ODD-DIMENSIONAL COHOMOLOGY WITH FINITE COEFFICIENTS AND ROOTS OF UNITY

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Abstract. We prove that the triviality of the Galois action on the suitably twisted *odd-dimensional* étale cohomogy group with finite coefficients of an absolutely irreducible smooth projective variety implies the existence of certain primitive roots of unity in the field of definition of the variety. This text was inspired by an exercise in Serre's Lectures on the Mordell–Weil theorem.

1. Introduction

We need to remind some basic facts about *cyclotomic characters*. Let K be a field, $\bar{K}$ its algebraic closure, $\mathrm{Gal}(K)=\mathrm{Aut}(\bar{K}/K)$ the absolute Galois group of K. Let n is a positive integer that is *not* divisible by $\mathrm{char}(K)$. We write $\mu_n \subset \bar{K}$ for the (cyclic) multiplicative group of nth roots of unity in K. We write

$$\bar{\chi}_n : \mathrm{Gal}(K) \to \mathrm{Aut}(\mu_n) = (\mathbb{Z}/n\mathbb{Z})^*$$

for the cyclotomic character that defines the Galois action on nth roots of unity. Clearly, $\mu_n \subset K$ if and only if

$$\bar{\chi}_n(g) = 1 \ \forall g \in \mathrm{Gal}(K).$$

Recall that the order of $(\mathbb{Z}/n\mathbb{Z})^*$ is $\phi(n)$ where ϕ is the *Euler function*. This implies that

$$\bar{\chi}_n^{\phi(n)}(g) = 1 \ \forall g \in \mathrm{Gal}(K).$$

Let $K(\mu_n) \subset \bar{K}$ be the nth cyclotomic extension of K. Then the degree $[K(\mu_n):K]$ of the (abelian) field extension $K(\mu_n)/K$ coincides with the order of the finite commutative Galois group $\mathrm{Gal}(K(\mu_n)/K)$ of this extension. By definition of $\bar{\chi}_n$, its kernel coincides with $\mathrm{Gal}(\bar{K}/K(\mu_n))$ and $\bar{\chi}_n$ coincides with the composition of the surjection

$$\mathrm{Gal}(K) \mapsto \mathrm{Gal}(K)/\mathrm{Gal}(\bar{K}/K(\mu_n)) = \mathrm{Gal}(K(\mu_n)/K)$$

and a certain embedding

$$\mathrm{Gal}(K(\mu_n)/K) =\hookrightarrow (\mathbb{Z}/n\mathbb{Z})^*,$$

which we continue to denote by $\bar{\chi}_n$, slightly abusing notation.

Remark 1.1. Clearly, the *exponent* $\exp(n,K)$ of $\mathrm{Gal}(K(\mu_n)/K)$ divides the order of $\mathrm{Gal}(K(\mu_n)/K)$, which, in turn, divides $\phi(n)$. In addition, if f is an integer then the character $\bar{\chi}_n^f$ is *trivial* if and only if f is divisible by $\exp(n,K)$. In particular, the character $\bar{\chi}_n^{\exp(n,K)}$ is *trivial*. On the other hand, if the degree of the field

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extension $K(\mu_n)/K$ is *even* then $\exp(n,K)$ is also even; this implies that if f is *odd* integer then the character $\bar{\chi}_n^f$ is *nontrivial*.

Remark 1.2. If m is (another) positive integer that is also *not* divisible by $\mathrm{char}(K)$ and relatively prime to n then the map

$$\mu_n \times \mu_m \to \mu_{nm}, \ (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2$$

is an isomorphism of groups (and even Galois modules). The natural map

$$\phi_{n,m} : \mathbb{Z}/nm\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, \quad c + nm\mathbb{Z} \mapsto (c + n\mathbb{Z}, c + m\mathbb{Z})$$

is a ring homomorphism and the group homomorphism

$$\bar{\chi}_{nm} : \mathrm{Gal}(K) \to (\mathbb{Z}/nm\mathbb{Z})^*$$

coincides with

$$g \mapsto (\bar{\chi}_n(g), \bar{\chi}_m(g)) \in (\mathbb{Z}/n\mathbb{Z})^* \times (\mathbb{Z}/m\mathbb{Z})^* \stackrel{\phi_{n,m}^{-1}}{\longrightarrow} (\mathbb{Z}/nm\mathbb{Z})^*.$$

If A is an abelian variety over K then we write $A[n]$ for the kernel of multiplication by n in $A(\bar{K})$. It is well known that is a finite Galois submodule of $A(\bar{K})$; if we forget about the Galois action then the commutative group $A[n]$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2\dim(A)$.

The following assertion is stated as an exercise (without a proof) in Serre's Lectures on the Mordell–Weil Theorem [7, Sect. 4.6, p. 55].

Theorem 1.3. *If* $\dim(A) > 0$ *and* $A[n] \subset A(K)$ *then* $\mu_n \subset K$.

Proof. First, it suffices to check the case when $n = \ell^r$ is a power of a prime $\ell \neq \mathrm{char}(K)$.

Second, if A^t is the dual of A then let us take a K-polarization $\lambda : A \to A^t$ of smallest possible degree. Then λ is not divisible by ℓ, i.e., $\ker(\lambda)$ does not contain the whole $A[\ell]$. (Otherwise, dividing λ by ℓ we get a K-polarization of lesser degree.)

Then the image $\lambda(A[\ell^r]) \subset A^t[\ell^r]$ contains a point of exact order ℓ^r, say Q. Otherwise,

$$\lambda(A[\ell^r]) \subset A^t[\ell^{r-1}]$$

and therefore $A[\ell] = \ell^{r-1}A[\ell^r]$ lies in the kernel of λ, which is not the case.

Since $A[\ell^r] \subset A[K]$ and λ is defined over K, the image $\lambda(A[\ell^r])$ lies in $A^t(K)$. In particular, Q is a K-rational point on A^t.

Third, there is a *nondegenerate* Galois-equivariant Weil pairing [5]

$$e_n : A[\ell^r] \times A^t[\ell^r] \to \mu_{\ell^r}.$$

I claim that there is a point $P \in A[\ell^r]$ such that $e_n(P,Q)$ is a primitive ℓ^rth root of unity. Indeed, otherwise

$$e_n(A[\ell^r], Q) \subset \mu_{\ell^{r-1}}$$

and therefore nonzero $\ell^{r-1}Q$ is orthogonal to the whole $A[\ell^r]$ with respect to e_n, which contradicts the nondegeneracy of e_n.

So,

$$\gamma := e_n(P,Q)$$

is a primitive ℓ^rth root of unity that lies in K, because both P and Q are K-points. Since cyclic μ_{ℓ^r} is generated by γ,

$$\mu_{\ell^r} \subset K.$$

□

The aim of this paper is to a prove a variant of Serre's exercise that deals with the Galois action on the twisted odd-dimensional étale cohomogy group with finite coefficients of a smooth projective varietiy (see Theorem 1.6 below). Our proof is based on the Hard Lefschetz Theorem [2] and the unimodularity of Poincaré duality [10].

1.4. If Λ is a commutative ring with 1 and without zero divisors and M is a Λ-module, then we write M_{tors} for its torsion submodule and M/tors for the quotient M/M_{tors}. Usually, we will use this notation when Λ is the ring $\mathbb{Z}_\ell$ of ℓ-adic integers.

If ℓ is a prime different from $\mathrm{char}(K)$ then we write $\mathbb{Z}_\ell(1)$ for the projective limit of the cyclic groups (Galois modules) μ_{ℓ^r} where the the transition map is raising to ℓth power. It is well known that $\mathbb{Z}_\ell(1)$ is a free $\mathbb{Z}_\ell$-module of rank 1 provided with natural continuous action of $\mathrm{Gal}(K)$ defined by the *cyclotomic character*

$$\chi_\ell : \mathrm{Gal}(K) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell^*.$$

There are canonical isomorphisms

$$\mathbb{Z}_\ell/\ell^r\mathbb{Z}_\ell = \mathbb{Z}/\ell^r\mathbb{Z},\ \mathbb{Z}_\ell(1)/\ell^r\mathbb{Z}_\ell(1) = \mu_{\ell^r};$$

in addition

$$\chi_\ell \bmod \ell^r = \bar{\chi}_{\ell^r}$$

for all positive integers r.

We write $\mathbb{Q}_\ell(1)$ for the one-dimensional $\mathbb{Q}_\ell$-vector space

$$\mathbb{Q}_\ell(1) = \mathbb{Z}_\ell(1) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

provided with the natural Galois action that is defined by the character χ_ℓ. For each integer a we will need the ath tensor power $\mathbb{Q}_\ell(a) := \mathbb{Q}_\ell(1)^{\otimes a}$, which is a one-dimensional $\mathbb{Q}_\ell$-vector space provided with the Galois action that is defined by the character χ_ℓ^a.

Let X be an absolutely irreducible smooth projective variety over K of positive dimension $d = \dim(X)$. We write $\bar{X}$ for the irreducible smooth projective d-dimensional variety $X \times_K \bar{K}$. Let ℓ be a prime $\ne \mathrm{char}(K)$ and a an integer. If $i \le 2d$ is a nonnegative integer then we write $H^i(\bar{X}, \mathbb{Z}_\ell(a))$ for the corresponding (twisted) ith étale ℓ-adic cohomology group. Recall that all the étale cohomology groups $H^i(\bar{X}, \mu_n^{\otimes a})$ are finite $\mathbb{Z}/n\mathbb{Z}$-modules and the $\mathbb{Z}_\ell$-modules $H^i(\bar{X}, \mathbb{Z}_\ell(a))$ are finitely generated (in particular, each $H^i(\bar{X}, \mathbb{Z}_\ell(a))/\mathrm{tors}$ is a free $\mathbb{Z}_\ell$-module of finite rank). These finiteness results are fundamental finiteness theorems in étale cohomology from **SGA 4, 4$\frac{1}{2}$, 5**, see [3] and [4, pp. 22–24] for precise references. All these groups are provided with the natural linear continuous actions of $\mathrm{Gal}(K)$. We also consider the corresponding finite-dimensional $\mathbb{Q}_\ell$-vector spaces

$$H^i(\bar{X}, \mathbb{Q}_\ell(a)) = H^i(\bar{X}, \mathbb{Z}_\ell(a)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

The Galois action on $H^i(\bar{X}, \mathbb{Z}_\ell(a))$ extends by $\mathbb{Q}_\ell$-linearity to $H^i(\bar{X}, \mathbb{Q}_\ell(a))$. There are natural isomorphisms of $\mathrm{Gal}(K)$-modules

$$H^i(\bar{X}, \mathbb{Q}_\ell(a+b)) = H^i(\bar{X}, \mathbb{Q}_\ell(a)) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(b)$$

for all integers a and b.

Remark 1.5. If a positive integer m is relatively prime to n (and $\mathrm{char}(K)$) then the splitting $\mu_{nm} = \mu_n \times \mu_m$ induces the splitting of Galois modules

$$H^i(\bar{X}, {\mu_{nm}}^{\otimes a}) = H^i(\bar{X}, {\mu_n}^{\otimes a}) \oplus H^i(\bar{X}, {\mu_m}^{\otimes a}).$$

The $\mathbb{Q}_\ell$-dimension of $H^i(\bar{X}, \mathbb{Q}_\ell(a))$ is denoted by $\mathbf{b}_i(\bar{X})$ and called the ith *Betti number* of $\bar{X}$: it does not depend on a choice of (a and) ℓ. In characteristic zero it follows from the comparison theorem between classical and étale cohomology [6]. In finite characteristic the independence follows from results of Deligne [1]. It is also known that $\mathbf{b}_i(\bar{X}) = 0$ if $i > 2d$ [4, 3].

Our main result is the following statement.

Theorem 1.6. *Let i be a nonnegative integer.*

(i) *Suppose that $i \le d-1$ and $\mathbf{b}_{2i+1}(\bar{X}) \ne 0$. If the Galois action on $H^{2i+1}(\bar{X}, {\mu_n}^{\otimes i})$ is trivial then $\mu_n \subset K$.*
(ii) *Suppose that $1 \le i \le d$ and $\mathbf{b}_{2i-1}(\bar{X}) \ne 0$. If the Galois action on $H^{2i-1}(\bar{X}, {\mu_n}^{\otimes i})$ is trivial then $\mu_n \subset K$.*

Applying Remarks 1.2 and 1.5, we conclude that Theorem 1.6 is an immediate corollary of its following special case.

Theorem 1.7. *Let i be a nonnegative integer and r is a positive integer.*

(i) *Suppose that $i \le d-1$ and $\mathbf{b}_{2i+1}(\bar{X}) \ne 0$. If the Galois action on $H^{2i+1}(\bar{X}, {\mu_{\ell^r}}^{\otimes i})$ is trivial then $\mu_{\ell^r} \subset K$.*
(ii) *Suppose that $1 \le i \le d$ and $\mathbf{b}_{2i-1}(\bar{X}) \ne 0$. If the Galois action on $H^{2i-1}(\bar{X}, {\mu_{\ell^r}}^{\otimes i})$ is trivial then $\mu_{\ell^r} \subset K$.*

Example 1.8. Let us take $i = 1$. Then the *Kummer theory* tells us that

$$H^{2i-1}(\bar{X}, {\mu_n}^{\otimes i}) = H^1(\bar{X}, \mu_n) = \mathrm{Pic}(\bar{X})[n]$$

is the kernel of multiplication by n in the Picard group $\mathrm{Pic}(\bar{X})$ of $\bar{X}$. On the other hand if B is an abelian variety over K that is the Picard variety of X [5] then $\dim(B) = \mathbf{b}_1(\bar{X})$ and $B[n]$ is a Galois sunbodule of $H^1(\bar{X}, \mu_n)$. If we know that the Galois action on $H^1(\bar{X}, \mu_n)$ is trivial then the same is true for its submodule $B[n]$. Now if $\mathbf{b}_1(\bar{X}) \ne 0$ then $B \ne \{0\}$ and Theorem 1.3 applied to B implies that $\mu_n \subset K$.

Remark 1.9. Clearly, Theorem 1.6 is equivalent to the following statement.

Theorem 1.10. *Let K be a field, n a positive integer that is not divisible by* $\mathrm{char}(K)$. *Suppose that K does not contain a primitive nth root of unity. Then for each positive odd integer j and an absolutely irreducible smooth projective variety X over K with $\mathbf{b}_j(\bar{X}) \ne 0$ the Galois group* $\mathrm{Gal}(K)$ *acts nontrivially on both $H^j(\bar{X}, \mu_n^{\otimes[(j+1)/2]})$ and $H^j(\bar{X}, \mu_n^{\otimes[(j-1)/2]})$.*

Theorem 1.10 is a special case of the following statement.

Theorem 1.11. *Let K be a field, n a positive integer that is not divisible by* $\mathrm{char}(K)$. *Suppose that K does not contain a primitive nth root of unity. Then for each positive odd integer j and an absolutely irreducible smooth projective variety X over K with $\mathbf{b}_j(\bar{X}) \ne 0$ the Galois group* $\mathrm{Gal}(K)$ *acts nontrivially on $H^j(\bar{X}, \mu_n^{\otimes a})$ where a is any integer such that $2a - j$ is relatively prime to $\phi(n)$.*

The next assertion covers (in particular) the case of *quadratic* $\bar{\chi}_n$ (e.g., when K is the maximal real subfield $\mathbb{Q}(\mu_n)^+$ of the nth cyclotomic field $\mathbb{Q}(\mu_n)$ of $\mathbb{Q}$.)

Theorem 1.12. *Let K be a field, n a positive integer that is not divisible by* $\mathrm{char}(K)$. *Suppose that the degree $[K(\mu_n) : K]$ is even. (E.g., $K(\mu_n)/K$ is a* *quadratic extension.) Then for each positive odd integer j, each integer a and every absolutely irreducible smooth projective variety X over K with $\mathbf{b}_j(\bar{X}) \ne 0$ the Galois group* $\mathrm{Gal}(K)$ *acts nontrivially on* $H^j(\bar{X}, \mu_n^{\otimes a})$.

Remark 1.13. The special case of Theorem 1.12 when $\bar{\chi}_n$ is a quadratic character follows directly from Theorem 1.10, because in this case the Galois module $H^j(\bar{X}, \mu_n^{\otimes a})$ is isomorphic either to $H^j(\bar{X}, \mu_n^{\otimes[(j+1)/2]})$ or to $H^j(\bar{X}, \mu_n^{\otimes[(j-1)/2]})$.

The paper is organized as follows. Section 2 contains auxiliary results about pairing between finitely generated modules over discrete valuation rings. We use them in Section 3, in order to proof the main results of the paper.

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2. Linear algebra

This section contains auxiliary results that will be used in the next section in order to prove Theorem 1.7.

2.1. Let E be a discrete valuation field, $\Lambda \subset E$ the corresponding discrete valuation ring with maximal ideal $\mathfrak{m}$. Let $\pi \in \mathfrak{m}$ be an uniformizer, i.e., $\mathfrak{m} = \pi\Lambda$.

If U is a finitely generated Λ-module then we write U_E for the corresponding (finite-dimensional) E-vector space $U \otimes_\Lambda E$. The kernel of the homomorphism of Λ-modules

$$\otimes 1 : U \to U \otimes_\Lambda E = U_E,\ x \mapsto x \otimes 1$$

coincides with U_{tors} while the image

$$\tilde{U} := \otimes 1(U) \subset U_E$$

is a Λ-lattice in V_E of (maximal) rank $\dim_E(U_E)$.

Let G be a group and

$$\chi : G \to \Lambda^* \subset E^*$$

is a homomorphism of G to the group Λ^* of invertible elements of Λ. If H is a nonzero finite-dimensional vector space over E and

$$\rho : G \to \mathrm{Aut}_E(V)$$

is a E-linear representation of G in H then H becomes a module over the group algebra $E[G]$ of G over E. Then

$$\rho \otimes \chi : G \to \mathrm{Aut}_E(H),\ \rho \otimes \chi(g) = \chi(g)\rho(g)\ \forall g \in G$$

is also a linear representation of G in H. We denote the corresponding $E[G]$-module by $H(\chi)$ and call it the *twist* of V by χ. Notice that V and $V(\chi)$ coincide as E-vector spaces. It is also clear that if T is a Λ-lattice in H then it is G-stable in $H(\chi)$ if and only if it is G-stable in (the $E[G]$-module) H. On the other hand, let L be a *one-dimensional* E-vector space provided with a structure of G-module defined by

$$gz := \chi(g)z\ \forall g \in G, z \in L.$$

Then the G-modules $H(\chi)$ and $H \otimes_E L$ are isomorphic (noncanonically).

Lemma 2.2. *Suppose that H_1 and H_2 are nonzero finite-dimensional E-vector spaces and*

$$\rho_1 : G \to \mathrm{Aut}_E(H_1),\ \rho_2 : G \to \mathrm{Aut}_E(H_2)$$

are isomorphic E-linear representations of G Suppose that T_1 is a G-stable Λ-lattice in H_1 of rank $\dim_E(H_1)$ and T_2 is a G-stable Λ-lattice in H_2 of rank $\dim_E(H_2)$. Then there is a isomorphism of $E[G]$-modules $u : H_1 \to H_2$ such that

$$u(T_1) \subset T_2,\ u(T_1) \not\subset \pi \cdot T_2.$$

Proof. Clearly,

$$H_2 = \bigcup_{j=1}^{\infty} \pi^{-j} \cdot T_2,\ \bigcap_{j=1}^{\infty} \pi^{j} \cdot T_2 = \{0\}.$$

Let $u_0 : H_1 \cong H_2$ be an isomorphism of $E[G]$-modules. Since H_1 is a finitely generated Λ-module, there exists an integer j such that $\pi^{-j} \cdot u_0(T_1) \subset T_2$. Let us take the smallest j that enjoys this property and put $u = \pi^{-j} u_0$. $\square$

Theorem 2.3. *Suppose that U and V are finitely generated Λ-modules provided with group homomorphisms*

$$G \to \mathrm{Aut}_\Lambda(U),\ G \to \mathrm{Aut}_\Lambda(V).$$

Let us assume that $U/\mathrm{tors} \ne \{0\}$, i.e., rank of U is positive.

Suppose that we are given a Λ-bilinear pairing

$$e : U \times V \to \Lambda$$

that enjoys the following properties.

(i)
$$e(gx, gy) = \chi(g) \cdot e(x, y)\ \forall g \in G; x \in U, y \in V.$$

(ii) *The Λ-bilinear pairing*
$$U/\mathrm{tors} \times V/\mathrm{tors} \to \Lambda$$
induced by e is perfect *(unimodular).*

(iii) *The $E[G]$-modules U_E and V_E are isomorphic.*

Let r be a positive integer such that the induced G-action on $U/\pi^r U$ is trivial, i.e.,

$$x - gx \in \pi^r U\ \forall g \in G, x \in U.$$

Then

$$\chi(g) \bmod \pi^r\Lambda = 1 \in \Lambda/\pi^r\Lambda\ \forall g \in G.$$

Proof. Clearly,

$$e(U_{\mathrm{tors}}, V) = \{0\} = e(U, V_{\mathrm{tors}}).$$

It is also clear that U_{tors} is a G-submodule of U and V_{tors} is a G-submodule of V. It is also clear that the G-module $[U/\mathrm{tors}]/\pi^r[U/\mathrm{tors}]$ is isomorphic to a quotient of the G-module $U/\pi^r U$. In particular, the G-action on $[U/\mathrm{tors}]/[\pi^r U/\mathrm{tors}]$ is (also) trivial. In the notation of Sect. 2.1, the natural homomorphisms

$$U/\mathrm{tors} = U/U_{tors} \to \tilde{U},\ x+U_{\mathrm{tors}} \mapsto x\otimes 1,\ V/\mathrm{tors} = V/V_{tors} \to \tilde{V},\ x+V_{\mathrm{tors}} \mapsto x\otimes 1$$

are G-equivariant isomorphisms of free Λ-modules of finite rank

$$U/\mathrm{tors} \cong \tilde{U},\ V/\mathrm{tors} \cong \tilde{V}$$

where $\tilde{U}$ and $\tilde{V}$ are G-stable lattices of maximal rank in U_E and V_E respectively. This implies that the G-action on $\tilde{U}/\pi^r\tilde{U}$ and e induces a Λ-bilinear *perfect* pairing

$$\tilde{e}: \tilde{U} \times \tilde{V} \to \Lambda$$

such that

$$\tilde{e}(gx, gy) = \chi(g) \cdot \tilde{e}(x, y) \ \forall g \in G; x \in \tilde{U}, y \in \tilde{V}.$$

Applying Lemma 2.2 to the isomorphic $E[G]$-modules U_E and V_E, we obtain an isomorphism of $E[G]$-modules $u : U_E \cong V_E$ such that

$$u(T_1) \subset T_2, \ u(T_1) \not\subset \pi T_2.$$

Let us pick $x_0 \in T_1$ with $y := u(x_0) \notin \pi T_2$. Since $x_0 \bmod \pi^r T_1 \in T_1/\pi^r T_1$ is G-invariant, its image

$$u(x) \bmod \pi^r T_2 = y \bmod \pi^r T_2 \in T_2/\pi^r T_2$$

is also G-invariant. Since y is *not* divisible in T_2, the Λ-submodule $\Lambda \cdot y$ is a direct summand of T_2. Since the pairing $\tilde{e}$ between T_1 and T_2 is perfect, there is $x \in T_1$ with $e(x, y) = 1$. This implies that

$$\chi(g) = \chi(g) \cdot 1 = \chi(g) \cdot \tilde{e}(x, y) = \tilde{e}(gx, gy),$$

i.e.,

$$\chi(g) = \tilde{e}(gx, gy) \ \forall g \in G.$$

On the other hand, since

$$x - gx \in \pi^r T_1, \ y - gy \in \pi^r T_2,$$

we have

$$\tilde{e}(gx, gy) - \tilde{e}(x, y) \in \pi^r \Lambda \ \forall g \in G.$$

This means that

$$\chi(g) - 1 = \tilde{e}(gx, gy) - \tilde{e}(x, y) \in \pi^r \Lambda \ \forall g \in G$$

and we are done. □

Theorem 2.4. *Suppose that U and V are finitely generated Λ-modules provided with group homomorphisms*

$$G \to \mathrm{Aut}_\Lambda(U), \ G \to \mathrm{Aut}_\Lambda(V).$$

Let us assume that $U/\mathrm{tors} \neq \{0\}$, i.e., rank of U is positive.

Suppose that we are given a Λ-bilinear pairing

$$e : U \times V \to \Lambda$$

that enjoys the following properties.

(i)
$$e(gx, gy) = e(x, y) \ \forall g \in G; x \in U, y \in V.$$

(ii) *The Λ-bilinear pairing*
$$U/\mathrm{tors} \times V/\mathrm{tors} \to \Lambda$$
induced by e is perfect *(unimodular).*

(iii) *The $E[G]$-modules U_E and $V_E(\chi)$ are isomorphic.*

Let r be a positive integer such that the induced G-action on $U/\pi^r U$ is trivial, i.e.,

$$x - gx \in \pi^r U \ \forall g \in G, x \in U.$$

Then

$$\chi(g) \bmod \pi^r\Lambda = 1 \in \Lambda/\pi^r\Lambda \ \forall g \in G.$$

Proof. Let

$$\rho_U : G \to \mathrm{Aut}_\Lambda(U), \ \rho_V : G \to \mathrm{Aut}_\Lambda(V)$$

be the structure homomorphisms that define the actions of G on U and V respectively. In this notation,

$$e(\rho_U(g)x, \rho_V(g)y) = e(x,y) \ \forall g \in G; x \in U, y \in V.$$

Let us twist ρ_V by considering the group homomorphism

$$\rho_{V(\chi)} : G \to \mathrm{Aut}_\Lambda(V), \ g \mapsto \chi(g)\rho(g).$$

We denote the resulting G-module by $V(\chi)$ and call it the *twist* of V by χ. Notice that V coincides with $V(\chi)$ as Λ-module. On the other hand, the $E[G]$-module $V(\chi)_E$ is canonically isomorphic to $V_E(\chi)$. The pairing e defines the Λ-bilinear pairing

$$e_\chi : U \times V(\chi) \to \Lambda, \ e_\chi(x,y) := e(x,y) \ \forall x \in U, y \in V = V(\chi)$$

of G-modules U and $V(\chi)$, which satisfies

$$e_\chi(\rho_U(g)x, \rho_{V(\chi)}(g)y) = e(\rho_U(g)x, \chi(g)\rho_V(g)y) = \chi(g)e(\rho_U(g)x, \rho_V(g)y) =$$

$$\chi(g)e(x,y) = \chi(g)e_\chi(x,y) \ \forall g \in G; x \in U, y \in V(\chi).$$

This implies that

$$e_\chi(\rho_U(g)x, \rho_{V(\chi)}(g)y) = \chi(g)e_\chi(x,y) \ \forall g \in G; x \in U, y \in V(\chi).$$

Now the result follows from Theorem 2.3 applied to U, $V(\chi)$ and e_χ. □

3. Proofs of main results

Let ℓ be a prime different from $\mathrm{char}(K)$ and r a positive integer. Let us put

$$E = \mathbb{Q}_\ell, \Lambda = \mathbb{Z}_\ell, \pi = \ell, G = \mathrm{Gal}(K).$$

We keep the notation and assumptions of Sect. 1.4. Recall that $d = \dim(X) \geq 1$.

Proposition 3.1. *Let j be a nonnegative integer with $j \leq 2d$ and $\mathbf{b}_j(\bar{X}) \neq 0$. Let a be an integer. Assume that the Galois action on $H^j(\bar{X}, \mu_{\ell^r}{}^{\otimes a})$ is trivial. Then*

$$\bar{\chi}_{\ell^r}^{2a-j}(g) = 1 \ \forall g \in G = \mathrm{Gal}(K).$$

Proof. Let us put $U := H^j(\bar{X}, \mathbb{Z}_\ell(a))$, which is provided with the natural structure of $G = \mathrm{Gal}(K)$-module. Then the universal coefficients theorem [6, Ch. V, Sect. 1, Lemma 1.11] gives us a canonical $\mathrm{Gal}(K)$-equivariant embedding

$$U/\ell^r U = H^j(\bar{X}, \mathbb{Z}_\ell(a))/\ell^r H^j(\bar{X}, \mathbb{Z}_\ell(a)) \hookrightarrow H^j(\bar{X}, \mu_n{}^{\otimes a}).$$

Since the Galois action on $H^j(\bar{X}, \mu_n{}^{\otimes a})$ is trivial, it is also trivial on $U/\ell^r U$. We have (in the notation of Sect. 2.1)

$$U_E = H^j(\bar{X}, \mathbb{Z}_\ell(a)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = H^j(\bar{X}, \mathbb{Q}_\ell(a)).$$

Let us put $V := H^{2d-j}(\bar{X}, \mathbb{Z}_\ell(d-a))$, which is provided with the natural structure of $G = \mathrm{Gal}(K)$-module and

$$V_E = H^{2d-j}(\bar{X}, \mathbb{Z}_\ell(d-a)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(d-a)).$$

The cup product pairing gives rise to a $\mathbb{Z}_\ell$-bilinear $\mathrm{Gal}(K)$-invariant pairing known as *Poincaré dualty* ([6, Ch. VI, Sect. 11, Cor. 11.2 on p. 276], [4, p. 23], [3, Ch. II, Sect. 1])

$$e : H^j(\bar{X}, \mathbb{Z}_\ell(a)) \times H^{2d-j}(\bar{X}, \mathbb{Z}_\ell(d-a)) \to\to H^{2d}(\bar{X}, \mathbb{Z}_\ell(d)) \cong \mathbb{Z}_\ell.$$

It is known [10] that the induced pairing of free $\mathbb{Z}_\ell$-modules of finite rank

$$e : H^j(\bar{X}, \mathbb{Z}_\ell(a))/\mathrm{tors} \times H^{2d-j}(\bar{X}, \mathbb{Z}_\ell(d-a))/\mathrm{tors} \to \mathbb{Z}_\ell$$

is *perfect/unimodular*.

Let $\mathcal{L}$ be an invertible very ample sheaf on X and let

$$h \in H^2(\bar{X}, \mathbb{Q}_\ell(1))^{\mathrm{Gal}(K)} \subset H^2(\bar{X}, \mathbb{Q}_\ell(1))$$

be its first ℓ-adic Chern class. If $j \le d$ then the Hard Lefschetz Theorem ([2], [3, Ch. IV, Sect. 5, pp. 274–275]) tells us that cup multiplication by $(d-j)$th power of h establishes an isomorphism between $\mathbb{Q}_\ell$-vector spaces $H^j(\bar{X}, \mathbb{Q}_\ell(a))$ and $H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(a+d-j))$. On the other hand, if $d \ge j$ then cup multiplication by $(j-d)$th power of h establishes an isomorphism between $\mathbb{Q}_\ell$-vector spaces $H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(a+d-j))$ and $H^j(\bar{X}, \mathbb{Q}_\ell(a))$. In both cases the Galois-invariance of h implies that the $\mathbb{Q}_\ell$-vector spaces $U_E = H^j(\bar{X}, \mathbb{Q}_\ell(a))$ and $H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(a+d-j))$ are isomorphic as $\mathrm{Gal}(K)$-modules. On the other hand, the $\mathrm{Gal}(K)$-module

$$H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(a+d-j)) = H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(d-a+2a-j)) =$$
$$H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(d-a)) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(2a-j) \cong V_E(\chi)$$

where

$$\chi := \chi_\ell^{2a-j} : G = \mathrm{Gal}(K) \to \mathbb{Z}_\ell^* = \Lambda^*.$$

So, the G-module U_E is isomorphic to $V_E(\chi)$ and Theorem 2.4 tells us that

$$\bar{\chi}_{\ell^r}^{2a-j}(g) = (\chi_\ell(g))^{2a-j} \bmod \ell^r\mathbb{Z}_\ell = \chi(g) \bmod \ell^r\mathbb{Z}_\ell = 1 \ \forall g \in G = \mathrm{Gal}(K).$$

□

Proof of Theorem 1.7. It follows from Proposition 3.1 that if $2a - j = \pm 1$ then

$$\bar{\chi}_{\ell^r}^{\pm 1}(g) = 1 \ \forall g \in \mathrm{Gal}(K),$$

i.e., $\bar{\chi}_{\ell^r}(g) = 1 \ \forall g \in \mathrm{Gal}(K)$ and therefore $\mu_{\ell^r} \subset K$. Now in order to finish the proof of Theorem 1.7, one has only to put $a = i$ and $j = 2a \pm 1 = 2i \pm 1$. □

Let n be a positive integer that is *not* divisible by $\mathrm{char}(K)$. Let ℓ be a prime dividing n and $\ell^{r_n(\ell)}$ be the exact power of ℓ that divides n. Applying Proposition 3.1 to all such ℓ with $r = r_n(\ell)$ and using Remarks 1.2 and 1.5, we obtain the following statement.

Theorem 3.2. *Let j be a nonnegative integer with $j \le 2d$ and $\mathbf{b}_j(\bar{X}) \ne 0$. Let a be an integer. Assume that the Galois action on $H^j(\bar{X}, {\mu_n}^{\otimes a})$ is trivial. Then*

$$\bar{\chi}_n^{2a-j}(g) = 1 \ \forall g \in G = \mathrm{Gal}(K).$$

Now we use Theorem 3.2 in order to prove Theorems 1.11 and 1.12

Proof of Theorem 1.11. Suppose that the Galois action on $H^j(\bar{X}, {\mu_n}^{\otimes a})$ is trivial for some absolutely irreducible smooth projective X with $\mathbf{b}_j(\bar{X}) \ne 0$. By Theorem 3.2, the character $\bar{\chi}_n^{2a-j}$ is trivial. We know that $\bar{\chi}_n^{\phi(n)}$ is trivial. Since $2a-j$ and $\phi(n)$ are relatively prime, $\bar{\chi}_n$ is trivial, i.e., K contains a primitive nth root of unity. This gives us a desired contradiction. □

Remark 3.3. In the course of the proof of Theorem 1.11 we did not use the assumption that j is *odd*. However, if we drop this assumption (while keeping all the other ones) and assume instead that j is *even* then $2a-j$ is also even and therefore $\phi(n)$ is odd, because it is relatively prime to $2a-j$. This implies that $n=2$ and therefore $\mathrm{char}(K) \ne 2$ and K does *not* contain a primitive square root of unity, i.e., K does *not* contain -1, which is absurd.

Remark 3.4. Theorem 1.11 (and its proof) remains true (valid) if in its statement we replace $\phi(n)$ by its divisor $\exp(n, K)$.

Proof of Theorem 1.12. Suppose that the Galois action on $H^j(\bar{X}, {\mu_n}^{\otimes a})$ is trivial for some absolutely irreducible smooth projective X with $\mathbf{b}_j(\bar{X}) \ne 0$. By Theorem 3.2, the character $\bar{\chi}_n^{2a-j}$ is trivial. On the other hand, since $f := 2a-j$ is *odd* and $[K(\mu_n) : K]$ is *even*, Remark 1.1 tells us that $\bar{\chi}_n^{2a-j}$ is *nontrivial*. This gives us a desired contradiction. □

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