UNIRULEDNESS CRITERIA AND APPLICATIONS

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1. Introduction

These are notes taken from an overview talk given at the 2012 Simons Symposium *Geometry Over Nonclosed Fields*. The talk focused on uniruledness criteria and their applications. One of the points made in the talk is that uniruledness criteria can be useful even in the study of varieties that are known a priori *not* to contain any rational curves. Two examples were given to illustrate this point.

- Assume we are given two normal, complex projective varieties X and Y, where Y is not uniruled. Perhaps somewhat surprisingly, uniruledness criteria apply to show that those components of $\mathrm{Hom}(X,Y)$ whose points correspond to surjective morphisms are Abelian varieties. If Y is smooth, their dimension can be bounded in terms of the Kodaira dimension $\kappa(Y)$.
- Given a smooth family $f : X \to Y$ of canonically polarised manifolds over a smooth quasi-projective base manifold Y, uniruledness criteria help to bound the variation of f in terms of the (logarithmic) Kodaira-dimension of Y.

The first item is discussed in detail in Section 2 below. We have chosen not to include any discussion of the moduli problems in this text because there are several surveys available, including [KS06, Sect. 5], [Keb11].

Next, it was shown how uniruledness criteria help to study the geometry of varieties that *are* uniruled or even rationally connected. In essence, we aim to decompose a given variety into parts depending on "density" of rational curves. More precisely, given a polarised projective manifold X, we show that the Harder-Narasimhan filtration of the tangent sheaf $\mathscr{T}_X$ induces a sequence of increasingly fine "partial rational quotients". This construction will be discussed in Section 3. We list a number of relatively new results pertaining to the dependence of the partial rational quotients on the choice of the polarisation, and to the relation between the partial rational quotients and the minimal model program. In spite of the progress made, a full understanding of the geometric meaning of the partial rational quotients is currently still missing.

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Miyaoka's uniruledness criterion and other criteria. The prototypical uniruledness criterion that we will be using most in this survey is Miyaoka's generic semipositivity.

Theorem 1.1 (Miyaoka's uniruledness criterion, [Miy87, Cor. 8.6])**.** *Let X be a normal, complex, projective variety of dimension* $\dim X \geq 2$*, and $C \subset X$ a general complete intersection curve. Then X is smooth along C, and either X is uniruled or $\Omega^1_X|_C$ is a nef vector bundle.* □

Miyaoka's criterion was shown using the Harder-Narasimhan filtration of $\mathscr{T}_X$ to exhibit a foliation along which the curve C can be deformed, in order to construct rational curves via bend-and-break. Over $\mathbb{C}$, an elementary introduction to "deformation along a subsheaf" is found in the expository papers [KKL10, JK11]. The key ingredient in the construction of the partial rational quotients is a criterion used to guarantee that leaves of a foliation are algebraic and rationally connected, Theorem 3.3. This can be seen as a generalisation and improvement of Miyaoka's Theorem 1.1.

Other uniruledness criteria. For completeness' sake, we mention a few other important criteria which were not discussed in the talk for lack of time. The most relevant is probably the result of Boucksom–Demailly–Păun–Peternell, [BDPP04], which asserts that the canonical bundle of any projective manifold is pseudo-effective, unless the manifold is uniruled. This result has recently been generalised by Campana–Peternell to higher tensor-powers of sheaves of differentials.

Theorem 1.2 (Pseudo-effectivity of quotients of pluri-forms, [CP11, Theorem 0.1])**.** *Let X be a complex, projective manifold, $m \in \mathbb{N}$ and $(\Omega^1_X)^{\otimes m} \to \mathscr{F}$ a torsion free coherent quotient. If X is not uniruled, then* $\det \mathscr{F}$ *is pseudo-effective. In particular, if X is not uniruled, then ω_X is pseudo-effective.* □

Theorem 1.2 has been generalised to reflexive differentials on singular varieties, see [GKP11, Prop. 5.6]. It plays a crucial role in recent generalisations of the Beauville-Bogomolov decomposition to varieties with trivial Chern class and singularities as they appear in minimal model theory. We refer to the paper [GKP11] for more details, and for an overview of this set of problems.

Disclaimer. The talk given at the Simons Symposium aimed to survey how uniruledness criteria are used in algebraic geometry today. It did not contain any new results. There exists some overlap between this overview and other survey papers, see for instance [Keb04, KS06]. The results discussed at the end of Section 3 have, however, not all been presented in public yet.

Throughout this survey, we work over the complex number field.

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2. Applications to non-uniruled spaces: Description of the Hom-scheme

Let $f : X \to Y$ be a surjective morphism between normal complex projective varieties. A classical problem of complex geometry asks for a criterion to guarantee the (non-)existence of deformations of the morphism f, with X and Y fixed. More generally, one would like to understand the geometry of the connected component $\mathrm{Hom}_f(X,Y) \subset \mathrm{Hom}(X,Y)$ of the space of morphisms. Somewhat surprisingly, using Miyaoka's uniruledness criterion, Theorem 1.1, we obtain a *very precise* description of $\mathrm{Hom}_f(X,Y)$ if the target manifold Y is *not* uniruled.

Theorem 2.1 (Description of the Hom-scheme, [HKP06, Theorem 1.2])**.** *Let $f : X \to Y$ be a surjective morphism between normal complex-projective varieties, and assume that Y is not uniruled. Then $\mathrm{Hom}_f(X,Y)$ is an Abelian variety.*

Remark 2.2. If $\dim \mathrm{Hom}_f(X,Y) = 0$, then Theorem 2.1 implies that $\mathrm{Hom}_f(X,Y)$ is a reduced point. Formulated in different terms, Theorem 2.1 implies that deformation-rigid morphisms are in fact infinitesimally rigid.

In fact, more is true. Up to a finite covering, the scheme $\mathrm{Hom}_f(X,Y)$ is the maximal connected subgroup of the automorphism group of a variety Z that admits a quasi-étale map[1] to Y. Since the existence of an Abelian variety in the automorphism group has strong implications for many other invariants, one obtains the following results as nearly immediate corollaries.

Corollary 2.3 (Deformations and Kodaira dimension, [HKP06, Corollary 1.3])**.** *In the setup of Theorem 2.1, if Y is smooth and has non-negative Kodaira dimension $\kappa(Y) \geq 0$, then $\dim \mathrm{Hom}_f(X,Y) \leq \dim Y - \kappa(Y)$.* □

Corollary 2.4 (Deformations and fundamental group, [HKP06, Corollary 1.5])**.** *In the setup of Theorem 2.1, let Y be a projective manifold which is not uniruled. If $\pi_1(Y)$ is finite, then $\mathrm{Hom}_f(X,Y)$ is a reduced point.* □

Corollary 2.5 (Deformations and top Chern class, [HKP06, Corollary 1.6])**.** *In the setup of Theorem 2.1, let Y be a projective n-dimensional manifold which is not uniruled. If $c_n(Y) \neq 0$, then $\mathrm{Hom}_f(X,Y)$ is a reduced point.* □

2.A. **Idea of proof.** We show only the much simpler assertion that $\mathrm{Hom}_f(X,Y)$ is smooth, that is, that every infinitesimal deformation of f is induced by a holomorphic one-parameter family of morphisms. To this end, assume we are given an infinitesimal deformation $\sigma \in H^0(X, f^*\mathscr{T}_Y)$.

To avoid technical difficulties and quickly come to the core of the argument, we make the following extra assumptions.

Additional Assumption 2.6. The varieties X and Y are smooth and the morphism f is finite. In particular, $f_*\mathscr{O}_X$ is a locally free sheaf on Y.

2.A.1. *Step 1: the composition morphism.* Recall that the automorphism group of the complex variety Y is a complex Lie group. Composing the morphism f with elements of its maximal connected subgroup $\mathrm{Aut}^0(Y)$, we obtain an injective composition morphism

$$f^\circ : \begin{array}{ccc} \mathrm{Aut}^0(Y) & \to & \mathrm{Hom}_f(X,Y). \\ g & \mapsto & g \circ f \end{array} \tag{2.6.1}$$

The tangent spaces to $\mathrm{Aut}^0(Y)$ and $\mathrm{Hom}_f(X,Y)$ are well understood. The derivative of f° at the identity $e \in \mathrm{Aut}^0(Y)$ thus yields a diagram

$$\begin{tikzcd} T_{\mathrm{Aut}^0(Y)}|_e \arrow[r, "df^\circ|_e"] \arrow[d, "\cong"'] & T_{\mathrm{Hom}}|_f \arrow[d, "\cong"] \\ H^0(Y, \mathscr{T}_Y) \arrow[r, "\text{pull-back}"'] & H^0(X, f^*\mathscr{T}_Y). \end{tikzcd} \tag{2.6.2}$$

The horizontal arrows in Diagram (2.6.2) are clearly injective. Since every infinitesimal deformation which is in the image of $df^\circ|_e$ is clearly induced by a one-parameter group in $\mathrm{Aut}^0(Y)$, we need to show that the infinitesimal deformation

[1] quasi-étale = finite and étale in codimension one

$\sigma \in H^0(X, f^*\mathscr{T}_Y)$ is obtained as the pull-back of a vector field on Y. We argue by contradiction and assume this is not the case.

Additional Assumption 2.7. The infinitesimal deformation $\sigma \in T_{\mathrm{Hom}}|_f$ is not the pull-back of a vector field on Y.

2.A.2. *Step 2: the splitting of $f_*\mathscr{O}_X$ and étale covers.* Fix an ample divisor $H \in \mathrm{Pic}(Y)$, and let $C \subset Y$ be an associated general complete intersection curve. We recall a few facts about the push-forward sheaf $f_*\mathscr{O}_X$ that are relevant in our context.

Fact 2.8 (Description of $f_*\mathscr{O}_X$, [Laz80] or [PS00, Theorem A])**.** *The trace map $tr : f_*\mathscr{O}_X \to \mathscr{O}_Y$ yields a natural splitting*

$$f_*\mathscr{O}_X \cong \mathscr{O}_Y \oplus \mathscr{E}^*.$$

where $\mathscr{E}$ is a locally free sheaf whose restriction $\mathscr{E}|_C$ is nef. The following conditions are equivalent.

(1) *The morphism f is branched.*
(2) *The morphism f is branched and its branch locus intersects the curve C.*
(3) *The degree of the restricted sheaf is positive,* $\deg(\mathscr{E}|_C) > 0$. □

Proposition 2.9 (Ampleness of $\mathscr{E}|_C$, [HKP06])**.** *The restricted sheaf $\mathscr{E}|_C$ is either ample, or there exists a non-trivial factorisation*

$$\begin{tikzcd} X \arrow[r, "\alpha,\, finite"'] \arrow[rr, bend left, "f"] & Z \arrow[r, "\beta,\, étale"'] & Y. \end{tikzcd}$$

Idea of proof. If the restricted sheaf $\mathscr{E}|_C$ is ample, we are done. If not, it there exists a term $\mathscr{A} \subset \mathscr{E}$ in the Harder-Narasimhan filtration of $\mathscr{E}$ such that the following holds.

- The restriction $\mathscr{A}|_C$ is an ample sub-vectorbundle of $\mathscr{E}|_C$.
- The quotient $\mathscr{E}|_C \big/ \mathscr{A}|_C$ has degree zero.

The second point follows from the fact that $\mathscr{E}|_C$ is nef, so that quotients will always have semi-positive degrees. Dualizing, we find a sheaf $\mathscr{B} \subset \mathscr{E}^*$ of slope zero whose quotient has negative slope.

Using that there is no map from a semistable sheaf of high slope to one of smaller slope, one observes that the subsheaf $\mathscr{B} \oplus \mathscr{O}_Y \subseteq f_*\mathscr{O}_X$ is closed under multiplication, hence forms a sheaf of $\mathscr{O}_Y$-algebras. The variety Z is obtained as Spec of that sheaf. □

To continue the proof of Theorem 2.1, assume we are in a situation where $\mathscr{E}|_C$ is *not* ample. Proposition 2.9 will then give a non-trivial decomposition of f, and it follows from the étaleness of β that the infinitesimal deformation σ can be interpreted as an infinitesimal deformation of the morphism α, that is

$$\sigma \in H^0(X, f^*\mathscr{T}_Y) = H^0(X, \alpha^*\mathscr{T}_Z).$$

To prove that σ comes from a one-parameter family of deformations of f, is clearly suffices to show that σ comes from a one-parameter family of deformations of α — composing with β will then give the deformation of f. Replacing Y by Z, iterating the argument, and using that f is of finite degree, we can therefore assume without loss of generality that the following holds.

Additional Assumption 2.10. The restricted sheaf $\mathscr{E}|_C$ is ample.

2.A.3. *Step 3: end of proof.* To end the proof, use the projection formula to obtain a decomposition

$$\begin{aligned} H^0(X, f^*\mathscr{T}_Y) &= H^0(Y, f_*(f^*\mathscr{T}_Y)) \\ &= H^0(Y, \mathscr{T}_Y) \oplus H^0(Y, \mathscr{E}^* \otimes \mathscr{T}_Y) \\ &= H^0(Y, \mathscr{T}_Y) \oplus \mathrm{Hom}_Y(\mathscr{E}, \mathscr{T}_Y) \end{aligned}$$

Consider the associated decomposition of σ. Since $\sigma \in T_{\mathrm{Hom}}|_f$ is not the pull-back of a vector field on Y by Assumption 2.7, we obtain a non-trivial morphism from $\mathscr{E}$ to $\mathscr{T}_Y$. Using that $\mathscr{E}|_C$ is ample, this implies that $\mathscr{T}_Y|_C$ has a positive subsheaf. Miyaoka's criterion, Theorem 1.1, therefore applies to show that Y is uniruled. This is in clear contradiction to the assumptions made in Theorem 2.1 and therefore ends the proof. □

2.B. **Further results. Questions.**

2.B.1. *Refinement of Stein factorisation.* The methods used to prove Theorem 2.1 show more than claimed above. With a little more work, the technique using the Harder-Narasimhan filtration of $f_*\mathscr{O}_X$ can be used to show that there exists a canonically defined refinement of Stein factorisation for any surjective morphism. The following definition summarises its main properties.

Definition 2.11 (Maximally étale factorization)**.** *Let $f : X \to Y$ be a surjective morphism between normal projective varieties, and assume we are given a factorisation*

(2.11.1)
$$\begin{tikzcd} X \arrow[r, "\alpha"'] \arrow[rr, bend left, "f"] & Z \arrow[r, "\beta"'] & Y \end{tikzcd}$$

where β is quasi-étale. We say that the factorisation (2.11.1) is maximally étale *if the following universal property holds: for any factorisation $f = \beta' \circ \alpha'$, where $\beta' : Z' \to Y$ is quasi-étale, there exists a morphism $\gamma : Z \to Z'$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\alpha"'] \arrow[rr, bend left, "f"] \arrow[d, equal] & Z \arrow[r, "\beta"'] \arrow[d, "\gamma"'] & Y \arrow[d, equal] \\ X \arrow[r, "\alpha'"] \arrow[rr, bend right, "f"'] & Z' \arrow[r, "\beta'"] & Y. \end{tikzcd}$$

Theorem 2.12 (Existence of a maximally étale factorisation, [KP08, Theorem 1.4])**.** *Let $f : X \to Y$ be a surjective morphism between normal projective varieties. Then there exists a maximally étale factorisation.* □

Remark 2.13. The universal properties of the maximally étale factorisation immediately imply that the maximally étale factorisation is unique up to unique isomorphism, and behaves extremely well under deformations of f, see [KP08, Sections 1.B, 4].

The natural refinement of Stein factorisation mentioned above is now an immediate corollary.

Corollary 2.14 (Refinement of Stein factorisation, [KP08, Section 1.A]). *Let $f : X \to Y$ be a surjective morphism between normal projective varieties. Then there exists a canonical refinement of Stein factorisation as follows,*

$$\begin{tikzcd} X \arrow[r, "\alpha_1\text{, conn. fibres}"'] \arrow[rrr, bend left, "f"] & W \arrow[r, "\alpha_2\text{, finite}"'] & Z \arrow[r, "\beta\text{, quasi-étale}"'] & Y, \end{tikzcd}$$

where β comes from the maximally étale factorisation of f. □

2.B.2. *Deformations of morphisms to uniruled varieties.* If $f : X \to Y$ is a surjective morphisms onto a rationally connected manifold, there is usually little we can say about the associated connected component of the Hom-scheme; partial results are found in [HM03, HM04, Hwa07b, Hwa07a]. With some extra work one can show, however, that the MRC quotient of Y induces a decomposition of $\mathrm{Hom}_f(X, Y)$ into an Abelian variety and a space that parametrises deformations *over the MRC quotient*. We refer to [KP08, Section 1.C] for a precise formulation of the somewhat involved result.

2.B.3. *Open problems.* We conjecture that Theorem 2.1 and its corollaries hold true when Y is a compact Kähler manifold of non-negative Kodaira dimension. Our proof needs the projectivity assumption because it employs Miyaoka's characterisation of uniruledness, Theorem 1.1.

3. Applications to uniruled manifolds: Partial rational quotients

Roughly speaking, the uniruledness criteria of Mori and Miyaoka can be summarised as "positivity properties of $\mathscr{T}_X$ imply the existence of rational curves on X". However, the precise relation between positivity and the geometric properties of the rational curves found by these criteria remains unclear.

- Does "more positivity" give "more rational curves"?
- If the tangent bundle contains a particularly positive subsheaf $\mathscr{F} \subseteq \mathscr{T}$, can we find rational curves whose geometry relates to $\mathscr{F}$? If so, how many?

Building on work of Miyaoka and Bogomolov–McQuillan, the present section aims to clarify at least some aspects of this relation. Given a projective manifold X, we will see that the terms in the Harder-Narasimhan filtration of $\mathscr{T}_X$ induce a canonically defined sequence of *partial rational quotients*. As of today, a precise geometric description of these partial rational quotients and their dependence on the choice of the polarisation is missing. We discuss some evidence which points to a strong connection between the partial rational quotients and the minimal model program.

3.A. Rationally connected foliations.

The key result of this section is a uniruledness criterion for *foliated* varieties. The following definition will be used.

Definition 3.1 (Foliation, singular foliation). *Let X be a normal variety and $\mathscr{F}$ a coherent subsheaf of the tangent sheaf $\mathscr{T}_X$. Let $X^\circ \subseteq X$ be the maximal open set where X is smooth, and $\mathscr{F}$ is a sub-vectorbundle of $\mathscr{T}_X$. We call $\mathscr{F}$ a (singular)* foliation *if the following two conditions hold.*

(1) *The sheaf $\mathscr{F}$ is a saturated subsheaf of $\mathscr{T}_X$. In other words, the quotient $\mathscr{T}_X/\mathscr{F}$ is torsion-free.*
(2) *The sheaf $\mathscr{F}$ is integrable, that is, the sub-vectorbundle $\mathscr{F}|_{X^\circ} \subseteq \mathscr{T}_X|_{X^\circ}$ is closed under Lie-bracket.*

The foliation $\mathscr{F}$ is regular *if $X^\circ = X$. A* leaf *of $\mathscr{F}$ is a connected, locally closed holomorphic submanifold $L \subset X^\circ$ such that $\mathscr{T}_L = \mathscr{F}|_L$. A leaf is called* algebraic *if it is open in its Zariski closure.*

Remark 3.2. In the setting of Definition 3.1, let $L \subset X^\circ$ be an algebraic leaf and $\overline{L} \subset X^\circ$ be its Zariski-closure. Then $\overline{L}$ is again a leaf.

The main result of this section asserts that positivity properties of $\mathscr{F}$ imply algebraicity of the leaves and rational connectedness of their closures. In particular, it gives a criterion for a manifold to be covered by rational curves.

Theorem 3.3 (Rationally connected foliations, [BM01, KST07]). *Let X be a normal complex projective variety, $C \subset X$ a complete curve which is entirely contained in the smooth locus X_{reg}, and $\mathscr{F} \subset T_X$ a (possibly singular) foliation which is regular in a neighbourhood of C. The restriction $\mathscr{F}|_C$ is then a vector bundle on C. If $\mathscr{F}|_C$ is ample, and if $x \in C$ is any point, then any leaf through x is algebraic. If $x \in C$ is general, the Zariski-closure of any leaf through x is a rationally connected subvariety of X.* □

The statement appeared first in the preprint [BM01] by Bogomolov and McQuillan, the first full proof was given in [KST07]. Methods used include a criterion of Hartshorne for a foliation to have algebraic leaves, the result of Graber-Harris-Starr, and bend-and-break arguments relying on a vanishing theorem in positive characteristic.

Remark 3.4. In Theorem 3.3, if $x \in C$ is any point, it is *not* generally true that the closure of a leaf through x is rationally connected. This was wrongly claimed in [BM01] and in the first preprint versions of [KST07].

The classical Reeb stability theorem for foliations [CLN85, Theorem IV.3], the fact that rationally connected manifolds are simply connected [Deb01, Corollary 4.18], and the openness of rational connectedness [KMM92, Corollary 2.4] immediately yield the following[2].

Theorem 3.5 (Rationally connected regular foliations, [KST07, Theorem 2]). *In the setup of Theorem 3.3, if $\mathscr{F}$ is regular and $L \subset X$ any leaf, then L is algebraic and its closure is a rationally connected submanifold.* □

Remark 3.6. In fact, a stronger statement holds, guaranteeing that most leaves are algebraic and rationally connected if there exists a single leaf whose closure does not intersect the singular locus of $\mathscr{F}$, see [KST07, Theorem 28].

The following characterisation of rational connectedness is a straightforward corollary of Theorem 3.3.

Corollary 3.7 (Criterion for rational connectedness). *Let X be a complex projective variety and let $f : C \to X$ be a curve whose image is contained in the smooth locus of X. If $\mathscr{T}_X|_C$ is ample, then X is rationally connected.* □

3.B. **Producing foliations using the Harder–Narasimhan filtration of $\mathscr{T}_X$.** The usefulness of Theorem 3.3 on rationally connected foliations depends on our ability to construct geometrically relevant foliations to which the theorem can be applied. In his work on uniruledness criteria and deformations along a foliation, Miyaoka noted that the subsheaves of $\mathscr{T}_X$ which appear in Harder–Narasimhan filtrations often satisfy this property.

Proposition 3.8 (Foliations coming from Harder–Narasimhan filtrations). *Let X be a normal n-dimensional projective variety and $H = \{H_1, \ldots, H_{n-1}\} \in \mathrm{Pic}(X)$ a polarisation by ample line bundles. Consider the associated Harder–Narasimhan filtration of $\mathscr{T}_X$,*

$$0 = \mathscr{F}_0 \subsetneq \mathscr{F}_1 \subsetneq \mathscr{F}_2 \subsetneq \cdots \subsetneq \mathscr{F}_{k-1} \subsetneq \mathscr{F}_k = \mathscr{T}_X$$

[2] Höring has independently obtained similar results, [Hör07].

and set

$$i_{\max} = \max\left\{0 < i < k \,|\, \mu_H\left(\mathscr{F}_i/\mathscr{F}_{i-1}\right) > 0\right\} \cup \{0\}.$$

Assume that $i_{\max} > 0$. *Given any index* $0 < i \leq i_{\max}$, *then* $\mathscr{F}_i$ *is a foliation in the sense of Definition 3.1.*

Idea of proof. The sheaf morphism induced by the Lie-bracket, $\mathscr{F}_i \times \mathscr{F}_i \to \mathscr{T}_X$, is by no means $\mathscr{O}_X$-bilinear. An elementary computation shows, however, that the induced map to the quotient,

$$N : \mathscr{F}_i \times \mathscr{F}_i \to \mathscr{T}_X/\mathscr{F}_i$$

is in fact bilinear. The claim then quickly follows from the well-known fact that in characteristic zero, semistability and slope are well-behaved under tensor product, and that there is no morphism from a semistable sheaf of high slope to one of lower slope. The map N must thus be trivial. □

Remark 3.9. If X is $\mathbb{Q}$-Fano, then $i_{\max} = k$.

Corollary 3.10 (Rational connectedness of foliations coming from HNFs)**.** *In the setting of Proposition 3.8, the leaves of the foliation* $\mathscr{F}_i$ *are algebraic. The general leaf is rationally connected.*

Proof. Let $C \subset X$ be a general complete intersection curve for the polarisation H. Then C is smooth, and entirely contained in the locus where both X is smooth and $\mathscr{F}_i$ is regular. The restriction $\mathscr{F}_i|_C$ is thus an ample vector bundle on C. Theorem 3.3 applies and yields both algebraicity and rational connectedness of leaves that intersect C. The claim holds for all leaves because deformations of C dominate X. □

3.C. **Applications: Sequences of partial rational quotients.** Corollary 3.8 allows to construct a rational map $X \dashrightarrow \mathrm{Chow}(X)$ by mapping general points of X to the closures of the associated leaves. In summary, we see that every polarised manifold is canonically equipped with a sequence of increasingly fine *partial rational quotients*.

Corollary 3.11 (Partial rational quotients associated to a polarisation)**.** *In the setting of Proposition 3.8, there exists a commutative diagram of rational maps,*

(3.11.1)
$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, dashed, "q_1"'] & X \arrow[r, equal] \arrow[d, dashed, "q_2"'] & \cdots \arrow[r, equal] & X \arrow[r, equal] \arrow[d, dashed, "q_{i_{\max}}"'] & X \arrow[d, dashed, "\text{\itshape MRC Quotient}"] \\
Q_1 \arrow[r, dashed] & Q_2 \arrow[r, dashed] & \cdots \arrow[r, dashed] & Q_{i_{\max}} \arrow[r, dashed] & Q,
\end{tikzcd}$$

with the following additional property: if $x \in X$ *is a general point, and* F_i *the closure of the* q_i*-fibre through* x*, then* F_i *is rationally connected, and its tangent space at* x *is exactly* $\mathscr{T}_{F_i}|_x = \mathscr{F}_i|_x$. □

While many people working in the field share the feeling that the Harder–Narasimhan filtrations should measure "density of rational curves with respect to the given polarisation", no convincing results have been obtained in this direction. The geometric meaning of the canonically given diagram (3.11.1) is not fully understood. The following Section 3.D discusses the known results and poses a few natural questions and conjectures.

3.D. **Interpretation of the partial rational quotients. Open problems.** The first questions that came to our mind when we first saw Corollary 3.11 and Diagram (3.11.1) were probably the following.

Questions 3.12. *Let X be a uniruled projective manifold or variety, equipped with a polarisation. Is the MRC quotient equal to the map $q_{i_{\max}}$?*

Questions 3.13. *Do the canonically defined morphisms q_i carry a deeper geometric meaning? Is Diagram (3.11.1) characterised by universal properties? To what extent does it depend on the polarisation chosen?*

Today these questions can be answered in special cases. We survey the known results in the remaining part of this section.

3.D.1. *The MRC quotient as a rationally connected foliation.* Perhaps somewhat surprisingly, the relative tangent sheaf of the MRC quotient does generally *not* appear as one of the terms of the Harder–Narasimhan filtration, unless X is a surface and the polarisation is particularly well-chosen.

Theorem 3.14 (MRC quotient not always equal to $q_{i_{\max}}$, [Eck08, Section 3])**.** *There exist elementary examples of polarised surfaces where the MRC quotient is not equal to the map $q_{i_{\max}}$.* □

Theorem 3.15 (MRC quotient equals $q_{i_{\max}}$ for good surface polarisation, [Neu09, Theorem 3.8])**.** *If X is a uniruled surface, then there exists a polarisation such that the MRC quotient equals the map $q_{i_{\max}}$.* □

In higher dimensions, we do not expect an analogue of Theorem 3.15 to hold true. There are, however, positive results when one is willing to generalise the notion of "polarisation" to include "polarisations by movable curve classes", as defined below.

Fact 3.16 (Polarisations by movable curve classes, [Neu10, Section 3])**.** *Let X be a projective manifold and $\alpha \in \overline{\mathrm{Mov}(X)} \subset N^1(X)_{\mathbb{R}}$ a non-trivial numerical curve class, contained in the closure of the movable cone. Define the slope of a coherent sheaf $\mathscr{F}$ as*

$$\mu_\alpha(\mathscr{F}) := \frac{\alpha.[\det \mathscr{F}]}{\operatorname{rank} \mathscr{F}} \in \mathbb{R}.$$

With this definition, a Harder–Narasimhan filtration exists exactly as in the case of an ample polarisation. The obvious analogue of Proposition 3.8 holds. □

Warning 3.17. For all we know, there is no analogue of the Mehta-Ramanathan theorem in the setting of Fact 3.16.

Theorem 3.18 (MRC quotient equals $q_{i_{\max}}$ for good movable polarisation, [SCT09, Theorem 1.1])**.** *Let X be a uniruled complex projective manifold, and let $\mathscr{F} \subseteq \mathscr{T}_X$ denote the foliation associated with its MRC quotient. Then there exists a numerical curve class class α, contained in the interior of the movable cone such that the following holds.*

1. *The class α is represented by a reduced movable curve C such that $\mathscr{F}|_C$ is ample.*
2. *The sheaf $\mathscr{F}$ appears as a term in the Harder–Narasimhan filtration of $\mathscr{T}_X$ with respect to α.* □

Remark 3.19. If X is a surface, then the interior of the movable cone equals the cone of general complete intersection curves.

Questions 3.20. *Let X be a uniruled projective manifold. Is there an ample polarisation such that the MRC quotient equals the map $q_{i_{\max}}$?*

Questions 3.21. *Is there an analogue of Corollary 3.10, "Rational connectedness of foliations coming from HNFs" when using movable curve classes to define a Harder–Narasimhan filtration? What if X is singular?*

3.D.2. *Decomposition of the cone of movable curve classes.* Let X be a projective manifold. Given a non-trivial numerical curve class $\alpha \in \overline{\mathrm{Mov}(X)} \subset N^1(X)_{\mathbb{R}}$, we are interested in the set of classes whose induced Harder–Narasimhan filtration of the tangent bundle agrees with that of α,

$$\Delta_\alpha := \left\{ \beta \in \overline{\mathrm{Mov}(X)} \mid \mathrm{HNF}(\alpha, \mathscr{T}_X) = \mathrm{HNF}(\beta, \mathscr{T}_X) \right\}.$$

The decomposition of the movable cone $\overline{\mathrm{Mov}(X)}$ into disjoint subsets of the form Δ_α, called "destabilising chambers", was studied in the 2010 Freiburg thesis of Sebastian Neumann. He obtained the following two results.

Theorem 3.22 (Decomposition of the moving cone, [Neu10, Theorem 3.3.4, Proposition 3.3.5])**.** *Let X be a projective manifold. The destabilising chambers are convex cones whose closures are locally polyhedral in the interior of* $\mathrm{Mov}(X)$*. The decomposition of the moving cone is locally finite in the interior of* $\mathrm{Mov}(X)$*. If we assume additionally that the cone of movable curves is polyhedral, then the chamber structure is finite.* □

Remark 3.23 (Movable cone of Fano manifolds, [Ara10, Cor. 1.2]). If X is a Fano manifold, then the closed cone of movable curves is polyhedral.

Theorem 3.24 (Relation to the minimal model program, [Neu10, Theorem 4.1])**.** *Let X be a Fano manifold of dimension three and $\alpha \in \overline{\mathrm{Mov}(X)} \subset N^1(X)_{\mathbb{R}}$ a non-trivial numerical curve class, with associated Harder–Narasimhan filtration*

$$0 = \mathscr{F}_0 \subsetneq \mathscr{F}_1 \subsetneq \mathscr{F}_2 \subsetneq \cdots \subsetneq \mathscr{F}_{k-1} \subsetneq \mathscr{F}_k = \mathscr{T}_X.$$

Then each term $\mathscr{F}_i$ is the relative tangent sheaf of a (not necessarily elementary) Mori fibration. □

Remark 3.25. The proof of Theorem 3.24 relies on the fine classification of Fano threefolds. It would be very interesting to understand the relation between the Harder–Narasimhan filtrations and minimal model theory in much greater detail.

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