we conclude $P_1(X,t) = P_1(\mathrm{Alb}(X), t)$. And finally, by Poincaré duality, we have $P_3(S,t) = P_1(S, qt)$. Thus, the "interesting" part of ℓ-adic cohomology and the zeta function is encoded in $P_2(S,t)$. We have also just seen that $(1 - qt)^{\rho(S)}$ divides $P_2(S,t)$.

Now, suppose for a moment that $S_{\overline{\mathbb{F}}_q}$ is Shioda-supersingular. After possibly replacing $\mathbb{F}_q$ by a finite extension, we may assume that all divisor classes of $S_{\overline{\mathbb{F}}_q}$ are defined over $\mathbb{F}_q$ and then, we have $P_2(S,t) = (1 - qt)^{b_2(S)}$. Moreover, if $\mathrm{Alb}(S)$ is trivial or a curve, then $Z(S,t)$ is equal to the zeta function of a birationally ruled surface. This fits perfectly to Conjecture 9.9. Also, one might expect that if a surface over $\mathbb{F}_q$ satisfies $P_2(S,t) = (1 - qt)^{b_2(S)}$, then it is Shioda-supersingular. This expectation would follow from the following, more general conjecture of Tate [Ta65]:

Conjecture 9.14 (Tate conjecture) *Let S be a smooth and projective surface over $\mathbb{F}_q$ and factor $P_2(S,t)$ as*

$$P_2(S,t) \;=\; \prod_{j=1}^{b_2(S)} (1-\alpha_{2,j}t), \qquad \text{with } \alpha_{2,j} \in \overline{\mathbb{Q}} \cap \mathbb{Z}.$$

Then, the Néron–Severi rank $\rho(S)$ is equal to the number of times q occurs among the $\alpha_{2,j}$.

For an overview, we refer to [Ul11, Lecture 2]. For a relation of Tate's conjecture with Igusa's inequality and a conjecture of Artin and Mazur on Frobenius eigenvalues on crystalline cohomology, we refer to [Ill79, Remarque II.5.13]. Also, Artin and Tate [Ta68, (C)] refined Tate's conjecture as follows: let $D_1, ..., D_\rho$ be independent classes in $\mathrm{NS}(S)$ and set $B := \sum_i \mathbb{Z}D_i$. Let $\#\mathrm{Br}(S)$ be the order of the Brauer group, which is conjecturally finite. Then

Conjecture 9.15 (Artin–Tate) *We have*

$$P_2(S,q^{-s}) \;\sim\; (-1)^{\rho(S)-1} \cdot \frac{\#\mathrm{Br}(S) \cdot \det(\{D_i \cdot D_j\}_{i,j})}{q^{\chi(\mathcal{O}_S)-1+b_1(S)} \cdot (\mathrm{NS}(S):B)^2} \cdot (1-q^{1-s})^{\rho(S)}$$

as s tends to 1

In fact, Conjectures 9.14 and 9.15 are equivalent, as shown up to p-power by Artin and Tate [Ta68] and the full equivalence was established by Milne [Mil75].

For elliptic surfaces, the Artin–Tate conjecture is a function field analog of the Birch–Swinnerton-Dyer conjecture, see [Ta68] and [A-S73]. For explicit examples, progress on this conjecture and interrelations, we refer to [Ul11].

The Tate conjecture is known in the following cases:

Theorem 9.16 (Tate [Ta66]) *The conjectures of Tate and Artin–Tate hold for Abelian varieties and products of curves over finite fields.*

Let us discuss what is known for K3 surfaces: for elliptic K3 surfaces, it was established by Artin and Swinnerton-Dyer [A-S73]. For ordinary K3 surfaces, it was established by Nygaard [Ny83], and for K3's with finite height of the formal Brauer group and $p \geq 5$ by Nygaard and Ogus [N-O85]. For K3 surfaces of infinite height (Artin-supersingular), equipped with a polarization of degree 2 it was established by Rudakov, Shafarevich and Zink [RSZ82], if p is large with respect to a polarization degree by Maulik [Ma12], and for $p \geq 5$ by Charles [Ch12]. Thus, we obtain

Theorem 9.17 *The conjectures of Tate and Artin–Tate hold for K3 surfaces in characteristic $p \geq 5$.*

By [LMS11], this implies that there exist only finitely many K3 surfaces defined over a fixed finite field of characteristic $p \geq 5$. This is similar to the situation for Abelian varieties: by [Za77], there exist only finitely many Abelian varieties of a fixed dimension over a fixed finite field.

Coming back to $P_2(S,t)$, we note that Poincaré duality implies that if β is among the $\alpha_{2,j}$, then so is q/β. For K3 surfaces, using the fact that $\deg P_2(S,t) = b_2(S) = 22$ is even, this has the following surprising consequence (see [BHT11, Theorem 13] for a proof)

Theorem 9.18 (Swinnerton-Dyer) *Let S be a K3 surface over $\mathbb{F}_q$, and assume that the Tate-conjecture holds for S. Then, the geometric Néron–Severi rank $\rho(S_{\overline{\mathbb{F}}_q})$ is even.*

Interestingly, there are more restrictions on $P_2(S,t)$ if S is a K3 surface, than those coming from the Weil conjectures, see [Za93] and [E-J10].

Let us finally note that if we have an $\alpha_{2,j}$ in the factorization of $P_2(S,t)$ of some surface S over $\mathbb{F}_q$ that is not of the form $\mu \cdot q$, where μ is a root of unity, then $S_{\overline{\mathbb{F}}_q}$ is *not* Shioda-supersingular, and thus, not unirational. For example, the zeta function of a Fermat surface $S_n \subset \mathbb{P}^3$ over $\mathbb{F}_p$ can be computed explicitly using Gauß- and Jacobi-sums. From this, one concludes that if $(S_n)_{\overline{\mathbb{F}}_p}$ is Shioda-supersingular, then there must exist a ν such that $p^\nu \equiv -1 \mod n$, see [K-S79] or Theorem 9.5.

10 Inseparable morphisms and foliations

In this section we study inseparable morphisms of height one in greater detail. On the level of function fields this is Jacobson's correspondence, a kind of Galois correspondence for purely inseparable field extensions. However, this correspondence is not via automorphisms but via derivations. On the level of geometry, this translates into p-closed foliations. For surfaces, it simplifies to p-closed vector fields. For other overviews, we refer to [Ek87] and [Miy97, Lecture III].

10.1 Jacobson's correspondence

Let us recall the classical Galois correspondence: given a field K and a finite and *separable* extension L, there exists a minimal Galois extension of K containing L, the Galois closure K_{gal} of L. By definition, the Galois group $G = \mathrm{Gal}(K_{\mathrm{gal}}/K)$ of this extension is the group of automorphism of K_{gal} over K, which is finite of degree equal to $[K_{\mathrm{gal}} : K]$. Finally, there is a bijective correspondence between subgroups of G and intermediate fields $K \subseteq M \subseteq K_{\mathrm{gal}}$. In particular, there are only finitely many fields between K and K_{gal}.

In Section 2.1 we encountered extensions of height one of a field K. It turns out that automorphism of purely inseparable extensions are trivial, and thus give no insight into these extensions. However, there does exist a Galois-type correspondence for such extensions, *Jacobson's correspondence* [Jac64, Chapter IV]. Instead of automorphisms, one studies *derivations* over K:

Namely, let L be a purely inseparable extension of height one of K, i.e., $K \subseteq L \subseteq K^{p^{-1}}$, or, equivalently, $L^p \subseteq K$. We remark that $K^{p^{-1}}$ plays the role of a Galois closure of L. Next, we consider the Abelian group

$$\mathrm{Der}(L) \;:=\; \{\delta : K^{p^{-1}} \to K^{p^{-1}}\,,\;\; \delta \text{ is a derivation and } \delta(L) = 0\}.$$

Since $\delta(x^p) = p \cdot x^{p-1} \cdot \delta(x) = 0$, these derivations are automatically K-linear and thus, $\mathrm{Der}(L)$ is a K-vector space. Also, $\mathrm{Der}(L)$ is a subvector space of $\mathrm{Der}(K)$. In case K is of finite transcendence degree n over some perfect field k, then $\mathrm{Der}(K)$ is n-dimensional.

Now, these vector spaces carry more structure: if δ and η are derivations, then in general their composition $\delta \circ \eta$ is no derivation, which is why one studies their *Lie bracket*, i.e., the commutator $[\delta, \eta] = \delta \circ \eta - \eta \circ \delta$, which is again a derivation. Now, over fields of positive characteristic p it turns out that the p-fold composite $\delta \circ ... \circ \delta$ is again a derivation. The reason is that expanding this composition the binomial coefficients occurring that usually prevent this composition from being a derivation are all divisible by p, i.e., vanish. This p-power operation is denoted by $\delta \mapsto \delta^{[p]}$. It turns out that the K-vector spaces $\mathrm{Der}(K)$ and $\mathrm{Der}(L)$ are closed under the Lie bracket, as well as the p-power operation.

Definition 10.1 *A* p-Lie algebra *or* restricted Lie algebra *is a Lie algebra over a field of characteristic p together with a p-power map $\delta \mapsto \delta^{[p]}$ satisfying the axioms in [Jac62, Definition 4 of Chapter V.7].*

We refer to [Jac62, Chapter V.7] for general results on p-Lie algebras.

So far, we have associated to every finite and purely inseparable extension L/K of height one a sub-p-Lie algebra of $\mathrm{Der}(K)$. Conversely, given such a Lie algebra $(V, -^{[p]})$, we may form the fixed set

$$(K^{p^{-1}})^{(V, -^{[p]})} \;:=\; \{x \in K^{p^{-1}} \,|\, \delta(x) = 0 \;\forall \delta \in V\}\,,$$

which is easily seen to be a field. Since elements of V are K-linear derivations, this field contains K. Moreover, by construction, it is contained in $K^{p^{-1}}$, i.e., of height one.

Theorem 10.2 (Jacobson) *There is a bijective correspondence*

$$\{ \textit{ height one extensions of } K \;\} \leftrightarrow \{\; \textit{sub-}p\textit{-Lie algebras of } \mathrm{Der}(K) \;\}\,.$$

Let us mention one important difference to Galois theory: suppose K is of transcendence degree n over an algebraically closed field k, e.g., the function field of an n-dimensional variety over k. Then, the extension $K^{p^{-1}}/K$ is finite of degree p^n. For $n \geq 2$ there are infinitely many sub-p-Lie algebras of $\mathrm{Der}(K)$ and thus, *infinitely* many fields between K and $K^{p^{-1}}$.

10.2 Curves

Let C be a smooth projective curve over a perfect field k with function field $K = k(C)$. Then, the purely inseparable field extension $K^p \subset K$ is of degree p and corresponds to the k-linear Frobenius morphism $F : C \to C^{(p)}$.

Since every purely inseparable extension L/K of degree p is of the form $L = K[\sqrt[p]{x}]$ for some $x \in L$, such extensions are of height one, i.e., $K \subseteq L \subseteq K^{p^{-1}}$. Simply for degree reasons, we see that the k-linear Frobenius morphism is the only purely inseparable morphism of degree p between normal curves. Since every finite purely inseparable field extension can be factored successively into extensions of degree p, we conclude

Proposition 10.3 *Let C and D be normal curves over a perfect field k and let $\varphi : C \to D$ be a purely inseparable morphism of degree p^n. Then, φ is the n-fold composite of the k-linear Frobenius morphism.*

10.3 Foliations

From dimension two on there are many more purely inseparable morphisms than just compositions of Frobenius. In fact, if X is an n-dimensional variety with $n \geq 2$ over an algebraically closed field k, then the k-linear Frobenius morphism has degree p^n and it factors over infinitely many height one morphisms.

To classify height one morphisms $\varphi : X \to Y$ from a fixed smooth variety X over a perfect field k, we geometrize Jacobson's correspondence as follows:

Definition 10.4 *A* (p-closed) foliation *on a smooth variety X is a saturated subsheaf $\mathcal{E}$ of the tangent sheaf Θ_X that is closed under the Lie bracket ($\mathcal{E}$ is* involutive*) and the p-power operation.*

Then, Jacobson's correspondence translates into

Theorem 10.5 *There is a bijective correspondence*

$$\left\{ \begin{array}{l} \textit{finite morphisms } \varphi : X \to Y \\ \textit{of height one with } Y \textit{ normal} \end{array} \right\} \leftrightarrow \{ \textit{ foliations in } \Theta_X \}$$

The saturation assumption is needed because an involutive and p-closed subsheaf and its saturation (which will also be involutive and p-closed) define the same extension of function fields, and thus, correspond to the same normal variety. We refer to [Ek87] or [Miy97, Lecture III] for details.

Let us also mention [Miy97, Lecture III.2], where a connection between p-closed foliations and non-stability of tangent bundles, and uniruledness of varieties (not only in positive characteristic, but also in characteristic zero!) is discussed.

10.4 Surfaces

In order to describe finite morphisms of height one $\varphi : X \to Y$ from a smooth surface onto a normal surface, we have to consider foliations inside Θ_X. The sheaf Θ_X and its zero subsheaf correspond to the k-linear Frobenius morphism and the identity, respectively. Thus, height one-morphisms of degree p correspond to foliations of rank one inside Θ_X.

To simplify our exposition, let us only consider $\mathbb{A}^2_k$, i.e., $X = \operatorname{Spec} R$ with $R = k[x,y]$ and assume that k is perfect. Then, Θ_X corresponds to the R-module generated by $\partial/\partial x$ and $\partial/\partial y$. Now, a finite morphism of height one $\varphi : X \to Y$ with Y normal corresponds to a ring extension

$$R^p = k[x^p, y^p] \subseteq S \subseteq R = k[x,y]\,,$$

where S is normal. By Jacobson's correspondence, giving S is equivalent to giving a foliation inside Θ_X, which will be of rank one if $S \neq R, R^p$. This amounts to giving a regular vector field

$$\delta \,=\, f(x,y)\,\frac{\partial}{\partial x} \,+\, g(x,y)\,\frac{\partial}{\partial y}$$

for some $f, g \in R$. Since the Lie bracket of a 1-dimensional Lie algebra is zero, every rank one subsheaf of Θ_X is involutive. Thus, we only have to check closedness under the p-power operation, which translates into

$$\delta^{[p]} \,=\, h(x,y)\cdot\delta \qquad \text{for some} \qquad h(x,y)\in R,$$

i.e., δ is a *p-closed vector field*.

We may assume that f and g are coprime. Then, the zero set of the ideal (f,g) is of codimension two and is called the *singular locus* of the vector field. It is not difficult to see that S is smooth over k outside the singular locus of δ, cf. [R-S76].

Finally, a purely inseparable morphism $\varphi : X \to Y$ is everywhere ramified, i.e., $\Omega_{X/Y}$ has support on the whole of X. Nevertheless, the canonical divisor classes of X and Y are related by a kind of Riemann–Hurwitz formula and the role of the ramification divisor is played by a divisor class that can be read off from the foliation, see [R-S76].

As an application, let us give the main result of [R-S76]: let S be a K3 surface over k, and suppose that we had $H^0(S, \Theta_S) \neq 0$. Then, there exists in fact a $0 \neq \delta \in H^0(S, \Theta_S)$ that is p-closed. As explained above, this δ gives rise to an inseparable morphism $S \to S/\delta$. A careful analysis of the hypothetical quotient S/δ and its geometry finally leads to a contradiction, and we conclude $H^0(S, \Theta_S) = 0$, which proves Theorem 7.3.

10.5 Quotients by group schemes

Let X be a smooth but not necessarily proper variety of any dimension over a perfect field k. We have seen that a global section $0 \neq \delta \in H^0(X, \Theta_X)$ gives rise to an inseparable morphism of degree p and height one if and only if δ is p-closed, i.e., $\delta^{[p]} = c \cdot \delta$ for some $c \in H^0(X, \mathcal{O}_X)$. Now, if X is proper over k, then $c \in H^0(X, \mathcal{O}_X) = k$, and after rescaling δ, we may in fact assume $c = 1$ or $c = 0$.

Definition 10.6 *A vector field δ is called* multiplicative *if $\delta^{[p]} = \delta$ and it is called* additive *if $\delta^{[p]} = 0$.*

Let δ be additive or multiplicative. Applying a (truncated) exponential series to δ, one obtains on X an action of some finite and flat group scheme G, which is infinitesimal of length p, see [Sch05, Section 1]. Then, the inseparable morphism $\varphi : X \to Y$ corresponding to δ is the quotient morphism $X \to X/G$. Moreover, the 1-dimensional p-Lie algebra generated by δ is the p-Lie algebra of G, i.e., the Zariski tangent space of G with p-power map coming from Frobenius. We recall from Theorem 2.8 that the only infinitesimal group schemes of length p are α_p and μ_p. Putting these observations together, we obtain

Proposition 10.7 *Additive (resp., multiplicative) vector fields correspond to purely inseparable morphisms of degree p that are quotients by α_p- (resp., μ_p-) actions.*

This also explains the terminology for these vector fields: α_p (resp., μ_p) is a subgroup scheme of the additive group $\mathbb{G}_a$ (resp., multiplicative group $\mathbb{G}_m$).

10.6 Singularities

Let us finally assume that X is a smooth surface and let δ be a multiplicative vector field. By [R-S76], such a vector field can be written near a singularity in local coordinates x, y as

$$\delta = x\frac{\partial}{\partial x} + a \cdot y\frac{\partial}{\partial y} \quad \text{for some} \quad a \in \mathbb{F}_p^\times .$$

Let $\varphi : X \to Y$ be the inseparable morphism corresponding to δ. In [Hir99b] it is shown that Y has toric singularities of type $\frac{1}{p}(1, a)$. Thus, quotients by μ_p behave very much like cyclic quotient singularities in characteristic zero. On the other hand, quotients by α_p are much more complicated - the singularities need not even be rational and we refer to [Li08a] for examples.

11 Witt vectors and lifting

This section deals with lifting to characteristic zero. There are various notions of lifting, and the nicest ones are are projective lifts over the Witt ring. For example, in the latter case Kodaira vanishing and degeneracy of the Frölicher spectral sequence hold true. Unfortunately, although such lifts exist for curves, they do not exist in general in dimension at least two.

11.1 Witt vectors

Let k be a field of positive characteristic p. Moreover, assume that k is perfect, e.g., algebraically closed or a finite field.

Then, one can ask whether there exist rings of characteristic zero having k as residue field. It turns out that there exists a particularly nice ring $W(k)$, the so-called *Witt ring*, or *ring of Witt vectors*, which has the following properties:

1. $W(k)$ is a discrete valuation ring of characteristic zero,
2. the unique maximal ideal $\mathfrak{m}$ of $W(k)$ is generated by p and the residue field $R/\mathfrak{m}$ is isomorphic to k,
3. $W(k)$ is complete with respect to the $\mathfrak{m}$-adic topology,
4. the Frobenius map $x \mapsto x^p$ on k lifts to a ring homomorphism of $W(k)$,
5. there exists an additive map $V : W(k) \to W(k)$, called *Verschiebung* (German for "shift"), which is zero on the residue field k and such that multiplication by p on $W(k)$ factors as $p = F \circ V = V \circ F$, and finally
6. every complete discrete valuation ring with quotient field of characteristic zero and residue field k contains $W(k)$ as subring.

We remark that the last property characterizes $W(k)$ up to isomorphism.

To obtain $W(k)$, one constructs successively rings $W_n(k)$, which are local Artin rings of length n with residue field k. One has $W_1(k) = k$ and surjective projection maps $W_{n+1}(k) \to W_n(k)$. By definition, $W(k)$ is the projective limit over the $W_n(k)$, cf. [Se68, Chapitre II.6]. The main example to bear in mind is the following:

Example 11.1 *For the finite field $\mathbb{F}_p$ we have $W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}$ and thus,*

$$W(\mathbb{F}_p) = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

is isomorphic to $\mathbb{Z}_p$, the ring of p-adic integers. The maximal ideal of $W(\mathbb{F}_p)$ is generated by p and $W(\mathbb{F}_p)$ is complete with respect to the p-adic topology. In this special case, F is the identity on $W(\mathbb{F}_p)$ and V is multiplication by p.

Witt's construction $W(-)$ makes sense for every commutative ring R. However, already $W(k)$ for a non-perfect field k is not Noetherian, and its maximal ideal is not generated by p. This is why we will assume k to be perfect for the rest of this section. We refer to [Se68, Chapitre II.6] and [Ha78] for more on Witt vectors.

11.2 Lifting over the Witt ring

Let X be a scheme of finite type over some perfect field k of positive characteristic p. Then, there are different notions of what it means to *lift X to characteristic zero*. To make it precise, let R be a ring of characteristic zero with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} \cong k$. For example, we could have $R = W(k)$ and $\mathfrak{m} = (p)$.

Definition 11.2 *A* lift (resp. formal lift) of X over R *is a scheme (resp. formal scheme)* $\mathcal{X}$ *of finite type and flat over* $\operatorname{Spec} R$ *(resp.* $\operatorname{Spf} R$*) with special fiber* X*.*

In case $R = W(k)$, i.e., if X admits a (formal) lift over the Witt ring, many "characteristic p pathologies" cannot happen. We have already encountered the following results in Section 3:

1. if X is of dimension $d \leq p$ and lifts over $W_2(k)$ then its Frölicher spectral sequence from Hodge to deRham-cohomology degenerates at E_1 by a result of Deligne and Illusie, see [D-I87] and [Ill02, Corollary 5.6],
2. if X is of dimension $d \leq p$ and lifts over $W_2(k)$, then ample line bundles satisfy Kodaira vanishing, see [D-I87] and [Ill02, Theorem 5.8], and
3. if X lifts over $W(k)$, then crystalline cohomology coincides with deRham-cohomology of $\mathcal{X}/W(k)$.

Actually, the last property is the starting point of crystalline cohomology, see the discussion in Section 3.8.

Example 11.3 *Smooth curves and birationally ruled surfaces lift over the Witt ring by Grothendieck's existence theorem [Ill05, Theorem 5.19].*

11.3 Lifting over more general rings

Let R be an integral ring with maximal ideal $\mathfrak{m}$, residue field $R/\mathfrak{m} \cong k$, and quotient field K of characteristic zero. Let X be a smooth projective variety over k, let $\mathcal{X}$ be a lift of X over Spec R, and denote its generic fiber by $\mathcal{X}_K \to \operatorname{Spec} K$.

After choosing a DVR dominating $(R, \mathfrak{m})$ and after passing to the $\mathfrak{m}$-adic completion, we may assume that $(R, \mathfrak{m})$ is a local and $\mathfrak{m}$-adically complete DVR. By the universal property of the Witt ring, R contains $W(k)$ and $\mathfrak{m}$ lies above $(p) \subset W(k)$. Thus, it makes sense to talk about the *ramification index*, usually denoted by e, of R over $W(k)$. This ramification index is an absolute invariant of R.

To give a flavor of the subtleties that occur when dealing with lifting problems, let us mention the following examples

1. Abelian varieties admit formal lifts over the Witt ring by an unpublished result of Grothendieck [Ill05, Theorem 5.23]. However, to obtain algebraic lifts, one would like to have an ample line bundle on a formal lift in order to apply Grothendiecks' existence theorem, see [Ill05, Theorem 4.10]. However, even if one succeeds in doing so, this is usually at the prize that this new formal lift (which then is algebraic) may exist over a *ramified* extension of the Witt ring only. For Abelian varieties, this was established by Mumford [Mu69b], and Norman and Oort [N-O80].

2. K3 surfaces have unobstructed deformations by Theorem 7.3, and thus, admit formal lifts over the Witt ring. Deligne [Del81a] has shown that one can lift with every K3 surface also an ample line bundle, which gives an algebraic lifting - again at the prize that this lift may exist over ramified extensions of the Witt ring only.
3. By results of Lang [La83b], Illusie [Ill79], Ekedahl and Shepherd-Barron [E-SB], and [Li10b], Enriques surfaces - even in characteristic 2 - lift to characteristic zero. However, the Frölicher spectral sequence of a supersingular Enriques surface in characteristic 2 does not degenerate at E_1 by [Ill79, Proposition II.7.3.8]. Thus, these latter surfaces only lift over *ramified* extensions of the Witt ring, but not over the Witt ring itself.
4. Lang [La95] gave examples of hyperelliptic surfaces that lift to a ramified extension of $W(k)$ of ramification index $e = 2$, but whose Frölicher spectral sequences do not degenerate at E_1. Thus, these surfaces do not lift over $W(k)$. Rather subtle examples of non-liftable smooth elliptic fibrations were given by Partsch [Pa10].

However, even if X lifts "only" over a ramified extension of the Witt ring, this does imply something: flatness of $\mathcal{X}$ over $\operatorname{Spec} R$ implies that $\chi(\mathcal{O})$ of special and generic fiber coincide, and smoothness of $\mathcal{X}$ over $\operatorname{Spec} R$ implies that the ℓ-adic Betti numbers of special and generic fiber coincide. For surfaces, we have additional results from [K-U85, Section 9]:

Theorem 11.4 (Katsura–Ueno) *Let $\mathcal{S}$ be a lift of the smooth projective surface S over* $\operatorname{Spec} R$ *with generic fiber $\mathcal{S}_K$. Then,*

$$\begin{array}{rclcrcl} b_i(S) &=& b_i(\mathcal{S}_K) & \qquad & c_2(S) &=& c_2(\mathcal{S}_K) \\ \chi(\mathcal{O}_S) &=& \chi(\mathcal{O}_{\mathcal{S}_K}) & & K_S^2 &=& K_{\mathcal{S}_K}^2 \\ \kappa(S) &=& \kappa(\mathcal{S}_K) & & & & \end{array}$$

Moreover, S is minimal if and only $\mathcal{S}_K$ is minimal.

If S is of general type then $P_n(S) = P_n(\mathcal{S}_K)$ for $n \geq 3$ since these numbers depend only on χ and K^2 by Riemann–Roch and [Ek88, Theorem II.1.7]. However, in general, $p_g(S)$ may differ from $p_g(\mathcal{S}_K)$, as the examples in [Se58a] and [Suh08] show. More precisely, Hodge invariants are semi-continuous, i.e., in general we have

$$h^{i,j}(S) \geq h^{i,j}(\mathcal{S}_K) \quad \text{for all} \quad i,j \geq 0\,.$$

In case of equality for all i, j, the Frölicher spectral sequence of S degenerates at E_1. Theorem 11.4 implies that from dimension two on there exist smooth projective varieties that do not admit any sort of lifting, namely:

Examples 11.5 *Let S be*

1. *a minimal surface of general type with $K_S^2 > 9\chi(\mathcal{O}_S)$, i.e., violating the Bogomolov–Miyaoka–Yau inequality (see Section 8.4), or*

2. a quasi-elliptic surface with $\kappa(S) = 1$ and $\chi(\mathcal{O}_S) < 0$ (see Section 8.2).

Then, S does not admit an algebraic lifting whatsoever, i.e., not even over a ramified extension of the Witt ring. The first example of such a smooth and projective variety that does not admit an algebraic lifting is due to Serre [Se61].

For this and related questions, see also [Ill05, Section 5F]. Moreover, we have the following highly non-explicit result: namely, "Murphy's law" holds for moduli spaces of surfaces of general type with very ample canonical sheaves [Va06]. Thus, we can find any kind of obstructed lifting behavior already on surfaces, for example:

Theorem 11.6 (Vakil) *For every integer $n > 0$ and every prime $p > 0$, there exists a smooth and projective surface over $\mathbb{F}_p$ that lifts over $W_n(\mathbb{F}_p)$ but not over $W_{n+1}(\mathbb{F}_p)$.*

11.4 Birational nature

One can also ask to what extent liftability is a birational invariant. If X and Y are smooth, proper, and birational varieties of dimension at most 2, then their lifting behavior is the same. However, in dimension ≥ 3, or when allowing canonical singularities in dimension 2, this is no longer the case. We refer to [L-S12] for details, some positive results, and (counter-)examples.

11.5 Canonical lifts

For an *ordinary* Abelian variety or K3 surface, there even exists a distinguished formal lift over the Witt ring, the *canonical lift*, or *Serre–Tate lift*. Quite generally, ordinary means that Newton- and Hodge-polygons on crystalline cohomology coincide, and we note that this property is open in equi-characteristic families. For a g-dimensional Abelian variety A over a field k of characteristic p, being ordinary is equivalent to $A[p](\overline{k}) \cong (\mathbb{Z}/p\mathbb{Z})^g$, which is the maximum possible (see also Definition 7.2). For a K3 surface S, being ordinary is equivalent to $h(\widehat{\mathrm{Br}}(S)) = 1$, see Section 9.8. We refer to [Me72] for details on canonical lifts of ordinary Abelian varieties. For ordinary Abelian varieties, this canonical lift is characterized by the property that the Frobenius morphism lifts. For the general case, we refer to [Del81b] and [Ka81]. Finally, K3 surfaces with $h(\widehat{\mathrm{Br}}(S)) < \infty$ still possess *quasi-canonical lifts*, which has been used to prove the Tate conjecture for them, see [N-O85] and Section 9.10.

12 Rational curves on K3 surfaces

In the final section we give an application of characteristic-p and lifting techniques to a characteristic zero conjecture. Namely, we show how infinitely many rational curves on complex projective K3 surfaces of odd Picard rank can be established by reduction modulo p, then finding the desired rational curves over finite fields, and eventually lifting cycles of them to characteristic zero.

12.1 Rational curves

Let C be a smooth projective curve of genus g over an algebraically closed field k. Then, the Riemann–Hurwitz formula implies that if there exists a non-constant map $\mathbb{P}^1 \to C$ then $g = 0$, i.e., $C \cong \mathbb{P}^1$. Similarly, one can ask about non-constant maps from $\mathbb{P}^1$ to higher dimensional varieties, i.e., whether they exist and if so, how many, whether they move in families, etc. First of all, let us introduce the following notion

Definition 12.1 *A* rational curve *on a variety X is a reduced and irreducible curve $C \subset X$ whose normalization is isomorphic to $\mathbb{P}^1$.*

Let us study rational curves on surfaces in detail: clearly, if S is a non-minimal surface, then every exceptional (-1)-curve is a rational curve.

Also, since surfaces with $\kappa(S) = -\infty$ are birationally ruled by Theorem 4.4, they contain moving families of rational curves. On the other extreme, Serge Lang [La97] conjectured that complex surfaces of general type contain only finitely many rational curves. However, we note that uniruled surfaces of general type in positive characteristic (see Section 9.6) contain infinitely many rational curves.

12.2 K3 surfaces

In between these extremes lie surfaces of Kodaira dimension zero. If S is an Abelian variety, then every map $\mathbb{P}^1 \to S$ factors over the Albanese variety of $\mathbb{P}^1$, which is a point. Thus, Abelian varieties contain no rational curves at all. On the other hand, there is the well-known

Conjecture 12.2 (Bogomolov) *A projective K3 surface contains infinitely many rational curves.*

In characteristic zero, rational curves cannot move inside their linear systems, for otherwise the K3 surface in question would have to be uniruled, which is impossible. But even in positive characteristic, where uniruled K3 surfaces do exist, they are rather special, namely supersingular by Theorem 9.8. The first important step towards Bogomolov's conjecture is to establish the existence of *at least one* rational curve, and we refer to [M-M93] for the following result:

Theorem 12.3 (Bogomolov–Mumford) *Let S be a projective K3 surface over an algebraically closed field, and let $\mathcal{L}$ be a non-trivial and effective invertible sheaf. Then, there exists a divisor $\sum_i n_i C_i$ inside $|\mathcal{L}|$, where $n_i \geq 1$ and the C_i are rational curves on S.*

For polarized K3 surfaces (S, H), say, of degree $H^2 = 2d$, there exists a moduli space $\mathcal{M}_{2d}$, which is smooth and irreducible over the complex numbers, see, for example [BHPV, Chapter VIII]. Using degenerations of K3 surfaces to unions of rational surfaces, Chen [Ch99] showed, among other things,

Theorem 12.4 (Chen) *A very general complex projective K3 surface in $\mathcal{M}_{2d}$ contains infinitely many rational curves.*

Here, very general is meant in the sense that there exists a countable union of analytic divisors inside $\mathcal{M}_{2d}$, outside of which the statement is true. Although this result strongly supports Conjecture 12.2, it does not give even a single example of a K3 surface containing infinitely many rational curves!

12.3 Explicit results

It is shown in [B-T00b, Section 4], or [BHT11, Example 5] that complex projective Kummer K3 surfaces contain infinitely many rational curves. In particular, since every complex K3 surface of Picard rank $\rho \geq 19$ is rationally dominated by a Kummer surface, these surfaces contain infinitely many rational curves.

In [B-T00a], elliptic K3 surfaces $S \to \mathbb{P}^1$ are studied. There, the authors define a *nt-multisection* to be a multisection M of the fibration such that for a general point $b \in \mathbb{P}^1$ there exist two points in the fiber $p_b, p_{b'} \in S_b \cap M$ such that the divisor $p_b - p_{b'}$, considered as a point of the Jacobian of S_b, is non-torsion. Establishing infinitely many nt-multisections that are rational curves, we find infinitely many rational curves on elliptic K3 surfaces of Picard rank $\rho \leq 19$, see [B-T00a, Corollary 3.28]. We note that K3 surfaces of Picard rank $\rho \geq 5$ are automatically elliptic: namely, in this case, by the theory of integral quadratic forms, there exists an isotropic vector in $\mathrm{Pic}(S)$, which gives rise to an elliptic fibration. Combining these results, we obtain the following

Theorem 12.5 (Bogomolov–Tschinkel) *Let S be a complex projective K3 surface that*

1. *carries an elliptic fibration, or*
2. *is a Kummer surface, or*
3. *has Picard rank $\rho \geq 5$.*

Then, S contains infinitely many rational curves.

Moreover, in case the effective cone of a K3 surface is not finitely generated, we find infinitely many rational curves using Theorem 12.3. Also, if the automorphism group is infinite, there are infinitely many rational curves. Combining these observations with the previous results, one can show that there are infinitely many rational curves for K3 surfaces of Picard rank $\rho \geq 4$, possibly with the exception of two Picard lattices of rank 4. We refer to [B-T00a, Section 4] and [BHT11, Section 2] for details, as well as to [B-T00a, Example 4.8] for an example of a K3 surface with $\rho = 4$, where infinity of rational curves is currently still unknown.

On the other hand, a very general K3 surface in $\mathcal{M}_{2d}$ has Picard rank $\rho = 1$, does not carry an elliptic fibration, and has a finite automorphism group. Thus, these are hard to come by, as they do not possess much geometric structure to work with.

12.4 Reduction modulo p

In [BHT11], Bogomolov, Hassett and Tschinkel gave an approach to the case of Picard rank $\rho = 1$, which uses reduction modulo finite characteristic. First, using degeneration techniques, they reduced to the number field case

Proposition 12.6 (Bogomolov–Hassett–Tschinkel) *Bogomolov's conjecture 12.2 holds for complex projective K3 surfaces if and only if it holds for K3 surfaces that are defined over number fields.*

Now, let S be a K3 surface over some number field K. Replacing K by a finite extension, we may assume that all divisor classes of $S_{\mathbb{C}}$ are already defined over K. Embedding S into some projective space $\mathbb{P}^N_K$, and taking the closure of its image inside $\mathbb{P}^N_{\mathcal{O}_K}$, we get a model of S over $\mathcal{O}_K$. After localizing at a finite set of places P depending on S and this embedding, we obtain a smooth projective model $\mathcal{S} \to \operatorname{Spec} \mathcal{O}_{K,P}$, i.e., a smooth projective scheme over $\mathcal{O}_{K,P}$ with generic fiber $\mathcal{S}_K \cong S$. In particular, for every prime ideal $\mathfrak{p}$ of $\mathcal{O}_{K,P}$, the reduction $\mathcal{S}_{\mathfrak{p}}$ of $\mathcal{S}$ modulo $\mathfrak{p}$ is a K3 surface over the finite field $\mathcal{O}_{K,P}/\mathfrak{p}$.

The crucial observations and strategy of [BHT11] are as follows: let (S, H) be a polarized K3 surface over K with geometric Picard rank $\rho = 1$, or, more generally, ρ odd. If $\operatorname{char}(\mathcal{O}_K/\mathfrak{p}) \geq 5$, then the Tate-conjecture holds for $\mathcal{S}_{\mathfrak{p}}$ by Theorem 9.17. In particular, if we denote by $\mathcal{S}_{\overline{\mathfrak{p}}}$ the base change of $\mathcal{S}_{\mathfrak{p}}$ to the algebraic closure of $\mathcal{O}_K/\mathfrak{p}$, then the Picard rank of $\mathcal{S}_{\overline{\mathfrak{p}}}$ is even by Theorem 9.18. On the other hand, the specialization map

$$\operatorname{Pic}(S) \cong \operatorname{Pic}(\mathcal{S}) \xrightarrow{\operatorname{sp}_{\mathfrak{p}}} \operatorname{Pic}(\mathcal{S}_{\mathfrak{p}})$$

is injective. Since ρ is odd, there exists for every prime $\mathfrak{p}$ not lying over 2 or 3 an invertible sheaf $\mathcal{L}_{\mathfrak{p}}$ on $\mathcal{S}_{\overline{\mathfrak{p}}}$ that does not lift to S. We may assume $\mathcal{L}_{\mathfrak{p}}$ to be effective, and then, by Theorem 12.3, we find an effective divisor in $|\mathcal{L}_{\mathfrak{p}}|$ that is a sum of rational curves. Since $\mathcal{L}_{\mathfrak{p}}$ does not lift, there is at least one rational curve $C_{\mathfrak{p}}$ in this sum that does not lift to S either. However, if $N_{\mathfrak{p}}$ is a sufficiently large integer, then $|N_{\mathfrak{p}}H - C_{\mathfrak{p}}|$ is effective, and by Theorem 12.3, there exist rational curves $R_{\mathfrak{p},i}$ on $\mathcal{S}_{\overline{\mathfrak{p}}}$ and positive integers n_i such that

$$C_{\mathfrak{p}} + \sum_i n_i R_{\mathfrak{p},i} \in |N_{\mathfrak{p}}H|\,. \tag{4}$$

This sum of rational curves can be represented by a stable map of genus zero and so, defines a point of the moduli space of stable maps $\mathcal{M}_0(\mathcal{S}_{\overline{\mathfrak{p}}}, N_{\mathfrak{p}}H)$. Next, we want this stable map to be *rigid*, i.e., the stable map allows at most infinitesimal deformations, i.e., the moduli space is zero-dimensional at this point.

The first problem is that rational curves can move on K3 surfaces in positive characteristic (in which case we might not be able to find a rigid representation). But then, the K3 surface is uniruled, and in particular, Artin-supersingular, see Section 9. By results of Bogomolov and Zarhin [B-Z09] (independently also obtained by Joshi and Rajan, but unpublished), we can always find infinitely many places $\mathfrak{p}$ such that $\mathcal{S}_{\mathfrak{p}}$ is not Artin-supersingular, which is sufficient for our application.

Now, take of these infinitely many primes of non-supersingular reduction and *suppose* (we comment on that below) that we can find a rigid stable map representing (4). We denote by k the algebraic closure of the finite field $\mathcal{O}_K/\mathfrak{p}$, let $W(k)$ be the Witt ring of k, and base-change the family $\mathcal{S} \to \operatorname{Spec} \mathcal{O}_{K,P}$ to $W(k)$. Then, dimension estimates of the relative formal moduli space $\mathcal{M}_0(\mathcal{S}, N_{\mathfrak{p}}H) \to \operatorname{Spf} W(k)$ imply that our stable map to $S_{\overline{\mathfrak{p}}}$ extends to a stable map to the family $\mathcal{S}$ (here, rigidity is crucial). Thus, the stable map lifts over a possibly ramified extension of $W(k)$, and in particular, there exists a rational curve on $S_{\mathbb{C}}$, whose reduction modulo $\mathfrak{p}$ contains $C_{\mathfrak{p}}$. Thus, for infinitely many $\mathfrak{p}$ we get rational curves on $S_{\mathbb{C}}$, and eventually obtain the following result [BHT11]:

Theorem 12.7 (Bogomolov–Hassett–Tschinkel) *Let S be a complex projective K3 surface with Picard group $\operatorname{Pic}(S) = \mathbb{Z} \cdot H$ such that $H^2 = 2$. Then, S contains infinitely many rational curves.*

The main issue is the representation of (4) by a *rigid* stable map, for otherwise it is not clear whether one can lift this sum of rational curves to characteristic zero.

For degree 2 and $\rho = 1$, such a rigid representation exists by exploiting the involution on K3 surfaces of degree 2, see [BHT11]. In general, this difficulty was overcome in [L-L12] by introducing *rigidifiers*: by definition, these are ample and irreducible rational curves with at worst nodal singularities. Then, every sum of rational curves can be represented by a rigid stable map after adding sufficiently many rigidifiers to them. Unfortunately, the surface $S_{\overline{\mathfrak{p}}}$ may not contain rigidifiers. However, surfaces containing rigidifiers are dense in the moduli space of polarized K3 surfaces. Using deformation techniques and rigidifiers, we obtained in [L-L12]

Theorem 12.8 (Li–Liedtke) *Let S be a complex projective K3 surface, whose Picard rank is odd. Then, S contains infinitely many rational curves.*

More generally, the method of proof works whenever a K3 surface S is defined over some field K, and we can find a DVR R with quotient field K, as well as infinitely many primes $\mathfrak{p}$ of R such that the geometric Picard rank of the reduction $S_{\mathfrak{p}}$ is strictly larger than that of S. For example, if S is a complex projective K3 surface that cannot be defined over a number field, then S can be realized as generic fiber of a non-isotrivial family $\mathcal{S} \to B$ over some positive dimensional base of characteristic zero. Using results on the jumping of Picard ranks of K3 surfaces in families from [BKPS98] or [Og03], we obtain

Theorem 12.9 *Let S be a complex projective K3 surface that cannot be defined over a number field. Then, S contains infinitely many rational curves.*

In view of these results and Theorem 12.5, it remains to deal with K3 surfaces of Picard rank $\rho = 2$ and $\rho = 4$ that are defined over number fields, in order to establish Conjecture 12.2 for all complex projective K3 surfaces. To apply the techniques of [BHT11] and [L-L12], we need jumping of Picard ranks for infinitely places of non-supersingular reduction. For example, such jumping results for certain classes of K3 surfaces with $\rho = 2$ and $\rho = 4$ over number fields were established in [Ch11].

We end by giving a heuristic reason why we always expect to find infinitely many places with non-supersingular reduction and jumping Picard rank (as in the case of odd rank), which would imply Conjecture 12.2. However, in view of the results in [M-P09] and [Ch11, Theorem 1], the situation may be more subtle than expected. In any case, here is our heuristic:

The universal polarized K3 surface has Picard rank $\rho = 1$. All its (non-supersingular) specializations to surfaces over finite fields have a larger geometric Picard rank, and the extra invertible sheaves extend (at least, formally) along divisors inside the moduli space. Also, these invertible sheaves must have unbounded intersection number with the polarization (otherwise some of them would lift to the universal K3 surface, which was excluded). Thus, the moduli space of polarized K3 surfaces over the integers is "flooded" by infinitely many divisors on which Picard ranks jump. It is likely that given a K3 surface over a number field, infinitely many of its non-supersingular reductions hit these divisors, establishing the desired jumping behavior of Picard ranks.

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