

PERVERSE SHEAVES OF CATEGORIES AND NON-RATIONALITY

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1. Introduction

In this paper we take a new look at the classical notions of rationality and stable rationality from the perspective of sheaves of categories.

Our approach is based on three recent developments:

- (1) The new striking approach to stable rationality introduced by Voisin and developed later by Colliot-Thélène Pirutka, Totaro, Tschinkel, Hassett, Kresch.
- (2) Recent breakthroughs made by Haiden, Katzarkov, Kontsevich, Pandit [HKKP], who introduced the theory of categorical Kähler metrics and moduli spaces of objects.
- (3) Developing the theory of categorical linear systems and sheaves of categories by Katzarkov and Liu.

An important part of our approach is developing of the correspondence between theory of Higgs Bundles and the theory of perverse sheaves of categories initiated in [KLa], [KLb]. In the rest of the paper most of the time we will omit perverse understanding that all our sheaves of categories are perverse. In the same way the moduli spaces of Higgs bundles record the homotopy type of projective and quasi-projective varieties sheaves of categories record the information of rationality of projective and quasi-projective varieties.

It was demonstrated in [KNPS15] and [KNPS13] that there is a correspondence between harmonic maps to buildings, and their singularities with stable networks and limiting stability conditions for degenerated categories, degenerated sheaves of categories. In this paper we take this correspondence to a new level. We describe this correspondence in the table below.

TABLE 1. Correspondence Higgs bundles \leftrightarrow Sheaves of categories

$\text{Func}(\Pi_1^{\leq}(X, s), \text{Vect})$ \swarrow \searrow groupoid Category of vector spaces	$\text{Func}(\Pi^{\leq\infty}(X, s), \text{dg Cat})$ \swarrow \searrow 2 category dg category
Higgs bundles	Perverse sheaves of categories
Complex var. Hodge structures	Classical LG models
Lyapunov exponents for Higgs Bundles	Lyapunov exponents for HH^* of categories

Shiffman truncation of HN filtration	Multiplier Ideal Sheaf $\alpha_n \subset \cdots \subset \alpha_0$, where α_i is the localization of α_{i+1}
Vol. \longleftrightarrow	Orlov spectra
Non-abelian MHS	Additional filtration

The analogue of the Harder-Narasimhan filtration are pieces of localized categories - subsheaves of categories which we denote by $\text{LG}_i(X^0)$. The idea is that every LG model (see [KLb]) splits into pieces. Each of these pieces is itself a sheaf of categories. Finding deformations of such a piece a sheaf of categories which has sheaves of Lagrangians with no sections is the main goal of this paper.

In this paper we describe some technology for finding such a “nice” $\text{LG}_i(X^0)$. This is done by deforming LG models as sheaves of categories. The main geometric outcomes from our work are:

Classical	Cateorical
$W = P$ equality for tropical varieties	“ $W = P$ ” for perverse sheaves of categories
Voisin theory of deformations	Sheaves of Lagrangians without sections

In this paper we give a few examples and outline a program. Full details will appear elsewhere. We introduce the notion of a good deformation of perverse sheaf of categories.

Definition 1.1 (Good deformation). *A deformation of PSH is **good** if*

- (1) $P = W$ is preserved;
- (2) Lattice conjecture is preserved.

These deformations control cohomologies and K -theory. Deformations used by [Voi15], [CTP14], [Tot15] are good.

The main conjecture we formulate in this paper is:

Conjecture 1.2. *Let $FS(PSC) \cong D^b(X)$, X be a Fano manifold. Assume there exists a good deformation of $FS(PSC)$ so that*

- (1) $L : K(T') \rightarrow HP_*(T')$ misses for H (vanishing cycles);
- (2) We create a basic class with support of $\text{codim} \leq 2$.

Then X is not rational.

We believe that this conjecture holds for examples in [Voi15], [CTP14], [Tot15]. In fact it seems the conditions of this conjecture are stronger than splitting of diagonal.

We start with deformations of sheaves of categories in section 2. Then we move to the parallel between local systems and sheaves of categories. Sections 4 and 5 contain our main examples. Concluding remarks are given in section 6.

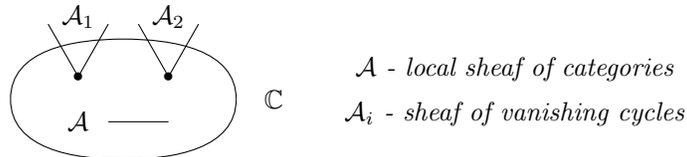
Acknowledgments:

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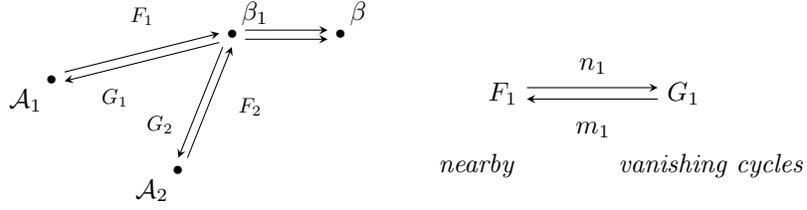
2. Perverse Sheaves of Categories

2.1. Definitions. In this section we develop the theory of sheaves of categories and their deformations. We start with a definition:

Definition 2.1 (Sheaves of categories over $\text{Sch}(\mathcal{A}, \mathcal{A}_1, \dots, \mathcal{A}_n)$).



$\mathcal{A}, \mathcal{A}_i$ - 2 categories
 F, G - 2 functors
 n, m - 2 natural transformations



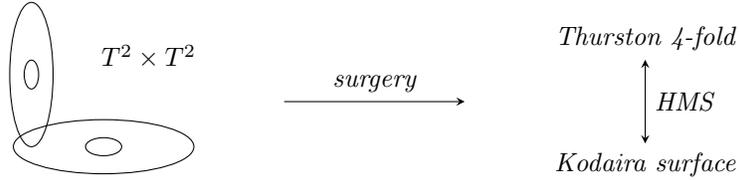
$$T_{G_1} = \beta_1 \text{Id}_{G_1} - m_1 n_1$$

Theorem 2.2. *The deformations of $\text{Sch}(\mathcal{A}, \mathcal{A}_1, \dots, \mathcal{A}_n)$ are described by :*

- (1) Adding a new category β .
- (2) Changes in natural transformations n_i, m_j .

We give some examples.

Example 2.3. *We start with a simple example $T^2 \times T^2$ - the product of two 2-dimensional tori.*



In [AAKO] the following theorem is proven.

Theorem 2.4. *The following categories are equivalent:*

$$D^b(T^2 \times T^2, \text{Gerbe}) \cong \text{Fuk}(\text{Thurston}) \cong D^b(\text{Kodaira}).$$

Example 2.5. *We generalize this construction to the case of LG models.*

<i>LG model</i>	<i>Dolg 2,3 surface</i>
<p style="margin: 0;"><i>Gerbe on the sheaf of categories</i></p>	<div style="display: flex; justify-content: space-around;"> <div style="text-align: center;"> <p style="margin: 0;"><i>Log 2</i></p> <p style="margin: 0;">2</p> </div> <div style="text-align: center;"> <p style="margin: 0;"><i>Log 3</i></p> <p style="margin: 0;">3</p> </div> </div>

Recall that *Dolg 2,3* is obtained from $\mathbb{P}_{p_1, \dots, p_9}^2$ by applying 2 surgeries with order 2,3.

Theorem 2.6. *The mirror of $\text{Dolg}_{2,3}$ is obtained from the LG model of $\widehat{\mathbb{P}}_{p_1, \dots, p_9}^2$ by adding a gerbe G on it corresponding to log transform. In other words:*

$$D^b(\text{Dolg } 2,3) = \text{FS}(LG(\widehat{\mathbb{P}}_{p_1, \dots, p_9}^2), G). \quad (2.1)$$

The simplest sheaves of categories are objects called *schobers* [KS16].

Definition 2.7. *Let $f : Y \rightarrow \mathbb{C}$ be a proper map. Then $f_* \mathcal{O}_Y$ as D -module defines a sheaf of categories associated with a function f (a *schober*).*

Example 2.8. *The sheaf of categories associated with the mirror of Dolg2, 3 is not a schober.*

2.2. Some simple examples. Consider fibration $\mathcal{F} \xrightarrow{f} \mathbb{C}$ with a multiple n -fiber over 0.

$$\begin{array}{ccc} E \times \mathbb{C} & \xrightarrow[\text{(\times l, \times \mathcal{E})}]{n:1} & \mathcal{F} \xrightarrow{f} \mathbb{C} \\ \downarrow \mathbb{Z}^n & & \\ \mathbb{C} & & nl = 0, \mathcal{E}^n = 1 \end{array}$$

Theorem 2.9. $\text{MF}(\mathcal{F} \xrightarrow{f} \mathbb{C})$ contains a quasi-phantom.

Proof. Indeed $H^*(\mathcal{F}, \text{vanishing cycles}) = 0$, since vanishing cycles are the elliptic curve E and $H^*(E, L) = 0, \forall L$ - nontrivial rank 1 local system.

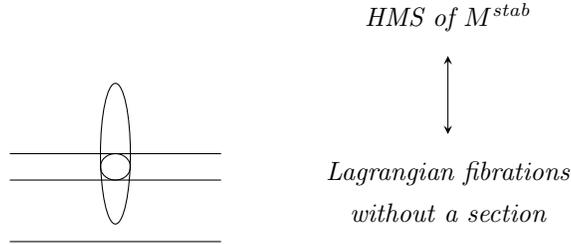
Also $K(\text{MF}(\mathcal{F} \rightarrow \mathbb{C})) = \mathbb{Z}_n$. □

Proposition 2.10. *There exists a moduli space of stable objects on $\text{MF}(\mathcal{F} \rightarrow \mathbb{C})$.*

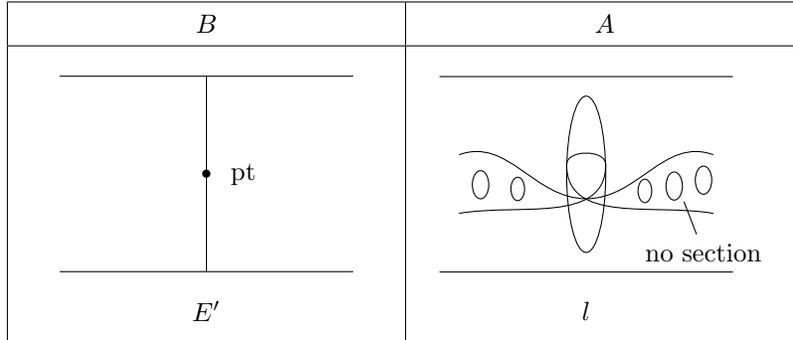
Proof. Indeed these are the \mathbb{Z}_n -equivalent objects on $E \times \mathbb{C}$. For example, we have $M^{\text{stab}} = E', E'$ - multiple fiber. □

Proposition 2.11. *HMS of $\text{MF}(\mathcal{F} \rightarrow \mathbb{C})$ is $\text{FS}(\mathcal{F} \rightarrow \mathbb{C})$.*

Theorem 2.12.



Definition 2.13. *The fibration over l is a global Lagrangian fibration with no section.*



We have the following correspondence on the level of sheaves of categories related with $\text{Fuk}(E) \cong \text{D}^b(E')$.

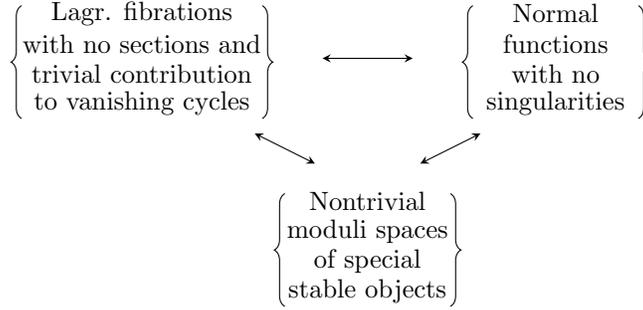
A	B
	<p>Log transform</p>
<p>Dolg surface</p>	<p>2 log transforms</p>
	<p>Enriques surface</p>

The fibration of Lagrangians with no sections can be interpreted as normal functions with no singularities.

$$\begin{array}{c} \text{HP}_*(\mathcal{F}) \\ \text{(normal function)} \end{array} \parallel \begin{array}{c} \mathcal{F} \\ \downarrow \\ \mathbb{C} \end{array}$$

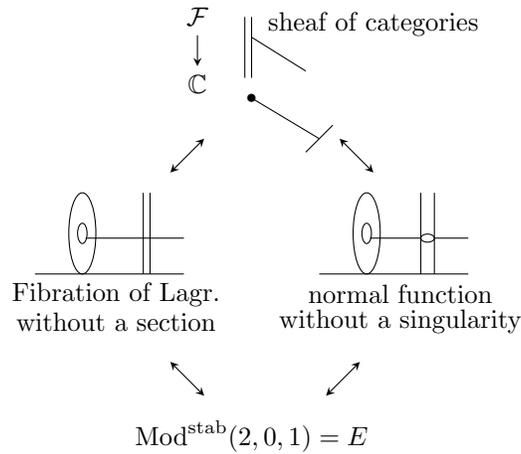
—————
No singularities

We have the following correspondences.



Theorem 2.14. *Let X be Dolgachev surface. Then the above diagram holds.*

Proof. Indeed we have



Indeed Donaldson proved that $\text{rk } \mathcal{E} = 2$ bundles with $c_2(\mathcal{E}) = 1$, $\bigwedge^2 \mathcal{E} = \mathcal{O}_{\text{Dolg}}$ is isomorphic to E . \square

Remark 2.15. Existence of M^{stab} is stronger obstruction to rationality than splitting of the diagonal. Indeed the diagonal splits for Dolgachev surface.

3. Sheaves of Categories and Local systems

In this section we discuss a parallel of the theory of local systems (Higgs Bundles) and sheaves of categories. Our goal is to apply this parallel to the study of Landau-Ginzburg models and their deformations.

We start with the following principal conjecture.

Conjecture 3.1. *There is a correspondence between the theory of local systems on topological spaces (projective varieties) and the theory of sheaves of categories on topological spaces.*

$\text{Funct}(\Pi_1^{\text{Group}}(X, s), \text{Vect}) \longleftrightarrow \text{Funct}(\Pi_{\leq \infty}^{\text{Sch}}(X, w, s), \text{dg Cat})$
$\text{Local system on } X \longleftrightarrow \text{Sheaves of Cat on a symplectic manifold } (X, w)$

Here s is a chosen base point.

This correspondence allows several generalizations. The first sequence of generalizations is connected with (X, w, s) .

- (1) We can make X a stratified space.
- (2) We can enhance it with a group action.

The second sequence of generalizations is connected with introducing Hitchin-Simpson theory on $\text{Funct}(\Pi_{<\infty}^{\text{Sch}}(X, \text{Strat}, s), \text{dg Cat})$. The last is connected with approach taken by Haiden-Katzarkov-Kontsevich-Pandit on defining Kähler, Kähler-Einstein, Hermit-Einstein categories. In this case we have:

Conjecture 3.2.

- (1) If (X, Strat, s) and dg Cat are holomorphic, then $\text{Funct}(\Pi_{<\infty}^{\text{Sch}}(X, \text{Strat}, s), \text{dg Cat})$ has a “Schober Hodge structure”.
- (2) The map $\text{Funct}(\Pi_{<\infty}^{\text{Sch}}(X_1, \text{Strat}_1, s), \text{dg Cat } 1) \rightarrow \text{Funct}(\Pi_{<\infty}^{\text{Sch}}(X_2, \text{Strat}_2, s), \text{dg Cat } 2)$ is functorial and satisfies strictness properties.

We now consider elements of $\text{Funct}(\Pi^{\text{Sch}}(X, \text{Strat}, s), \text{dg Cat})$ more closely. We start with definition of A_∞ 2-category - our sheaves of categories. An approach to 2-categories is as follows:

- $\forall x, y \in \mathcal{C}$ - assign an ∞ -category A_{xy} .
- $\forall x, y, z$ a 3-module M

$$A_{xy} \overset{M(x,y,z)}{\otimes} \begin{matrix} A_{yz} \\ A_{zx} \end{matrix} \longrightarrow M(x, y, z)[\]$$

- $\forall x, y, z, t$ a 4-module M

$$A_{xy} \overset{M(x,y,z,t)}{\otimes} \begin{matrix} A_{zt} \\ A_{xy} \\ A_{yz} \end{matrix} \longrightarrow M(x, y, z, t)[\]$$

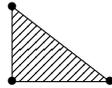
- for x_1, \dots, x_n, \dots

This can be done via homomorphisms x_0, \dots, x_n .

$$m_n : \text{Hom}(x_0, x_1) \rightarrow \dots \rightarrow \text{Hom}(x_{n-1}, x_n) \rightarrow \text{Hom}(x_0, x_n)[2 - n].$$

(This will lead to a description of 2 categories of sheaves of categories.)

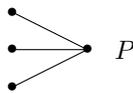
- We add n functors.
- And high natural transformations on Stasheff polygons.



Now we move to deformations of sheaves of categories.

Theorem 3.3. *Deformations of sheaves of categories can be described by:*

- (1) Adding a new vertex



P-Module Coefficients of the sheaf of categories.

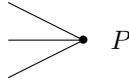
- (2) Changing the high natural transformations associated with Stasheff polygons.

In fact this phenomenon has been already studied in [BDF⁺13].

Theorem 3.4. *Deformations of the sheaf of categories of classical LG models of (see [BDF⁺13]) 3-dimensional Fano varieties contains the moduli space of LG models.*

The space $\text{Map}(\mathbb{P}^1, K3)$ can be considered as analogue of CVHS in Simpson's theory.

Theorem 3.5. *Let $D^b(X)$ be a category with exceptional collection (E_1, \dots, E_n) . Assume that there exists a coefficient module P*



so that:

$$\text{NHH}^\bullet(P) \rightarrow \text{HH}^\bullet(E_1, \dots, E_n, P) \rightarrow \text{HH}^\bullet(E_1, \dots, E_n) \rightarrow$$

- (1) Normal cohomologies $\text{NHH}(P) \neq 0$.
(2) $K^0(E_1, \dots, E_n) = K^0(E_1, \dots, E_n, P)$.

Then P is a phantom category.

Proof. See [Kuz10a] □

According to the previous section we can describe the deformations of sheaves of categories in at least three ways.

- (1) Changing the Stasheff polytope structure.
- (2) Changing the natural transformations and high natural transformations.
- (3) Changing the coefficient module which amounts to changing the coefficient in the Fukaya Seidel category.

As a consequence from [KLb], there exists a correspondence between multiplier ideal sheaves and change of coefficients in sheaves of categories:

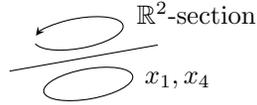
Commutative	Noncommutative
$J_n \subset J_{n-1} \subset \dots \subset J_0 \subset \mathcal{O}_X$ multiplier ideal sheaf	$\begin{array}{ccc} \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \dots \subset \mathcal{F} & & \\ \downarrow & \downarrow & \downarrow \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \end{array}$ sequence of sheaf of category $\text{FS}(\mathbb{C}, \mathcal{F}_n) \text{ FS}(\mathbb{C}, \mathcal{F}_{n-1}) \text{ FS}(\mathbb{C}, \mathcal{F})$ sequence of Fukaya-Seidel categories with coefficients \mathcal{F}_j
Okounkov bodies	Big Stasheff polytope
test configuration	$\bigcup_i (\text{Stasheff polytope})^j$

The changes in the Stasheff polytope produce dramatic deformations - changes of high natural transformations via change of sheaves of Lagrangians. Let us give some examples.

Example 3.6 (Kodaira surface). *We will start with $T^2 \times T^2$.*

$$\begin{array}{ccc}
 T^2 \times T^2 & \xleftarrow{\text{Mirror}} & T'^2 \times T'^2 \\
 \uparrow \text{Change of sheaf} & & \uparrow \text{surgery} \\
 \text{Fuk}(T^2 \times T^2, \beta) & \cong & \text{D}^b(\text{Kodaira surface}) \\
 \cap & & \\
 \mathbb{H}^2(\mathcal{O}^*) & &
 \end{array}$$

Let x_1, \dots, x_4 be coordinates in $T'^2 \times T'^2$ with $(x_1, x_2, x_3, x_4) \rightarrow (x_1 + 1, x_2, x_3, x_4 + x_3)$ invariance. We have a new sheaf of Lagrangians.

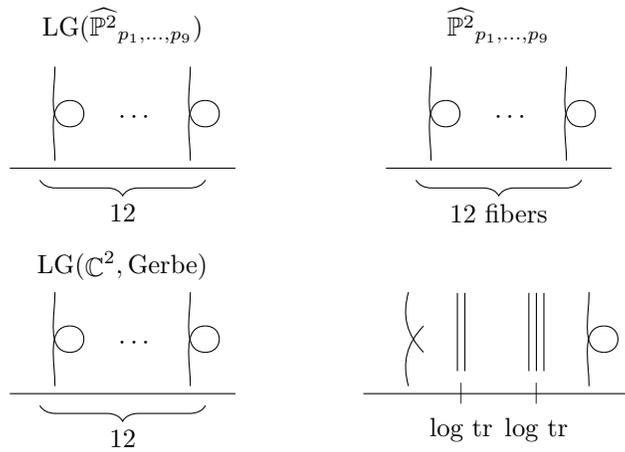


Theorem 3.7. *In the case of Fukaya Seidel categories the change of module coefficients of the sheaf of categories β is recorded by a new sheaf of Lagrangians (with or without a section).*

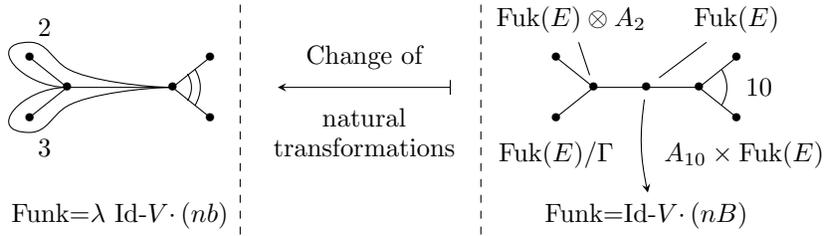
Example 3.8 (Dolgachev surface). *We move to the example of Dolgachev surface $\text{Dolg}_{2,3}$.*

Theorem 3.9. *The log transforms involved in the construction of $\text{Dolg}_{2,3}$ correspond to introducing a gerbe on the LG model of the rational elliptic surface.*

Proof.



In the language of Theorem 3.3, we are changing the structure of the sheaf of categories.



□

3.1. “Good” deformations. In this section, we define good deformations of perverse sheaf of categories. For this we need to introduce two new notions:

- (1) Lattice conjecture for perverse sheaves of categories;
- (2) $W = P$ conjecture for perverse sheaves of categories.

3.2. Lattice conjecture. In [KKP08] we have formulated the following:

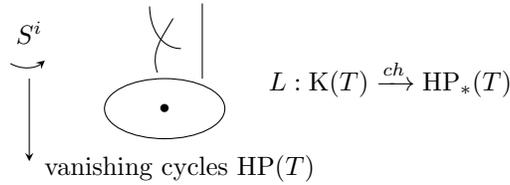
Conjecture 3.10. *Let T be a smooth and compact category. Then*

$$\text{Image of } L = \text{HP}_*(T),$$

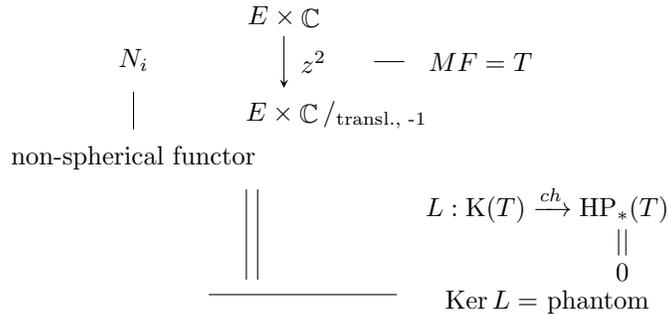
here L is the lattice map:

$$L : \text{K}(T) \xrightarrow{ch} \text{HP}_*(T).$$

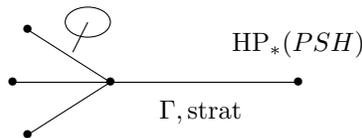
We extend this map to sheaves of categories. We start locally:



The lattice map is defined as a map to vanishing cycles, associated with spherical functors. We have such a lattice map for non-spherical functors.



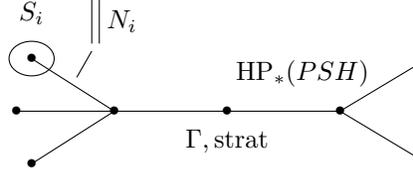
After that we globalize:



We have global monodromy map

$$\mu : \pi_1(\Gamma, \text{strat}) \rightarrow \text{HP}_*(PSC),$$

here PSC - perverse sheaf of categories and Γ, strat is the symplectic base.



We have monodromy equation

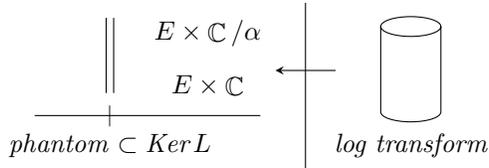
$$S_1^{n_1} \dots S_r^{n_r} = N_1^{l_1} \dots N_p^{l_p} \quad (**)$$

between spherical and non-spherical functors.

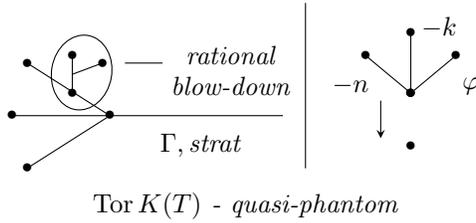
The main result of the operator given below is change of **.

Example 3.11.

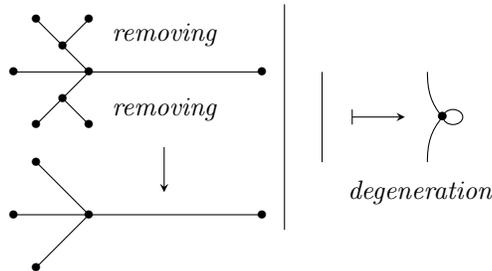
(1) *Mirror of surgery*



(2) *Mirror of rational blow-down.*



(3) *Mirror of degenerations*



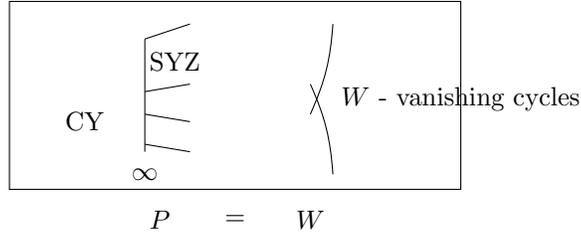
create $\text{Tor } H^*$ (vanishing cycles).

Recall that

$$H^*(\text{vanishing cycles}) \otimes \mathbb{C} = HP.$$

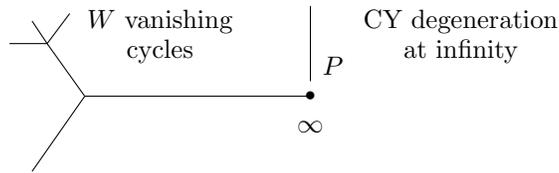
Consider a LG model with a gerb on it, \mathcal{F} .

In [II] it was proven that for toric varieties: $W = P$. For LG models of Fanos:



The P filtration of $\infty = W$ of vanishing cycles.

Similarly



we can define $W = P$ conjecture for perverse sheaves of CY categories.

We need a notion of a good deformation of perverse sheaf of categories.

Definition 3.14 (Good deformation). *A deformation of PSH is **good** if*

- (1) $P = W$ is preserved;
- (2) Lattice conjecture is preserved.

These deformations control cohomologies and K -theory. Deformations used by [Voi15], [CTP14], [Tot15] are good.

The main conjecture:

Conjecture 3.15. *Let $FS(PSC) \cong D^b(X)$, X - Fano. Assume there exists a good deformation of $FS(PSC)$ so that*

- (1) $L : K(T') \rightarrow HP_*(T')$ misses for $H(\text{vanishing cycles})$;
- (2) We create a basic class with support of $\text{codim} \leq 2$.

Then X is not rational.

We believe that this conjecture holds for examples in [Voi15], [CTP14], [Tot15]. In fact it seems this conjecture is stronger than splitting of diagonal.

4. Landau-Ginzburg model computations for threefolds

In this section we connect the program described in previous sections to birational geometry and theory of LG models.

We recall some inspiration from birational geometry stemming from the work of Voisin [Voi15], Colliot-Thélène and Pirutka [CTP14]. A variety X is called stably non-rational if $X \times \mathbb{P}^n$ is non-rational for all n . It is known that if a variety over \mathbb{C} is stably rational then for any field L containing \mathbb{C} , the Chow group $\text{CH}_0(X_L)$ is isomorphic to \mathbb{Z} . Under this condition, $\text{CH}_0(X)$ is said to be universally trivial. Voisin has shown that universal non-triviality of $\text{CH}_0(X)$ can be detected by deformation arguments, in particular [Voi15, Theorem 1.1] says that if we have a smooth variety \mathcal{X} fibered over a smooth curve B so that $\mathcal{X}_t = X$ and so that \mathcal{X}_0 has only mild singularities, then if X has universally trivial $\text{CH}_0(X)$ then so does any projective model of \mathcal{X}_0 . Alternatively, this means that if one can prove that \mathcal{X}_0 is

not stably rational, then neither is X . If V is a threefold, then one can detect stable non-rationality by showing that there exists torsion in $H^3(V, \mathbb{Z})$ (i.e. there exists torsion in the Brauer group). As an example, we may look at the classical Artin-Mumford example [AM72] which takes a degeneration of a quartic double solid to a variety which is a double cover of \mathbb{P}^3 ramified along a quartic with ten nodes. It is then proven in [AM72] that the resolution of singularities of this particular quartic double solid V has a $\mathbb{Z}/2$ in $H^3(V, \mathbb{Z})$. Voisin uses this to conclude that a general quartic double solid is not stably rational, whereas Artin and Mumford could only conclude from this that their specific quartic double solid is not rational.

The main idea that we explore in this section is that the approach of Voisin to stable non-rationality should have generalization to deformations or degenerations of $D^b(\text{coh } X)$. Via mirror symmetry, this should translate to a question about deformations or degenerations of sheaves of categories associated to the corresponding LG model of X . Mirror symmetry for Fano threefolds should exchange

$$\begin{aligned} H^{\text{even}}(X, \mathbb{Z}) &\cong H^{\text{odd}}(\text{LG}(X), S; \mathbb{Z}) \\ H^{\text{odd}}(X, \mathbb{Z}) &\cong H^{\text{even}}(\text{LG}(X), S; \mathbb{Z}) \end{aligned}$$

where S is a smooth generic fiber of the LG model of X . See [KKP14] for some justification for this relationship. This is analogous to the case where X is a Calabi-Yau threefold (see [Gro01, Gro98]). The deformations of the sheaf of categories associated to $\text{LG}(X)$ that we will produce are not deformations of LG models in the usual geometric sense, but they are produced by blowing up or excising subvarieties from X , as described in section 3. We then show that we find torsion in $H^2(U, S; \mathbb{Z})$ for U our topologically modified LG model. We propose that this torsion is mirror dual to torsion in the K_0 of some deformation of the corresponding category. By the relation above, the torsion groups appearing in the following subsections should be mirror categorical obstructions to stable rationality of the quartic double solid and the cubic threefold. Another way to view the constructions in the following sections is that we are identifying a subcategory of the Fukaya-Seidel category of the LG model of X and performing computations using this subcategory.

4.1. The LG model of a quartic double solid. Here we review a description of the LG models of several Fano threefolds in their broad strokes. We begin with the following situation. Let X be a Fano threefold of one of the following types. Recall that V_7 denotes the blow-up of \mathbb{P}^3 at a single point.

- (1) X is a quartic double solid.
- (2) X is a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2, 2)$.
- (3) X is a double cover of V_7 with branch locus an anticanonical divisor.
- (4) X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with ramification locus of degree $(2, 2, 2)$.

Then the singular fibers of the LG model of X take a specific form which is independent of X . The construction described here appears in [DHNT15] for the case of quartic double solids. There are several fibers of each LG model which are simply nodal K3 surfaces, and there is one fiber which is a more complicated. We assume the complicated fiber is the fiber over 0 in \mathbb{C} and we will denote it Y_0 . Monodromy about this complicated fiber has order 2, and the fiber itself has a single smooth rational component with multiplicity 2 and a number of rational components with

multiplicity 1. We will henceforward denote the LG model by Y , and it will be equipped with a regular function w .

A natural way to understand Y_0 is to take base-change along the map $t = s^2$ where s is a parameter on the base \mathbb{C}_t of the original LG model Y . Performing this base-change and taking normalization, we obtain a (possibly) singular family of K3 surfaces \widehat{Y} with a map $\widehat{w} : \widehat{Y} \rightarrow \mathbb{C}_s$. The (possible) singularities of \widehat{Y} are contained in the fiber $\widehat{w}^{-1}(0) = \widehat{Y}_0$, which is a K3 surface with several A_1 singularities.

Furthermore, there is an involution ι on \widehat{Y} from which we may recover the original LG model Y . This quotient map sends no fiber to itself except for \widehat{Y}_0 . On this fiber, the automorphism ι acts as a non-symplectic involution on \widehat{Y}_0 and fixes a number of rational curves.

In the Landau-Ginzburg model Y , given as the resolved quotient of \widehat{Y}/ι , the fiber Y_0 is described as follows. In the quotient \widehat{Y}/ι , the fiber over 0 is scheme-theoretically 2 times the preimage of 0 under the natural map. Furthermore, there are a number of curves of cA_1 singularities. We resolve these singularities by blowing up along these loci in sequence, since there is non-trivial intersection between them. This blow-up procedure succeeds in resolving the singularities of \widehat{Y}/ι and that the relative canonical bundle of the resolved threefold is trivial. Let E_1, \dots, E_n denote the exceptional divisors obtained in Y under this resolution of singularities.

4.2. Torsion in cohomology of the LG model. We will now denote by U the manifold obtained from Y by removing components of Y_0 with multiplicity 1, in other words, $U = Y \setminus (\cup_{i=1}^n E_i)$ where E_1, \dots, E_n are the exceptional divisors described in the previous paragraph. Another way to describe this threefold is as follows. Take the threefold \widehat{Y} described above, and excise the fixed locus of ι , calling the resulting threefold \widehat{U} . Note that this is the complement of a union of smooth codimension 2 subvarieties. The automorphism ι extends to a fixed-point free involution on \widehat{U} and the quotient \widehat{U}/ι is U . Let us denote by w_U the restriction of w to U . Our goal is to show that if S is a generic smooth fiber of w_U , then there is $\mathbb{Z}/2$ torsion in $H^2(U, S; \mathbb{Z})$.

The group $H^2(U, S; \mathbb{Z})$ should be the Hochschild homology of some subcategory of the Fukaya-Seidel category of $\text{LG}(X)$ equipped with an appropriate integral structure.

Proposition 4.1. *The manifold \widehat{U} is simply connected.*

Proof. First, let \widetilde{Y} be a small analytic resolution of singularities of \widehat{Y} and let \widetilde{w} be the natural map $\widetilde{w} : \widetilde{Y} \rightarrow \mathbb{A}_s^1$. Then, since the fixed curves of ι contain the singular points of \widehat{Y} , the variety \widehat{U} can be written as the complement in \widetilde{Y} of the union of the exceptional curves of the resolution $\widetilde{Y} \rightarrow \widehat{Y}$ and the proper transform of the fixed locus of the involution ι on \widehat{Y} . This is all to say that \widehat{U} is the complement of a codimension 2 subvariety of the smooth variety \widetilde{Y} . Thus it follows by general theory that $\pi_1(\widehat{U}) = \pi_1(\widehat{Y})$, and so it is enough to show that $\pi_1(\widehat{Y})$ is simply connected.

At this point, we may carefully apply the van Kampen theorem and the fact that ADE singular K3 surfaces are simply connected to prove that \widetilde{Y} is simply connected. Begin with a covering $\{V_i\}_{i=1}^m$ of \mathbb{A}^1 so that the following holds:

- (1) Each V_i is contractible,
- (2) Each $\widetilde{w}^{-1}(V_i)$ contains at most one singular fiber of \widetilde{w} ,

- (3) For each pair of indices i, j , the intersection $V_i \cap V_j$ is contractible, connected,
- (4) For each triple of indices i, j, k , the intersection $V_i \cap V_j \cap V_k$ is empty.

(it is easy to check that such a covering can be found). Then by the Clemens contraction theorem tells us that $Y_i := \tilde{w}^{-1}(V_i)$ is homotopic to the unique singular fiber (if V_i contains no critical point, then Y_i is homotopic to a smooth K3 surface). Since ADE singular K3 surfaces are simply connected, then Y_i is simply connected. The condition that $V_i \cap V_j$ is connected then allows us to use the Siefert-van Kampen theorem to conclude that \tilde{Y} is simply connected. \square

As a corollary to this proposition, we have that

Corollary 4.2. *The free quotient $U = \hat{U}/\iota$ has fundamental group $\mathbb{Z}/2$ and hence $H^2(U, \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}^n$ for some positive integer n .*

Now, finally, we show that this implies that there is torsion $\mathbb{Z}/2$ in the cohomology group $H^2(U, S; \mathbb{Z})$.

Theorem 4.3. *We have an isomorphism $H^2(U, S; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^m$ for some positive integer m .*

Proof. We compute using the long exact sequence in relative cohomology,

$$\dots \rightarrow H^1(S, \mathbb{Z}) \rightarrow H^2(U, S; \mathbb{Z}) \rightarrow H^2(U, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z}) \rightarrow \dots$$

Since S is a smooth K3 surface, we know that $H^1(S, \mathbb{Z}) = 0$, and that the subgroup $\mathbb{Z}/2$ of $H^2(U, \mathbb{Z})$ must be in the kernel of the restriction map $H^2(U, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z})$. Thus it follows that there is a copy of $\mathbb{Z}/2$ in $H^2(U, S; \mathbb{Z})$, and furthermore, that $H^2(U, S; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^m$ for some integer m . \square

Here U is a $\text{LG}_i(X^0)$ as in Conjecture 1.1.

Remark 4.4. In the case of the LG model of the quartic double solid, this construction is strikingly similar to the example of Artin and Mumford described in [AM72]. The authors degenerate the quartic double solid until it obtains ten nodes and then take a small resolution of these nodes, which is topologically equivalent to contracting ten copies of S^3 and replacing them with copies of S^2 . These ten copies of S^3 span a subspace H of $H^3(X, \mathbb{Z})$ for X the quartic double solid, so that $H^3(X, \mathbb{Z})/H \cong \mathbb{Z}^2 \oplus \mathbb{Z}^m$. In the LG model, we have removed ten rational divisors so that the quotient of $H^2(Y, \mathbb{Z})$ modulo the subgroup spanned by these ten divisors contains an order 2 torsion class. However, we do not know whether this is truly mirror to the Artin-Mumford example.

4.3. The cubic threefold. A very similar construction can be performed in the case of the LG model of the cubic threefold with some minor modifications. The details of the construction of the LG model of the cubic threefold that are relevant are contained in [GKR12]¹. There is a smooth log Calabi-Yau LG model of the cubic threefold, which we denote (Y, w) with the following properties:

- (1) The generic fiber is a K3 surface with Picard lattice $M_6 = E_8^2 \oplus U \oplus \langle -6 \rangle$.
- (2) There are three fibers with nodes.

¹In the most recent versions of [GKR12], these details have been removed, so we direct the reader to versions 1 and 2 of [GKR12] on the arXiv

- (3) The fiber over 0 which is a union of 6 rational surfaces whose configuration is described in [GKR12]. Monodromy around this fiber is of order 3.

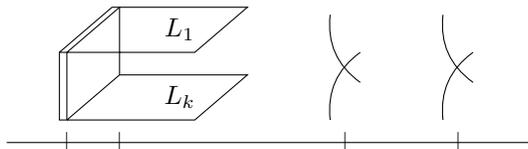
By taking base change of Y along the map $g : \mathbb{C} \rightarrow \mathbb{C}$ which assigns λ to μ^3 , and resolving g^*Y , we obtain a threefold \hat{Y} which is K3 fibered over \mathbb{C} , but now has only 6 singular fibers, each with only a node. This means that there is a birational automorphism ι on \hat{Y} of order 3 so that \hat{Y}/ι is birational to Y . Explicitly, in [GKR12] it is shown that the automorphism ι is undefined on nine pairs of rational curves, each pair intersecting in a single point and all of these pairs of curves are in the fiber of \hat{Y} over 0. We can contract these A_2 configurations of rational curves to get a threefold \tilde{Y} on which ι acts as an automorphism, but which is singular. The automorphism ι fixes six rational curves in the fiber of \tilde{Y} over 0. After blowing up sequentially along these six rational curves to get \tilde{Y}' , the automorphism ι continues to act biholomorphically, and no longer has fixed curves. The quotient \tilde{Y}'/ι is smooth, according to [GKR12], and there are seven components, the image of the six exceptional divisors, and a single component $R \cong \mathbb{P}^1 \times \mathbb{P}^1$ of multiplicity three. The rational surfaces coming from exceptional divisors meet R along three vertical and three horizontal curves. The divisor R can be contracted onto either one of its \mathbb{P}^1 factors. Performing one of these two contractions, we recover Y .

Now let $U = (\tilde{Y}'/\iota) \setminus \{S_1, \dots, S_6\}$. Note that this can be obtained by blowing up Y in the curve which is the intersection of three components of the central fiber and removing all of the other components. Then a proof almost identical to that of Theorem 4.3 shows that, if S is a generic fiber of w , then

Theorem 4.5. *There is an isomorphism $H^2(U, S; \mathbb{Z}) \cong \mathbb{Z}/3 \oplus \mathbb{Z}^m$ for some positive integer m .*

Therefore, if X is the cubic threefold, then there should exist a non-commutative deformation of $D^b(\text{coh } X)$ with torsion in its periodic cyclic cohomology obstructing stable rationality of X . Again, the cohomology group $H^2(U, S; \mathbb{Z})$ should be related to the Hochschild homology of some subcategory of the Fukaya category of the LG model of the cubic threefold.

Example 4.6. *We give an interpretation of the above considerations from the prospective of sheaves of categories. Consider the LG model of quartic double solid. The fiber over 0 is made up of 11 rational surfaces. One of these surfaces has multiplicity two and the others (labelled L_1, \dots, L_{10}) have multiplicity 1, as shown in the following diagram.*



Remove the rational surfaces L_1, \dots, L_{10} from Y , and the map w is no longer proper.

4.4. Cubic fourfolds and their mirrors. In this section, we will look at the LG models of cubic fourfolds and cubic fourfolds containing one or two planes. Since cubic fourfolds containing one or two planes are still topologically equivalent to a generic cubic fourfold, this is a somewhat subtle problem which we avoid by instead

obtaining LG models for cubic fourfolds containing planes which are blown up in the relevant copies of \mathbb{P}^2 .

It is known (see [Kuz10b]) that a general cubic has bounded derived category of coherent sheaves $D^b(X)$ which admits a semi-orthogonal decomposition

$$\langle \mathcal{A}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{O}_X(3) \rangle.$$

When X contains a plane, $\mathcal{A}_X = D^b(S, \beta)$ is the bounded derived category of β twisted sheaves on a K3 surface S for β an order 2 Brauer class. It is known [Has99, Lemma 4.5] that the lattice T in $H^4(X, \mathbb{Z})$ orthogonal to the cycles $[H]^2$ and $[P]$ where H is the hyperplane class and P is the plane contained in X , is isomorphic to

$$E_8^2 \oplus U \oplus \begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

which is not the transcendental lattice of any K3 surface. It is expected that such cubic fourfolds are non-rational. When X contains two planes, it is known that X is then rational. According to Kuznetsov [Kuz10b], we then have that the category \mathcal{A} is the derived category of a K3 surface S , and by work of Hassett [Has99], we have that the orthogonal complement of the classes $[H]^2, [P_1], [P_2]$ where P_1 and P_2 are the planes contained in X is isomorphic to

$$U \oplus E_8^2 \oplus \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix},$$

which is the transcendental lattice of a K3 surface S , and generically $\mathcal{A}_X = D^b(\text{coh } S)$ and $S^{[2]}$ is the Fano variety of lines in X .

Our goal in this section is to describe the mirror side of this story. In particular, we want to observe in the three cases above, how rationality and non-rationality can be detected using symplectic characteristics of LG models. We will construct smooth models of smooth models of

- (1) The LG model of a cubic fourfold (which we call Z_0).
- (2) The LG model of a cubic fourfold containing a plane P blown up in P (which we call Z_1).
- (3) The LG model of a cubic fourfold containing a pair of disjoint planes P_1 and P_2 blown up in $P_1 \cup P_2$ (which we call Z_2).

According to a theorem of Orlov [Orl92], the bounded derived categories of Z_1 and Z_2 admit semi-orthogonal decompositions with summands equal to the underlying cubics. Therefore, homological mirror symmetry predicts that the derived categories of coherent sheaves of the underlying cubics should be visible in the Fukaya-Seidel (or directed Fukaya) categories of the LG models of Z_1 and Z_2 . In particular, we should be able to see $D^b(\text{coh } S, \beta)$ in the Fukaya-Seidel category of $\text{LG}(Z_1)$ and $D^b(\text{coh } Z_2)$ in the Fukaya-Seidel category of $\text{LG}(Z_2)$.

It is conjectured by Kuznetsov [Kuz10b] that a cubic fourfold X is rational if and only if \mathcal{A}_X is the bounded derived category of a geometric K3 surface, thus in the case where X contains a single plane, the gerbe β is an obstruction to rationality of X . Such gerbes arise naturally in mirror symmetry quite commonly. If we have a special Lagrangian fibration on a manifold M over a base B , and assume that there is a special Lagrangian multisection of π and no special Lagrangian section, then mirror symmetry is expected assign to a pair (L, ∇) in the Fukaya category of

M a complex of α -twisted sheaves on the mirror for α some non-trivial gerbe. We will see this structure clearly in the LG models of Z_0, Z_1 and Z_2 .

4.5. The general cubic fourfold. Let us now describe the LG model of the general cubic fourfold in a such a way that a nice smooth resolution becomes possible. Givental [Giv98] gives a description of constructions of mirrors of toric complete intersections. A more direct description of Givental’s construction is described in [HD15].

We begin with the polytope Δ corresponding to \mathbb{P}^5 given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Using Givental’s construction, we get a LG model with total space

$$Y^0 = \{z + w + u = 1\} \subseteq (\mathbb{C}^\times)^5$$

equipped with the function

$$w(x, y, z, w, u) = x + y + \frac{1}{xyzwu}.$$

We will express Y^0 as a fibration over \mathbb{C}^3 by elliptic curves. Then we will use work of Miranda [Mir83] to resolve singularities of this fibration and thus obtain a smooth model of Y^0 . This is necessary, since there are singularities “at infinity” in the LG model provided by Givental. A more uniform construction of smooth compactifications of the LG models constructed by Givental can be found in [Har16, Chapter 3].

To carry do this, we decompose w into three different functions

$$w_1 = x, \quad w_2 = y, \quad w_3 = \frac{1}{xyzwu}.$$

Then Y^0 is birational to a variety fibered by affine curves written as

$$w_1 w_2 w_3 z w (z + w - 1) - 1 = 0$$

where w_1, w_2, w_3 are treated as coordinates on \mathbb{C}^3 . This is can be rearranged into Weierstrass form as

$$y^2 = x^3 + w_2^2 w_1^2 w_3^2 x^2 + 8w_3^3 w_2^3 w_1^3 x + 16w_1^4 w_2^4 w_3^4.$$

The discriminant locus of this fibration over \mathbb{C}^3 has four components, and for a generic point in each component we can give a description of the structure of the resolution of singularities over that point in terms of Kodaira’s classification of the singular fibers of elliptic fibrations.

- Singular fibers of type IV^* along $\{w_i = 0\}$ for $i = 1, 2, 3$,
- Singular fibers of type I_1 along the divisor cut out by the equation $w_1 w_2 w_3 - 27 = 0$.

The loci $w_i = 0$ intersect each other of course, but D_{I_1} does not intersect any $\{w_i = 0\}$, thus we must only worry about singularities at $(0, 0, 0)$ and $w_i = w_j = 0$ for $i, j = 1, 2, 3$ and $i \neq j$. We blow up sequentially at these loci and describe the fibers over the exceptional divisors. We will use Kodaira’s conventions for describing the minimal resolution of singular fibers of an elliptic fibration.

- Blow up the base \mathbb{C}^3 at $(0,0,0)$. Call the associated blow-up map $f_1 : T_1 \rightarrow \mathbb{C}^3$ and call the exceptional divisor Q . As before, if π_1 is the induced elliptic fibration on T_1 , then on Q there are just smooth fibers away from the intersection of the strict transform of $\{w_i = 0\}$.
- Blow up the intersections $\{w_i = w_j = 0\}$ for $i, j = 1, 2, 3$ and $i \neq j$. Call the associated map $f_2 : T_2 \rightarrow T_1$ and call the exceptional divisors $E_{i,j}$. Let π_2 be the induced elliptic fibration on T_2 . The fibration π_2 has fibers with resolutions of type IV over R_{ij} .
- Blow up at the intersections of R_{ij} and the strict transforms of $\{w_i = 0\}$ and $\{w_j = 0\}$. Call the associated map $f_3 : T_3 \rightarrow T_2$ and call the exceptional divisors $R_{ij,i}$ and $R_{ij,j}$ respectively. Let π_3 be the induced elliptic fibration over T_3 , then the fibration π_3 has smooth fibers over the divisors $R_{ij,i}$ and $R_{ij,j}$.

Thus we have a fibration over T_3 with discriminant locus a union of divisors, and none of these divisors intersect one another. Thus we may resolve singularities of the resulting Weierstrass form elliptic fourfold by simply blowing up repeatedly the singularities along these loci. Call this fourfold $\text{LG}(Z_0)$. By composing the elliptic fibration π_3 of $\text{LG}(Z_0)$ over T_3 with the contraction of T_3 onto \mathbb{C}^3 we get a map which we call $w_1 + w_2 + w_3$ from $\text{LG}(Z_0)$ to \mathbb{C}^3 . We will describe explicitly the fibers over points of $w_1 + w_2 + w_3$.

- If p is a point in the complement of the strict transform of $\{w_1 = 0\} \cup \{w_2 = 0\} \cup \{w_3 = 0\} \cup \{w_1 w_2 w_3 - 27 = 0\}$ then the fiber over p is smooth.
- If p is in $\{w_1 = 0\}, \{w_2 = 0\},$ or $\{w_3 = 0\}$ then the fiber over p is of type IV*. If p is a point in $\{w_1 w_2 w_3 - 27 = 0\}$, then the fiber over p is a nodal elliptic curve.
- If $p \in \{w_1 = w_2 = 0\}, \{w_1 = w_3 = 0\}$ or $\{w_2 = w_3 = 0\}$ then the fiber over p is of dimension 2.
- If $p = (0, 0, 0)$, then the fiber is a threefold. This threefold is precisely the restriction of the fibration π_3 to the strict transform of the exceptional \mathbb{P}^2 obtained by blowing up $(0, 0, 0)$.

Now we will let $\text{LG}(Z_0)$ be the smooth resolution of the elliptically fibered threefold over T_3 described above. We compose the fibration map π_3 with the map $(z_1, z_2, z_3) \mapsto z_1 + z_2 + z_3$ from \mathbb{C}^3 to \mathbb{C} , then we recover the map w on the open set that $\text{LG}(Z_0)$ and Y^0 have in common. Then we obtain a nice description of the fiber in $\text{LG}(Z_0)$ of w over 0 as a union of two elliptically fibered threefolds, one component being the threefold fiber over $(0, 0, 0)$ in Y , and the other being the natural elliptically fibered threefold obtained by taking the preimage of the line $w_1 + w_2 + w_3 = 0$ in $\text{LG}(Z_0)$ under the elliptic fibration map. These two threefolds intersect along a surface S which is naturally elliptically fibered. This surface can be described by taking the subvariety of the exceptional divisor $Q = \mathbb{P}^2$ given by a the natural fibration over a hyperplane in \mathbb{P}^2 . This is an elliptically fibered surface over \mathbb{P}^2 with three singular fibers of type IV* and a order 3 torsion section.

Proposition 4.7. *The smooth K3 surface S of Picard rank 20 with transcendental lattice isomorphic to the (positive definite) root lattice A_2 , which has Gram matrix*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

This can be proved without much difficulty using the techniques described in [HT15].

4.6. Cubic fourfolds blown up in a plane. We will apply a similar approach to describe the LG model of the cubic fourfold blown up in a plane. We start by expressing this as a toric hypersurface. Blowing up \mathbb{P}^5 in the intersection of three coordinate hyperplanes is again a smooth toric Fano variety \mathbb{P}_Δ which is determined by the polytope Δ with vertices given by points ρ_1, \dots, ρ_7 given by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

The vertices of this polytope (determined by the columns of the above matrix) determine torus invariant Cartier divisors in \mathbb{P}_Δ , and the cubic blown up in a plane is linearly equivalent to $D_{\rho_3} + D_{\rho_4} + D_{\rho_5}$. Thus, following the prescription of Givental [Giv98] (or more precisely, [HD15]), one obtains the Landau-Ginzburg model with

$$Y^0 = \left\{ z + w + u + \frac{a}{xyz} = 1 \right\} \subseteq (\mathbb{C}^\times)^5$$

equipped with potential given by restriction of

$$w(x, y, z, w, u) = x + y + \frac{1}{xyzwu}$$

to Y^0 . We may decompose w into the three potentials

$$w_1 = x, \quad w_2 = y, \quad w_3 = \frac{1}{xyzwu}.$$

so that $w = w_1 + w_2 + w_3$. Therefore, if we take the map $\pi : Y^0 \rightarrow \mathbb{C}^3$ given by (w_1, w_2, w_3) , this can be compactified to a family of elliptic curves with fiber

$$w_1 w_2 w_3 z w (z + w - 1) + 1 + a w_3 w = 0.$$

This can be written as a family of elliptic curves in Weierstrass form as

$$y^2 = x^3 + w_1 w_2^2 w_3 (w_1 w_3 - 4a) x^2 + 8w_1^3 w_2^3 w_3^3 x + 16w_1^4 w_2^4 w_3^4.$$

Away from $(0, 0, 0)$, the singularities of this fibration can be resolved.

- I_1^* along $w_1 = 0$ and $w_2 = 0$
- IV^* along $w_3 = 0$
- I_1 along

$$(a w_1^2 w_2^2 w_3^2 - 8a^2 w_1 w_2 w_3^2 + w_1^2 w_2^2 w_3 + 16a^3 w_3^2 - 36a w_1 w_2 w_3 - 27 w_1 w_2) = 0$$

We first blow up the base \mathbb{C}^3 at $(0, 0, 0)$ to obtain a fibration with smooth fibers over the exceptional divisor. We cannot yet resolve singularities of this fibration, since the fibers over the intersection of any two coordinate hyperplanes do not have known resolutions. Following work of Miranda [Mir83], we may blow up the base of this fibration again several times in order to produce a fibration over a threefold which has a fiber-wise blow-up which resolves singularities.

We blow up the base along the lines $R_{ij} = \{w_i = w_j = 0\}$ to get three exceptional surfaces R_{ij} over which there are singular fibers generically of type IV. Blowing

up again in all lines of intersection between R_{ij} and $w_j = 0$ and R_{ij} and $w_i = 0$, calling the resulting exceptional divisors $R_{ij,j}$ and $R_{ij,i}$, we get an elliptic fibration over this blown up threefold so that:

- I_1^* along $w_1 = 0$ and $w_2 = 0$
- IV^* along $w_3 = 0$
- IV along R_{ij} .
- I_0 (i.e. smooth) along $R_{ij,j}$ and $R_{ij,i}$.
- I_1 along some divisor which does not intersect $w_1 = 0, w_2 = 0, w_3 = 0$ or $R_{ij} = 0$.

Therefore, one may simply resolve singularities of this fibration in the same way as one would in the case of surfaces – blowing up repeatedly in sections over divisors in the discriminant locus. Let us refer to this elliptically fibered fourfold as $LG(Z_1)$. There is an induced map from $LG(Z_1)$ to \mathbb{C} which we call w essentially comes from the composition of the fibration on $LG(Z_1)$ by elliptic curves with its contraction onto \mathbb{C}^3 along with the addition map $(z_1, z_2, z_3) \mapsto z_1 + z_2 + z_3$ from \mathbb{C}^3 to \mathbb{C} . This is the superpotential on $LG(Z_1)$, and $LG(Z_1)$ is a partially compactified version of the Landau-Ginzburg model of the cubic fourfold blown up in a plane.

The fiber of w over 0 is the union of two elliptically fibered smooth threefolds, one being the induced elliptic fibration over the proper transform of the exceptional divisor obtained when we blew up $(0, 0, 0)$ in \mathbb{C}^3 . The other is the proper transform in $LG(Z_1)$ of the induced elliptic fibration over the surface $z_1 + z_2 + z_3 = 0$ in \mathbb{C}^3 .

These two threefolds meet transversally along a smooth K3 surface S . This K3 surface is equipped naturally with an elliptic fibration structure over \mathbb{P}^1 and inherits two singular fibers of type I_1^* , a singular fiber of type IV^* and two singular fibers of type I_1 .

Proposition 4.8. *The orthogonal complement of the Picard lattice in $H^2(S, \mathbb{Z})$ is isomorphic to*

$$\begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix},$$

for a generic K3 surface S appearing as in the computations above.

To prove this, one uses a concrete model of S and shows that there is another elliptic fibration on S so that the techniques in [HT15] can be applied to show that there is a lattice polarization on a generic such S by the lattice

$$E_8^2 \oplus \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}. \quad (4.1)$$

Then one shows that the complex structure on the surface S varies nontrivially as the parameter a varies, thus a generic such S has Picard lattice equal to exactly the lattice in Equation (4.1). Then applying standard results of Nikulin [Nik80], one obtains the proposition.

4.7. Cubic threefolds blown up in two planes. Here we begin with the toric variety \mathbb{P}^5 blown up at two disjoint planes, which is determined by the polytope Δ

with vertices at the columns ρ_1, \dots, ρ_8 of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

The cubic blown up along two disjoint planes is then linearly equivalent to the torus invariant divisor $D_{\rho_3} + D_{\rho_4} + D_{\rho_5} + D_{\rho_7}$, therefore, by the prescription of Givental, we may write the associated LG model as

$$Y^0 = \left\{ z + w + u + \frac{a}{xyz} \right\} \subseteq (\mathbb{C}^\times)^5$$

equipped with the function

$$w(x, y, z, w, u) = x + y + \frac{1}{xyzwu} + bxyz.$$

We split this into the sum of three functions,

$$w_1 = x + bxyz, \quad w_2 = y, \quad w_3 = \frac{1}{xyzwu}.$$

The fibers of the map (w_1, w_2, w_3) from Y to \mathbb{C}^3 are written as a family of affine cubics

$$(z + w - 1)w_1w_2w_3zw + (1 + bw_2z)(1 + aw_3w) = 0$$

which are open elliptic curves. We may write this in Weierstrass form and use Tate's algorithm to show that, the singular fibers of this fibration are of types:

- I_1^* along $w_3 = 0$ and $w_2 = 0$
- I_5 along $w_1 = 0$
- I_1 along a divisor determined by a complicated equation in w_1, w_2 and w_3 .

Elsewhere, the fibers of this map can be compactified to smooth elliptic curves.

In order to obtain a smooth model of this fibration, we will first blow up \mathbb{C}^3 at $(0, 0, 0)$. The induced elliptic fibration is generically smooth over this exceptional divisor, which we call Q . In order to obtain a model of this elliptic fibration which we may resolve by sequentially blowing up in singular fibers, we must now blow up along the line $w_2 = w_3 = 0$. We will call the exceptional surface under this blow-up R_{23} . We obtain a singular elliptically fibered fourfold over this new threefold base so that the fibers over the divisor R_{23} are generically of Kodaira type IV. Blowing up again at the intersections of R_{23} and $w_2 = 0$ and at the intersection of R_{23} and $w_3 = 0$ (calling the exceptional divisors $R_{23,2}$ and $R_{23,3}$ respectively) we obtain a fibration which can be resolved by blowing up curves of divisors in the fibers over $R_{23}, w_1 = 0, w_2 = 0$ and $w_3 = 0$, and by taking resolution over curves in $w_1 = w_2 = 0$ and $w_1 = w_3 = 0$ (following [Mir83, Table 14.1]). Call the resulting fibration $\text{LG}(Z_2)$ and let π be the fibration map onto the blown up threefold. We have singular fibers of types:

- I_1^* along $w_3 = 0$ and $w_2 = 0$
- I_5 along $w_1 = 0$
- IV along R_{23}
- Fibers over $w_1 = w_2 = 0$ and $w_1 = w_3 = 0$ of the type determined by Miranda [Mir83] and described explicitly in [Mir83, Table 14.1].

- I_1 along a complicated divisor which does not intersect any of the divisors above.

and smooth fibers otherwise.

The variety $\text{LG}(Z_2)$ admits a non-proper elliptic fibration over \mathbb{C}^3 obtained by composing π with the blow-up maps described above. Then the fiber in $\text{LG}(Z_2)$ over $(0, 0, 0)$ is an elliptic threefold over a blown-up \mathbb{P}^2 base. Composing this non-proper elliptic fibration with the map $(w_1, w_2, w_3) \mapsto w_1 + w_2 + w_3$ from \mathbb{C}^3 to \mathbb{C} recovers the potential w . The fiber over 0 of the map w from $\text{LG}(Z_2)$ to \mathbb{C} has two components, each an elliptically fibered threefold meeting along a smooth K3 surface. This K3 surface, which we call S , admits an elliptic fibration over \mathbb{P}^1 canonically with two singular fibers of type I_1^* , a singular fiber of type I_5 and five singular fibers of type I_1 .

Proposition 4.9. *The orthogonal complement of the Picard lattice in $H^2(S, \mathbb{Z})$ is isomorphic to*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

for a generic K3 surface S appearing as in the computations above.

Again, this result is obtained by finding an appropriate alternative elliptic fibration on S and demonstrating that an appropriate lattice embeds into its Picard lattice, then combining results of Nikulin [Nik80] and the fact that there is a non-trivial 2-dimensional deformation of S obtained by letting the parameters a and b vary to see that indeed, this is the transcendental lattice of a generic such S .

Remark 4.10. In the last three sections, we have glossed over the issue of providing an appropriate relative compactification of our LG models with respect to w . Indeed, one wants to produce a relatively compact partial compactification of the LG models above whose total space is smooth and has at least trivial canonical class. In the cases that we have described above, this can be done by taking a relative compactification of \mathbb{C}^3 with respect to the map $(w_1, w_2, w_3) \mapsto w_1 + w_2 + w_3$ and writing $\text{LG}(Z_i)$ as an elliptically fibered fourfold over this variety. Performing the same procedure as above (blowing up the base of this fibration until a global resolution can be obtained by simply blowing up in fibers or taking small resolutions as described by Miranda [Mir83], one can produce a partial compactification of $\text{LG}(Z_i)$ so that the fibers of w are compact. Using the canonical bundle formula in [Mir83], one can then show that this compactification is indeed appropriate. We note that, strictly speaking, Miranda's work only applies to three dimensional elliptic fibrations. However, since we do not have to deal with intersections of more than two divisors in our discriminant locus, and all of our intersections are transverse, the arguments of [Mir83] still may be applied.

4.8. Special Lagrangian fibrations. In the case of hyperkähler surfaces, special Lagrangian fibrations can be constructed with relatively little difficulty. The procedure is outlined in work of Gross and Wilson [GW97]. We review their work in the following section and apply it to our examples.

Definition 4.11. *A K3 surface S is lattice polarized by a lattice L if there is a primitive embedding of L into $\text{Pic}(S)$ whose image contains a pseudo-ample class.*

For a given lattice L of signature $(1, \rho - 1)$ for $\rho \leq 20$ which may be embedded primitively into $H^2(S, \mathbb{Z})$ for a K3 surface, there is a $(20 - \rho)$ -dimensional space of complex structures on S corresponding to K3 surfaces which admit polarization by L . A generic L -polarized K3 surface will then be a general enough choice of complex structure in this space.

We will follow the notation of Gross and Wilson [GW97] from here on. We choose I to be a complex structure on a K3 surface S and let g be a compatible Kähler-Einstein metric. Since S is hyperkähler, there is an S^2 of complex structures on S which are compatible with g . We will denote by I, J and K the complex structures from which all of these complex structures are obtained. The complex 2-form associated to the complex structure I is written as $\Omega(u, v) = g(J(u), v) + ig(K(u), v)$ for u and v sections of T_S . The associated Kähler form is given, as usual, by $\omega(u, v) = g(I(u), v)$. Similarly, one may give formulas for the holomorphic 2-form and Kähler forms associated to the complex structures J and K easily in terms of the real and imaginary parts of Ω and ω as described in [GW97, pp. 510].

A useful result that Gross and Wilson attribute to Harvey and Lawson [HL82, pp. 154] is:

Proposition 4.12 ([GW97, Proposition 1.2]). *A two-dimensional submanifold Y of S is a special Lagrangian submanifold of S with respect to the complex structure I if and only if it is a complex submanifold with respect to the complex structure K .*

Using the same notation as in [GW97], we will let S_K be the complex K3 surface with complex structure K , which then has holomorphic 2-form given by $\Omega_K = \text{Im}\Omega + i\omega$ where ω and Ω are as before. If this vanishes when restricted to a submanifold E of S , then we must have $\omega|_E = 0$ as well. If ω is chosen generically enough in the Kähler cone of S (so that $\omega \cap L = 0$) then this forces E to be in L^\perp . One can show that a complex elliptic curve E on a K3 surface satisfies $[E]^2 = 0$ therefore, since L^\perp has no isotropic elements, S_K cannot contain any complex elliptic curves and thus S has no special Lagrangian fibration. Therefore, we have proven that:

Proposition 4.13. *If L is a lattice so that L^\perp contains no isotropic element, then a generic L -polarized K3 surface with a generic choice of Kähler-Einstein metric g has no special Lagrangian fibration.*

We will use this to prove a theorem regarding K3 surfaces which appeared in the previous sections. Let us recall that the transcendental lattices of the K3 surface appearing as the intersection of the pair of divisors in $\text{LG}(Z_0), \text{LG}(Z_1)$ and $\text{LG}(Z_2)$ are

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

In the first case, it is clear that the lattice is positive definite, therefore it cannot represent 0, and thus Proposition 4.13 shows that in this case there is no special Lagrangian fibration on this specific K3 surface. In the third case, we can use [GW97, Proposition 1.3] to see that there is a special Lagrangian fibration with numerical special Lagrangian section for a generic choice of Kähler-Einstein metric g .

In the second case, the discriminant of the lattice (which we will call M) is -8 , and its discriminant group, which is just M^\vee/M , is isomorphic to $\mathbb{Z}/8$ and has generator with square $3/8$. Using a result of Nikulin [Nik80], it follows that this is not equivalent to the lattice $\langle -8 \rangle \oplus U$. At the same time, one can conclude that this is not the lattice $\langle -2 \rangle \oplus U(2)$, and therefore, we cannot directly apply [GW97, Proposition 1.3] to obtain a special Lagrangian fibration on such a K3 surface.

However, applying the method used in the proofs of [GW97, Proposition 1.1] and [GW97, Proposition 1.3], one obtains a special Lagrangian fibration on S for a generic choice of g so that there is no special Lagrangian section, but there is a numerical special Lagrangian 2-section. To do this, we use the fact that $(1, -1, 1)$ is isotropic in this lattice.

Putting all of this together, we obtain the following theorem:

Theorem 4.14. *Let S be a generic K3 surface appearing as the intersection of the two components of the fiber over 0 of the LG models of a generic cubic Z_0 , a cubic blown up in a plane Z_1 , and a cubic blown up in two disjoint planes Z_2 . Let ω be a generic Kähler class on S and Ω the corresponding holomorphic 2-form on S . Then:*

- (1) *In the case where $S \subseteq \text{LG}(Z_0)$, then S admits no special Lagrangian torus fibration.*
- (2) *In the case where $S \subseteq \text{LG}(Z_1)$, then S admits a special Lagrangian torus fibration with no Lagrangian section but a (numerical) Lagrangian 2-section.*
- (3) *In the case where $S \subseteq \text{LG}(Z_2)$, then S admits a special Lagrangian torus fibration with a (numerical) Lagrangian section.*

The first statement in Theorem 4.14 is mirror dual to the fact that the subcategory \mathcal{A}_X of $D^b(\text{coh } X)$ for X a generic cubic fourfold is not the derived category of a K3 surface. The second statement corresponds to the fact that $\mathcal{A}_X \cong D^b(S, \beta)$ for β an order 2 Brauer class on S for X a general cubic fourfold containing a plane. The third case corresponds to the fact that when X contains two disjoint planes, $\mathcal{A}_X \cong D^b(S)$ for S a K3 surface.

According to [AAK12, Corollary 7.8], there is an embedding of the (derived) Fukaya category of the K3 surface S appearing in Theorem 4.14 as a subcategory of the derived version of the Fukaya-Seidel category of the LG model of Z_0, Z_1 and Z_2 respectively. The objects in the Fukaya-Seidel category of an LG model are so-called admissible Lagrangians, which are, roughly, Lagrangian submanifolds L of the LG model with (possible) boundary in a fiber V of w . In the case where w is a Lefschetz fibration, it is well-known (see [Sei01]) that such Lagrangians (so-called Lagrangian thimbles) can be produced by taking appropriate paths between V and p for p a critical value of w and tracing the image of the vanishing cycle at $w^{-1}(p)$ along this path.

A similar construction using vanishing cycles will suffice to produce an embedding of the Fukaya category of S into the Fukaya-Seidel category of $\text{LG}(X_i)$. In the case that we are interested in, where the central fiber of our degeneration is simply a union of two smooth varieties meeting transversally in a K3 surface, the vanishing cycle is simply an S^1 bundle over the critical locus of the degenerate fiber. In our case, this is simply an S^1 bundle over a K3 surface, which is then homotopic to $S^1 \times \text{K3}$. Thus, along any path approaching 0 in \mathbb{C} which doesn't intersect itself, we have a vanishing thimble homotopic to $D^2 \times \text{K3}$ where D^2 is the two-dimensional disc. This, of course, cannot be a Lagrangian in $\text{LG}(Z_i)$, but if instead we take all

points in $D^2 \times \text{K3}$ which converge to a Lagrangian ℓ in the K3 surface (in some appropriate sense), then there exists a Lagrangian thimble L_ℓ whose restriction to $w^{-1}(0)$ is ℓ . In this way, Lagrangians in S extend to admissible Lagrangians in $\text{LG}(X)$ and in particular induce a faithful A_∞ functor from the Fukaya category of S into the Fukaya-Seidel category of $\text{LG}(Z_i)$, both with appropriate symplectic forms. In particular, we have that

- (1) There is no admissible Lagrangian L in $\text{LG}(Z_0)$ so that $L|_{w^{-1}(0)}$ is a special Lagrangian torus.
- (2) There is no pair of admissible Lagrangians L_1 and L_2 in $\text{LG}(Z_1)$ so that $(L_1)|_{w^{-1}(0)}$ is a special Lagrangian torus and $(L_2)|_{w^{-1}(0)}$ is a special Lagrangian section of a special Lagrangian fibration on S .

These statements should be viewed as interpretations of Theorem 4.14 in terms of the Fukaya-Seidel category of Z_0, Z_1 and Z_2 . As claimed in section 3, the non-existence of a family appropriate Lagrangians in the LG models of Z_0 and Z_1 therefore corresponds to the conjectural fact that Z_0 and Z_1 are non-rational.

This section gives another example of a piece of $\text{LG}_i(X^0)$ of stable objects for HN filtration for a limiting stability conditions which serves as Conjecture 1.1 suggests as obstruction to stable rationality.

4.9. Lagrangian fibrations and the fibers of the LG models of the cubic fourfold. Here we will discuss the possibility of the existence of Lagrangian fibrations on the fibers of the LG models of the cubic fourfolds containing one and two planes. We note that the discussion here is related to the constructions in [DHT16]. A Tyurin degeneration, defined in [Tyu03] is a projective degeneration of d -dimensional Calabi-Yau manifolds V to a normal crossing union of two smooth varieties X_1 and X_2 meeting along a smooth Calabi-Yau anticanonical divisor Z in both X_1 and X_2 so that for $i = 1$ and 2 , we have that $H^i(X_i, \mathcal{O}_{X_i}) = 0$. Such varieties X_1 and X_2 are called quasi-Fano varieties. It is expected that there are often (special) Lagrangian torus fibrations π_1 and π_2 on X_1 and X_2 over the open ball B_d by real d -dimensional tori.

One expects that upon taking a real analytic blow-up of each X_i , along Z , which we call \overline{X}_1 and \overline{X}_2 , this Lagrangian torus fibration extends to a Lagrangian torus fibrations $\overline{\pi}_1$ and $\overline{\pi}_2$ over the d -dimensional disc D^d . If we let S_i be the S^1 bundle over Z homotopic to the normal bundle N_{Z/X_i} with the zero section removed. Then $\overline{\pi}_i$ is just the T^d bundle over S^{d-1} obtained by composing the bundle U_i over Z with a Lagrangian fibration on Z over S^{d-1} .

If V is the general fiber of a Tyurin degeneration, then X_1 and X_2 must satisfy $N_{Z/X_1} \otimes N_{Z/X_2} = \mathcal{O}_Z$ and therefore, we may glue the boundaries of \overline{X}_1 and \overline{X}_2 together. The result is homotopic to V , and, assuming the picture described above regarding the special Lagrangian fibrations on $X_1 \setminus Z$ and $X_2 \setminus Z$ is true, then V should be equipped with a topological T^d fibration over S^d , where the base is the pair of D^d s glued to one another along their boundaries. This construction is due to Auroux and appears in [Aur08].

Assuming that one can build such Lagrangian fibrations on $X_1 \setminus Z$ and $X_2 \setminus Z$, then the dual special Lagrangian fibration on V to build the mirror dual Calabi-Yau variety.

The situation that we are interested in is the Tyurin degenerations which appear in the central fiber of the LG model of the cubic fourfolds blown up in a plane. In

this situation, we will let U_1 and U_2 be the two threefold components of the central fiber which meet in a smooth K3 surface S . We will now compile some evidence that both U_1 and U_2 admit Lagrangian fibrations without sections but with 2-sections.

Calabrese and Thomas [CT15] have shown that there is a K3 fibered Calabi-Yau threefold which is twisted derived equivalent to the anticanonical hypersurface in the cubic fourfold blown up in a plane. To construct this twisted derived equivalent threefold, one takes a pair of cubic fourfolds passing through the same plane in \mathbb{P}^5 . This gives fiber space of cubic fourfolds over \mathbb{P}^1 . Each of the fibers of this pencil is a cubic fourfold containing a plane, so the derived category \mathcal{A}_{X_t} described in Section 4.4 is the derived category of a twisted K3 surface, where α is some order 2 Brauer class α_t on a K3 surface S_{X_t} . There is then an associated K3 surface fibration over \mathbb{P}^1 whose fibers are S_{X_t} , whose total space has a smooth resolution which is a Calabi-Yau threefold V' . There is a global Brauer class α on V' so that if V is the base locus of the pencil of cubic threefolds containing a plane (given by $X_{t_1} \cap X_{t_2}$ for $t_1 \neq t_2$), then $\mathbf{D}^b(V', \alpha)$ is equivalent to $\mathbf{D}^b(V)$.

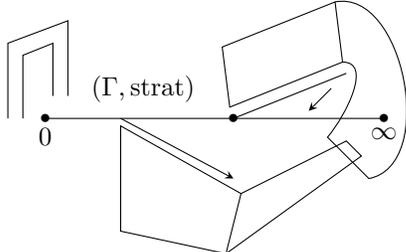
We expect that there are special Lagrangian fibrations on \bar{U}_1 and \bar{U}_2 over D^3 so that the resulting glued torus fibration over S^3 (as described in the paragraphs above) admits no section but a 2-section. Furthermore, this fibration should be not just topological, but actually a special Lagrangian fibration on a fiber W of the LG model close enough to 0. At such a point, taking the SYZ dual should yield the Calabi-Yau threefold V' , not the threefold V , and the fact that there is no section but a 2-section should imply that V' is not homologically mirror dual to W , but that there is a gerbe α so that

$$\mathbf{D}^b \mathcal{Fuk}(W) \cong \mathbf{D}^b(V', \alpha) \cong \mathbf{D}^b(V).$$

The fact that this occurs is corroborated by the fact that we observed in Section 4.4, that there is no section on any SYZ fibration on the K3 surface R obtained as the intersection of U_1 and U_2 . Since W can be viewed topologically as a gluing of \bar{U}_1 to \bar{U}_2 along an S^1 bundle over R , we expect that the 2-section above induces a numerical 2-section on the induced SYZ fibration on R .

5. Justification for Conjecture 1.2

We move to analyzing the examples from the perspective of degenerations of perverse sheaves of categories. We start with 3-dimensional examples. In examples 6.2, 6.3 in the previous section we have a degeneration $X \rightarrow X_0$, where X_0 has rational singularities.

Sheaf of categories	B side
	$X \rightarrow X_0$

The deformation described here is of the type described in Theorem 2.2. Therefore we have the following proposition.

Proposition 5.1. *The deformation described above is a good deformation.*

Indeed we need to check that

- (1) $W = P$ holds after deformation;
- (2) $\text{Im } L = \text{HP}_*$ holds after deformation.

But both $(W = P)$ and $\text{Im } L = \text{HP}$ hold for the LG sheaves of categories we remove so this is a good deformation.

We create a $\text{Tor} = \text{H}^*(\text{Vanishing cycles})/\text{HP}_*(T)$ in $\text{H}^*(\text{Vanishing cycles})$. This is a basic class with a small codimension in LG model. So we have the conditions of conjecture 1.2 satisfied. Below we give an interpretation of Theorems 4.3, 4.5.

This phenomenon holds for many 3-dimensional Fano's. In fact in [KP12] we prove:

Theorem 5.2. *Let X be a 3-dim Fano and $\neq \mathbb{P}^3$. Let μ_0 be the monodromy of $\text{LG}(X)$ at 0. We have the following correspondence:*

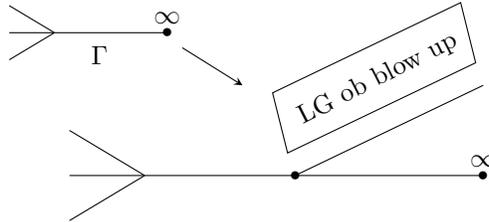
$$\{\mu_0 \text{ strictly quasi unipotent}\} \longleftrightarrow \{X \text{ not rational}\}.$$

We now give a conceptual proof of this statement.

Proof. Indeed as our observation shows

$$\{\mu_0 \text{ strictly quasi unipotent}\} \longleftrightarrow \{\text{Tor H}(\text{vanishing cycles}) \neq 1\}.$$

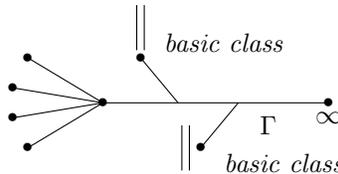
We just need to show that blow-ups and blow-downs do not create basic classes of small codimension. Indeed the effect of blow-up on perverse sheaf of categories is:



This transformation preserves $W = P$ and $\text{Im } L = \text{HP}_*$. Also no torsion in $\text{H}^*(\text{vanishing cycles})$ is created. □

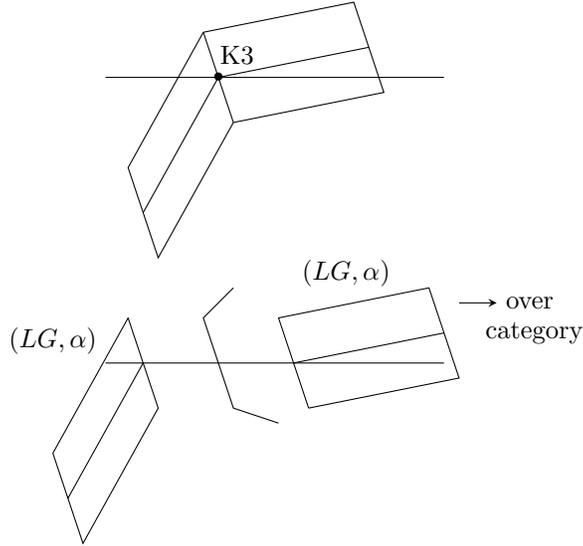
Observe that in higher dimension we can create basic classes by blowing up.

Example 5.3 ($\widehat{\mathbb{P}^N}(\text{Dolg})$).



But in this case the codimension of basic classes is very big.

We give a perverse sheaf of categories interpretation of Theorem 4.14 - the case of 4-dim cubics. In this case the perverse sheaf looks like:



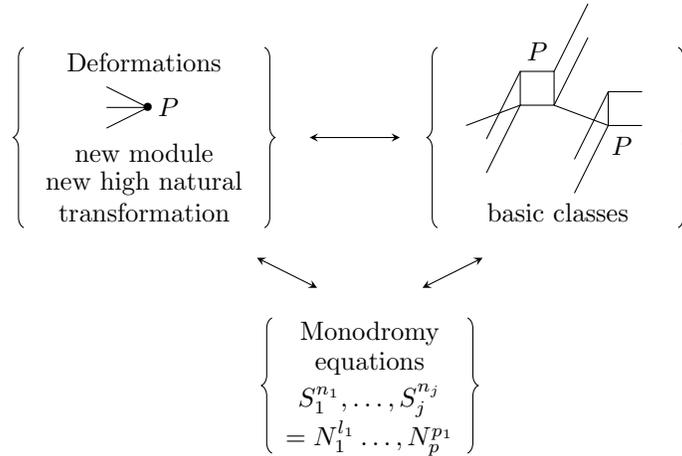
The deformation here is described in Theorem 2.2. Therefore we have:

Proposition 5.4. *The deformation described above is a good deformation.*

Indeed this deformation amounts to creating $\text{FS}(LG_1, \alpha)$ and $\text{FS}(LG_2, \alpha)$ - two over categories with LG models with a gerb. Since $W = P$ and $\text{Im } L = \text{HP}_*$ are preserved, this is a good deformation which creates a torsion in $K(T)$. This torsion corresponds to Lagrangian fibration on K3, with no section but with 2-section - see Theorem 4.14. Therefore we are in the situation of conjecture 1.2. We have a basic class of codimension 2.

6. Conclusions

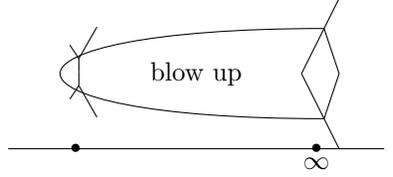
This suggests the following correspondence:



Here N_i are the non-spherical functors.

Proposition 6.1. *The modules P which determine deformations correspond to basic classes and non-spherical functors.*

Recall that blow-ups



Basic class with $\text{codim} > 2 \in \text{Ker } L$

produce spherical functors.

Proposition 6.2. *Basic classes, non-spherical functors with big ($\text{codim} \leq 2$) do not correspond to blow-ups.*

In [KLb] we define categorical multiplier ideal sheaf. We have a correspondence:

$$\{\mathcal{J}_k \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{T}\} \longleftrightarrow \{\cdots \subset \mathcal{T}/\text{Ker } s_1 s_2 \subset \mathcal{T}/\text{Ker } s_i\}.$$

This suggests a conjecture:

Conjecture 6.3. *We have a correspondence:*

$$\{N_1^{l_{j_1}} \dots, N_r^{j_r} S_1^{i_1}, \dots, S_k^{i_k}\} \longleftrightarrow \{\text{Orlov spectra}\}.$$

We propose the following conjecture:

Conjecture 6.4. *Let X be a Fano s.t. there exists a deformation*

- (1) $X \rightarrow X_0$ and $\text{Br}(X_0) \neq 1$;
- (2) $\Delta \subset X \times X$ does not split.

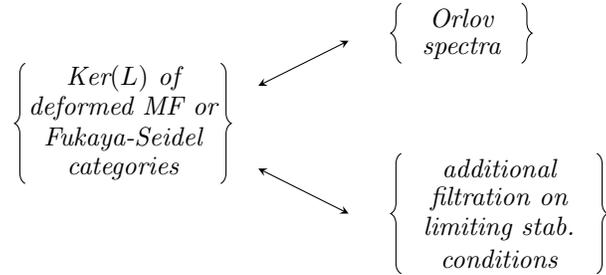
Then there exists a good deformation T of $D^b(X)$ and a basic class B , $\text{codim } B \leq 2$.

The main point of this paper is to point out that categorical deformations, deformations of perverse sheaf of categories give more deformation options. This could be a powerful tool for Fukaya categories and matrix factorizations. Indeed $\text{Ker } L$ is trivial for MF and Fukaya categories since they are CY categories. From another point good deformations of MF and Fukaya categories need not to be CY.

Example 6.5. $\text{Ker } L(E \times \mathbb{C}/\sigma) = \mathbb{Z}_2$.

Of course the question remains how to produce an invariant from $\text{Ker}(L)$ for deformations. We have the following conjecture:

Conjecture 6.6.



We explore this correspondence in the future work.

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