ALGEBRAIC VARIETIES WITH MANY RATIONAL POINTS

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Introduction

Let $f \in \mathbb{Z}[t, x_1, \dots, x_n]$ be a polynomial with coefficients in the integers. Consider

$$f(t, x_1, \dots, x_n) = 0,$$

as an equation in the unknowns $t, x_1, \dots, x_n$ or as an algebraic family of equations in $x_1, \dots x_n$ parametrized by t. We are interested in integer solutions: in their existence and distribution. Sometimes the emphasis is on individual equations, e.g.,

$$x^n + y^n = z^n,$$

sometimes we want to understand a *typical* equation, i.e., a general equation in some family. To draw inspiration (and techniques) from different branches of algebra it is necessary to consider solutions with values in other rings and fields, most importantly, finite fields $\mathbb{F}_q$, finite extensions of $\mathbb{Q}$, or the function fields $\mathbb{F}_p(t)$ and $\mathbb{C}(t)$. While there is a wealth of *ad hoc* elementary approaches to individual equations,

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and deep theories focussing on their visible or hidden symmetries, our primary approach here will be via geometry.

Basic geometric objects are the affine space $\mathbb{A}^n$ and the projective space $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m$, the quotient by the diagonal action of the multiplicative group. Concretely, affine algebraic varieties $X^{\mathrm{affine}} \subset \mathbb{A}^n$ are defined by systems of polynomial equations with coefficients in some base ring R; their solutions with values in R, $X^{\mathrm{affine}}(R)$, are called R-integral points. Projective varieties are defined by homogeneous equations, and their R-points are equivalence classes of solutions, with respect to diagonal multiplication by nonzero elements in R. If F is the fraction field of R, then $X^{\mathrm{projective}}(R) = X^{\mathrm{projective}}(F)$, and these points are called F-rational points. The geometric advantages of working with "compact" projective varieties translate to important technical advantages in the study of equations, and the theory of rational points is currently much better developed.

The sets $X(F)$ reflect on the one hand the geometric and algebraic complexity of X (e.g., the dimension of X), and on the other hand the structure of the ground field F (e.g., its topology, analytic structure). It is important to consider the variation of $X(F')$, as F' runs over extensions of F, either algebraic extensions, or completions. It is also important to study projective and birational invariants of X, its birational models, automorphisms, fibration structures, deformations. Each point of view contributes its own set of techniques, and it is the interaction of ideas from a diverse set of mathematical cultures that makes the subject so appealing and vibrant.

The focus in these notes will be on smooth projective varieties X defined over $\mathbb{Q}$, with *many* $\mathbb{Q}$-rational points. Main examples are varieties $\mathbb{Q}$-birational to $\mathbb{P}^n$ and hypersurfaces in $\mathbb{P}^n$ of low degree. We will study the relationship between the global *geometry* of X over $\mathbb{C}$ and the distribution of rational points in Zariski topology and with respect to *heights*. Here are the problems we face:

- Existence of solutions: local obstructions, the Hasse principle, global obstructions;
- Density in various topologies: Zariski density, weak approximation;
- Distribution with respect to heights: bounds on smallest points, asymptotics.

Here is the roadmap of the paper. Section 1 contains a summary of basic terms from complex algebraic geometry: main invariants of algebraic varieties, classification schemes, and examples most relevant to arithmetic in dimension ≥ 2. Section 2 is devoted to the existence of rational and integral points, including aspects of decidability, effectivity, local and global obstructions. In Section 3 we discuss Lang's conjecture and its converse, focussing on varieties with nontrivial endomorphisms and fibration structures. Section 4 introduces heights, counting functions, and height zeta functions. We explain conjectures of Batyrev, Manin, Peyre and their refinements. The remaining sections are devoted to geometric and analytic techniques employed in the proof of these conjectures: universal torsors, harmonic analysis on adelic groups, p-adic integration and "estimates".

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1. Geometry background

We discuss basic notions and techniques of algebraic geometry that are commonly encountered by number theorists. For most of this section, F is an algebraically closed field of characteristic zero. Geometry over algebraically closed fields of positive characteristic, e.g., algebraic closure of a finite field, differs in several aspects: difficulties arising from inseparable morphisms, "unexpected" maps between algebraic varieties, additional symmetries, lack (at present) of *resolution of singularities*. Geometry over nonclosed fields, especially number fields, introduces new phenomena: varieties may have *forms*, not all constructions *descend* to the ground field, parameter counts do not suffice. In practice, it is "equivariant geometry for finite groups", with Galois-symmetries acting on all geometric invariants and special loci. The case of surfaces is addressed in detail in [Has08].

1.1. **Basic invariants.** Let X be an algebraic variety over F. We may assume that X is projective and smooth. We seek to isolate invariants of X that are most relevant for arithmetic investigations.

There are two natural types of invariants: *birational* invariants, i.e., invariants of the function field $F(X)$, and *projective geometry* invariants, i.e., those arising from a concrete representation of X as a subvariety of $\mathbb{P}^n$. Examples are the *dimension* $\dim(X)$, defined as the transcendence degree $F(X)$ over F, and the *degree* of X in the given projective embedding. For hypersurfaces $X_f \subset \mathbb{P}^n$ the degree is simply the degree of the defining homogeneous polynomial. In general, it is defined via the Hilbert function of the homogeneous ideal, or geometrically, as the number of intersection points with a general hyperplane of codimension $\dim(X)$.

The degree alone is not a sensitive indicator of the complexity of the variety: Veronese embeddings of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ exhibit it as a curve of degree n. In general, we may want to consider all possible projective embeddings of a variety X. Two such embeddings can be "composed" via the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$, where $N = nm + n + m$. For example, we have the standard embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, with image a smooth quadric. In this way, projective embeddings of X form a "monoid"; the corresponding abelian group is the *Picard group* $\mathrm{Pic}(X)$. Alternatively, it is the group of isomorphism classes of *line bundles* on X. Cohomologically,

$$\mathrm{Pic}(X) = \mathrm{H}^1_{et}(X, \mathbb{G}_m),$$

where $\mathbb{G}_m$ is the sheaf of invertible functions. Yet another description is

$$\mathrm{Pic}(X) = \mathrm{Div}(X) / \left(\mathbb{C}(X)^*/\mathbb{C}^*\right),$$

where $\mathrm{Div}(X)$ is the free abelian group generated by codimension one subvarieties of X, and $\mathbb{C}(X)^*$ is the multiplicative group of rational functions of X, each $f \in \mathbb{C}(X)^*$ giving rise to a *principal* divisor $\mathrm{div}(f)$ (divisor of zeroes and poles of f). Sometimes it is convenient to identify divisors with their classes in $\mathrm{Pic}(X)$. Note that Pic is a *contraviariant functor*: a morphism $\tilde{X} \to X$ induces a homomorphism of abelian groups $\mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$. There is an exact sequence

$$1 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 1$$

where $\mathrm{Pic}^0(X)$ is the connected component of the identity in $\mathrm{Pic}(X)$ and $\mathrm{NS}(X)$ is the *Néron-Severi* group of X. In most applications in this paper, $\mathrm{Pic}^0(X)$ is trivial.

Given a projective variety $X \subset \mathbb{P}^n$, via an explicit system of homogeneous equations, we can easily write down at least one divisor on X, a hyperplane section L in this embedding. Another divisor, the divisor of zeroes of a differential form of top degree on X, can also be computed from the equations. Its class $K_X \in \mathrm{Pic}(X)$, i.e., the class of the line bundle $\Omega_X^{\dim(X)}$, is called the *canonical class*. In general, it is not known how to write down effectively divisors whose classes are not proportional to linear combinations of K_X and L. This can be done for some varieties over $\mathbb{Q}$, e.g., smooth cubic surfaces in $X_3 \subset \mathbb{P}^3$ (see Section 1.9), but is already open for smooth quartics $X_4 \subset \mathbb{P}^4$ (for some partial results in this direction, see Section 1.10).

Elements in $\mathrm{Pic}(X)$ corresponding to projective embeddings generate the *ample* cone $\Lambda_{\mathrm{ample}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$; ample divisors arise as hyperplane sections of X in a projective embedding. The closure of $\Lambda_{\mathrm{ample}}(X)$ in $\mathrm{Pic}(X)_{\mathbb{R}}$ is called the *nef* cone. An *effective divisor* is a sum with nonnegative coefficients of irreducible subvarieties of codimension one. Their classes span the *effective cone* $\Lambda_{\mathrm{eff}}(X)$. Divisors giving rise to embeddings of some Zariski open subset of X form the *big* cone. To summarize we have

$$\Lambda_{\mathrm{ample}}(X) \subseteq \Lambda_{\mathrm{nef}}(X) \quad \text{and} \quad \Lambda_{\mathrm{big}}(X) \subseteq \Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}.$$

These cones and their combinatorial structure encode important geometric information. For example, for all divisors $D \in \Lambda_{\mathrm{nef}}(X)$ and all curves $C \subset X$, the intersection number $D.C \geq 0$ [Kle66]. Divisors on the boundary of $\Lambda_{\mathrm{ample}}(X)$ give rise to fibration structures on X; we will discuss this in more detail in Section 1.4.

Example 1.1.1. Let X be a smooth projective variety, $Y \subset X$ a smooth subvariety and $\pi : \tilde{X} = \mathrm{Bl}_Y(X) \to X$ the *blowup* of X in Y, i.e., the complement in $\tilde{X}$ to the *exceptional divisor* $E := \pi^{-1}(Y)$ is isomorphic to $X \setminus Y$, and E itself can be identified with the projectivized tangent cone to X at Y. Then

$$\mathrm{Pic}(\tilde{X}) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}E$$

and

$$K_{\tilde{X}} = \pi^*(K_X) + \mathcal{O}((\mathrm{codim}(Y) - 1)E)$$

(see [Har77, Exercise 8.5]). Note that

$$\pi^*(\Lambda_{\mathrm{eff}}(X)) \subset \Lambda_{\mathrm{eff}}(\tilde{X}),$$

but that pullbacks of ample divisors are not necessarily ample.

Example 1.1.2. Let $X \subset \mathbb{P}^n$ be a hypersurface of dimension ≥ 3 and degree d. Then $\mathrm{Pic}(X) = \mathrm{NS}(X) = \mathbb{Z}L$, generated by the class of the hyperplane section, and

$$\Lambda_{\text{ample}}(X) = \Lambda_{\text{eff}}(X) = \mathbb{R}_{\geq 0}L.$$

The canonical class is

$$K_X = -(n+1-d)L.$$

Example 1.1.3. If X is a smooth cubic surface over an algebraically closed field, then $\mathrm{Pic}(X) = \mathbb{Z}^7$. The anticanonical class is proportional to the sum of 27 exceptional curves (lines):

$$-K_X = \frac{1}{9}(D_1 + \cdots + D_{27}).$$

The effective cone $\Lambda_{\text{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$ is spanned by the classes of the lines.

On the other hand, the effective cone of a minimal resolution of the singular cubic surface

$$x_0x_3^2 + x_1^2x_3 + x_2^3 = 0$$

is a simplicial cone (in $\mathbb{R}^7$) [HT04].

Example 1.1.4. Let G be a connected solvable linear algebraic group, e.g., the additive group $G = \mathbb{G}_a$, an algebraic torus $G = \mathbb{G}_m^d$ or the group of upper-triangular matrices. Let X be an equivariant compactification of G, i.e., the action of G on itself extends to X. Using equivariant resolution of singularieties, if necessary, we may assume that X is smooth projective and that the boundary

$$X \setminus G = D = \cup_{i\in\mathcal{I}}\, D_i, \quad \text{with} \quad D_i \text{ irreducible},$$

is a divisor with normal crossings. Every divisor D on X is equivalent to a divisor with support in the boundary since it can be "moved" there by the action of G. Thus $\mathrm{Pic}(X)$ is generated by the components D_i, and the relations are given by functions with zeroes and poles supported in D, i.e., by the characters $\mathfrak{X}^*(G)$. We have an exact sequence

$$0 \to \mathfrak{X}^*(G) \to \oplus_{i\in\mathcal{I}}\, \mathbb{Z}D_i \xrightarrow{\pi} \mathrm{Pic}(X) \to 0 \tag{1.1}$$

The cone of effective divisors $\Lambda_{\text{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$ is the image of the simplicial cone $\oplus_{i\in\mathcal{I}}\, \mathbb{R}_{\geq 0}D_i$ under the projection π. The anticanonical class is

$$-K_X = \oplus_{i\in\mathcal{I}}\, \kappa_i D_i, \quad \text{with} \quad \kappa_i \geq 1, \quad \text{for all } i.$$

For unipotent G one has $\kappa_i \geq 2$, for all i [HT99].

For higher-dimensional varieties without extra symmetries, the computation of the ample and effective cones, and of the position of K_X with respect to these cones, is a difficult problem. A sample of recent papers on this subject is: [CS06], [Far06], [FG03], [Cas07], [HT03], [HT02], [GKM02]. However, we have the following fundamental result (see also Section 1.4):

Theorem 1.1.5. *Let X be a smooth projective variety with $-K_X \in \Lambda_{\mathrm{ample}}(X)$. Then $\Lambda_{\mathrm{nef}}(X)$ is a finitely generated rational cone. If $-K_X$ is big and nef then $\Lambda_{\mathrm{eff}}(X)$ is finitely generated.*

Finite generation of the nef cone goes back to Mori (see [CKM88] for an introduction). The result concerning $\Lambda_{\mathrm{eff}}(X)$ has been proved in [Bat92] in dimension ≤ 3, and in higher dimensions in [BCHM06] (see also [Ara05], [Leh08]).

1.2. **Classification schemes.** In some arithmetic investigations (e.g., Zariski density or rational points) we rely mostly on birational properties of X; in others (e.g., asymptotics of points of bounded height), we need to work in a fixed projective embedding.

Among *birational* invariants, the most important are those arising from a comparison of X with a projective space:

(1) *rationality*: there exists a birational isomorphism $X \sim \mathbb{P}^n$, i.e., the is an isomorphisms of function fields $F(X) = F(\mathbb{P}^n)$, for some $n \in \mathbb{N}$;
(2) *unirationality*: there exists a dominant map $\mathbb{P}^n \dashrightarrow X$;
(3) *rational connectedness*: for general $x_1, x_2 \in X(F)$ there exists a morphism $f : \mathbb{P}^1 \to X$ such that $x_1, x_2 \in f(\mathbb{P}^1)$.

It is easy to see that

$$(1) \Rightarrow (2) \Rightarrow (3).$$

These properties are equivalent in dimension two, but diverge in higher dimensions. First examples of unirational but not rational threefolds were constructed in [IM71] and [CG72]. The approach of [IM71] was to study of the group $\mathrm{Bir}(X)$ of birational automorphisms of X; finiteness of $\mathrm{Bir}(X)$, i.e., *birational rigidity*, implies nonrationality. No examples of smooth projective rationally connected but not unirational varieties are known.

Interesting unirational varieties arise as quotients V/G, where $V = \mathbb{A}^n$ is a representation space for a faithful action of a linear algebraic group G. For example, the moduli space $\mathcal{M}_{0,n}$ of n points on $\mathbb{P}^1$ is birational to $(\mathbb{P}^1)^n/\mathrm{PGL}_2$. Moduli spaces of degree d hypersurfaces $X \subset \mathbb{P}^n$

are naturally isomorphic to $\mathbb{P}(\mathrm{Sym}^d(\mathbb{A}^{n+1}))/\mathrm{PGL}_{n+1}$. Rationality of V/G is known as *Noether's problem.* It has a positive solution for G being the symmetric group $\mathfrak{S}_n$, the group PGL_2 [Kat82], [BK85], and in many other cases [SB89], [SB88]. Counterexamples for some *finite* G were constructed in [Sal84], [Bog87]; nonrationality is detected by the *unramified Brauer group*, $\mathrm{Br}_{\mathrm{un}}(V/G)$, closely related to the Brauer group of the function field $\mathrm{Br}(F(V/G)) = \mathrm{H}^2_{et}(F(V/G), \mathbb{G}_m)$.

Now we turn to invariants arising from projective geometry, i.e., ample line bundles on X. For smooth curves C, an important invariant is the *genus* $\mathsf{g}(C) := \dim(\mathrm{H}^0(X, K_X))$. In higher dimensions, one considers the *Kodaira dimension*

$$\kappa(X) := \limsup \frac{\log(\dim(\mathrm{H}^0(X, nK_X)))}{\log(n)}, \tag{1.2}$$

and the related graded *canonical* section ring

$$R(X, K_X) = \oplus_{n\geq 0} \mathrm{H}^0(X, nK_X). \tag{1.3}$$

A fundamental theorem is that this ring is finitely generated [BCHM06]. The Kodaira dimension is the dimension of the variety $\mathrm{Proj}(R(X, K_X))$, or equivalently, the dimension of the image of X under the map

$$X \to \mathbb{P}(\mathrm{H}^0(X, nK_X)),$$

for sufficiently large n. For K_X ample one has $\kappa(X) = \dim(X)$.

A very rough classification of smooth algebraic varieties is based on the position of the anticanonical class with respect to the cone of ample divisors. Numerically, this is reflected in the value of the Kodaira dimension. There are three main cases:

- *Fano:* $-K_X$ ample;
- *general type:* K_X ample;
- *intermediate type:* none of the above.

The qualitative behavior of rational points on X mirrors this classification (see Section 3). In our arithmetic applications we will mostly encounter Fano varieties and varieties of intermediate type.

For curves, this classification can be read off from the genus: curves of genus 0 are of Fano type, of genus 1 of intermediate type, and of genus ≥ 2 of general type. Other examples of varieties in each group are:

- Fano: $\mathbb{P}^n$, smooth degree d hypersurfaces $X_d \subset \mathbb{P}^n$, with $d \leq n$;

- general type: hypersurfaces $X_d \subset \mathbb{P}^n$, with $d \geq n+2$, moduli spaces of curves of high genus and abelian varieties of high dimension;
- intermediate type: $\mathbb{P}^2$ blown up in 9 points, abelian varieties, Calabi-Yau varieties.

There are only finitely many families of smooth Fano varieties in each dimension [KMM92]. On the other hand, the universe of varieties of general type is boundless and there are many open classification questions already in dimension 2.

In finer classification schemes such as the *Minimal Model Program* (MMP) it is important to take into account fibration structures and mild singularities (see [KMM87] and [Cam04]). Indeed, recall that $R(X, K_X)$ is finitely generated and put $Y = \mathrm{Proj}(R(X, K_X))$. Then the general fiber of the rational projection

$$X \dashrightarrow Y = \mathrm{Proj}(R(X, K_X)),$$

is a (possibly singular) Fano variety. For example, a surface of Kodaira dimension 1 is birational to a Fano fiber space over a curve of genus ≥ 1.

Analogously, in many arithmetic questions, the passage to fibrations is inevitable (see Section 4.14). These often arise from the section rings

$$R(X, L) = \oplus_{n \geq 0} \mathrm{H}^0(X, nL). \tag{1.4}$$

Consequently, one considers the *Iitaka dimension*

$$\kappa(X, L) := \limsup \frac{\log(\dim(\mathrm{H}^0(X, nL)))}{\log(n)}. \tag{1.5}$$

Finally, a pair (X, D), where X is smooth projective and D is a divisor in X, gives rise to another set of invariants: the *log Kodaira dimension* $\kappa(X, K_X + D)$ and the *log canonical ring* $R(X, K_X + D)$, whose finite generation is also known in many cases [BCHM06]. Again, one distinguishes

- *log Fano:* $\kappa(X, -(K_X + D)) = \dim(X)$;
- *log general type:* $\kappa(X, K_X + D) = \dim(X)$;
- *log intermediate type:* none of the above.

This classification has consequences for the study of *integral* points on the open variety $X \setminus D$.

1.3. **Singularities.** Assume that X is $\mathbb{Q}$-Cartier, i.e., there exists an integer m such that mK_X is a Cartier divisor. Let $\tilde{X}$ be a normal variety and $f : \tilde{X} \to X$ a proper birational morphism. Denote by E the f-exceptional divisor and by e its generic point. Let $g = 0$ be a local equation of E. Locally, we can write

$$f^*(\text{generator of } \mathcal{O}(mK_X)) = g^{md(E)}(dy_1 \wedge \ldots \wedge dy_n)^m$$

for some $d(E) \in \mathbb{Q}$ such that $md(E) \in \mathbb{Z}$. If, in addition, $K_{\tilde{X}}$ is a line bundle (e.g., $\tilde{X}$ is smooth), then $mK_{\tilde{X}}$ is linearly equivalent to

$$f^*(mK_X) + \sum_i m \cdot d(E_i)E_i; \quad E_i \text{ exceptional},$$

and numerically

$$K_{\tilde{X}} \sim f^*(K_X) + \sum_i d(E_i)E_i.$$

The number $d(E)$ is called the *discrepancy* of X at the exceptional divisor E. The discrepancy $\mathrm{discr}(X)$ of X is

$$\mathrm{discr}(X) := \inf\{d(E) \,|\, \text{all } f, E\}$$

If X is smooth then $\mathrm{discr}(X) = 1$. In general,

$$\mathrm{discr}(X) \in \{-\infty\} \cup [-1, 1].$$

Definition 1.3.1. The singularities of X are called

- *terminal* if $\mathrm{discr}(X) > 0$ and
- *canonical* if $\mathrm{discr}(X) \geq 0$.

It is essential to remember that *terminal* = smooth in codimension 2 and that for surfaces, *canonical* means *Du Val* singularities.

Canonical isolated singularities on surfaces are classified via Dynkin diagrams: Let $f : \tilde{X} \to X$ be the *minimal* desingularization. Then the submodule in $\mathrm{Pic}(\tilde{X})$ spanned by the classes of exceptional curves (with the restriction of the intersection form) is isomorphic to the root lattice of the corresponding Dynkin diagram (exceptional curves give simple roots).

Canonical singularities don't influence the expected asymptotic for rational points on the complement to all exceptional curves: for (singular) Del Pezzo surfaces X we still expect an asymptotic of points of bounded anticanonical height of the shape $\mathsf{B}\log(\mathsf{B})^{9-d}$, where d is the degree of X, just like in the smooth case (see Section 4.10). This fails when the singularities are worse than canonical.

Example 1.3.2. Let $w = (w_0, \dots, w_n) \in \mathbb{N}^n$, with $\gcd(w_0, \dots, w_n) = 1$ and let

$$X = X(w) = \mathbb{P}(w_0, \dots, w_n)$$

be a *weighted projective space*, i.e., we have a quotient map

$$(\mathbb{A}^{n+1} \setminus 0) \xrightarrow{\mathbb{G}_m} X,$$

where the torus $\mathbb{G}_m$ acts by

$$\lambda \cdot (x_0, \dots, x_{n+1}) \mapsto (\lambda^{w_0} x_0, \dots, \lambda^{w_n} x_n).$$

For $w = (1, \dots, 1)$ it is the usual projective space, e.g., $\mathbb{P}^2 = \mathbb{P}(1, 1, 1)$. The weighted projective plane $\mathbb{P}(1, 1, 2)$ has a canonical singularity and the singularity of $\mathbb{P}(1, 1, m)$, with $m \geq 3$, is worse than canonical.

For a discussion of singularieties on general weighted projective spaces and so called *fake* weighted projective spaces see, e.g., [Kas08].

1.4. **Minimal Model Program.** Here we recall basic notions from the Minimal Model Program (MMP) (see [CKM88], [KM98], [KMM87], [Mat02] for more details). The starting point is the following fundamental theorem due to Mori [Mor82]:

Theorem 1.4.1. *Let X be a smooth Fano variety of dimension n. Then there exists an integer $d \leq n+1$ such that through every geometric point x of X there passes a rational curve of $-K_X$-degree $\leq d$.*

These rational curves move in families. Their specializations are rational curves, which may move again, and again, until one arrives at "rigid" rational curves.

Theorem 1.4.2 (Cone theorem)**.** *Let X be a smooth Fano variety. Then the closure of the cone of (equivalence classes of) effective curves in* $\mathrm{H}_2(X, \mathbb{R})$ *is finitely generated by classes of rational curves.*

The generating rational curves are called *extremal rays*, they correspond to codimension-1 faces of the dual cone of nef divisors. Mori's Minimal Model Program links the convex geometry of the nef cone $\Lambda_{\mathrm{nef}}(X)$ with birational transformations of X. Pick a divisor D on the face dual to an extremal ray $[C]$. It is not ample anymore, but it still defines a map

$$X \to \mathrm{Proj}(R(X, D)),$$

which contracts the curve C to a point. The map is one of the following:

- a fibration over a base of smaller dimension, and the restriction of D to a general fiber proportional to the anticanonical class of the fiber, which is a (possibly singular) Fano variety,

- a birational map contracting a divisor,
- a contraction of a subvariety in codimension ≥ 2 (a *small* contraction).

The image could be singular, as in Example 1.3.2, and one of the most difficult issues of MMP was to develop a framework which allows to maneuver between birational models with singularities in a restricted class, while keeping track of the modifications of the Mori cone of curves. In arithmetic applications, for example proofs of the existence of rational points as in, e.g., [CTSSD87a], [CTSSD87b], [CTS89], one relies on the *fibration method and descent*, applied to some auxiliary varieties. Finding the "right" fibration is an art. Mori's theory gives a systematic approach to these questions.

A variant of Mori's theory, the *Fujita program*, analyzes fibrations arising from divisors on the boundary of the *effective* cone $\Lambda_{\mathrm{eff}}(X)$. In this case, the restriction of D to a general fiber is a perturbation of (some positive rational multiple of) the anticanonical class of the fiber by a *rigid* effective divisor. This theory turns up in the analysis of height zeta functions in Section 6 (see also Section 4.13).

Let X be smooth projective with $\mathrm{Pic}(X) = \mathrm{NS}(X)$ and a finitely generated effective cone $\Lambda_{\mathrm{eff}}(X)$. For a line bundle L on X define

$$a(L) := \min(a \mid aL + K_X \in \Lambda_{\mathrm{eff}}(X)). \tag{1.6}$$

We will also need the notion of the *geometric hypersurface of linear growth*:

$$\Sigma_X^{geom} := \{L \in \mathrm{NS}(X)_{\mathbb{R}} \mid a(L) = 1\} \tag{1.7}$$

Let $b(L)$ be the maximal codimension of the face of $\Lambda_{\mathrm{eff}}(X)$ containing $a(L)L + K_X$. In particular,

$$a(-K_X) = 1 \quad \text{and} \quad b(-K_X) = \mathrm{rk}\,\mathrm{Pic}(X).$$

These invariants arise in Manin's conjecture in Section 4.10 and the analysis of analytic properties of height zeta functions in Section 6.1.

1.5. **Campana's program.** Recently, Campana developed a new approach to classification of algebraic varieties with the goal of formulating necessary and sufficient conditions for *potential density* of rational points, i.e., Zariski density after a finite extension of the ground field. The key notions are: the *core* of an algebraic variety and *special* varieties. Special varieties include Fano varieties and Calabi–Yau varieties. They are conjectured to have a potentially dense set of rational points. This program is explained in [Abr08].

1.6. **Cox rings.** Again, we assume that X is a smooth projective variety with $\mathrm{Pic}(X) = \mathrm{NS}(X)$. Examples are Fano varieties, equivariant compactifications of algebraic groups and holomorphic symplectic varieties. Fix line bundles $L_1, \ldots, L_r$ whose classes generate $\mathrm{Pic}(X)$. For $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{Z}^r$ we put

$$L^\nu := L_1^{\nu_1} \otimes \ldots \otimes L_r^{\nu_r}.$$

The *Cox* ring is the multigraded section ring

$$\mathrm{Cox}(X) := \oplus_{\nu \in \mathbb{Z}^r} \mathrm{H}^0(X, L^\nu).$$

The nonzero graded pieces of $\mathrm{Cox}(X)$ are in bijection with effective divisors of X. The key issue is finite generation of this ring. This has been proved under quite general assumptions in [BCHM06, Corollary 1.1.9]. Assume that $\mathrm{Cox}(X)$ is finitely generated. Then both $\Lambda_{\mathrm{eff}}(X)$ and $\Lambda_{\mathrm{nef}}(X)$ are finitely generated polyhedral (see [HK00, Proposition 2.9]). Other important facts are:

- X is a toric variety if and only if $\mathrm{Cox}(X)$ is a polynomial ring [Cox95], [HK00, Corollary 2.10]; Cox rings of some equivariant compatifications of other semi-simple groups are computed in [Bri07];
- $\mathrm{Cox}(X)$ is multigraded for $\mathrm{NS}(X)$, in particular, it carries a natural action of the dual torus T_{NS}.

1.7. **Universal torsors.** We continue to work over an algebraically closed field. Let G be a linear algebraic group and X an algebraic variety. A G-torsor over X is a principal G-bundle $\pi : \mathcal{T}_X \to X$. Basic examples are GL_n-torsors, they arise from vector bundles over X; for instance, each line bundle L gives rise to a $\mathrm{GL}_1 = \mathbb{G}_m$-torsor over X. Up to isomorphism, G-torsors are classified by $\mathrm{H}^1_{et}(X, G)$; line bundles are classified by $\mathrm{H}^1_{et}(X, \mathbb{G}_m) = \mathrm{Pic}(X)$. When G is commutative, $\mathrm{H}^1_{et}(X, G)$ is a group.

Let $G = \mathbb{G}_m^r$ be an algebraic torus and $\mathfrak{X}^*(G) = \mathbb{Z}^r$ its character lattice. A G-torsor over an algebraic variety X is determined, up to isomorphism, by a homomorphism

$$\chi : \mathfrak{X}^*(G) \to \mathrm{Pic}(X). \tag{1.8}$$

Assume that $\mathrm{Pic}(X) = \mathrm{NS}(X) = \mathbb{Z}^r$ and that χ is in fact an isomorphism. The arising G-torsors are called *universal*. The introduction of universal torsors is motivated by the fact that over *nonclosed* fields they "untwist" the action of the Galois group on the Picard group of X (see

Sections 1.13 and 2.5). The "extra dimensions" and "extra symmetries" provided by the torsor add crucial freedom in the analysis of the geometry and arithmetic of the underlying variety. Examples of applications to rational points will be presented in Sections 2.5 and 5. This explains the surge of interest in explicit equations for universal torsors, the study of their geometry: singularities and fibration structures.

Assume that $\mathrm{Cox}(X)$ is finitely generated. Then $\mathrm{Spec}\,(\mathrm{Cox}(X))$ contains a universal torsor $\mathcal{T}_X$ of X as an open subset. More precisely, let

$$\overline{\mathcal{T}}_X := \mathrm{Spec}\,(\mathrm{Cox}(X)).$$

Fix an ample class $L^\nu \in \mathrm{Pic}(X)$ and let $\chi_\nu \in \mathfrak{X}^*(T_{\mathrm{NS}})$ be the corresponding character. Then

$$X = \mathrm{Proj}(\oplus_{n\geq 0}\mathrm{H}^0(X, \mathcal{O}(nL^\nu))) = \overline{\mathcal{T}}_X /\!\!/ T_{\mathrm{NS}},$$

the geometric invariant theory quotient linearized by χ_ν. The *unstable locus* is

$$Z_\nu := \{t \in \overline{\mathcal{T}}_X \mid f(t) = 0 \ \ \forall f \in \mathrm{Cox}(X)_{n\nu}, \ \ n > 0\}$$

Let W_ν be the set of $t \in \overline{\mathcal{T}_X}$ such that the orbit of t is not closed in $\overline{\mathcal{T}_X} \setminus Z_\nu$, or such that t has a positive-dimensional stabilizer. Geometric invariant theory implies that

$$\overline{\mathcal{T}}_X \setminus W_\nu =: \mathcal{T}_X \to X$$

is a geometric quotient, i.e., $\mathcal{T}_X$ is a T_{NS}-torsor over X.

1.8. **Hypersurfaces.** We now turn from the general theory to specific varieties. Let $X = X_f \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. We have already described some of its invariants in Example 1.1.2, at least when $\dim(X) \geq 3$. In particular, in this case $\mathrm{Pic}(X) \simeq \mathbb{Z}$ and $T_{\mathrm{NS}} = \mathbb{G}_m$. The universal torsor is the hypersurface in $\mathbb{A}^{n+1} \setminus 0$ given by the vanishing of the defining polynomial f.

In dimension two, there are more possibilities. The most interesting cases are $d = 2, 3$, and 4. A quadric X_2 is isomorpic to $\mathbb{P}^1 \times \mathbb{P}^1$ and has Picard group $\mathrm{Pic}(X_2) \simeq \mathbb{Z} \oplus \mathbb{Z}$. A cubic has Picard group of rank 7. These are examples of *Del Pezzo surfaces* discussed in Section 1.9. They are birational to $\mathbb{P}^2$. A smooth quartic $X_4 \subset \mathbb{P}^3$ is an example of a *K3 surface* (see Section 1.10). We have $\mathrm{Pic}(X_4) = \mathbb{Z}^r$, with r between 1 and 20. They are not rational and, in general, do not admit nontrivial fibrations.

Cubic and quartic surfaces have a rich geometric structure, with large "hidden" symmetries. This translates into many intricate arithmetic issues.

1.9. **Del Pezzo surfaces.** A smooth projective surface X with ample anticanonical class is called a *Del Pezzo surface*. Standard examples are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Over algebraically closed ground fields, all other Del Pezzo surfaces X_r are obtained as blowups of $\mathbb{P}^2$ in $r \leq 8$ points in general position (e.g., no three on a line, no 6 on a conic). The number $d = 9 - r$ is the anticanonical *degree* of X_r. Del Pezzo surfaces of low degree admit the following realizations:

- $d = 4$: intersection of two quadrics in $\mathbb{P}^4$;
- $d = 3$: hypersurface of degree 3 in $\mathbb{P}^3$;
- $d = 2$: hypersurface of degree 4 in the weighted projective space $\mathbb{P}^2(1, 1, 1, 2)$ given by
$$w^2 = f_4(x, y, z), \quad \text{with } \deg(f_4) = 4.$$
- $d = 1$: hypersurface of degree 6 in $\mathbb{P}(1, 1, 2, 3)$ given by
$$w^2 = t^3 + f_4(x, y)t + f_6(x, y), \quad \text{with } \deg(f_i) = i.$$

Visually and mathematically most appealing are, perhaps, *cubic surfaces* with $d = 3$. Note that for $d = 1$, the anticanonical linear series has one *base point*, in particular, $X_8(F) \neq \emptyset$, over *any* field F.

Let us compute the geometric invariants of a Del Pezzo surface of degree d, expanding the Example 1.1.3. Since $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}L$, the hyperplane class, we have

$$\mathrm{Pic}(X_r) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_r,$$

where E_i are the full preimages of the blown-up points. The canonical class is computed as in Example 1.1.1

$$K_{X_r} = -3L + (E_1 + \cdots E_r).$$

The intersection pairing defines a quadratic form on $\mathrm{Pic}(X_r)$, with $L^2 = 1$, $L \cdot E_i = 0$, $E_i \cdot E_j = 0$, for $i \neq j$, and $E_j^2 = -1$. Let W_r be the subgroup of $\mathrm{GL}_{r+1}(\mathbb{Z})$ of elements preserving K_{X_r} and the intersection pairing. For $r \geq 2$ there are other classes with square -1, e.g.,

$$L - (E_i + E_j), \quad 2L - (E_1 + \cdots + E_5), \quad \text{etc.}$$

The classes whose intersection with K_{X_r} is also -1 are called (classes of) *exceptional curves*, these curves are *lines* in the anticanonical embedding. Their number $n(r)$ can be found in the table below. We have

$$-K_{X_r} = c_r \sum_{j=1}^{n(r)} E_j,$$

the sum over all exceptional curves, where $c_r \in \mathbb{Q}$ can be easily computed, e.g., $c_6 = 1/9$. The effective cone is spanned by the $n(r)$ classes of exceptional curves, and the nef cone is the cone dual to $\Lambda_{\mathrm{eff}}(X_r)$ with respect to the intersection pairing on $\mathrm{Pic}(X_r)$. Put

$$\alpha(X_r) := \mathrm{vol}\left(\Lambda_{\mathrm{nef}}(X_r) \cap \{C \mid (-K_{X_r}, C) = 1\}\right). \tag{1.9}$$

This "volume" of the nef cone has been computed in [Der07a]:

r	1	2	3	4	5	6	7	8
$n(r)$	1	3	6	10	16	27	56	240
$\alpha(X_r)$	1/6	1/24	1/72	1/144	1/180	1/120	1/30	1

Given a Del Pezzo surface over a number field, the equations of the lines can be computed effectively. For example, this is easy to see for the diagonal cubic surface

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

Writing

$$x_i^3 + x_j^3 = \prod_{r=1}^{3}(x_i + \zeta_3^r x_j) = x_\ell^3 + x_k^3 = \prod_{r=1}^{3}(x_\ell + \zeta_3^r x_k)0,$$

with $i, j, k, l \in [0, \ldots, 3]$, and permuting indices we get all 27 lines. In general, equations for the lines can be obtained by solving the corresponding equations on the Grassmannian of lines.

Degenerations of Del Pezzo surfaces are also interesting and important. Typically, they arise as special fibers of fibrations, and their analysis is unavoidable in the theory of *models* over rings such as $\mathbb{Z}$, or $\mathbb{C}[t]$. A classification of singular Del Pezzo surfaces can be found in [BW79], [DP80]. Models of Del Pezzo surfaces over curves are discussed in [Cor96]. Volumes of nef cones of singular Del Pezzo surfaces are computed in [DJT08].

We turn to Cox rings of Del Pezzo surfaces. Smooth Del Pezzo surfaces of degree $d \geq 6$ are toric and their Cox rings are polynomial rings on $12 - d$ generators. The generators and relations of the Cox rings of Del Pezzo surfaces have been computed [BP04], [Der06], [STV06], [TVAV08], [SX08]. For $r \in \{4, 5, 6, 7\}$ the generators are the nonzero sections from exceptional curves and the relations are induced by fibration structures on X_r (rulings). In degree 1 two extra generators are needed, the independent sections of $\mathrm{H}^0(X_8, -K_{X_8})$.

It was known for a long time that the (affine cone over the) Grassmannian $\mathrm{Gr}(2,5)$ is a universal torsor for the unique (smooth) degree 5 Del Pezzo surface (this was used in [SD72] and [Sko93] to prove that every Del Pezzo surface of degree 5 has a rational point). Batyrev conjectured that universal torsors of other Del Pezzo surfaces should embedd into other Grassmannians, and this is why:

One of the most remarkable facts of the theory of Del Pezzo surfaces is the "hidden" symmetry of the collection of exceptional curves in the Picard lattice. Indeed, for $r = 3, 4, 5, \ldots, 8$, the group W_r is the *Weyl group* of a *root system*:

$$\mathsf{R}_r \in \{\mathsf{A}_1 \times \mathsf{A}_2, \mathsf{A}_4, \mathsf{D}_5, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8\}, \tag{1.10}$$

and the root lattice itself is the orthogonal to K_{X_r} in $\mathrm{Pic}(X_r)$, the *primitive* Picard group. Let G_r be the simply-connected Lie group with the corresponding root system. The embedding $\mathrm{Pic}(X_{r-1}) \hookrightarrow \mathrm{Pic}(X_r)$ induces an embedding of root lattices $\mathsf{R}_{r-1} \hookrightarrow \mathsf{R}_r$, and identifies a unique simple root α_r in the set of simple roots of R_r, as the complement of simple roots from R_{r-1}. This defines a parabolic subgroup $P_r \subset G_r$. Batyrev's conjecture was that the flag variety G_r/P_r contains a universal torsor of X_r.

Recent work on Cox rings of Del Pezzo surfaces established this *geometric* connection between smooth Del Pezzo surfaces and Lie groups with root systems of the corresponding type: $r = 5$ was treated in [Pop01] and $r = 6, 7$ in [Der07b], via explicit manipulations with defining equations. The papers [SS07] and [SS08] give conceptual, representation-theoretic proofs of these results. It would be important to extend this to singular Del Pezzo surfaces.

Example 1.9.1 (Degree four). Here are some examples of singular degree four Del Pezzo surfaces $X = \{Q_0 = 0\} \cap \{Q = 0\} \subset \mathbb{P}^4$, where $Q_0 = x_0x_1 + x_2^2$ and Q is given in the table below. Let $\tilde{X}$ be the minimal desingularization of X. In all cases below the Cox ring is given by

$$\mathrm{Cox}(\tilde{X}) = F[\eta_1, \ldots, \eta_9]/(f)$$

with *one* relation f [Der07b]. Note that the Cox ring of a smooth degree 4 Del Pezzo surface has 16 generators and 20 relations (see Example 5.3.2).

Singularities	Q	f
$3\mathsf{A}_1$	$x_2(x_1+x_2)+x_3x_4$	$\eta_4\eta_5+\eta_1\eta_6\eta_7+\eta_8\eta_9$
$\mathsf{A}_1+\mathsf{A}_3$	$x_3^2+x_4x_2+x_0^2$	$\eta_6\eta_9+\eta_7\eta_8+\eta_1\eta_3\eta_4^2\eta_5^3$
A_3	$x_3^2+x_4x_2+(x_0+x_1)^2$	$\eta_5\eta_9+\eta_1\eta_4^2\eta_7+\eta_3\eta_6^2\eta_8$
D_4	$x_3^2+x_4x_1+x_0^2$	$\eta_3\eta_5^2\eta_8+\eta_4\eta_6^2\eta_9+\eta_2\eta_7^2$
D_5	$x_1x_2+x_0x_4+x_3^2$	$\eta_3\eta_7^2+\eta_2\eta_6^2\eta_9+\eta_4\eta_5^2\eta_8^2.$

Example 1.9.2 (Cubics). Here are some singular cubic surfaces $X \subset \mathbb{P}^3$, given by the vanishing of the corresponding cubic form:

$4\mathsf{A}_1$	$x_0x_1x_2+x_1x_2x_3+x_2x_3x_0+x_3x_0x_1$
$2\mathsf{A}_1+\mathsf{A}_2$	$x_0x_1x_2=x_3^2(x_1+x_2+x_3)$
$2\mathsf{A}_1+\mathsf{A}_3$	$x_0x_1x_2=x_3^2(x_1+x_2)$
$\mathsf{A}_1+2\mathsf{A}_2$	$x_0x_1x_2=x_1x_3^2+x_3^3$
$\mathsf{A}_1+\mathsf{A}_3$	$x_0x_1x_2=(x_1+x_2)(x_3^2-x_1^2)$
$\mathsf{A}_1+\mathsf{A}_4$	$x_0x_1x_2=x_3^2x_2+x_3x_1^2$
$\mathsf{A}_1+\mathsf{A}_5$	$x_0x_1x_2=x_1^3-x_3^2x_2$
$3\mathsf{A}_2$	$x_0x_1x_2=x_3^3$
A_4	$x_0x_1x_2=x_2^3-x_3x_1^2+x_3^2x_2$
A_5	$x_3^3=x_1^3+x_0x_3^2-x_2^2x_3$
D_4	$x_1x_2x_3=x_0(x_1+x_2+x_3)^2$
D_5	$x_0x_1^2+x_1x_3^2+x_2^2x_3$
E_6	$x_3^3=x_1(x_1x_0+x_2^2).$

Further examples of Cox rings of singular Del Pezzo surfaces can be found in [Der06] and [DT07]. In practice, most geometric questions are easier for smooth surfaces, while most arithmetic questions turn out to be easier in the singular case. For a survey of arithmetic problems on rational surfaces, see Sections 2.4 and 3.4, as well as [MT86].

Example 1.9.3. In some applications, torsors for subtori of T_{NS} are also used. Let X be the diagonal cubic surface

$$x_0^3+x_1^3+x_2^3+x_3^3=0.$$

The following equations were derived in [CTKS87]:

$$\mathcal{T}_X := \left\{ \begin{array}{lcl} x_{11}x_{12}x_{13}+x_{21}x_{22}x_{23}+x_{31}x_{32}x_{33} & = & 0 \\ x_{11}x_{21}x_{31}+x_{12}x_{22}x_{32}+x_{13}x_{23}x_{33} & = & 0 \end{array} \right\} \subset \mathbb{P}^8.$$

This is a torsor for $G=\mathbb{G}_m^4$.

1.10. **K3 surfaces.** Let X be a smooth projective surface with trivial canonical class. There are two possibilities: X could be an abelian surface or a K3 surface. In the latter case, X is simply-connected and $\mathrm{h}^1(X, \mathcal{O}_X) = 0$. The Picard group $\mathrm{Pic}(X)$ of a K3 surface X is a torsion-free $\mathbb{Z}$-module of rank ≤ 20 and the intersection form on $\mathrm{Pic}(X)$ is even, i.e., the square of every class is an even integer. K3 surfaces of with polarizations of small degree can be realized as complete intersections in projective space. The most common examples are K3 surfaces of degree 2, given explicitly as double covers $X \to \mathbb{P}^2$ ramified in a curve of degree 6; or quartic surfaces $X \subset \mathbb{P}^3$.

Example 1.10.1. The Fermat quartic

$$x^4 + y^4 + z^4 + w^4 = 0$$

has Picard rank 20 over $\mathbb{Q}(\sqrt{-1})$. The surface X given by

$$xy^3 + yz^3 + zx^3 + w^4 = 0$$

has $\mathrm{Pic}(X_{\mathbb{Q}}) = \mathbb{Z}^{20}$ (see [Ino78] for more explicit examples). All such K3 surfaces are classified in [Sch08].

The surface

$$w(x^3+y^3+z^3+x^2z+xw^2) = 3x^2y^2+4x^2yz+x^2z^2+xy^2z+xyz^2+y^2z^2$$

has geometric Picard rank 1, i.e., $\mathrm{Pic}(X_{\bar{\mathbb{Q}}}) = \mathbb{Z}$ [vL07].

Other interesting examples arise from abelian surfaces as follows: Let

$$\begin{array}{rccc} \iota : & A & \to & A \\ & a & \mapsto & -a \end{array}$$

be the standard involution. Its fixed points are the 2-torsion points of A. The quotient A/ι has 16 singularieties (the images of the fixed points). The minimal resolution of these singularities is a K3 surface, called a *Kummer surface*. There are several other finite group actions on abelian surfaces such that a similar construction results in a K3 surface, a *generalizied* Kummer surface (see [Kat87]).

The nef cone of a polarized K3 surface (X, g) admits the following characterization: h is ample if and only if $(h, C) > 0$ for each class C with $(g, C) > 0$ and $(C, C) \geq -2$. The *Torelli theorem* implies an intrinsic description of automorphisms: every automorphism of the lattice $\mathrm{Pic}(X)$ preserving the nef cone arises from an automorphisms of X. There is an extensive literature devoted to the classification of possible automorphism groups [Nik81], [Dol08]. These automorphisms give examples of interesting algebraic dynamical systems [McM02], [Can01];

they can be used to propagate rational points and curves [BT00], and to define canonical heights [Sil91], [Kaw08].

1.11. **Threefolds.** The classification of smooth Fano threefolds was a major achievement completed in the works of Iskovskikh [Isk79], [IP99a], and Mori–Mukai [MM86]. There are more than 100 families. Among them, for example, cubics $X_3 \subset \mathbb{P}^4$, quartics $X_4 \subset \mathbb{P}^4$ or double covers of $W_2 \to \mathbb{P}^3$, ramified in a surface of degree 6. Many of these varieties, including the above examples, are not rational. Unirationality of cubics can be seen directly: projecting from a line on X_3 we get a cubic surface fibration, which *splits* after base change. The nonrationality of cubics was proved in [CG72] using *intermediate Jacobians*. Nonrationality of quartics was proved by establishing *birational rigidity*, i.e., showing triviality of the group of birational automorphisms, via an analysis of *maximal* singularities of such maps [IM71]. This technique has been substantially developed within the Minimal Model Program (see [Isk01], [Puk98], [Puk07], [Che05]). Some quartic threefolds are also unirational, e.g., the diagonal, Fermat type, quartic

$$\sum_{i=0}^{4} x_i^4 = 0.$$

It is expected that the *general* quartic is *not* unirational. However, it admits an elliptic fibration: fix a line $\mathfrak{l} \in X_4 \subset \mathbb{P}^4$ and consider a plane in $\mathbb{P}^4$ containing this line, the residual plane curve has degree three and genus 1. A general double cover W_2 does not admit an elliptic or abelian fibration, even birationally [CP07].

1.12. **Holomorphic symplectic varieties.** Let X be a smooth projective simply-connected variety. It is called *holomorphic symplectic* if it carries a unique, modulo constants, nondegenerate holomorphic two-form. Typical examples are K3 surfaces X and their Hilbert schemes $X^{[n]}$ of zero-dimensional length-n subschemes. Another example is the variety of lines of a smooth cubic fourfold, it is deformation equivalent to $X^{[2]}$ of a K3 surface [BD85].

These varieties are interesting for the following reasons:

- The symplectic forms allows to define a *quadratic* form on $\mathrm{Pic}(X)$, the Beauville–Bogomolov form. The symmetries of the lattice carry rich geometric information.
- There is a *local* Torelli theorem, connecting the symmetries of the Picard lattice with symmetries of the variety.

- there is a *conjectural* characterization of the ample cone and of abelian fibration structures, at least in dimension 4 [HT01].

Using this structure as a compass, one can find a plethora of examples with (Lagrangian) abelian fibrations over $\mathbb{P}^n$ or with infinite *endomorphisms*, resp. *birational* automorphisms, which are interesting for arithmetic and algebraic dynamics.

1.13. **Nonclosed fields.** There is a lot to say: F-rationality, F-unirationality, Galois actions on $\mathrm{Pic}(X_{\bar{F}})$, $\mathrm{Br}(X_{\bar{F}})$, algebraic points, special loci, *descent* of Galois-invariant structures to the ground field etc. Here we touch on just one aspect: the effective computation of the Picard group as a Galois-module, for Del Pezzo and K3 surfaces.

Let $X = X_r$ be a Del Pezzo surface over F. A *splitting field* is a normal extension of the ground field over which each exceptional curve is defined. The action of the Galois group Γ factors through a subgroup of the group of symmetries of the exceptional curves, i.e., W_r. In our arithmetic applications we need to know

- $\mathrm{Pic}(X)$ as a Galois module, more specifically, the Galois cohomology

$$\mathrm{H}^1(\Gamma, \mathrm{Pic}(X_{\bar{F}})) = \mathrm{Br}(X)/\mathrm{Br}(F);$$

  this group is an obstruction to F-rationality, and also a source of obstructions to the Hasse principle and weak approximation (see Section 2.4);
- the effective cone $\Lambda_{\mathrm{eff}}(X_F)$.

For Del Pezzo surfaces, the possible values of $\mathrm{H}^1(\Gamma, \mathrm{Pic}(X_{\bar{F}}))$ have been computed [SD93], [KST89], [Ura96], [Cor07]. This information alone does not suffice. Effective Chebotarev theorem [LO77] implies that, given equations defining an Del Pezzo surface, the Galois action on the exceptional curves, i.e., the image of the Galois group in the Weylgroup W_r, can be computed in principle. The cone $\Lambda_{\mathrm{eff}}(X_F)$ is spanned by the Galois orbits on these curves.

It would be useful to have a `Magma` implementation of an algorithm computing the Galois representation on $\mathrm{Pic}(X)$, for X a Del Pezzo surface over $\mathbb{Q}$.

Example 1.13.1. The Picard group may be smaller over nonclosed fields: for $X/\mathbb{Q}$ given by

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

$\mathrm{Pic}(X_{\mathbb{Q}}) = \mathbb{Z}^4$. It has a basis e_1, e_2, e_3, e_4. such that $\Lambda_{\mathrm{eff}}(X)$ is spanned by

$$e_2, \quad e_3, \quad 3e_1 - 2e_3 - e_4, \quad 2e_1 - e_2 - e_3 - e_4, \quad e_1 - e_4,$$
$$4e_1 - 2e_2 - 2e_3 - e_4, \quad e_1 - e_2, \quad 2e_1 - 2e_2 - e_4, \quad 2e_1 - e_3$$

(see [PT01]).

Example 1.13.2 (Maximal Galois action). Let $X/\mathbb{Q}$ be the cubic surface

$$x^3 + 2xy^2 + 11y^3 + 3xz^2 + 5y^2w + 7zw^2 = 0$$

Then the Galois group acting on the 27 lines is $\mathsf{W}(\mathsf{E}_6)$ [EJ08a] (see [Eke90], [Ern94], [VAZ08], and [Zar08], for more examples).

No algorithms for computing even the rank, or the geometric rank of a K3 surface over a number field are known at present. There are infinitely many possibilities for the Galois action on the Picard lattice.

Example 1.13.3. Let X be a K3 surface over a number field $\mathbb{Q}$. Fix a model $\mathcal{X}$ over $\mathbb{Z}$. For primes p of good reduction we have an injection

$$\mathrm{Pic}(X_{\bar{\mathbb{Q}}}) \hookrightarrow \mathrm{Pic}(X_{\bar{\mathbb{F}}_p}).$$

The rank of $\mathrm{Pic}(X_{\bar{\mathbb{F}}_p})$ is always even. In some examples, it can be computed by counting points over $\mathbb{F}_{p^r}$, for several r, and by using the Weil conjectures.

This local information can sometimes be used to determine the rank of $\mathrm{Pic}(X_{\bar{\mathbb{Q}}})$. Let p, q be distinct primes of good reduction such that the corresponding local ranks are ≤ 2 and the discriminants of the lattices $\mathrm{Pic}(X_{\bar{\mathbb{F}}_p})$, $\mathrm{Pic}(X_{\bar{\mathbb{F}}_q})$ do not differ by a square of a rational number. Then the rank of $\mathrm{Pic}(X_{\bar{\mathbb{Q}}})$ equals 1. This idea has been used in [vL07].

2. Existence of points

2.1. **Projective spaces and their forms.** Let F be a field and $\bar{F}$ an algebraic closure of F. A projective space over F has many rational points: they are dense in Zariski topology and in the adelic topology. Varieties F-birational to a projective space inherit these properties.

Over nonclosed fields F, projective spaces have *forms*, so called *Brauer–Severi* varieties. These are isomorphic to $\mathbb{P}^n$ over $\bar{F}$ but not necessarily over F. They can be classified via the nonabelian cohomology group $\mathrm{H}^1(F, \mathrm{Aut}(\mathbb{P}^n))$, where $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}$ is the group of algebraic automorphisms of $\mathbb{P}^n$. The basic example is a conic $C \subset \mathbb{P}^2$, e.g.,

$$ax^2 + by^2 + cz^2 = 0, \quad \text{with} \quad a, b, c \in \mathbb{Z}. \tag{2.1}$$

It is easy to verify solvability of this equation in $\mathbb{R}$ and modulo p, for $p \nmid abc$. Legendre proved that (2.1) has nontrivial solutions in $\mathbb{Z}$ if and only if it has nontrivial solutions in $\mathbb{R}$ and modulo p, for all primes p. This is an instance of a local-to-global principle that will be discussed in Section 2.4.

Checking solvability modulo p is a finite problem which gives a finite procedure to verify solvability in $\mathbb{Z}$. Actually, Legendre's proof provides effective bounds for the size of the smallest solution, e.g.,

$$\max(|x|, |y|, |z|) \le abc,$$

which gives another approach to checking solvability - try all $x, y, z \in \mathbb{N}$ subject to the inequality. If $C(\mathbb{Q}) \neq \emptyset$, then the conic is $\mathbb{Q}$-isomorphic to $\mathbb{P}^1$: draw lines through a $\mathbb{Q}$-point in C.

One could also ask about the number $\mathsf{N}(\mathsf{B})$ of triples of nonzero coprime square-free integers

$$(a, b, c) \in \mathbb{Z}^3, \quad \max(|a|, |b|, |c|) \le \mathsf{B}$$

such that Equation (2.1) has a nontrivial solution. It is [Guo95]:

$$\mathsf{N}(\mathsf{B}) \sim \frac{9}{7\,\Gamma(\frac{3}{2})^3} \prod_p \left(1 - \frac{1}{p}\right)^{3/2} \left(1 + \frac{3}{2p}\right) \frac{\mathsf{B}}{\log(\mathsf{B})^{3/2}}, \quad \mathsf{B} \to \infty.$$

In general, forms of $\mathbb{P}^n$ over number fields satisfy the local-to-global principle. Moreover, Brauer–Severi varieties with at least one F-rational point are *split* over F, i.e., isomorphic to $\mathbb{P}^n$ over F. It would be useful to have a routine (in `Magma`) that would check efficiently whether or not a Brauer–Severi variety of small dimension over $\mathbb{Q}$, presented by explicit equations, is split, and to find the smallest solution. The frequency of split fibers in families of Brauer–Severi varieties is studied in [Ser90b].

2.2. **Hypersurfaces.**

Algebraically, the simplest examples of varieties are hypersurfaces, defined by a single homogeneous equation $f(\mathbf{x}) = 0$. Many classical diophantine problems reduce to the study of rational points on hypersurfaces. Below we give two proofs and one heuristic argument to motivate the idea that hypersurfaces of low degree should have *many* rational points.

Theorem 2.2.1 (Chevalley-Warning, Abh. M. Sem. Hamb. (1936))**.** *Let $X = X_f \subset \mathbb{P}^n$ be a hypersurface over a finite field F given by the equation $f(\mathbf{x}) = 0$. If $\deg(f) \le n$ then $X(F) \neq \emptyset$.*

Proof. We reproduce a textbook argument [BS66], for $F = \mathbb{F}_p$.
Step 1. Consider the δ-function

$$\sum_{x=1}^{p-1} x^d = \begin{cases} -1 & \mod p \quad \text{if } p-1 \mid d \\ 0 & \mod p \quad \text{if } p-1 \nmid d \end{cases}$$

Step 2. Apply it to a (not necessarily homogeneous) polynomial $\phi \in \mathbb{F}_p[x_0, \dots, x_n]$, with $\deg(\phi) \le n(p-1)$. Then

$$\sum_{x_0,\dots,x_n} \phi(x_0, \dots, x_n) = 0 \mod p.$$

Indeed, for monomials, we have

$$\sum_{x_0,\dots,x_m} x_1^{d_1} \cdots x_n^{d_n} = \prod(\sum x_j^{d_j}), \quad \text{with } \; d_0 + \ldots + d_n \le n(p-1).$$

For some j, we have $0 \le d_j < p-1$.

Step 3. For $\phi(x) = 1 - f(x)^{p-1}$ we have $\deg(\phi) \le \deg(f)\cdot(p-1)$. Then

$$\mathsf{N}(f) := \#\{x \mid f(x) = 0\} = \sum_{x_0,\dots,x_n} \phi(x) = 0 \mod p,$$

since $\deg(f) \le n$.

Step 4. The equation $f(\mathbf{x}) = 0$ has a trivial solution. It follows that

$$\mathsf{N}(f) > 1 \quad \text{and} \quad X_f(\mathbb{F}_p) \neq \emptyset.$$

□

A far-reaching generalization is the following theorem.

Theorem 2.2.2. [Esn03] *If X is a Fano variety over a finite field $\mathbb{F}_q$ then*

$$X(\mathbb{F}_q) \neq \emptyset.$$

Now we pass to the case in which $F = \mathbb{Q}$. Given a form $f \in \mathbb{Z}[x_0, \dots, x_n]$, homogeneous of degree d, we ask how many solutions $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ to the equation $f(\mathbf{x}) = 0$ should we expect? Primitive solutions with $\gcd(x_0, \dots, x_n) = 1$, up to diagonal multiplication with ± 1, are in bijection to rational points on the hypersurface $X_f \subset \mathbb{P}^n$. We have $|f(x)| = O(\mathsf{B}^d)$, for $\|x\| := \max_j(|x_j|) \le \mathsf{B}$. We may argue that f takes values $0, 1, 2, \dots$ with equal probability, so that the the probability of $f(\mathbf{x}) = 0$ is B^{-d}. There are B^{n+1} "events" with

$\|x\| \leq \mathsf{B}$. In conclusion, we expect B^{n+1-d} solutions with $\|x\| \leq \mathsf{B}$. There are three cases:

- $n+1 < d$: as $\mathsf{B} \to \infty$ we should have *fewer and fewer* solutions, and, eventually, none!
- $n+1-d$: this is the *stable* regime, we get no information in the limit $\mathsf{B} \to \infty$;
- $n+1-d$: the expected number of solutions grows.

We will see many instances when this heuristic reasoning fails. However, it is reasonable, as a first approximation, at least when

$$n + 1 - d \gg 0.$$

Diagonal hypersurfaces have attracted the attention of computational number theorists (see `http://euler.free.fr`). A sample is given below:

Example 2.2.3.

- There are no rational points (with non-zero coordinates) on the Fano 5-fold $x_0^6 = \sum_{j=1}^6 x_j^6$ with height $\leq 2.6 \cdot 10^6$.
- There are 12 (up to signs and permutations) rational points on $x_0^6 = \sum_{j=1}^7 x_j^6$ of height $\leq 10^5$ (with non-zero coordinates).
- The number of rational points (up to signs, permutations and with non-zero coordinates) on the Fano 5-fold $x_0^6+x_1^6 = \sum_{j=2}^6 x_j^6$ of height $\leq 10^4$ (resp. $2 \cdot 10^4, 3 \cdot 10^4$) is 12 (resp. 33, 57).

Clearly, it is difficult to generate solutions when the $n - d$ is small. On the other hand, we have the following theorem:

Theorem 2.2.4. [Bir62] *If* $n \geq (\deg(f) - 1) \cdot 2^{\deg(f)}$, *and* f *is smooth then the number* $\mathsf{N}(f, \mathsf{B})$ *of solutions* $\mathbf{x} = (x_i)$ *with* $\max(|x_i|) \leq \mathsf{B}$ *is*

$$\mathsf{N}(f, \mathsf{B}) \sim \prod_p \tau_p \cdot \tau_\infty \mathsf{B}^{n+1-d}, \quad \text{as} \quad \mathsf{B} \to \infty,$$

where τ_p, τ_∞ *are the* p*-adic, resp. real, densities. The Euler product converges provided local solutions exist for all* p *and in* $\mathbb{R}$.

We sketch the method of a proof of this result in Section 4.6.

Now we assume that $X = X_f$ is a hypersurface over a function field in one variable $F = \mathbb{C}(t)$. We have

Theorem 2.2.5. *If* $\deg(f) \leq n$ *then* $X_f(\mathbb{C}(t)) \neq \emptyset$.

Proof. It suffices to count parameters: Insert $x_j = x_j(t) \in \mathbb{C}[t]$, of degree e, into

$$f = \sum_J f_J x^J = 0,$$

with $|J| = \deg(f)$. This gives a system of $e \cdot \deg(f) + \text{const}$ equations in $e(n+1)$ variables. This system is solvable for $e \gg 0$, provided $\deg(f) \leq n$. □

2.3. **Decidability.** Hilbert's 10th problem has a negative solution:

Theorem 2.3.1 (see [Mat00], [Mat06])**.** *Let $f \in \mathbb{Z}[t, z_1, \ldots, x_n]$ be polynomial. The set of $t \in \mathbb{Z}$ such that $f(t, \ldots, x_n) = 0$ is solvable in $\mathbb{Z}$ is not decidable, i.e., there is no algorithm to decide whether or not a diophantine equation is solvable in integers.*

Theorem 2.3.2. [Cha94] *The set of $t \in \mathbb{Z}$ such that $f_t = 0$ has infinitely many primitive solutions is algorithmically random* [1].

There are many results concerning indecidability of general diophantine equations over other rings and fields (for a recent survey, see [Poo08b]). The case of rational points, over a number field, is open; even for a cubic surface we cannot decide, at present, whether or not there are rational points.

2.4. **Obstructions.** As we have just said, there is no hope of finding an algorithm which would determine the solvability of a diophantine equation in integers, i.e., there is no algorithm to test for the existence of *integral* points on quasi-projective varieties. The corresponding question for *homogeneous* equations, i.e., for *rational* points, is still open. It is reasonable to expect that at least for certain classes of algebraic varieties, for example, for Del Pezzo surfaces, the existence question can be answered. In this section we survey some recent results in this direction.

[1]The author's abstract: "One normally thinks that everything that is true is true for a reason. I've found mathematical truths that are true for no reason at all. These mathematical truths are beyond the power of mathematical reasoning because they are accidental and random. Using software written in Mathematica that runs on an IBM RS/6000 workstation, I constructed a perverse 200-page algebraic equation with a parameter t and 17,000 unknowns. For each whole-number value of the parameter t, we ask whether this equation has a finite or an infinite number of whole number solutions. The answers escape the power of mathematical reason because they are completely random and accidental."

Let X_B be a scheme over a base scheme B. We are looking for obstructions to the existence of points $X(B)$, i.e., sections of the structure morphism $X \to B$. Each morphism $B' \to B$ gives rise to a base-change diagram, and each section $x : B \to X$ provides a section $x' : B' \to X_{B'}$.

$$
\begin{array}{ccc}
X & \longleftarrow & X_{B'} \\
\downarrow & & \downarrow \\
B & \longleftarrow & B'
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \longleftarrow & X_{B'} \\
\uparrow{\scriptstyle x} & & \uparrow{\scriptstyle x'} \\
B & \longleftarrow & B'
\end{array}
$$

This gives rise to a *local* obstruction, since it is sometimes easier to check that $X_{B'}(B') = \emptyset$. In practice, B could be a curve and B' a cover, or an analytic neighborhood of a point on B. In the number-theoretic context, $B = \mathrm{Spec}\,(F)$ and $B' = \mathrm{Spec}\,(F_v)$, where v is a valuation of the number field F and F_v the v-adic completion of F. One says that the *local-global principle*, or the *Hasse principle*, holds, if the existence of F-rational points is implied by the existence of v-adic points in all completions.

Example 2.4.1. The Hasse principle holds for:
(1) smooth quadrics $X_2 \subset \mathbb{P}^n$;
(2) Brauer–Severi varieties;
(3) Del Pezzo surfaces of degree ≥ 5;
(4) Chatelet surfaces $y^2 - az^2 = f(x_0, x_1)$, where f is an irreducible polynomial of degree ≤ 4 [CTSSD87b];
(5) hypersurfaces $X_d \subset \mathbb{P}^n$, for $n \gg d$ (see Theorem 4.6.1).

The Hasse principle may fail for cubic curves, e.g.,

$$3x^3 + 4y^3 + 5z^3 = 0.$$

In topology, there is a classical obstruction theory to the existence of sections. An adaptation to algebraic geometry is formulated as follows: Let $\mathfrak{C}$ be a contravariant functor from the category of schemes over a base scheme B to the category of abelian groups. Applying the functor $\mathfrak{C}$ to the diagrams above, we have

$$
\begin{array}{ccc}
\mathfrak{C}(X) & \longrightarrow & \mathfrak{C}(X_{B'}) \\
\uparrow & & \uparrow \\
\mathfrak{C}(B) & \longrightarrow & \mathfrak{C}(B')
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathfrak{C}(X) & \longrightarrow & \mathfrak{C}(X_{B'}) \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x'} \\
\mathfrak{C}(B) & \longrightarrow & \mathfrak{C}(B')
\end{array}
$$

If for all sections x', the image of x' in $\mathfrak{C}(B')$ is nontrivial in the cokernel of the map $\mathfrak{C}(B) \to \mathfrak{C}(B')$, then we have a problem, i.e., an obstruction to the existence of B-points on X. So far, this is still a version of a local obstruction. However, a *global obstruction* may arise, when we vary B'.

We are interested in the case when $B = \mathrm{Spec}\,(F)$, for a number field F, with B' ranging over all completions F_v. A global obstruction is possible whenever the map

$$\mathfrak{C}(\mathrm{Spec}\,(F)) \to \prod_v \mathfrak{C}(\mathrm{Spec}\,(F_v))$$

has a nontrivial cokernel. What are sensible choices for $\mathfrak{C}$? Basic contravariant functors on schemes are $\mathfrak{C}(-) := \mathrm{H}^i_{et}(-, \mathbb{G}_m)$. For $i = 1$, we get the Picard functor, introduced in Section 1.1. However, by Hilbert's theorem 90,

$$\mathrm{H}^1_{et}(F, \mathbb{G}_m) := \mathrm{H}^1_{et}(\mathrm{Spec}\,(F), \mathbb{G}_m) = 0,$$

for all fields F, and this won't generate an obstruction. For $i = 2$, we get the (cohomological) Brauer group $\mathrm{Br}(X) = \mathrm{H}^2_{et}(X, \mathbb{G}_m)$, classifying sheaves of central simple algebras over X, modulo equivalence (see [Mil80, Chapter 4]). By class class field theory, we have an exact sequence

$$0 \to \mathrm{Br}(F) \to \bigoplus_v \mathrm{Br}(F_v) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0, \tag{2.2}$$

where $\mathrm{inv}_v : \mathrm{Br}(F_v) \to \mathbb{Q}/\mathbb{Z}$ is the *local invariant*. We apply it to the diagram and obtain:

$$\begin{array}{ccccccccc} & & \mathrm{Br}(X_F) & \longrightarrow & \bigoplus_v \mathrm{Br}(X_{F_v}) & & & & \\ & & \downarrow x & & \downarrow (x_v)_v & & & & \\ 0 & \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \bigoplus_v \mathrm{Br}(F_v) & \xrightarrow{\sum_v \mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0, \end{array}$$

Define

$$X(\mathbb{A}_F)^{\mathrm{Br}} := \cap_{A \in \mathrm{Br}(X)} \{(x_v)_v \in X(\mathbb{A}_F) \mid \sum_v \mathrm{inv}(A(x_v)) = 0\}. \tag{2.3}$$

Let $\overline{X(F)}$ be the closure of $X(F)$ in $X(\mathbb{A}_F)$, in the adelic topology. One says that X satisfies *weak approximation* over F if $\overline{X(F)} = X(\mathbb{A}_F)$. We have

$$X(F) \subset \overline{X(F)} \subseteq X(\mathbb{A}_F)^{\mathrm{Br}} \subseteq X(\mathbb{A}_F).$$

From this we derive the *Brauer–Manin* obstruction to the Hasse principle and weak approximation:

- if $X(\mathbb{A}_F) \neq \emptyset$ but $X(\mathbb{A}_F)^{\mathrm{Br}} = \emptyset$ then $X(F) = \emptyset$, i.e., the Hasse principle fails;
- if $X(\mathbb{A}_F) \neq X(\mathbb{A}_F)^{\mathrm{Br}}$ then weak approximation fails.

Del Pezzo surfaces of degree ≥ 5 satisfy the Hasse principle and weak approximation. Arithmetically most interesting are Del Pezzo surfaces of degree $4, 3$, and 2: these may fail the Hasse principle:

- deg $= 4$: $z^2 + w^2 = (x^2 - 2y^2)(3y^2 - x^2)$ [Isk71];
- deg $= 3$: $5x^3 + 12y^3 + 9z^3 + 10w^3 = 0$ [CG66];
- deg $= 2$: $w^2 = 2x^4 - 3y^4 - 6z^4$ [KT04a].

One says that the Brauer–Manin obstruction to the existence of rational point is the only one, if $X(\mathbb{A}_F)^{\mathrm{Br}} \neq \emptyset$ implies that $X(F) \neq \emptyset$. This holds for:

(1) certain curves of genus ≥ 2 (see, e.g., [Sto07]);
(2) principal homogeneous spaces for a connected linear algebraic group;
(3) Del Pezzo surfaces of degree ≥ 3 admitting a conic bundle structure defined over the ground field F;
(4) conjecturally(!), for all geometrically rational surfaces.

However, the Brauer–Manin obstruction is not the only one, in general. Here is a heuristic argument: a smooth hypersurface in $\mathbb{P}^4$ has trivial $\mathrm{Br}(X)/\mathrm{Br}(F)$. It is easy to satisfy local local conditions, so that for a positive proportion of hypersurfaces one has $X(\mathbb{A}_F) \neq \emptyset$ (see [PV04]). Consider X of *very large* degree. Lang's philosophy (see Conjecture 3.1.1) predicts that there are very few rational points over *any* finite extension of the ground field. Why should there be points over F? This was made precise in [SW95]. The first unconditional result in this direction was [Sko99]: there exist surfaces X with empty Brauer–Manin obstructions with étale covers $\tilde{X}$ which acquire new Brauer group elements producing nontrivial obstructions on $\tilde{X}$ and *a posteriori* on X. These type of "multiple-descent", nonabelian, obstructions were systematically studied in [HS05], [HS02], [Sko01] (see also [Har08], and [Pey05], [Har04]).

Insuffiency of these nonabelian obstructions for threefolds was established in [Poo08a]. The counterexample is a fibration $\phi : X \to C$, defined over $\mathbb{Q}$, such that

- C is a curve of genus ≥ 2 with $C(\mathbb{Q}) \neq \emptyset$ (e.g., a Fermat curve);

- every fiber X_c, for $c \in C(\mathbb{Q})$, is the counterexample

$$z^2 + w^2 = (x^2 - 2y^2)(3y^2 - x^2)$$

  from [Isk71], i.e., $X_c(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$, and $X_c(\mathbb{Q}) = \emptyset$;
- $\mathrm{Br}(X) \simeq \mathrm{Br}(C)$, and the same holds for any base change under an étale map $\tilde{C} \to C$.

Then $\tilde{X}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$, for every étale cover $\tilde{X} \to X$, while $X(\mathbb{Q}) = \emptyset$.

2.5. **Descent.** Let T be an algebraic torus, considered as a group scheme, and X a smooth projective variety over a number field F. We assume that $\mathrm{Pic}(X_{\bar{F}}) = \mathrm{NS}(X_{\bar{F}})$. The F-isomorphisms classes of T-torsors

$$\pi \, : \, \mathcal{T} \to X$$

are parametrized by $\mathrm{H}^1_{et}(X, T)$. A rational point $x \in X(F)$ gives rise to the specialization homomorphism

$$\sigma_x \, : \, \mathrm{H}^1_{et}(X, T) \to \mathrm{H}^1_{et}(F, T),$$

a finite set. Thus the partition:

$$X(F) = \cup_{\tau \in \mathrm{H}^1_{et}(F,T)} \pi_\tau(\mathcal{T}_\tau(F)), \tag{2.4}$$

exhibiting $\mathcal{T}_\tau$ as *descent* varieties.

We now consider the $\Gamma = \mathrm{Gal}(\bar{F}/F)$-module $\mathrm{NS}(X_{\bar{F}})$ and the dual torus T_{NS}. The classifying map in Equation 1.8 is now

$$\chi \, : \, \mathrm{H}^1(X, T_{\mathrm{NS}}) \to \mathrm{Hom}_\Gamma(\mathrm{NS}(X_{\bar{F}}), \mathrm{Pic}(X_{\bar{F}})),$$

a T_{NS}-torsor $\mathcal{T}$ is called *universal* if $\chi([\mathcal{T}]) = \mathrm{Id}$. These may not exist over the ground field F. When they do, their F-equivalence classes form a principal homogeneous space under $\mathrm{H}^1_{et}(F, T)$. The main reasons for working with universal torsors, rather than other torsors are:

- the Brauer–Manin obstruction on X translates to local obstructions on universal torsors, i.e.,

$$X(\mathbb{A}_F)^{\mathrm{Br}} = \cup_{\tau \in \mathrm{H}^1_{et}(F,T)} \pi_\tau(\mathcal{T}_\tau(\mathbb{A}_F));$$

- the Brauer–Manin obstruction on universal torsors vanishes.

The foundations of the theory are in [CTS87] and in the book [Sko01].

2.6. **Effectivity.** In light of the discussion in Section 2.3 it is important to know whether or not the Brauer–Manin obstruction can be computed, effectively in terms of the coefficients of the defining equations. There is an extensive literature on such computations for curves (see the recent papers [Fly04], [BBFL07] and references therein) and for surfaces (e.g., [CTKS87], [BSD04], [Cor07], [KT04b]).

Effective computability of the Brauer-Manin obstruction for all Del Pezzo surfaces over number fields has been proved in [KT08]. The main steps are as follows:

(1) Computation of the equations of the exceptional curves and of the action of the Galois group Γ of a splitting field on these curves as in Section 1.13. One obtains the exact sequence of Γ-modules
$$0 \to \mathrm{Relations} \to \oplus \mathbb{Z}E_j \to \mathrm{Pic}(\overline{X}) \to 0.$$
(2) We have
$$\mathrm{Br}(X)/\mathrm{Br}(F) = \mathrm{H}^1(\Gamma, \mathrm{Pic}(\overline{X})).$$
Using the equations for exceptional curves and functions realizing relations between the curves classes in the Picard group one can compute explicitly Azumaya algebras $\{\mathcal{A}_i\}$ representing the classes of $\mathrm{Br}(X)/\mathrm{Br}(F)$.
(3) The local points $X(F_v)$ can be effectively decomposed into a *finite* union of subsets such that each $\mathcal{A}_i$ is constant on each of these subsets. This step uses an effective version of the arithmetic Hilbert Nullstellensatz.
(4) It remains to compute the local invariants.

3. Density of points

3.1. **Lang's conjecture.** One of the main principles underlying arithmetic geometry is the expectation that the trichotomy in the classification of algebraic varieties via the Kodaira dimension in Section 1.2 has an arithmetic manifestation. The broadly accepted form of this is

Conjecture 3.1.1 (Lang's conjecture)**.** Let X be a variety of general type, i.e., a smooth projective variety with ample canonical class, defined over a number field F. Then $X(F)$ is not Zariski dense.

What about a converse? The obvious necessary condition for Zariski density of rational points, granted Conjecture 3.1.1, is that X does not dominate a variety of general type. This condition is not enough, as was shown in [CTSSD97]: there exist surfaces which do not dominate curves of general type but which have étale covers dominating curves of general type. By the Chevalley–Weil theorem (see, e.g., [Abr08] in this volume), these covers would have a dense set of rational points, over some finite extension of the ground field, contradicting Conjecture 3.1.1.

As a first approximation, one expects that rational points are potentially dense on Fano varieties, on rationally connected varieties, and on Calabi–Yau varieties. Campana formulated precise conjectures characterizing varieties with potentially dense rational points via the notion of *special* varieties (see Section 1.5). In the following sections we survey techniques for proving density of rational points and provide representative examples illustrating these. For a detailed discussion of geometric aspects related to potential density see [Abr08], and [Has03].

3.2. **Zariski density over fixed fields.** Here we address Zariski density of rational points in the "unstable" situation, when the density of points is governed by subtle number-theoretical properties, rather than geometric considerations. We have the following fundamental result:

Theorem 3.2.1. *Let C be a smooth curve of genus $\mathsf{g} = \mathsf{g}(C)$ over a number field F. Then*

- *if $\mathsf{g} = 0$ and $C(F) \neq \emptyset$ then $C(F)$ is Zariski dense;*
- *if $\mathsf{g} = 1$ and $C(F) \neq \emptyset$ then $C(F)$ is an abelian group (the Mordell-Weil group) and there is a constant c_F (independent of C) bounding the order of the torsion subgroup $C(F)_{\mathrm{tors}}$ of $C(F)$* [Maz77], [Mer96]*; in particular, if there is an F-rational point of infinite order then $C(F)$ is Zariski dense;*
- *if $\mathsf{g} \geq 2$ then $C(F)$ is finite* [Fal83], [Fal91].

In higher dimensions we have:

Theorem 3.2.2. *Let X be an algebraic variety over a number field F. Assume that $X(F) \neq \emptyset$ and that X is one of the following*

- *X is a Del Pezzo surface of degree 2 and has a point on the complement to exceptional curves;*
- *X is a Del Pezzo surface of degree ≥ 3;*
- *X is a Brauer-Severi variety.*

Then $X(F)$ is Zariski dense.

The proof of the first claims can be found in [Man86].

Remark 3.2.3. Let X/F be a Del Pezzo surface of degree 1 (it always contains an F-rational point, the base point of the anticanonical linear series) or a conic bundle $X \to \mathbb{P}^1$, with $X(F) \neq \emptyset$. It is unknown whether or not $X(F)$ is Zariski dense.

Theorem 3.2.4. [Elk88] *Let $X \subset \mathbb{P}^3$ be the quartic K3 surface given by*

$$x_0^4 + x_1^4 + x_2^4 = x_3^4. \tag{3.1}$$

Then $X(\mathbb{Q})$ is Zariski dense.

The trivial solutions $(1:0:0:1)$ etc are easily seen. The smallest nontrivial solution is

$$(95\,800, 217\,519, 414\,560, 422\,481).$$

Geometrically, over $\bar{\mathbb{Q}}$, the surface given by (3.1) is a Kummer surface, with many elliptic fibrations.

Example 3.2.5. [EJ06] Let $X \subset \mathbb{P}^3$ be the quartic given by

$$x^4 + 2y^4 = z^4 + 4t^4.$$

The obvious $\mathbb{Q}$-rational points are given by $y = t = 0$ and $x = \pm z$. The next smallest solution is

$$1484801^4 + 2 \cdot 1203120^4 = 1169407^4 + 4 \cdot 1157520^4.$$

3.3. **Potential density: techniques.** Here is a (short) list of possible strategies to propagate points:

- use the group of automorphisms $\mathrm{Aut}(X)$, if it is infinite;
- try to find a dominant map $\tilde{X} \to X$ where $\tilde{X}$ satisfies potential density (for example, try to prove *unirationality*);
- try to find a fibration structure $X \to B$ where the fibers satisfy potential density in some *uniform way* (that is, the field extensions needed to insure potential density of the fibers V_b can be uniformly controlled).

In particular, it is important for us keep track of minimal conditions which would insure Zariski density of points on varieties. A fundamental result is

Example 3.3.1. Let $\pi : X \to \mathbb{P}^1$ be a conic bundle, defined over a field F. Then rational points on X are potentially dense. Indeed, by Tsen's theorem, π has section $s : \mathbb{P}^1 \to X$ (which is defined over some finite extension F'/F), each fiber has an F'-rational point and it suffices to apply Theorem 3.2.1. Potential density for conic bundles over higher-dimensional bases is an open problem.

If X is an abelian variety then there exists a finite extension F'/F and a point $P \in X(F')$ such that the cyclic subgroup of $X(F')$ generated by P is Zariski dense.

Example 3.3.2. If $\pi : X \to \mathbb{P}^1$ is a Jacobian nonisotrivial elliptic fibration (π admits a section and the j-invariant is nonconstant), then potential density follows from a strong form of the Birch/Swinnerton-Dyer conjecture [GM97], [Man95]. The key problem is to control the variation of the *root number* (the sign of the functional equation of the L-functions of the elliptic curve) (see [GM97]).

On the other hand, rational points on certain elliptic fibration with *multiple* fibers are not potentially dense [CTSSD97].

Example 3.3.3. One geometric approach to Zariski density of rational points on (certain) elliptic fibration can be summarized as follows:

Case 1. Let $\pi : X \to B$ be a Jacobian elliptic fibration and $e : B \to X$ its zero-section. Suppose that we have another section s which is nontorsion in the Mordell-Weil group of $X(F(B))$. Then a specialization argument implies that the restriction of the section to infinitely many fibers of π gives a nontorsion point in the Mordell-Weil group of the corresponding fiber (see [Ser90a], 11.1). In particular, $X(F)$ is Zariski dense, provided $B(F)$ is Zariski dense in B.

Case 2. Suppose that $\pi : X \to B$ is an elliptic fibration with a *multisection* M (an irreducible curve surjecting onto the base B). After a basechange $X \times_B M \to M$ the elliptic fibration acquires the identity section Id (the image of the diagonal under $M \times_B M \to V \times_B M$) *and* a (rational) section

$$\tau_M := d\mathrm{Id} - \mathrm{Tr}(M \times_B M),$$

where d is the degree of $\pi : M \to B$ and $\mathrm{Tr}(M \times_B M)$ is obtained (over the generic point) by summing all the points of $M \times_B M$. We will say that M is *nontorsion* if τ_M is nontorsion.

If M is nontorsion and if $M(F)$ is Zariski dense then the same holds for $X(F)$ (see [BT99]).

Remark 3.3.4. Similar arguments work for abelian fibrations [HT00b]. The difficulty here is to formulate some simple geometric conditions insuring that a (multi)section leads to points which are not only of infinite order in the Mordell-Weil groups of the corresponding fibers, but in fact generate Zariski dense subgroups.

3.4. **Potential density for surfaces.** By Theorem 3.2.1, potential density holds for curves of genus $\mathsf{g} \leq 1$. It holds for surfaces which become rational after a finite extension of the ground field, e.g., for all Del Pezzo surfaces. The classification theory in dimension 2 gives us the following list of surfaces of Kodaira dimension 0:

- abelian surfaces;
- bielliptic surfaces;
- Enriques surfaces;
- K3 surfaces.

Potential density for the first two classes follows from Theorem 3.2.2. The classification of Enriques surfaces X implies that either $\mathrm{Aut}(X)$ is infinite or X is dominated by a K3 surface $\tilde{X}$ with $\mathrm{Aut}(\tilde{X})$ infinite [Kon86]. Thus we are reduced to the study of K3 surfaces.

Theorem 3.4.1. [BT00] *Let X be a K3 surface over* any *field of characteristic zero. If X is elliptic or admits an infinite group of automorphisms then rational points on X are potentially dense.*

Sketch of the proof. One needs to find sufficiently nondegenerate rational or elliptic multisections of the elliptic fibration $X \to \mathbb{P}^1$. These are produced using deformation theory. One starts with special K3 surfaces which have rational curves $C_t \subset X_t$ in the desired homology class (for example, Kummer surfaces) and then deforms the pair. This deformation technique has to be applied to twists of the original elliptic surface. □

Example 3.4.2. A smooth hypersurface $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(2, 2, 2)$ is a K3 surface with $\mathrm{Aut}(X)$ infinite.

Example 3.4.3. Every smooth quartic surface $S_4 \subset \mathbb{P}^3$ which contains a line is an elliptic K3 surface. Indeed, let M be this line and assume that both S_4 and M are defined over a number field F. Consider the 1-parameter family of planes $\mathbb{P}^2_t \subset \mathbb{P}^3$ containing M. The residual curve in the intersection $\mathbb{P}^2_t \cap S_4$ is a plane cubic intersecting M in 3 points. This gives a fibration $\pi : S_4 \to \mathbb{P}^1$ with a *rational* tri-section M.

To apply the strategy of Section 6.1 we need to insure that M is nontorsion. A sufficient condition, satisfied for generic quartics S_4, is that the restriction of π to M ramifies in a smooth fiber of $\pi : X \to \mathbb{P}^1$. Under this condition $X(F)$ is Zariski dense.

Theorem 3.4.4. [HT00a] *Let $X \subset \mathbb{P}^3$ be a quartic K3 surface containing a line defined over a field F. If X is general, then $X(F)$ is Zariski dense. In all cases, there exists a finite extension F'/F such that $X(F')$ is Zariski dense.*

Theorem 3.4.5. [BT00] *Let X be an elliptic K3 surface over a field F. Then rational points are potentially dense.*

Remark 3.4.6. No examples of K3 surfaces X over $\mathbb{Q}$ with geometric Picard number 1, $X(\mathbb{Q}) \neq \emptyset$ and $X(\mathbb{Q})$ not Zariski dense are known at present.

3.5. **Potential density in dimension ≥ 3.** Potential density holds for unirational varieties. Classification of (smooth) Fano threefolds and the detailed study of occuring families implies unirationality for all but three cases:

- X_4: quartics in $\mathbb{P}^4$;
- V_1: double covers of a cone over the Veronese surface in $\mathbb{P}^5$ famified in a surface of degree 6;
- W_2: double covers of $\mathbb{P}^3$ ramified in a surface of degree 6.

We now sketch the proof of potential density for quartics from [HT00a], the case of V_1 is treated by similar techniques in [BT99].

The threefold X_4 contains a 1-parameter family of lines. Choose a line M (defined over some extension of the ground field, if necessary) and consider the 1-parameter family of hyperplanes $\mathbb{P}^3_t \subset \mathbb{P}^4$ containing M. The generic hyperplane section $S_t := \mathbb{P}^3_t \cap X_4$ is a quartic surface with a line. Now we would like to argue as in the Example 3.4.3. We need to make sure that M is nontorsion in S_t for a dense set of $t \in \mathbb{P}^1$. This will be the case for general X_4 and M. The analysis of all exceptional cases requires care.

Remark 3.5.1. It would be interesting to have further (nontrivial) examples of birationally rigid Fano varieties. Examples of Calabi–Yau varieties over functions fields of curves, with geometric Picard number one and dense sets of rational points have been constructed in [HT08]; no nontrivial examples Calabi-Yau threefolds with a potentially dense set of rational points are known at present.

Theorem 3.5.2. [HT00b] *Let X be a K3 surface over a field F, of degree $2(n-1)$. Then rational points on $X^{[n]}$ are potentially dense.*

The proof relies on the existence of an abelian fibration

$$Y := X^{[n]} \to \mathbb{P}^n,$$

with a nontorsion multisection which has a potentially dense set of rational points. Numerically, such fibrations are predicted by square-zero classes in the Picard group $\mathrm{Pic}(Y)$, with respect to the Beauville–Bogomolov form (see Section 1.12). Geometrically, the fibration is the degree n Jacobian fibration associated to hyperplane sections of X.

Theorem 3.5.3. [AV07] *Let Y be the Fano variety of lines on a general cubic fourfold $X_3 \subset \mathbb{P}^5$ over a field of characteristic zero. Then rational points on Y are potentially dense.*

Sketch of proof. The key tool is a rational endomorphism $\phi\colon Y \to Y$ analyzed in [Voi04]: let $\mathfrak{l}$ on $X_3 \subset \mathbb{P}^5$ be a general line and $\mathbb{P}^2_{\mathfrak{l}} \subset \mathbb{P}^5$ the unique plane everywhere tangent to $\mathfrak{l}$. Let $[\mathfrak{l}] \in Y$ be the corresponding point and put $\phi([\mathfrak{l}]) := [\mathfrak{l}']$, where $\mathfrak{l}'$ is the residual line in X_3.

Generically, one can expect that the orbit $\{\phi^n([\mathfrak{l}])\}_{n\in\mathbb{N}}$ is Zariski dense in Y. This was proved by Amerik and Campana in [AC08], over *uncountable* ground fields. Over countable fields, one faces the difficulty that the countably many exceptional loci could cover all algebraic points of Y. Amerik and Voisin were able to overcome this obstacle over number fields. Rather than proving density of $\{\phi^n([\mathfrak{l}])\}_{n\in\mathbb{N}}$ they find surfaces $\Sigma \subset Y$, birational to abelian surfaces, whose orbits are dense in Y. The main effort goes into showing that one can choose sufficiently general Σ defined over $\bar{\mathbb{Q}}$, provided that Y is sufficiently general and still defined over a number field. In particular, Y has geometric Picard number one. A case by case geometric analysis excludes the possibility that the Zariski closure of $\{\phi^n(\Sigma)\}_{n\in\mathbb{N}}$ is a proper subvariety of F. □

Theorem 3.5.4. [HT] *Let Y be the variety of lines of a cubic fourfold $X_3 \subset \mathbb{P}^5$ which contains a cubic scroll T. Assume that the hyperplane section of X_3 containing T has exactly 6 double points in linear general position and that X_3 does not contain a plane. If X_3 and T are defined over a field F then $Y(F)$ is Zariski dense.*

Remark 3.5.5. In higher dimensions, (smooth) hypersurfaces $X_d \subset \mathbb{P}^n$ of degree d represent a major challenge. The circle method works well when

$$n \gg 2^d$$

while the geometric methods for proving unirationality require at least a super-exponential growth of n (see [HMP98] for a contruction of a unirational parametrization).

3.6. **Approximation.** Let X be smooth and projective. Assume that $X(F)$ is dense in each $X(F_v)$. A natural question is whether or not $X(F)$ is dense in the adeles $X(\mathbb{A}_F)$. This *weak approximation* may be obstructed globally, by the Brauer–Manin obstruction, as explained in Section 2.4. There are examples of such obstructions for Del Pezzo surfaces in degree ≤ 4, for conic bundles over $\mathbb{P}^2$ [Har96], and for K3 surfaces as in the following example.

Example 3.6.1. [Wit04] Let $E \to \mathbb{P}^1$ be the elliptic fibration given by

$$y^2 = x(x-g)(x-h) \quad \text{where} \quad g(t) = 3(t-1)^3(t+3) \quad \text{and} \quad h = g(-t).$$

Its minimal proper regular model X is an elliptic K3 surface that fails weak approximation. The obstruction comes from transcendental classes in the Brauer group of X.

The theory is parallel to the theory of the Brauer–Manin obstruction to the Hasse principle, up to a certain point. The principal new feature is:

Theorem 3.6.2. [Min89] *Let X be a smooth projective variety over a number field with a nontrivial geometric fundamental group. Then weak approximation fails for X.*

This applies to Enriques surfaces [HS05].

Among open questions, of particular interest are varieties which are unirational over the ground field F. For example, weak approximation is unknown for the diagonal cubic surface

$$x^3 + y^3 + z^3 + dt^3 = 0, \quad d \in F,$$

even for $F = \mathbb{C}(B)$, the function field of a curve. Other natural examples are quotients V/G, where G is a group and V a G-representation, discussed in Section 1.2.

4. Counting problems

Here we consider projective algebraic varieties $X \subset \mathbb{P}^n$ defined over a number field F. We assume that $X(F)$ is Zariski dense. We seek to understand the distribution of rational points with respect to heights.

4.1. **Heights.** First we assume that $F = \mathbb{Q}$. Then we can define a *height* integral (respectively rational points) on the affine (respectively projective space) as follows

$$\begin{array}{rcc} \mathsf{H}_{\text{affine}} : \mathbb{A}^n(\mathbb{Z}) = \mathbb{Z}^n & \to & \mathbb{R}_{\geq 0} \\ x = (x_1, \ldots, x_n) & \mapsto & \max_j(|x_j|) \\ \mathsf{H} : \mathbb{P}^n(\mathbb{Q}) = (\mathbb{Z}^{n+1}_{\text{prim}} \setminus 0)/\pm & \to & \mathbb{R}_{>0} \\ x = (x_0, \ldots, x_n) & \mapsto & \max_j(|x_j|). \end{array}$$

This induces heights on points of subvarieties of affine or projective spaces. In some problems it is useful to work with equivalent norms, e.g., $\sqrt{\sum x_j^2}$ instead of $\max_j(|x_j|)$. Such choices are referred to as a *change of metrization*. A more conceptual definition of heights and adelic metrizations is given in Section 4.8

4.2. **Counting functions.** For a subvariety $X \subset \mathbb{P}^n$ put

$$\mathsf{N}(X, \mathsf{B}) := \#\{x \in X(\mathbb{Q}) \,|\, \mathsf{H}(x) \leq \mathsf{B}\}.$$

What can be said about

$$\mathsf{N}(X, \mathsf{B}), \quad \text{for} \ \ \mathsf{B} \to \infty \ ?$$

Main questions here concern:

- (uniform) upper bounds,
- asymptotic formulas,
- geometric interpretation of the asymptotics.

By the very definition, $\mathsf{N}(X, \mathsf{B})$ depends on the projective embedding of X. For $X = \mathbb{P}^n$ over $\mathbb{Q}$, with the standard embedding via the line bundle $\mathcal{O}(1)$, we get

$$N(\mathbb{P}^n, \mathsf{B}) \sim \frac{1}{\zeta(n+1)} \cdot \tau_\infty \cdot \mathsf{B}^{n+1}, \ \ B \to \infty.$$

But we may also consider the Veronese re-embedding

$$\begin{array}{ccc} \mathbb{P}^n & \to & \mathbb{P}^N \\ x & \mapsto & x^I, \ \ |I| = d, \end{array}$$

e.g.,

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^2 \\ (x_0 : x_1) & \mapsto & (x_0^2 : x_0 x_1 : x_1^2) \end{array}$$

The image $y_0 y_2 = y_1^2$ has $\sim \mathsf{B}$ points of height $\leq \mathsf{B}$. Similarly, the number of rational points on height $\leq \mathsf{B}$ in the $\mathcal{O}(d)$ embedding of $\mathbb{P}^n$ will be $\sim \mathsf{B}^{(n+1)/d}$.

More generally, if $F/\mathbb{Q}$ is a finite extension, put

$$\begin{array}{ccc} \mathbb{P}^n(F) & \to & \mathbb{R}_{>0} \\ x & \mapsto & \prod_v \max(|x_j|_v). \end{array}$$

Theorem 4.2.1. [Sch79]

$$N(\mathbb{P}^n(F), \mathsf{B}) \sim \frac{h_F R_F (n+1)^{r_1+r_2-1}}{w_F \zeta_F(n+1)} \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{\mathrm{disc}(F)}} \right)^{n+1} \mathsf{B}^{n+1}, \tag{4.1}$$

where

- h_F *is the class number of* F*;*
- R_F *the regulator;*
- r_1 *(resp.* r_2*) the number of real (resp. pairs of complex) embeddings of* F*,*
- $\mathrm{disc}(F)$ *the discriminant;*
- w_F *the number of roots of 1 in* F*;*
- ζ_F *the zeta function of* F*.*

With this starting point, one may try to prove asymptotic formulas of similar precision for arbitrary projective algebraic varieties X, at least under some natural geometric conditions. This program was initiated in [FMT89] and it has rapidly grown in recent years.

4.3. **Upper bounds.** A first step in understanding growth rates of rational points of bounded height is to obtain uniform upper and lower bounds, with effective control of error terms. Results of this type are quite valuable in arguments using fibration structures. Here is a sample:

- [BP89], [Pil96]: Let $X \subset \mathbb{A}^2$ be a geometrically irreducible affine curve. Then

  $$\#\{x \in X(\mathbb{Z}) \,|\, \mathsf{H}(x) \le \mathsf{B}\} \ll_{\deg(X)} \mathsf{B}^{\frac{1}{\deg(X)}} \log(\mathsf{B})^{2\deg(X)+3}.$$

- [EV05]: Let $X \subset \mathbb{P}^2$ be a geometrically irreducible curve of genus ≥ 1. Then there is a $\delta > 0$ such that

  $$\mathsf{N}(X(\mathbb{Q}), \mathsf{B}) \ll_{\deg(X),\delta} \mathsf{B}^{\frac{2}{\deg(X)}-\delta}.$$

Fibering and using estimates for lower dimensional varieties, one has:

Theorem 4.3.1. [Pil95] *Let* $X \subset \mathbb{P}^n$ *be a geometrically irreducible variety, and* $\epsilon > 0$*. Then*

$$\mathsf{N}(X(\mathbb{Q}), \mathsf{B}) \ll_{\deg(X),\dim(X),\epsilon} \mathsf{B}^{\dim(X)+\frac{1}{\deg(X)}+\epsilon}$$

The next breakthrough was accomplished in [HB02]; further refinements combined with algebro-geometric tools lead to

Theorem 4.3.2 ([BHBS06], [Sal07]). *Let $X \subset \mathbb{P}^n$ be a geometrically irreducible variety, and $\epsilon > 0$. Then*

$$\mathsf{N}(X(\mathbb{Q}), \mathsf{B}) \ll_{\deg(X),\dim(X),\epsilon} \begin{cases} \mathsf{B}^{\dim(X)-\frac{3}{4}+\frac{5}{3\sqrt{3}}+\epsilon} & \deg(X) = 3 \\ \mathsf{B}^{\dim(X)-\frac{2}{3}+\frac{3}{2\sqrt{\deg(X)}}+\epsilon} & \deg(X) = 4, 5 \\ \mathsf{B}^{\dim(X)+\epsilon} & \deg(X) \geq 6 \end{cases}$$

A survey of results on upper bounds, with detailed proofs, is in [HB06].

4.4. **Lower bounds.** Let X be a projective variety over a number field F and let L be a very ample line bundle on X. This gives an embedding $X \hookrightarrow \mathbb{P}^n$. We fix a height H on $\mathbb{P}^n(F)$ and consider the counting function

$$\mathsf{N}(X(F), -K_X, \mathsf{B}) := \#\{x \in X(F) \,|\, \mathsf{H}_L(x) \leq \mathsf{B}\},$$

with respect to the induced height H_L (see Section 4.8 for more explanations on heights).

Lemma 4.4.1. *Let X be a smooth Fano variety over a number field F and $Y := \mathrm{Bl}_Z(X)$ a blowup in a smooth subvariety $Z = Z_F$ of codimension ≥ 2. If $\mathsf{N}(X^\circ(F), -K_X, \mathsf{B}) \gg \mathsf{B}^1$, for all dense Zariski open $X^\circ \subset X$ then the same holds for Y:*

$$\mathsf{N}(Y^\circ(F), -K_Y, \mathsf{B}) \gg \mathsf{B}^1.$$

Proof. Let $\pi : Y \to X$ be the blowup. We have

$$-K_Y = \pi^*(-K_X) - D$$

with $\mathrm{supp}(D) \subset E$, the exceptional divisor. It remains to use the fact that $\mathsf{H}_D(x)$ is uniformly bounded from below on $(X \setminus D)(F)$ (see, e.g., [BG06, Proposition 2.3.9]), so that

$$\mathsf{N}(\pi^{-1}(X^\circ)(F), -K_Y, \mathsf{B}) \geq \mathsf{c} \cdot \mathsf{N}(X^\circ(F), -K_X, \mathsf{B}),$$

for some constant $\mathsf{c} > 0$ and an appropriate Zariski open $X^\circ \subset X$. □

In particular, *split* Del Pezzo surfaces X_r satisfy the lower bound of Conjecture 4.10.1

$$\mathsf{N}(X_r(F), -K_{X_r}, \mathsf{B}) \gg \mathsf{B}^1.$$

Finer lower bounds, in some nonsplit cases have been proved in [SSD98]:

$$\mathsf{N}(X_6^\circ(F), -K_{X_6}, \mathsf{B}) \gg \mathsf{B}^1 \log(\mathsf{B})^{r-1},$$

if X_6 is a cubic surface with at least two skew lines defined over F. This gives support to Conjecture 4.10.2. The following theorem provides evidence for Conjecture 4.10.1 in dimension 3.

Theorem 4.4.2. [Man93] *Let X be a Fano threefold over a number field F_0. For every Zariski open subset $X^\circ \subset X$ there exists a finite extension F/F_0 such that*

$$\mathsf{N}(X(F), -K_X, \mathsf{B}) \gg \mathsf{B}^1$$

This relies on the classification of Fano threefolds (cf. [IP99b], [MM82], [MM86]). One case was missing from the classification when [Man93] was published; the Fano threefold obtained as a blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a curve of tri-degree $(1, 1, 3)$ [MM03]. Lemma 4.4.1 proves the expected lower bound in this case as well.

4.5. **Finer issues.** At the next level of precision we need to take into account more refined arithmetic and geometric data. Specifically, we need to analyze the possible sources of failure of the heuristic $\mathsf{N}(\mathsf{B}) \sim \mathsf{B}^{n+1-d}$ in Section 2.2:

- *Local or global obstructions*: as in $x_0^2 + x_1^2 + x_2^2 = 0$ or $x_0^3 + 4x_1^3 + 10x_2^3 + 25x_3^3 = 0$;
- *Singularities*: the surface $x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2 = x_0x_1x_2x_3$ has $\sim \mathsf{B}^{3/2}$ points of height $\le \mathsf{B}$, on every Zariski open subset, too many!
- *Accumulating subvarieties*: On $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ there are $\sim \mathsf{B}^2$ points on $\mathbb{Q}$-lines and and provably $O(\mathsf{B}^{4/3+\epsilon})$ points in the complement [HB97]. The expectation is $\mathsf{B}\log(\mathsf{B})^3$, over $\mathbb{Q}$. Similar effects persist in higher dimensions. A quartic $X_4 \subset \mathbb{P}^4$ contains a 1-parameter family of lines, each contributing $\sim \mathsf{B}^2$ to the asymptotic, while the expectation is $\sim \mathsf{B}$. Lines on a cubic $X_3 \subset \mathbb{P}^4$ are parametrized by a surface, which is of general type. We expect $\sim \mathsf{B}^2$ points of height $\le \mathsf{B}$ on the cubic threefold, and on each line. In [BG06, Theorem 11.10.11] it is shown that

  $$\mathsf{N}_{\text{lines}}(\mathsf{B}) \sim c\,\mathsf{B}^2, \quad \text{as} \quad \mathsf{B} \to \infty,$$

  where the count is over F-rational points on lines defined over F, and the constant c is a *converging* sum of leading terms of contributions from each line of the type (4.1). In particular, each line contributes a positive density to the main term. On the other hand, one expects the same asymptotic $\sim \mathsf{B}^2$ on the complement of lines, with the leading term a product of local densities. How to reconcile this? The forced compromise is to discard such *accumulating* subvarieties and to hope that for some Zariski open subset $X^\circ \subset X$, the asymptotic of points of bounded height does reflect the geometry of X.

These finer issues are particularly striking in the case of K3 surfaces. They may have local and global obtructions to the existence of rational points, they may fail the heuristic asymptotic, and they may have accumulating subvarieties, even infinitely many:

Conjecture 4.5.1. Let X be a K3 surface over a number field F. Let L be a polarization, $\epsilon > 0$ and $Y = Y(\epsilon, L)$ be the union of all F-rational curves $C \subset X$ (i.e., curves that are isomorphic to $\mathbb{P}^1$ over F) and have L-degree $\leq 2/\epsilon$. Then

$$\mathsf{N}(X, L, \mathsf{B}) = \mathsf{N}(Y, L, \mathsf{B}) + O(\mathsf{B}^{\epsilon}), \quad \text{as} \quad B \to \infty.$$

Theorem 4.5.2. [McK00] *Let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be a double cover ramified over a curve of bidegree $(4, 4)$. Then there exists an open cone $\Lambda \subset \Lambda_{\mathrm{ample}}(X)$ such that for every $L \in \Lambda$ there exists a $\delta > 0$ such that*

$$\mathsf{N}(X, L, \mathsf{B}) = \mathsf{N}(Y, L, \mathsf{B}) + O(\mathsf{B}^{2/d-\delta}), \quad \textit{as} \quad \mathsf{B} \to \infty,$$

where d is the minimal L-degree of a rational curve on X and Y is the union of all F-rational curves of degree d.

This theorem exhibits the first layer of an arithmetic stratification predicted in Conjecture 4.5.1.

4.6. **The circle method.** Let $f \in \mathbb{Z}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree d such that the hypersurface $X_f \subset \mathbb{P}^n$ is nonsingular. Let

$$\mathsf{N}_f(\mathsf{B}) := \#\{\mathbf{x} \in \mathbb{Z}^n \mid f(\mathbf{x}) = 0 \ \ \|\mathbf{x}\| \leq \mathsf{B}\}$$

be the counting function. In this section we sketch a proof of the following

Theorem 4.6.1. [Bir62] *Assume that $n \geq 2^d(d+1)$. Then*

$$\mathsf{N}_f(\mathsf{B}) = \Theta \cdot \mathsf{B}^{n+1-d}(1 + o(1)) \ \ \mathsf{B} \to \infty, \tag{4.2}$$

where

$$\Theta = \prod_p \tau_p \cdot \tau_\infty > 0,$$

provided $f(\mathbf{x}) = 0$ is solvable in $\mathbb{Z}_p$, for all p, and in $\mathbb{R}$.

The constants τ_p and τ_∞ admit an interpretation as *local densitities*; these are explained in a more conceptual framework in Section 4.12.

Substantial efforts have been put into reducing the number of variables, especially for low degrees. Another direction is the extension of the method to systems of equations [Sch85] or to more general number fields [Ski97].

We now outline the main steps of the proof of the asymptotic formula 4.2. The first step is the introduction of a "delta"-function: for $x \in \mathbb{Z}$ we have

$$\int_0^1 e^{2\pi i\alpha x} d\alpha = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

Now we can write

$$\mathsf{N}_f(\mathsf{B}) = \int_0^1 \mathsf{S}(\alpha) d\alpha \tag{4.3}$$

where

$$\mathsf{S}(\alpha) := \sum_{\mathbf{x}\in\mathbb{Z}^{n+1},\, \|\mathbf{x}\|\leq \mathsf{B}} e^{2\pi i\alpha f(\mathbf{x})}.$$

The function $\mathsf{S}(\alpha)$ is wildly oscillating (see Figure 4.6), with peaks at $\alpha = a/q$, with small q. Indeed, the probability that $f(\mathbf{x})$ is divisible by q is higher for small q, and each such term contributes 1 to $\mathsf{S}(\alpha)$. The idea of the circle method is to analyze the asymptotic of the integral in equation 4.3, for $\mathsf{B} \to \infty$, by extracting the contributions of α close to rational numbers a/q with small q, and finding appropriate bounds for integrals over the remaining intervals.

More precisely, one introduces the *major arcs*

$$\mathfrak{M} := \bigcup_{(a,q)=1,\, q\leq \mathsf{B}^{\Delta}} \mathfrak{M}_{a,q},$$

where $\Delta > 0$ is a parameter to be specified, and

$$\mathfrak{M}_{a,q} := \{\alpha \,|\, |\alpha - \frac{a}{q}| \leq \mathsf{B}^{-d+\delta}\}.$$

The *minor arcs* are the complement:

$$\mathfrak{m} := [0,1] \setminus \mathfrak{M}.$$

The goal is to prove the bound

$$\int_{\mathfrak{m}} \mathsf{S}(\alpha) d\alpha = O(\mathsf{B}^{n-d-\epsilon}), \quad \text{for some } \ \epsilon > 0, \tag{4.4}$$

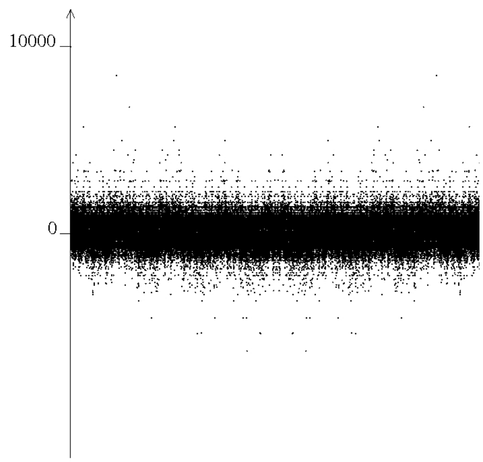

FIGURE 1. Oscillations of $\mathsf{S}(\alpha)$

and the asymptotic

$$\int_{\mathfrak{M}} \mathsf{S}(\alpha)d\alpha \sim \prod_p \tau_p \cdot \tau_\infty \cdot \mathsf{B}^{n+1-d} \quad \text{for} \quad \mathsf{B} \to \infty. \tag{4.5}$$

Remark 4.6.2. Modern refinements employ "smoothed out" intervals, i.e., the delta function of an interval in the major arcts is replaced by a smooth bell curve with support in this interval. In Fourier analysis, "rough edges" translate into bad bounds on the dual side, and should be avoided. An implementation of this idea, leading to savings in the number of variables can be found in [HB83].

There are various approaches to proving upper bounds in equation 4.4; most are a variant or refinement of Weyl's bounds (1916) [Wey16]. Weyl considered the following exponential sums

$$\mathsf{s}(\alpha) := \sum_{0 \le x \le \mathsf{B}} e^{2\pi i \alpha x^d},$$

The main observation is that $|\mathsf{s}(\alpha)|$ is "small", when $|\alpha - a/q|$ "large". This is easy to see when $d = 1$; summing the geometric series we get

$$|\mathsf{s}(\alpha)| = |\frac{1-e^{2\pi i\alpha(\mathsf{B}+1)}}{1-e^{2\pi i\alpha}}| \ll \frac{1}{\langle\!\langle\alpha\rangle\!\rangle},$$

where $\langle\!\langle\alpha\rangle\!\rangle$ is the distance to the nearest integer. In general, Weyl's differencing technique is applied to reduce the degree, to eventually arrive at a geometric series.

We turn to major arcs. Let

$$\alpha = \frac{a}{q} + \beta$$

with β *very small*, and getting smaller as a function of B. Here we will assume that $|\beta| \le \mathsf{B}^{-d+\delta'}$, for some small $\delta' > 0$. We put $\mathbf{x} = q\mathbf{y} + \mathbf{z}$, with $\mathbf{z}$ the corresponding residue class modulo q, and obtain

$$\begin{aligned}
\mathsf{S}(\alpha) &= \sum_{\mathbf{x}\in\mathbb{Z}^{n+1},\, \|\mathbf{x}\|\le\mathsf{B}} e^{2\pi i\frac{a}{q}f(\mathbf{x})}e^{2\pi i\beta f(\mathbf{x})} \\
&= \sum_{\|\mathbf{x}\|\le\mathsf{B}} e^{2\pi i\frac{a}{q}f(q\mathbf{y}+\mathbf{z})}e^{2\pi i\beta f(\mathbf{x})} \\
&= \sum_{\mathbf{z}} e^{2\pi i\frac{a}{q}f(\mathbf{z})}(\sum_{\|\mathbf{y}\|\le\mathsf{B}/q} e^{2\pi i\beta f(\mathbf{x})}) \\
&= \sum_{\mathbf{z}} e^{2\pi i\frac{a}{q}f(\mathbf{z})}\int_{\|\mathbf{y}\|\le\mathsf{B}/q} e^{2\pi i\beta f(\mathbf{x})}d\mathbf{y} \\
&= \sum_{\mathbf{z}} \frac{e^{2\pi i\frac{a}{q}f(\mathbf{z})}}{q^{n+1}}\int_{\|\mathbf{x}\|\le\mathsf{B}} e^{2\pi i\beta f(\mathbf{x})}d\mathbf{x},
\end{aligned}$$

where $d\mathbf{y} = q^{n+1}d\mathbf{x}$. The passage $\sum \mapsto \int$ is justified for our choice of small β - the difference will be adsorbed in the error term in (4.2). We have obtained

$$\int_0^1 \mathsf{S}(\alpha)d\alpha \sim \sum_{a,q}\sum_{\mathbf{z}} \frac{e^{2\pi i\frac{a}{q}f(\mathbf{z})}}{q^{n+1}} \cdot \int_{|\beta|\le\mathsf{B}^{-d+\delta}}\int_{\|\mathbf{x}\|\le\mathsf{B}} e^{2\pi i\beta f(\mathbf{x})}d\mathbf{x}d\beta.$$

We first deal with the integral on the right, called the *singular integral*. Put $\beta' = \beta\mathsf{B}^d$ and $\mathbf{x}' = \mathbf{x}/\mathsf{B}$. The change of variables leads to

$$\int_{|\beta|\le\frac{1}{\mathsf{B}^{d-\delta}}} d\beta\int_{\|\mathbf{x}\|\le\mathsf{B}} e^{2\pi i\beta\mathsf{B}^d f(\frac{\mathbf{x}}{\mathsf{B}})}\mathsf{B}^{n+1}d(\frac{\mathbf{x}}{\mathsf{B}}) = \mathsf{B}^{n+1-d}\int_{|\beta'|\le\mathsf{B}^{\delta}}\int_{\|\mathbf{x}'\|\le 1} e^{2\pi i\beta' f(\mathbf{x}')}d(\mathbf{x}').$$

We see the appearance of the main term B^{n-d} and the density

$$\tau_\infty := \int_0^1 d\beta' \int_{\|\mathbf{x}\|\leq 1} e^{2\pi i \beta' f(\mathbf{x}')} d\mathbf{x}'.$$

Now we analyze the *singular integral*

$$\sigma_Q := \sum_{a,q} \sum_{\mathbf{z}} \frac{e^{2\pi i \frac{a}{q} f(\mathbf{z})}}{q^n},$$

where the outer sum runs over positive coprime integers a, q, $a < q$ and $q < Q$, and the inner sum over residue classes $\mathbf{z} \in (\mathbb{Z}/q)^{n+1}$. This sum has the following properties

(1) multiplicativity in q, in particular we have

$$\sigma := \prod_p (\sum_{i=0}^{\infty} A(p^i)).$$

with $\sigma_Q \to \sigma$, for $Q \to \infty$, (with small error term),

(2)

$$\sum_{i=0}^{k} \frac{A(p^i)}{p^{i(n+1)}} = \frac{\varrho(f, p^k)}{p^{kn}},$$

where

$$\varrho(f, p^k) := \#\{\, \mathbf{z} \mod p^k \mid f(\mathbf{z}) = 0 \mod p^k \,\}.$$

Here, a discrete version of equation (4.3) comes into play:

$$\#\{\text{solutions} \mod p^k\} = \frac{1}{p^k} \sum_{a=0}^{p^k-1} \sum_{\mathbf{z}} e^{2\pi i \frac{a f(\mathbf{z})}{p^k}}$$

However, our sums run over a with $(a,p)=1$. A rearranging of terms leads to

$$\begin{aligned}\frac{\varrho(f,p^k)}{p^{kn}} &= \sum_{i=0}^{k}\sum_{(a,p^k)=p^i}\sum_{\mathbf{z}}\frac{1}{p^{k(n+1)}}e^{2\pi i\frac{a}{p^k}f(\mathbf{z})}\\ &= \sum_{i=0}^{k}\sum_{(\frac{a}{p^i},p^{k-i})=1}\sum_{\mathbf{z}}\frac{1}{p^{k(n+1)}}e^{2\pi i\frac{a/p^i}{p^{k-i}}f(\mathbf{z})}\cdot p^{(n+1)i}\\ &= \sum_{i=0}^{k}\frac{1}{p^{(n+1)(k-i)}}\sum_{(a,p^{k-i})=1}\sum_{\mathbf{z}}e^{2\pi i\frac{a}{p^{k-i}}f(\mathbf{z})}\\ &= \sum_{i=0}^{k}\frac{1}{p^{(n+1)(k-i)}}\cdot A(p^{k-i}).\end{aligned}$$

In conclusion,

$$\sigma=\prod_p \tau_p, \quad \text{where} \quad \tau_p=\lim_{k\to\infty}\frac{\varrho(f,p^k)}{p^{nk}}. \tag{4.6}$$

As soon as there is at least one (nonsingular) solution $f(\mathbf{z})=0 \mod p$, $\tau_p\neq 0$, and in fact, for almost all p,

$$\frac{\varrho(f,p^k)}{p^{nk}}=\frac{\varrho(f,p)}{p^n},$$

by Hensel's lemma. Moreover, if $\tau_p\neq 0$ for all p, the Euler product in equation (4.6) converges.

Let us illustrate this in the example of Fermat type equations

$$f(\mathbf{x})=a_0x^d+\cdots+a_nx_n^d=0.$$

Using properties of Jacobi sums one can show that

$$\varrho(f,p)=p^n+E, \quad \text{with} \quad E=O((p-1)p^{(n-1)/2}),$$

so that

$$\left|\frac{\varrho(f,p)}{p^{n-1}}-1\right|\leq\frac{C}{p^{(n+1)/2}}.$$

The corresponding Euler product

$$\prod_p\frac{\varrho(f,p)}{p^n}\ll\prod_p(1+\frac{C}{p^{(n+1)/2}})$$

is convergent.

Some historical background: the circle method was firmly established in the series of papers of Hardy and Littlewood *Partitio numerum.* They comment: *"A method of great power and wide scope, applicable to almost any problem concerning the decomposition of integers into parts of a particular kind, and to many against which it is difficult to suggest any other obvious method of attack."*

4.7. **Function fields: heuristics.** Here we present Batyrev's heuristic arguments from 1987, which lead to Conjecture 4.10.4.

Let p be a prime and put $q = p^n$. Let Λ be a convex n-dimensional cone in $\mathbb{R}^n$ with vertex at 0. Let

$$f_1, f_2 \,:\, \mathbb{R}^n \to \mathbb{R}$$

be two linear functions such that

- $f_i(\mathbb{Z}^n) \subset \mathbb{Z}$;
- $f_2(x) > 0$ for all $x \in \Lambda \setminus \{0\}$;
- there exists an $x \in \Lambda \setminus \{0\}$ such that $f(x) > 0$.

Put

$$M = \cup_\lambda M_\lambda, \;\; \lambda \in \mathbb{Z}^n \cap \Lambda$$

and

$$|M_\lambda| := q^{\max(0, f_1(\lambda))}, \;\; \varphi(m) := q^{f_2(\lambda)}, \;\text{for}\; m \in M_\lambda.$$

Then the series

$$F(s) = \sum_{\lambda \in \Lambda \cap \mathbb{Z}^n} \frac{|M_\lambda|}{q^{s f_2(\lambda)}}$$

converges for

$$\Re(s) > a := \max_{x \in \Lambda}(f_1(x)/f_2(x)) > 0.$$

What happens around $s = a$? Choose an $\epsilon > 0$ and decompose the cone

$$\Lambda := \Lambda_\epsilon^+ \cup \Lambda_\epsilon^-,$$

where

$$\begin{array}{rcl} \Lambda_\epsilon^+ & := & \{x \in \Lambda \,|\, f_1(x)/f_2(x) \geq a - \epsilon\} \\ \Lambda_\epsilon^- & := & \{x \in \Lambda \,|\, f_1(x)/f_2(x) < a - \epsilon\} \end{array}$$

Therefore,

$$F(s) = F_\epsilon^+ + F_\epsilon^-,$$

where F_ϵ^- converges absolutely for $\Re(s) > a - \epsilon$.

Now we make some assumptions concerning Λ: suppose that for all $\epsilon \in \mathbb{Q}_{>0}$, the cone Λ_ϵ^+ is a rational finitely generated polyhedral cone. Then

$$\Lambda_\epsilon^a := \{x \,|\, f_1(x)/f_2(x) = a\}$$

is a face of Λ_ϵ^+, and thus also finitely generated polyhedral.

Lemma 4.7.1. *There exists a function $G_\epsilon(s)$, holomorphic for $\Re(s) > a - \epsilon$, such that*

$$F_\epsilon(s) = \frac{G_\epsilon(s)}{(s-a)^b},$$

where b is the dimension of the face Λ_ϵ^a.

Proof. For $y \in \mathbb{Q}_{>0}$ we put

$$P(y) := \{x \mid x \in \Lambda,\ f_2(x) = y\}.$$

Consider the expansion

$$F(s) = \sum_{y \in \mathbb{N}} \sum_{\lambda \in P(y) \cap \mathbb{Z}^n} q^{f_1(\lambda) - s f_2(\lambda)}.$$

Replacing by the integral, we obtain (with $w = yz$)

$$\begin{aligned}
&= \int_0^\infty dy \left(\int_{P(y)} q^{f_1(w) - s f_2(w)} dw \right) \\
&= \int_0^\infty dy \left(\int_{P(1)} y^{n-1} q^{f_1(w) - s f_2(y)} dz \right) \\
&= \int_{P(1)} dz \int_0^\infty y^{n-1} q^{(f_1(w)-s)y} dy \qquad (4.7) \\
&= \int_{P(1)} dz \frac{1}{(s - f_1(z))^n} \int_0^\infty u^{n-1} q^{-u} du \\
&= \frac{\Gamma(n)}{(\log(q))^n} \int_{P(1)} \frac{1}{(s - f_1(z))^n} dz.
\end{aligned}$$

It is already clear that we get a singularity at $s = \max(f_1(z))$ on $P(1)$, which is a. In general, let f be a linear function and

$$\Phi(s) := \int_\Delta (s - f(x))^{-n} d\Omega$$

where Δ is a polytope of dimension $n-1$. Then Φ is a rational function in s, with an asymptotic at $s = a$ given by

$$\mathrm{vol}_{f,a} \frac{(b-1)!}{(n-1)!} (s-a)^{-b},$$

where $\Delta_{f,a}$ is the polytope $\Delta \cap \{f(x) = a\}$, $\mathrm{vol}_{f,a}$ is its volume and $b = 1 + \dim(\Delta_{f,a})$. □

Let C be a curve of genus g over the finite field $\mathbb{F}_q$ and F its function field. Let X be a variety over $\mathbb{F}_q$ of dimension n. Then $V := X \times S$ is a variety over F. Every F-rational point x of V gives rise to a section $\tilde{x}$ of the map $V \to S$. We have a pairing

$$\mathrm{A}^1(V) \times \mathrm{A}^n(V) \to \mathbb{Z}$$

between the groups of (numerical) equivalence classes of codimension 1-cycles and codimension n-cycles. We have

$$\mathrm{A}^n(V) = \mathrm{A}^n(X) \otimes \mathrm{A}^1(S) \oplus \mathrm{A}^{n-1}(X) \otimes \mathrm{A}^0(S)$$

and

$$\begin{array}{rcl} \mathrm{A}^1 & = & \mathrm{A}^1(X) \oplus \mathbb{Z} \\ L & = & (L_X, \ell) \\ -K_V & = & (-K_X, 2-2\mathsf{g}). \end{array}$$

Let L be a very ample line bundle on V. Then

$$q^{(L,\tilde{x})}$$

is the height of the point x with respect to L. The height zeta function takes the form

$$\begin{aligned} \mathsf{Z}(s) &= \sum_{x \in V(F)} q^{-(L,\tilde{x})s} \\ &= \sum_{y \in \mathrm{A}^n(X)} \tilde{\mathsf{N}}(q) a^{-[(L_X,y)+\ell]s}, \end{aligned}$$

where

$$\tilde{\mathsf{N}}(q) := \#\{x \in V(F) \,|\, \mathrm{cl}(x) = y\}.$$

We proceed to give some *heuristic*(!) bound on $\tilde{\mathsf{N}}(q)$. The cycles in a given class y are parametrized by an algebraic variety M_y and

$$\dim(M_{y(\tilde{x})}) \geq \chi(\mathcal{N}_{V|\tilde{x}})$$

(the Euler characteristic of the normal bundle). More precisely, the local ring on the moduli space is the quotient of a powerseries ring with $\mathrm{h}^0(\mathcal{N}_{V|\tilde{x}})$ generators by $\mathrm{h}^1(\mathcal{N}_{V|\tilde{x}})$ relations. Our main heuristic assumption is that

$$\tilde{\mathsf{N}}(q) \sim q^{\dim(M_y)} \sim q^{\chi(\mathcal{N}_{V|\tilde{x}})}.$$

This assumption fails, for example, for points contained in "exceptional" (accumulating) subvarieties.

By the short exact sequence

$$0 \to \mathcal{T}_{\tilde{x}} \to \mathcal{T}_{V|\tilde{x}} \to \mathcal{N}_{V|\tilde{x}} \to 0$$

we have

$$\chi(\mathcal{T}_{V|\tilde{x}}) = (-K_V, \tilde{x}) + (n+1)\chi(\mathcal{O}_{\tilde{x}})$$
$$\chi(\mathcal{N}_{V|\tilde{x}}) = (-K_X, \mathrm{cl}(x)) + n\chi(\mathcal{O}_{\tilde{x}})$$

From now on we consider a *modified* height zeta function

$$\mathsf{Z}_{\mathrm{mod}}(s) := \sum q^{\chi(\mathcal{N}_{V|\tilde{x}}) - (L,\tilde{x})s}.$$

We observe that its analytic properties are determined by the ratio between two linear functions

$$(-K_X, \cdot) \text{ and } (L, \cdot).$$

The relevant cone Λ is the cone spanned by classes of (maximally moving) effective curves. The *finite* generation of this cone for Fano varieties is one of the main results of Mori's theory. We conclude that

$$\mathsf{N}(X^\circ, L, \mathsf{B}) \sim \mathsf{B}^a (\log(\mathsf{B}))^{b-1},$$

where

$$a = a(L) = \max_{z \in \Lambda}((-K_X, z)/(L, z))$$

and $b = b(L)$ is the dimension of the face of the cone where this maximum is achieved.

4.8. **Metrizations of line bundles.** In this section we discuss a refined theory of height functions, based on the notion of an adelically metrized line bundle.

Let F be a number field and $\mathrm{disc}(F)$ the discriminant of F (over $\mathbb{Q}$). The set of places of F will be denoted by $\mathrm{Val}(F)$. We shall write $v|\infty$ if v is archimedean and $v \nmid \infty$ if v is nonarchimedean. For any place v of F we denote by F_v the completion of F at v and by $\mathfrak{o}_v$ the ring of v-adic integers (for $v \nmid \infty$). Let q_v be the cardinality of the residue field $\mathbb{F}_v$ of F_v for nonarchimedean valuations. The local absolute value $|\cdot|_v$ on F_v is the multiplier of the Haar measure, i.e., $d(ax_v) = |a|_v dx_v$ for some Haar measure dx_v on F_v. We denote by $\mathbb{A} = \mathbb{A}_F = \prod_v' F_v$ the adele ring of F. We have the *product formula*

$$\prod_{v \in \mathrm{Val}(F)} |a|_v = 1, \quad \text{for all } a \in F^*.$$

Definition 4.8.1. Let X be an algebraic variety over F and L a line bundle on X. A v-adic metric on L is a family $(\|\cdot\|_x)_{x \in X(F_v)}$ of v-adic Banach norms on the fibers L_x such that for all Zariski open subsets $X^\circ \subset X$ and every section $\mathsf{f} \in \mathrm{H}^0(X^\circ, L)$ the map

$$X^\circ(F_v) \to \mathbb{R}, \;\; x \mapsto \|\mathsf{f}\|_x,$$

is continuous in the v-adic topology on $X^\circ(F_v)$.

Example 4.8.2. Assume that L is generated by global sections. Choose a basis $(\mathsf{f}_j)_{j\in[0,\dots,n]}$ of $\mathrm{H}^0(X,L)$ (over F). If f is a section such that $\mathsf{f}(x)\neq 0$ then define

$$\|\mathsf{f}\|_x := \max_{0\le j\le n}(|\frac{\mathsf{f}_j}{\mathsf{f}}(x)|_v)^{-1},$$

otherwise $\|0\|_x := 0$. This defines a v-adic metric on L. Of course, this metric depends on the choice of $(\mathsf{f}_j)_{j\in[0,\dots,n]}$.

Definition 4.8.3. Assume that L is generated by global sections. An adelic metric on L is a collection of v-adic metrics, for every $v\in\mathrm{Val}(F)$, such that for all but finitely many $v\in\mathrm{Val}(F)$ the v-adic metric on L is defined by means of some *fixed* basis $(\mathsf{f}_j)_{j\in[0,\dots,n]}$ of $\mathrm{H}^0(X,L)$.

We shall write $\|\cdot\|_{\mathbb{A}} := (\|\cdot\|_v)$ for an adelic metric on L and call a pair $\mathcal{L}=(L,\|\cdot\|_{\mathbb{A}})$ an adelically metrized line bundle. Metrizations extend naturally to tensor products and duals of metrized line bundles, which allows to define adelic metrizations on arbitrary line bundles L (on projective X): represent L as $L=L_1\otimes L_2^{-1}$ with very ample L_1 and L_2. Assume that L_1,L_2 are adelically metrized. An adelic metrization of L is any metrization which for all but finitely many v is induced from the metrizations on L_1,L_2.

Definition 4.8.4. Let $\mathcal{L}=(L,\|\cdot\|_{\mathbb{A}})$ be an adelically metrized line bundle on X and f an F-rational section of L. Let $X^\circ\subset X$ be the maximal Zariski open subset of X where f is defined and does not vanish. For all $x=(x_v)_v\in X^\circ(\mathbb{A})$ we define the local

$$H_{\mathcal{L},\mathsf{f},v}(x_v) := \|\mathsf{f}\|_{x_v}^{-1}$$

and the global *height function*

$$H_{\mathcal{L}}(x) := \prod_{v\in\mathrm{Val}(F)} H_{\mathcal{L},\mathsf{f},v}(x_v).$$

By the product formula, the restriction of the global height to $X^\circ(F)$ does not depend on the choice of f.

Example 4.8.5. For $X=\mathbb{P}^1=(x_0:x_1)$ one has $\mathrm{Pic}(X)=\mathbb{Z}$, spanned by the class $L=[(1:0)]$. For all $\mathsf{f}=x_0/x_1\in\mathbb{G}_a(\mathbb{A})$ we define

$$H_{\mathcal{L},\mathsf{f},v}(x_v)=\max(1,|\mathsf{f}|_v).$$

The restriction of $H_{\mathcal{L}}=\prod_v H_{\mathcal{L},\mathsf{f},v}$ to $\mathbb{G}_a(F)\subset\mathbb{P}^1$ is the usual height on $\mathbb{P}^1$ (with respect to the usual metrization of $\mathcal{L}=\mathcal{O}(1)$).

Example 4.8.6. Let X be an equivariant compactification of a unipotent group G and L a very ample line bundle on X. The space $\mathrm{H}^0(X, L)$, a representation space for G, has a *unique* G-invariant section f, modulo scalars. Indeed, if we had two nonproportional sections, their quotient would be a character of G, which are trivial.

Fix such a section. We have $\mathsf{f}(g_v) \neq 0$, for all $g_v \in G(F_v)$. Put

$$\mathsf{H}_{\mathcal{L},\mathsf{f},v}(g_v) = \|\mathsf{f}(g_v)\|^{-1} \quad \text{and } \mathsf{H}_{\mathcal{L},\mathsf{f}} = \prod_v \mathsf{H}_{\mathcal{L},\mathsf{f},v}.$$

By the product formula, the global height is independent of the choice of f.

4.9. **Height zeta functions.** Let X be an algebraic variety over a global field F, $\mathcal{L} = (L, \|\cdot\|_{\mathbb{A}})$ an adelically metrized ample line bundle on X, $\mathsf{H}_{\mathcal{L}}$ a height function associated to $\mathcal{L}$, X° a subvariety of X, $a_{X^\circ}(\mathcal{L})$ the convergence abscissa of the height zeta function

$$\mathsf{Z}_{X^\circ}(\mathcal{L}, s) := \sum_{x \in X^\circ(F)} \mathsf{H}_{\mathcal{L}}(x)^{-s}.$$

Proposition 4.9.1.

(1) *The value of $a_{X^\circ}(\mathcal{L})$ depends only on the class of L in* $\mathrm{NS}(X)$.
(2) *Either $0 \leq a_{X^\circ}(\mathcal{L}) < \infty$, or $a_{X^\circ}(\mathcal{L}) = -\infty$, the latter possibility corresponding to the case of finite $X^\circ(F)$. If $a_{X^\circ}(\mathcal{L}) > 0$ for one ample L then this is so for every ample L.*
(3) *$a_{X^\circ}(\mathcal{L}^m) = \frac{1}{m} a_{X^\circ}(\mathcal{L})$. In general, $a_{X^\circ}(\mathcal{L})$ extends uniquely to a continuous function on $\Lambda_{\mathrm{nef}}(X)^\circ$, which is inverse linear on each half-line unless it identically vanishes.*

Proof. All statements follow directly from the standard properties of heights. In particular,

$$a_{X^\circ}(\mathcal{L}) \leq a(\mathbb{P}^n(F), \mathcal{O}(m)) = \frac{n+1}{m}$$

for some n, m. If $\mathsf{Z}_{X^\circ}(\mathcal{L}, s)$ converges at some negative s, then it must be a finite sum. Since for two ample heights H, H' we have

$$\mathsf{c}\mathsf{H}^m < \mathsf{H}' < \mathsf{c}'\mathsf{H}^n, \;\; \mathsf{c}, \mathsf{c}', m, n > 0,$$

the value of a can only be simultaneously positive or zero. Finally, if L and L' are close in the (real) topology of $\mathrm{NS}(V)_{\mathbb{R}}$, then $L - L'$ is a linear combination of ample classes with small coefficients, and so $a_{X^\circ}(\mathcal{L})$ is close to $a_{X^\circ}(\mathcal{L}')$. $\square$

Notation 4.9.2. By Property (1) of Proposition 4.9.1, we may write $a_{X^\circ}(\mathcal{L}) = a_{X^\circ}(L)$.

Example 4.9.3. For an abelian variety X and ample line bundle L we have

$$\mathsf{H}_{\mathcal{L}}(x) = \exp(\mathsf{q}(x) + \mathsf{l}(x) + O(1)),$$

where q is a positive definite quadratic form on $X(F) \otimes \mathbb{Q}$ and l is a linear form. It follows that $a_X(L) = 0$, although $X(F)$ may well be Zariski dense in X. Also

$$\mathsf{N}(X, \mathcal{L}, \mathsf{B}) \sim \log(\mathsf{B})^{r/2}$$

where $r = \operatorname{rk} X(F)$. Hence for $a = 0$, the power of $\log(\mathsf{B})$ in principle cannot be calculated geometrically: it depends on the arithmetic of X and F. The hope is that for $a > 0$ the situation is more stable.

Definition 4.9.4. The *arithmetic hypersurface of linear growth* is

$$\Sigma_{X^\circ}^{arith} := \{L \in \mathrm{NS}(X)_{\mathbb{R}} \,|\, a_{X^\circ}(L) = 1\}.$$

Proposition 4.9.5.

- *If $a_{X^\circ}(L) > 0$ for some L, then $\Sigma_{X^\circ}^{arith}$ is nonempty and intersects each half-line in $\Lambda_{\text{eff}}(X)^\circ$ in exactly one point.*
- *$\Sigma_{X^\circ}^{<} := \{L \,|\, a_{X^\circ}(L) < 1\}$ is convex.*

Proof. The first statement is clear. The second follows from the Hölder inequality: if

$$0 < \sigma, \sigma' \leq 1 \quad \text{and} \quad \sigma + \sigma' = 1$$

then

$$\mathsf{H}_{\mathcal{L}}^{-\sigma}(x)\mathsf{H}_{\mathcal{L}}^{\sigma'}(x) \leq \sigma \mathsf{H}_{\mathcal{L}}(x)^{-1} + \sigma' \mathsf{H}_{\mathcal{L}}(x)^{-1}$$

so that from $L, L' \in \Sigma_{X^\circ}^{<}$ it follows that $\sigma L + \sigma' L' \in \Sigma_{X^\circ}^{<}$. ☐

When $\operatorname{rk} \mathrm{NS}(X) = 1$, Σ_{X° is either empty, or consists of one point. Schanuel's theorem 4.2.1 implies that for $\mathbb{P}^n(F)$, this point is the anticanonical class.

Definition 4.9.6. A subvariety $Y \subset X^\circ \subset X$ is called *point accumulating*, or simply *accumulating* (in X° with respect to L), if

$$a_{X^\circ}(L) = a_Y(L) > a_{X^\circ \setminus Y}(L).$$

It is called *weakly accumulating*, if

$$a_{X^\circ}(L) = a_Y(L) = a_{X^\circ \setminus Y}(L).$$

Example 4.9.7. If we blow up an F-point of an abelian variety X, the exceptional divisor will be an accumulating subvariety in the resulting variety, although to prove this we must analyze the height with respect to the exceptional divisor, which is not quite obvious.

If $X := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, with $n_j > 0$, then every fiber of a partial projection is weakly accumulating with respect to the anticanonical class.

The role of accumulating subvarieties is different for various classes of varieties, but we will generally try to pinpoint them in a geometric way. For example, on Fano varieties we need to to remove the $-K_X$-accumulating subvarieties to ensure stable effects, e.g., the *linear growth conjecture*. Weakly accumulating subvarieties sometimes allow to obtain lower bounds for the growth rate of $X(F)$ by analyzing subvarieties of smaller dimension (as in Theorem 4.4.2).

4.10. **Manin's conjecture.** The following picture emerged from the analysis of examples such as $\mathbb{P}^n$, flag varieties, complete intersections of small degree [FMT89], [BM90].

Let X be a smooth projective variety with ample anticanonical class over a number field F_0. The conjectures below describe the asymptotic of rational points of bounded height in a *stable* situation, i.e., after a sufficiently large finite extension F/F_0 and passing to a sufficiently small Zariski dense subset $X^\circ \subset X$.

Conjecture 4.10.1 (Linear growth conjecture)**.** One has

$$\mathsf{B}^1 \ll \mathsf{N}(X^\circ(F), -K_X, \mathsf{B}) \ll \mathsf{B}^{1+\epsilon}. \tag{4.8}$$

Conjecture 4.10.2 (The power of log)**.**

$$\mathsf{N}(X^\circ(F), -K_X, \mathsf{B}) \sim \mathsf{B}^1 \log(\mathsf{B})^{r-1}, \tag{4.9}$$

where $r = \mathrm{rk}\,\mathrm{Pic}(X_F)$.

Conjecture 4.10.3 (General polarizations / linear growth)**.** Every smooth projective X with $-K_X \in \Lambda_{\mathrm{big}}(X)$ has a dense Zariski open subset X° such that

$$\Sigma_{X^\circ}^{arith} = \Sigma_X^{geom},$$

(see Definitions 4.9.4, (1.7)).

The next level of precision requires that $\Lambda_{\mathrm{eff}}(X)$ is a finitely generated polyhedral cone. By Theorem 1.1.5, this holds when X is Fano.

Conjecture 4.10.4 (General polarizations / power of log)**.** Let L be an ample line bundle and $a(L), b(L)$ the constants defined in Section 1.4. Then

$$\mathsf{N}(X^\circ(F), L, \mathsf{B}) \sim \mathsf{B}^{a(L)} \log(\mathsf{B})^{b(L)-1}, \quad \mathsf{B} \to \infty. \tag{4.10}$$

4.11. **Counterexamples.** At present, no counterexamples to Conjecture 4.10.1 are known. However, Conjecture 4.10.2 fails in dimension 3. The geometric reason for this failure comes from Mori fiber spaces, more specifially from "unexpected" jumps in the rank of the Picard group in fibrations.

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface. We know, by Lefschetz, that $\mathrm{Pic}(X) = \mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$, for $n \geq 4$. However, this may fail when X has dimension 2. Moreover, the variation of the rank of the Picard group in a family of surfaces X_t over a number field F may be nontrivial, even when *geometrically*, i.e., over the algebraic closure $\bar{F}$ of F, the rank is constant.

Concretely, consider a hypersurface $X \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{P}^3_{\mathbf{y}}$ given by a form of bidegree (1,3):

$$\sum_{j=0}^{3} x_j y_j^3 = 0.$$

By Lefschetz, the Picard group $\mathrm{Pic}(X) = \mathbb{Z}^2$, with the basis of hyperplane sections of $\mathbb{P}^3_{\mathbf{x}}$, resp. $\mathbb{P}^3_{\mathbf{y}}$, and the anticanonical class is computed as in Example 1.1.2

$$-K_X = (3, 1).$$

Projection onto $\mathbb{P}^3_{\mathbf{y}}$ exhibits X as a $\mathbb{P}^2$-fibration over $\mathbb{P}^3$. The second Mori fiber space structure on X is given by projection to $\mathbb{P}^3_{\mathbf{x}}$, with fibers diagonal cubic surfaces. The restriction of $-K_X$ to each (smooth) fiber $X_{\mathbf{x}}$ is the anticanonical class of the fiber.

The rank $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbf{x}})$ varies between 1 and 7. For example, if F contains $\sqrt{-3}$, then the $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbf{x}}) = 7$ whenever all x_j are cubes in F. The lower bounds in Section 4.4 show that

$$\mathsf{N}(X^\circ_{\mathbf{x}}, -K_{X_{\mathbf{x}}}, \mathsf{B}) \sim \mathsf{B} \log(\mathsf{B})^3$$

for all such fibers, all dense Zariski open subsets $X^\circ_{\mathbf{x}}$ and all F. On the other hand, Conjecture 4.10.2 implies that

$$\mathsf{N}(X^\circ, -K_X, \mathsf{B}) \sim \mathsf{B} \log(\mathsf{B}),$$

for some Zariski open $X^\circ \subset X$, over a sufficiently large number field F. However, every Zariski open subset $X^\circ \subset X$ intersects infinitely many fibers $X_{\mathbf{x}}$ with $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbf{x}}) = 7$ in a dense Zariski open subset. This is a contradiction.

4.12. **Peyre's refinement.** The refinement concerns the conjectured asymptotic formula (4.9). Fix a metrization of $-\mathcal{K}_X = (-K_X, \|\cdot\|_{\mathbb{A}})$. The expectation is that

$$\mathsf{N}(X^\circ(F), -\mathcal{K}_X, \mathsf{B}) = c(-\mathcal{K}_X)\cdot \mathsf{B}^1 \log(\mathsf{B})^{r-1}(1+o(1)), \quad \text{as} \quad \mathsf{B}\to\infty,$$

with $r = \mathrm{rk}\,\mathrm{Pic}(X)$. Peyre's achievement was to give a conceptual interpretation of the constant $c(-\mathcal{K}_X)$. Here we explain the key steps of his construction.

Let F be a number field and F_v its v-adic completion. Let X be a smooth algebraic variety over F of dimension d equipped with an adelically metrized line bundle $\mathcal{K} = \mathcal{K}_X = (K_X, \|\cdot\|_{\mathbb{A}})$. Fix a point $x \in X(F_v)$ and let $x_1, \ldots, x_d$ be local analytic coordinates in an analytic neighborhood U_x of x giving a homeomorphism

$$\phi : U \xrightarrow{\sim} F_v^d.$$

Let $dy_1 \wedge \ldots \wedge dy_d$ be the standard differential form on F_v^d and $\mathsf{f} := \phi^*(dy_1 \wedge \ldots \wedge dy_d)$ its pullback to U. Note that f is a local section of the canonical sheaf K_X and that a v-adic metric $\|\cdot\|_v$ on K_X gives rise to a norm $\|\mathsf{f}(u)\|_v \in \mathbb{R}_{>0}$, for each $u \in U_x$. Let $d\mu_v = dy_1 \cdots dy_d$ be the standard Haar measure, normalized by

$$\int_{\mathfrak{o}_v^d} d\mu_v = \frac{1}{\mathfrak{d}_v{}^{d/2}},$$

where $\mathfrak{d}_v$ is the local different (which equals 1 for almost all v).

Define the local v-adic measure $\tilde{\omega}_{\mathcal{K},v}$ on U_x via

$$\int_W \tilde{\omega}_{\mathcal{K},v} = \int_{\phi(W)} \|\mathsf{f}(\phi^{-1}(y))\|_v d\mu_v,$$

for every open $W \subset U_x$. This local measure glues to a measure $\tilde{\omega}_{\mathcal{K},v}$ on $X(F_v)$.

Let $\mathcal{X}$ be a model of X over the integers $\mathfrak{o}_F$ and let v be a place of good reduction. Let $\mathbb{F}_v = \mathfrak{o}_v/\mathfrak{m}_v$ be the corresponding finite field and put $q_v = \#\mathbb{F}_v$. Since X is projective, we have

$$X(F_v) = \mathcal{X}(\mathfrak{o}_v) \to \mathcal{X}(\mathbb{F}_v).$$

We have

$$\begin{aligned}\int_{X(F_v)} \tilde{\omega}_{\mathcal{K},v} &= \sum_{\bar{x}_v \in X(\mathbb{F}_v)} \int \tilde{\omega}_{\mathcal{K},v} \\ &= \frac{X(\mathbb{F}_v)}{q_v^d} \\ &= 1 + \frac{\mathrm{Tr}_v(\mathrm{H}^{2d-1}_{et}(X_{\bar{\mathbb{F}}_v}))}{\sqrt{q_v}} + \frac{\mathrm{Tr}_v(\mathrm{H}^{2d-2}_{et}(X_{\bar{\mathbb{F}}_v}))}{q_v} + \cdots + \frac{1}{q_v^d},\end{aligned}$$

where Tr_v is the trace of the v-Frobenius on the ℓ-adic cohomology of X. Trying to integrate the product measure over $X(\mathbb{A})$ is problematic, since the Euler product

$$\prod_v \frac{X(\mathbb{F}_v)}{q_v^d}$$

diverges. In all examples of interest to us, the cohomology group $\mathrm{H}^{2d-1}_{et}(X_{\bar{\mathbb{F}}_v}, \mathbb{Q}_\ell)$ vanishes. For instance, this holds if the anticanonical class is ample. Still the product diverges, since the $1/q_v$ term does not vanish, for projective X. There is a standard regularization procedure: Choose a finite set $S \subset \mathrm{Val}(F)$, including all $v \mid \infty$ and all places of bad reduction. Put

$$\lambda_v = \begin{cases} \mathsf{L}_v(1, \mathrm{Pic}(X_{\bar{\mathbb{Q}}})) & v \notin S \\ 1 & v \in S \end{cases},$$

where $\mathsf{L}_v(s, \mathrm{Pic}(X_{\bar{\mathbb{Q}}}))$ is the local factor of the Artin L-function associated to the Galois representation on the geometric Picard group. Define the regularized *Tamagawa measure*

$$\omega_{\mathcal{K},v} := \lambda_v^{-1} \tilde{\omega}_{\mathcal{K},v}.$$

Write $\omega_{\mathcal{K},S} := \prod_v \omega_{\mathcal{K},v}$ and define

$$\tau(-\mathcal{K}_X) := \mathsf{L}^*_S(1, \mathrm{Pic}(X_{\bar{\mathbb{Q}}})) \cdot \int_{\overline{X(F)}} \omega_{\mathcal{K},S}, \tag{4.11}$$

where

$$\mathsf{L}^*_S(1, \mathrm{Pic}(X_{\bar{\mathbb{Q}}})) := \lim_{s \to 1} (s-1)^r \mathsf{L}^*_S(s, \mathrm{Pic}(X_{\bar{\mathbb{Q}}}))$$

and r is the rank of $\mathrm{Pic}(X_F)$.

Example 4.12.1. Let G be a linear algebraic group over F. It carries an F-rational d-form ω, where $d = \dim(G)$. This form is unique, modulo multiplication by nonzero constants. Fixing ω, we obtain an isomorphism $K_X \simeq \mathcal{O}_G$, the structure sheaf, which carries a natural adelic metrization $(\|\cdot\|_{\mathbb{A}})$.

Let (A, Λ) be a pair consisting of a lattice and a strictly convex (closed) cone in $A_{\mathbb{R}}$: $\Lambda \cap -\Lambda = 0$. Let $(\check{A}, \check{\Lambda})$ the pair consisting of the dual lattice and the dual cone defined by

$$\check{\Lambda} := \{\check{\lambda} \in \check{A}_{\mathbb{R}} \mid \langle \lambda', \check{\lambda} \rangle \geq 0, \ \ \forall \lambda' \in \Lambda\}.$$

The lattice $\check{A}$ determines the normalization of the Lebesgue measure $d\check{a}$ on $\check{A}_{\mathbb{R}}$ (covolume =1). For $a \in A_{\mathbb{C}}$ define

$$\mathcal{X}_{\Lambda}(a) := \int_{\check{\Lambda}} e^{-\langle a, \check{a} \rangle} d\check{a}. \tag{4.12}$$

The integral converges absolutely and uniformly for $\Re(a)$ in compacts contained in the interior Λ° of Λ.

Definition 4.12.2. Assume that X is smooth, $\mathrm{NS}(X) = \mathrm{Pic}(X)$ and that $-K_X$ is in the interior of $\Lambda_{\mathrm{eff}}(X)$. We define

$$\alpha(X) := \mathcal{X}_{\Lambda_{\mathrm{eff}}(X)}(-K_X).$$

Remark 4.12.3. This constant measures the volume of the polytope obtained by intersecting the affine hyperplane $(-K_X, \cdot) = 1$ with the *dual* to the cone of effective divisors $\Lambda_{\mathrm{eff}}(X)$ in the dual to the Néron-Severi group. The explicit determination of $\alpha(X)$ can be a serious problem. For Del Pezzo surfaces, these volumes are given in Section 1.9. For example, let X be the moduli space $\bar{\mathcal{M}}_{0,6}$ (see Example **??**). The dual to the cone $\Lambda_{\mathrm{eff}}(X)$ has 3905 generators (in a 16-dimensional vector space), forming 25 orbits under the action of the symmetric group $\mathbb{S}_6$.

Conjecture 4.12.4 (Leading constant)**.** Let X be a Fano variety over a number field F with an adelically metrized anticanonical line bundle $-\mathcal{K}_X = (-K_X, \|\cdot\|_{\mathbb{A}})$. Assume that $X(F)$ is Zariski dense. Then there exists a Zariski open subset $X^{\circ} \subset X$ such that

$$\mathsf{N}(X^{\circ}(F), -\mathcal{K}_X, \mathsf{B}) \sim c(-\mathcal{K}_X)\mathsf{B}^1 \log(\mathsf{B})^{r-1}, \tag{4.13}$$

where $r = \mathrm{rk}\,\mathrm{Pic}(X_F)$ and

$$c(-\mathcal{K}_X) = \alpha(X)\beta(X)\tau(-\mathcal{K}_X), \tag{4.14}$$

with $\beta(X) := \mathrm{Br}(X)/\mathrm{Br}(F)$, and $\tau(-\mathcal{K}_X)$ the constant defined in equation 4.11.

4.13. **General polarizations.** I follow closely the exposition in [BT98]. Let E/F be some finite Galois extension such that all of the following constructions are defined over E. Let $(X, \mathcal{L})$ be a smooth *quasi-projective* d-dimensional variety together with a metrized very ample line bundle $\mathcal{L}$ which embeds X in some projective space $\mathbb{P}^n$. We denote by $\overline{X}^{\mathcal{L}}$ the normalization of the projective closure of $X \subset \mathbb{P}^n$. In general, $\overline{X}^{\mathcal{L}}$ is singular. We will introduce several notions relying on a resolution of singularities

$$\rho \,:\, X \to \overline{X}^{\mathcal{L}}.$$

Naturally, the defined objects will be independent of the choice of the resolution.

For $\Lambda \subset \mathrm{NS}(X)_{\mathbb{R}}$ we define

$$a(\Lambda, \mathcal{L}) := a(\Lambda, \rho^*\mathcal{L}).$$

We will always assume that $a(\Lambda_{\mathrm{eff}}(X), \mathcal{L}) > 0$.

Definition 4.13.1. A pair $(X, \mathcal{L})$ is called *primitive* if $a(\Lambda_{\mathrm{eff}}(X), \mathcal{L}) \in \mathbb{Q}_{>0}$ and if there exists a resolution of singularities

$$\rho \,:\, X \to \overline{X}^{\mathcal{L}}$$

such that for some $k \in \mathbb{N}$

$$((\rho^*\mathcal{L})^{\otimes a(\Lambda_{\mathrm{eff}}(X), \mathcal{L})} \otimes K_X)^{\otimes k} = \mathcal{O}(D),$$

where D is a *rigid* effective divisor ($\mathrm{h}^0(X, \mathcal{O}(\nu D)) = 1$ for all $\nu \gg 0$).

Example 4.13.2. of a primitive pair: $(X, -\mathcal{K}_X)$, where X is a smooth projective Fano variety and $-\mathcal{K}_X$ is a metrized anticanonical line bundle.

Let $k \in \mathbb{N}$ be such that $a(\Lambda, \mathcal{L})k \in \mathbb{N}$ and consider

$$R(\Lambda, \mathcal{L}) := \oplus_{\nu \geq 0} \mathrm{H}^0(X, (((\rho^*\mathcal{L})^{a(\Lambda, \mathcal{L})} \otimes K_X)^{\otimes k})^{\otimes \nu}).$$

In both cases ($\Lambda = \Lambda_{\mathrm{ample}}$ or $\Lambda = \Lambda_{\mathrm{eff}}$) it is expected that $R(\Lambda, \mathcal{L})$ is finitely generated and that we have a fibration

$$\pi = \pi_{\mathcal{L}} \,:\, X \to Y^{\mathcal{L}},$$

where $Y^{\mathcal{L}} = \mathrm{Proj}(R(\mathcal{L}, \Lambda))$. For $\Lambda = \Lambda_{\mathrm{eff}}(X)$ the generic fiber of π is (expected to be) a primitive variety in the sense of Definition 4.13.1. More precisely, there should be a diagram:

$$\begin{array}{cccc} \rho: & X & \to & \overline{X}^{\mathcal{L}} \supset X \\ & \downarrow & & \\ & Y^{\mathcal{L}} & & \end{array}$$

such that:

- $\dim(Y^{\mathcal{L}}) < \dim(X)$;

• there exists a Zariski open $U \subset Y^{\mathcal{L}}$ such that for all $y \in U(\mathbb{C})$ the pair $(X_y, \mathcal{L}_y)$ is primitive (here $X_y = \pi^{-1}(y) \cap X$ and $\mathcal{L}_y$ is the restriction of $\mathcal{L}$ to X_y);
• for all $y \in U(\mathbb{C})$ we have $a(\Lambda_{\mathrm{eff}}(X), \mathcal{L}) = a(\Lambda_{\mathrm{eff}}(X_y), \mathcal{L}_y)$;
• For all $k \in \mathbb{N}$ such that $a(\Lambda_{\mathrm{eff}}(X), \mathcal{L})k \in \mathbb{N}$ the vector bundle

$$\mathcal{L}_k := \mathrm{R}^0\pi_*(((\rho^*\mathcal{L})^{\otimes a(\Lambda_{\mathrm{eff}}(X),\mathcal{L})} \otimes K_X)^{\otimes k})$$

is in fact an ample line bundle on $Y^{\mathcal{L}}$.

Such a fibration will be called an *$\mathcal{L}$-primitive fibration.* A variety may admit several primitive fibrations.

Example 4.13.3. Let $X \subset \mathbb{P}_1^n \times \mathbb{P}_2^n$ $(n \geq 2)$ be a hypersurface given by a bi-homogeneous form of bi-degree (d_1, d_2). Both projections $X \to \mathbb{P}_1^n$ and $X \to \mathbb{P}_2^n$ are $\mathcal{L}$-primitive, for appropriate $\mathcal{L}$. In particular, for $n = 3$ and $(d_1, d_2) = (1, 3)$ there are *two* distinct $-\mathcal{K}_X$-primitive fibrations: one onto a point and another onto $\mathbb{P}_1^3$.

4.14. **Tamagawa numbers.** For smooth projective Fano varieties X with an adelically metrized anticanonical line bundle Peyre defined in [Pey95] a Tamagawa number, generalizing the classical construction for linear algebraic groups (see Section 4.12). We need to further generalize this to primitive pairs.

Abbreviate $a(\mathcal{L}) = a(\Lambda_{\mathrm{eff}}(X), \mathcal{L})$ and let $(X, \mathcal{L})$ be a primitive pair such that

$$\mathcal{O}(D) := ((\rho^*\mathcal{L})^{\otimes a(\mathcal{L})} \otimes K_X)^{\otimes k},$$

where k is such that $a(\mathcal{L})k \in \mathbb{N}$ and D is a rigid effective divisor as in Definition 4.13.1. Choose an F-rational section $g \in \mathrm{H}^0(X, \mathcal{O}(D))$; it is unique up to multiplication by F^*. Choose local analytic coordinates $x_{1,v}, \dots, x_{d,v}$ in a neighborhood U_x of $x \in X(F_v)$. In U_x the section g has a representation

$$g = f^{ka(\mathcal{L})}(dx_{1,v} \wedge \ldots \wedge dx_{d,v})^k,$$

where f is a local section of L. This defines a local v-adic measure in U_x by

$$\omega_{\mathcal{L},g,v} := \|f\|_{x_v}^{a(\mathcal{L})} dx_{1,v} \cdots dx_{d,v},$$

where $dx_{1,v} \cdots dx_{d,v}$ is the Haar measure on F_v^d normalized by $\mathrm{vol}(\mathfrak{o}_v^d) = 1$. A standard argument shows that $\omega_{\mathcal{L},g,v}$ glues to a v-adic measure on $X(F_v)$. The restriction of this measure to $X(F_v)$ does not depend on the choice of the resolution $\rho : X \to \overline{X}^{\mathcal{L}}$. Thus we have a measure on $X(F_v)$.

Denote by $(D_j)_{j\in\mathcal{J}}$ the irreducible components of the support of D and by

$$\mathrm{Pic}(X,\mathcal{L}) := \mathrm{Pic}(X \setminus \cup_{j\in\mathcal{J}} D_j).$$

The Galois group Γ acts on $\mathrm{Pic}(X,\mathcal{L})$. Let S be a finite set of places of bad reduction for the data (ρ, D_j, etc.), including the archimedean places. Put $\lambda_v = 1$ for $v \in S$, $\lambda_v = \mathsf{L}_v(1, \mathrm{Pic}(X,\mathcal{L}))$ (for $v \notin S$) and

$$\omega_{\mathcal{L}} := \mathsf{L}^*_S(1, \mathrm{Pic}(X,\mathcal{L}))|\mathrm{disc}(F)|^{-d/2} \prod_v \lambda_v^{-1} \omega_{\mathcal{L},g,v}.$$

(Here L_v is the local factor of the Artin L-function associated to the Γ-module $\mathrm{Pic}(X,\mathcal{L})$ and $\mathsf{L}^*_S(1, \mathrm{Pic}(X,\mathcal{L}))$ is the residue at $s = 1$ of the partial Artin L-function.) By the product formula, the measure does not depend on the choice of the F-rational section g. Define

$$\tau_{\mathcal{L}}(X) := \int_{\overline{X(F)}} \omega_{\mathcal{L}},$$

where $\overline{X(F)} \subset X(\mathbb{A})$ is the closure of $X(F)$ in the direct product topology. The convergence of the Euler product follows from

$$\mathrm{h}^1(X, \mathcal{O}_X) = \mathrm{h}^2(X, \mathcal{O}_X) = 0.$$

We have a map

$$\tilde{\rho} : \mathrm{Pic}(X)_{\mathbb{R}} \to \mathrm{Pic}(X,\mathcal{L})_{\mathbb{R}}$$

and we denote by

$$\Lambda_{\mathrm{eff}}(X,\mathcal{L}) := \tilde{\rho}(\Lambda_{\mathrm{eff}}(X)) \subset \mathrm{Pic}(X,\mathcal{L})_{\mathbb{R}}.$$

Definition 4.14.1. Let $(X,\mathcal{L})$ be a primitive pair as above. Define

$$c(X,\mathcal{L}) := \mathcal{X}_{\Lambda_{\mathrm{eff}}(X,\mathcal{L})}(\tilde{\rho}(-K_X)) \cdot |\mathrm{H}^1(\Gamma, \mathrm{Pic}(X,\mathcal{L}))| \cdot \tau_{\mathcal{L}}(X).$$

If $(X,\mathcal{L})$ is not primitive then some Zariski open subset $U \subset X$ admits a primitive fibration: there is a diagram

$$\begin{array}{ccc} X & \to & \overline{X}^{\mathcal{L}} \\ \downarrow & & \\ Y^{\mathcal{L}} & & \end{array}$$

such that for all $y \in Y^{\mathcal{L}}(F)$ the pair $(U_y, \mathcal{L}_y)$ is primitive. Then

$$c(U,\mathcal{L}) := \sum_{y\in Y^0} c(U_y, \mathcal{L}_y), \tag{4.15}$$

where the right side is a (possibly infinite, conjecturally converging (!)) sum over the subset $Y^0 \subset Y^{\mathcal{L}}(F)$ of all those fibers U_y where

$$a(\mathcal{L}) = a(\mathcal{L}_y) \text{ and } \operatorname{rk}\operatorname{Pic}(X, \mathcal{L})^{\mathbb{G}_a} = \operatorname{rk}\operatorname{Pic}(X_y, \mathcal{L}_y)^{\mathbb{G}_a}.$$

In Section 4.1 we will see that even if we start with pairs $(X, \mathcal{L})$ where X is a smooth projective variety and $\mathcal{L}$ is a very ample adelically metrized line bundle on X we still need to consider singular varieties.

4.15. **Tamagawa number as a height.** Why does the right side of Formula (4.15) converge? The natural idea is to interpret it as a height zeta function, i.e., to think of the Tamagawa numbers of the fibers of an $\mathcal{L}$-primitive fibration as "heights". One problem with this guess is that the "functorial" properties of these notions under field extensions are quite different: Let U_y be a fiber defined over the ground field. The local and global heights of U_y don't change under extensions. The local Tamagawa factors, however, take into account information about $\mathbb{F}_q$-points of U_y, i.e., the density

$$\tau_v = \#U_y(\mathbb{F}_{q_v})/q_v^{\dim(U_y)},$$

for almost all v, which may vary nontrivially.

In absence of conclusive arguments, let us look at examples. For $\mathbf{a} \in \mathbb{P}^3(\mathbb{Q})$, let $X_{\mathbf{a}} \subset \mathbb{P}^3$ be the diagonal cubic surface fibration

$$a_0x_0^3 + a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = 0, \tag{4.16}$$

considered in Section 4.11. Let $\mathsf{H} : \mathbb{P}^3(\mathbb{Q}) \to \mathbb{R}_{>0}$ be the standard height as in Section 4.1.

Theorem 4.15.1. [EJ08b] *For all $\epsilon > 0$ there exists a constant $\mathsf{c} = \mathsf{c}(\epsilon)$ such that*

$$\frac{1}{\tau(X_{\mathbf{a}})} \geq \mathsf{c}\,\mathsf{H}\left(\frac{1}{a_0} : \ldots : \frac{1}{a_3}\right)^{1/3-\epsilon}$$

In particular, we have the following fundamental finiteness property: for $\mathsf{B} > 0$ there are finitely many $\mathbf{a} \in \mathbb{P}^3(\mathbb{Q})$ such that $\tau(S_{\mathbf{a}}) > \mathsf{B}$.

A similar result holds for 3 dimensional quartics.

Theorem 4.15.2. [EJ07] *Let $X_{\mathbf{a}}$ be the family of quartic threefolds*

$$a_0x_0^4 + a_1x_1^4 + a_2x_2^4 + a_3x_3^4 + a_4x_4^4 = 0,$$

with $a_0 < 0$ *and* $a_1, \dots, a_4 > 0$, $a_i \in \mathbb{Z}$. *For all* $\epsilon > 0$ *there exists a constant* $\mathsf{c} = \mathsf{c}(\epsilon)$ *such that*

$$\frac{1}{\tau(X_{\mathbf{a}})} \geq \mathsf{c}\,\mathsf{H}\left(\frac{1}{a_0} : \dots : \frac{1}{a_4}\right)^{1/4-\epsilon}.$$

4.16. **Smallest points.** Let X be a smooth Fano variety over a number field F. What is the smallest height $\mathsf{H}_{\min}(X(F))$ of an F-rational points on X? For a general discussion of bounds of diophantine equations in terms of the *height* of the equation, see [Mas02]. A sample result in this direction is [Pit71], [NP89]: Let

$$\sum_{i=0}^{n} a_i x_i^d = 0, \tag{4.17}$$

with d odd and let $\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ be a vector with nonzero coordinates. For $n \gg d$ (e.g., $n = 2^d + 1$) and any $\epsilon > 0$ there exists a constant c such that (4.17) has a solution $\mathbf{x}$ with

$$\sum_{i=0}^{n} |a_i x_i^d| < \mathsf{c} \prod |a_i|^{d+\epsilon}.$$

For $d \geq 12$, one can work with $n \sim 4d^2 \log(d)$. There have been a several improvements of this result for specific values of d, e.g. [Cas55], [Die03] for quadrics and [Bak89], [Brü94] for $d = 3$.

In our setup, the expectation

$$\mathsf{N}(X^\circ(F), -\mathcal{K}_X, \mathsf{B}) \sim \alpha\beta\tau(-\mathcal{K}_X)\mathsf{B}^1 \log(\mathsf{B})^{r-1}$$

where $r = \operatorname{rk}\operatorname{Pic}(X)$, and the hope that the points are equidistributed with respect to the height lead to the guess that $\mathsf{H}_{\min}(X(F))$ is inversely related to $\tau(-\mathcal{K}_X)$, rather than the height of the defining equations. The figure shows the distribution of smallest points in comparison with the Tamagawa number on a sample of smooth quartic threefolds of the form

$$ax^4 = by^4 + z^4 + u^4 + w^4, \quad a, b = 1, \dots, 1000.$$

On the other hand, there is the following result:

Theorem 4.16.1. [EJ07], [EJ08b] *Let* $X_a \subset \mathbb{P}^4$ *be the quartic threefold given by*

$$ax^4 = x_1^4 + x_2^4 + x_3^4 + x_4^4, \quad a \in \mathbb{N}.$$

Then there exist no constant c *such that*

$$\mathsf{H}_{\min}(X(\mathbb{Q})) < \frac{\mathsf{c}}{\tau(-\mathcal{K}_{X_a})}.$$

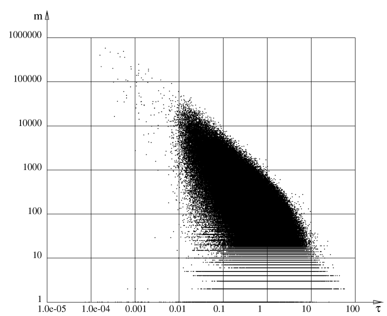

FIGURE 2. Smallest point versus the Tamagawa number

Let $X_a \subset \mathbb{P}^3$ be the cubic surface given by

$$ax_0^3 + 4x_1^3 + 2x_2^3 + x_3^3 = 0, \quad a \in \mathbb{N}.$$

Assume the Generalized Riemann Hypothesis. Then there does not exist a constant $\mathsf{c} > 0$ such that

$$\mathsf{H}(X_a(\mathbb{Q})) \leq \frac{\mathsf{c}}{\tau(-\mathcal{K}_{X_a})},$$

for all $a \in \mathbb{Z}$.

It may still be the case that

$$\mathsf{H}_{\min}(X(F)) \sim \frac{\mathsf{c}(\epsilon)}{\tau(-\mathcal{K}_X)^{1+\epsilon}}.$$

5. COUNTING POINTS VIA UNIVERSAL TORSORS

5.1. **The formalism.** We explain the basic elements of the point counting technique on universal torsors developed in [Pey04], [Sal98]. The prototype is the projective space:

$$\mathbb{A}^{n+1} \setminus 0 \xrightarrow{\mathbb{G}_m} \mathbb{P}^n.$$

The bound $\mathsf{H}(\mathbf{x}) \leq \mathsf{B}$ translates to a bound on $\mathbb{A}^{n+1}(\mathbb{Z})$, it remains to replace the lattice point count on $\mathbb{A}^{n+1}(\mathbb{Z})$ by the volume of the domain. The coprimality on the coordinates leads to the product of local densities formula

$$\mathsf{N}(\mathsf{B}) \sim \frac{1}{\zeta(n+1)} \cdot \tau_\infty \cdot \mathsf{B}^{n+1}, \quad \mathsf{B} \to \infty,$$

where τ_∞ is the volume of the unit ball with respect to the norm at infinity.

The lift of points in $\mathbb{P}^n(\mathbb{Q})$ to primitive integral vectors in $\mathbb{Z}^{n+1} \setminus 0$, modulo ± 1 admits a generalization to the context of torsors

$$\mathcal{T}_X \xrightarrow{T_{\mathrm{NS}}} X.$$

Points in $X(\mathbb{Q})$ can be lifted to certain integral points on $\mathcal{T}_X$, uniquely, modulo the action of $T_{\mathrm{NS}}(\mathbb{Z})$ (the analog of the action by ± 1). The height bound on $X(\mathbb{Q})$ lifts to a bound on $\mathcal{T}_X(\mathbb{Z})$. The issue then is to prove, for $\mathsf{B} \to \infty$, that

$$\#\text{ lattice points } \sim \text{volume of the domain }.$$

The setup for the generalization is as follows. Let X be a smooth projective variety over a number field F. We assume that

- $\mathrm{H}^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$;
- $\mathrm{Pic}(X_{\bar{F}}) = \mathrm{NS}(X_{\bar{F}})$ is torsion free;
- $\Lambda_{\mathrm{eff}}(X)$ is a finitely generated rational cone;
- $-K_X$ is in the interior of $\Lambda_{\mathrm{eff}}(X)$;
- $X(F)$ is Zariski dense;
- there is a Zariski open subset without strongly or weakly accumulating subvarieties;
- all universal torsors over X satisfy the Hasse principle and weak approximation.

For simplicity of exposition we will first ignore the Galois actions and assume that $\mathrm{NS}(X_F) = \mathrm{NS}(X_{\bar{F}})$. Fix a line bundle L on X and consider the map

$$\begin{array}{ccc} \mathbb{Z} & \to & \mathrm{NS}(X) \\ 1 & \mapsto & [L] \end{array}$$

By duality, we get a homomorphism $\phi_L : T_{\mathrm{NS}} \to \mathbb{G}_m$ and the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\psi_L} & L^* \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

compatible with the $T_{\rm NS}$-action (where $L^* = L \setminus 0$). Fix a point $t_0 \in \mathcal{T}(F)$ and an adelic metrization $\mathcal{L} = (L, \|\cdot\|_v)$ of L. For each v, we get a map

$$\begin{array}{rcl} \mathsf{H}_{\mathcal{L},\mathcal{T},v} : \mathcal{T}(F_v) & \longrightarrow & \mathbb{R}_{>0} \\ t_v & \mapsto & \|\psi_L(t)\|_v / \|\psi_L(t_0)\|_v \end{array}$$

Fix an adelic height system $\mathsf{H} = \prod_v \mathsf{H}_v$ on X as in Section 4.8, i.e., a basis $L_1, \dots, L_r$ of $\mathrm{Pic}(X)$ and adelic metrizations of these line bundles. This determines compatible adelic metrizations on all $L \in \mathrm{Pic}(X)$. Define

$$\mathcal{T}_{\mathsf{H}_v}(\mathfrak{o}_v) := \{t \in \mathcal{T}(F_v) \,|, \mathsf{H}_{\mathcal{L},\mathcal{T},v}(t) \le 1 \quad \forall L \in \Lambda_{\rm eff}(X)\}.$$

Let

$$\mathcal{T}_{\mathsf{H}}(\mathbb{A}) := \prod_v \mathcal{T}(F_v)$$

be the *restricted product* with respect to the collection

$$\{\mathcal{T}_{\mathsf{H}_v}(\mathfrak{o}_v)\}_v.$$

This space does not depend on the choice of the points t_0 or on the choice of adelic metrizations.

The next step is the definition of local *Tamagawa measures* on $\mathcal{T}(F_v)$, whose product becomes a global Tamagawa measure on $\mathcal{T}_{\mathsf{H}}(\mathbb{A})$. The main insight is that

- *locally*, in v-adic topology, $\mathcal{T}(F_v) = X(F_v) \times T_{\rm NS}(F_v)$;
- both factors carry a local Tamagawa measure (defined by the metrizations of the corresponding canonical line bundles);
- the regularizing factor (needed to globalize the measure to the adeles, see Equation 4.11) on $X(F_v)$ is $\lambda_v = \mathsf{L}_v(1, \mathrm{Pic}(X_{\bar{F}})$, for almost all v, and the regularizing factor on $T_{\rm NS}(F_v)$ is λ_v^{-1};
- the regularizing factors cancel and the product measure is integrable over the adelic space $\mathcal{T}_{\mathsf{H}}(\mathbb{A}_F)$.

One chooses a fundamental domain for the action of units $T_{\rm NS}(\mathfrak{o})/W$ (where W is the group of torsion elements), establishes a bijection between the set of rational points $X(F)$ and certain integral points on $\mathcal{T}$ (integral with respect to the unstable locus for the action of $T_{\rm NS}$)

in this domain and compares a lattice point count, over these integral points, with the adelic integral, over the space $\mathcal{T}_{\mathsf{H}}(\mathbb{A})$. *If* the difference between these counts goes into the error term, then Conjecture 4.12.4 holds.

5.2. **Toric Del Pezzo surfaces.** Notation and terminology regarding toric varieties are explained in Section 6.5.

Example 5.2.1. Let $X = \mathrm{Bl}_Y(\mathbb{P}^2)$ be the blowup of the projective plane in the subscheme

$$Y := (1:0:0) \cup (0:1:0) \cup (0:0:1),$$

a toric Del Pezzo surface of degree 6. We can realize it as a subvariety $X \subset \mathbb{P}^1_{\mathbf{x}} \times \mathbb{P}^1_{\mathbf{y}} \times \mathbb{P}^1_{\mathbf{z}}$ given by $x_0y_0z_0 = x_1y_1z_1$. The anticanonical height is given by

$$\max(|x_0|, |x_1|) \times \max(|y_0|, |y_1|) \times \max(|z_0|, |z_1|).$$

There are six exceptional curves: the preimages of the 3 points and the preimages of lines joining two of these points.

Example 5.2.2 (Degree four). There are 3 toric Del Pezzo surfaces of degree 4, given by $X = \{Q_0 = 0\} \cap \{Q = 0\} \subset \mathbb{P}^4$, with $Q_0 = x_0x_1 + x_2^2$ and Q as in the table below.

Singularities	Q
$4\mathsf{A}_1$	$x_3x_4 + x_2^2$
$2\mathsf{A}_1 + \mathsf{A}_2$	$x_1x_2 + x_3x_4$
$2\mathsf{A}_1 + \mathsf{A}_3$	$x_0^2 + x_3x_4$

Example 5.2.3 (Degree three). The unique toric cubic surface is given by

$$xyz = w^3.$$

The corresponding *fan* is spanned in $\mathbb{Z}^2$ by $(1,1), (1,-2), (-2,1)$. Let X° be the complement to lines, i.e., the locus with $w \neq 0$. The asymptotic

$$\mathsf{N}(X^\circ(F), \mathsf{B}) = c\,\mathsf{B}^1 \log(\mathsf{B})^6(1 + o(1)), \quad \mathsf{B} \to \infty,$$

has been established in [BT98] using harmonic analysis (see Section 6.5) and in [?], [?], [?] using the torsor approach.

Example 5.2.4. The toric quartic surface

$$x^2yz = w^4$$

is given by the fan $(2,-1),(0,1),(-2,-1)$. Let X° be the complement to $w=0$. By Theorem **??**, we have

$$\mathsf{N}(X^\circ(F),\mathsf{B}) = c\mathsf{B}^1\log(\mathsf{B})^5(1+o(1)), \quad \mathsf{B}\to\infty.$$

This is more than suggested by the naive heuristic in Section 2.2.

The torsor approach has been successfully implemented for toric varieties over $\mathbb{Q}$ in [**?**] and [**?**].

5.3. **Torsors over Del Pezzo surfaces.**

Example 5.3.1. A quartic Del Pezzo surface X with two singularities of type A_1 can be realized as a blow-up of the following points

$$\begin{array}{rcl} p_1 &=& (0:0:1)\\ p_2 &=& (1:0:0)\\ p_3 &=& (0:1:0)\\ p_4 &=& (1:0:1)\\ p_5 &=& (0:1:1) \end{array}$$

in $\mathbb{P}^2=(x_0:x_1:x_2)$. The anticanonical line bundle embedds X into $\mathbb{P}^4$:

$$(x_0^2x_1 : x_0x_1^2 : x_0x_1x_2 : x_0x_2(x_0+x_1-x_2) : x_1x_2(x_0+x_1-x_2)).$$

The Picard group is spanned by

$$\mathrm{Pic}(X) = \langle L, E_1,\cdots,E_5\rangle$$

and $\Lambda_{\mathrm{eff}}(X)$ by

$$\begin{array}{c} E_1,\cdots,E_5\\ L-E_2-E_3, L-E_3-E_4, L-E_4-E_5, L-E_2-E_5\\ L-E_1-E_3-E_5, L-E_1-E2-E_4. \end{array}$$

The universal torsor is given by the following equations

$$\begin{array}{c} (23)(3)-(1)(124)(4)+(25)(5)=0\\ (23)(2)-(1)(135)(5)+(34)(4)=0\\ (124)(1)(2)-(34)(3)+(45)(5)=0\\ (25)(2)-(135)(1)(3)+(45)(4)=0\\ (23)(45)+(34)(25)-(1)^2(124)(135)=0. \end{array}$$

(with variables labeled by the corresponding exceptional curves). Introducing additional variables

$$(24)' := (1)(124), \ \ (35)' := (1)(135)$$

we see that the above equations define a $\mathbb{P}^1$-bundle over a codimension one subvariety of the Grassmannian $\mathrm{Gr}(2,5)$.

We need to estimate the number of 11-tuples of nonzero integers, satisfying the equations above and subject to the inequalities:

$$\begin{gathered} |(135)(124)(23)(1)(2)(3)| \leq \mathsf{B} \\ |(135)(124)(34)(1)(3)(4)| \leq \mathsf{B} \\ \cdots \end{gathered}$$

By symmetry, we can assume that $|(2)| \geq |(4)|$ and write $(2) = (2)'(4) + r_2$. Now we weaken the first inequality to

$$|(135)(124)(23)(1)(4)(2)'(3)| \leq \mathsf{B}.$$

There are $O(\mathsf{B}\log(\mathsf{B})^6)$ 7-tuples of integers satisfying this inequality.

Step 1. Use equation $(23)(3) - (1)(124)(4) + (25)(5)$ to reconstruct $(25), (5)$ with ambiguity $O(\log(\mathsf{B}))$.

Step 2. Use $(25)(2) - (135)(1)(3) + (45)(4) = 0$ to reconstruct the residue r_2 modulo (4). Notice that (25) and (4) are "almost" coprime since the corresponding exceptional curves are disjoint.

Step 3. Reconstruct (2) and (45).

Step 4. Use $(23)(2) - (1)(135)(5) + (34)(4)$ to reconstruct (34).

In conclusion, if $X^\circ \subset X$ is the complement to exceptional curves then

$$\mathsf{N}(X^\circ, -K_X, \mathsf{B}) = O(\mathsf{B}\log(\mathsf{B})^7).$$

We expect that

$$\mathsf{N}(X^\circ, -K_X, \mathsf{B}) = \mathsf{B}\log(\mathsf{B})^5(1 + o(1))$$

as $B \to \infty$, where is the constant defined in Chapter **??**.

Example 5.3.2. The universal torsor of a smooth quartic Del Pezzo surface, given as a blow up of the five points

$$\begin{aligned} p_1 &= (1:0:0) \\ p_2 &= (0:1:0) \\ p_3 &= (0:0:1) \\ p_4 &= (1:1:1) \\ p_5 &= (1:a_2:a_3), \end{aligned}$$

assumed to be in general position, is given by the vanishing of

$(14)(23)$	$+$	$(12)(34)$	$-$	$(13)(24)$	$(00)(05) - (12)(34) + (13)(24) - (14)(23)$
$(00)(05)$	$+$	$a_3(a_2-1)(12)(34)$	$-$	$a_2(a_3-1)(13)(24)$	
$(23)(03)$	$+$	$(24)(04)$	$-$	$(12)(01)$	$(12)(01) - (23)(03) + (24)(04) - (25)(05)$
$a_2(23)(03)$	$+$	$(25)(05)$	$-$	$(12)(01)$	
$(12)(35)$	$-$	$(13)(25)$	$+$	$(15)(23)$	$(00)(04) - (12)(35) + (13)(25) - (15)(23)$
$(a_2-1)(12)(35)$	$+$	$(00)(04)$	$-$	$(a_3-1)(13)(25)$	
$(12)(45)$	$+$	$(14)(25)$	$-$	$(15)(24)$	$(00)(03) - (12)(45) + (14)(25) - (15)(24)$
$(00)(03)$	$+$	$a_3(14)(25)$	$-$	$(15)(24)$	
$(13)(45)$	$+$	$(14)(35)$	$-$	$(15)(34)$	$(00)(02) - (13)(45) + (14)(35) - (15)(34)$
$(00)(02)$	$+$	$a_2(14)(35)$	$-$	$(15)(34)$	
$(23)(45)$	$+$	$(24)(35)$	$-$	$(25)(34)$	$(00)(01) - (23)(45) + (24)(35) - 25)(34)$
$(00)(01)$	$+$	$a_2(24)(35)$	$-$	$a_3(25)(34)$	
$(04)(34)$	$+$	$(02)(23)$	$-$	$(01)(13)$	$(13)(01) - (23)(02) + (34)(04) + 35)(05)$
$(05)(35)$	$+$	$a_3(02)(23)$	$-$	$(01)(13)$	
$(a_2-1)(03)(34)$	$+$	$(05)(45)$	$-$	$(a_3-1)(02)(24)$	$(14)(01) - (24)(02) + (34)(03) - (45)(05)$
$(03)(34)$	$+$	$(01)(14)$	$-$	$(02)(24)$	
$(04)(14)$	$+$	$(03)(13)$	$-$	$(02)(12)$	$(12)(02) - (13)(03) + (14)(04) - (15)(05)$
$(05)(15)$	$+$	$a_2(03)(13)$	$-$	$a_3(02)(12)$	
$a_3(02)(25)$	$-$	$a_2(03)(35)$	$-$	$(01)(15)$	$(15)(01) - (25)(02) + (35)(03) - (45)(04)$
$(a_3-1)(02)(25)$	$-$	$(a_2-1)(03)(35)$	$-$	$(04)(45)$	

Connection to the D_5-Grassmannian

Example 5.3.3. The Cayley cubic is the unique cubic hypersurface in $X \subset \mathbb{P}^3$ with 4 double points (A_1-singularities), the maximal number of double points on a cubic surface. It can be given by the equation

$$y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3 = 0.$$

The double points correspond to

$$(1:0:0:0), (0:1:0:0), (0:0:1:0), (0:0:0:1).$$

It can be realized as the blow-up of $\mathbb{P}^2 = (x_1 : x_2 : x_3)$ in the points

$$q_1 = (1:0:0), q_2 = (0:1:0), q_3 = (0:0:1), q_4 = (1:-1:0), q_5 = (1:0:-1), q_6 = (0:1:-1)$$

The points lie on a rigid configuration of 7 lines

$x_1 = 0$	$(12)(13)(14)(1)$
$x_2 = 0$	$(12)(23)(24)(2)$
$x_3 = 0$	$(13)(23)(34)(3)$
$x_4 = x_1 + x_2 + x_3$	$(14)(24)(34)(4)$
$x_1 + x_3 = 0$	$(13)(24)(13, 24)$
$x_2 + x_3 = 0$	$(23)(14)(14, 23)$
$x_1 + x_2 = 0$	$(12)(34)(12, 34).$

The proper transform of the line x_j is the (-2)-curve corresponding to (j). The curves corresponding to (ij), (ij,kl) are (-1)-curves. The accumulating subvarieties are exceptional curves. The (anticanonical) embeddding $X \hookrightarrow \mathbb{P}^3$ is given by the linear system:

$$\begin{array}{rcl} s_1 & = & x_1x_2x_3 \\ s_2 & = & x_2x_3x_4 \\ s_3 & = & x_1x_3x_4 \\ s_4 & = & x_1x_2x_4 \end{array}$$

The counting problem is: estimate

$$\mathsf{N}(\mathsf{B}) = \#\{(x_1,x_2,x_3) \in \mathbb{Z}^3_{\mathrm{prim}}/\pm, |\ \max i(|s_i|)/\gcd(s_i) \le \mathsf{B}\},$$

where the triple x_j is subject to the conditions

$$x_i \neq 0,\ (i = 1,\dots,4)\ \ x_j + x_i \neq 0 (1 \le i < j \le 3).$$

We expect $\sim \mathsf{B}\log(\mathsf{B})^6$ solutions. After dividing by the coordinates by their gcd, we obtain

$$\begin{array}{rcl} s'_1 & = & (1)(2)(3)(12)(13)(23) \\ s'_2 & = & (2)(3)(4)(23)(24)(34) \\ s'_3 & = & (1)(3)(4)(13)(14)(34) \\ s'_4 & = & (1)(2)(4)(12)(14)(24) \end{array}$$

(These are special sections in the anticanonical series, other decomposable sections are: $(1)(2)(12)^2(12,34)$ and $(12,34)(13,24)(14,23)$, for example.) The conic bundles on X produce the following equations for the universal torsor:

$$\begin{array}{crcrcl} \mathrm{I} & (1)(13)(14) & + & (2)(23)(24) & = & (34)(12,34) \\ \mathrm{II} & (1)(12)(14) & + & (3)(23)(34) & = & (24)(13,24) \\ \mathrm{III} & (2)(12)(24) & + & (3)(13)(34) & = & (14)(14,23) \\ \mathrm{IV} & -(3)(13)(23) & + & (4)(14)(24) & = & (12)(12,34) \\ \mathrm{V} & -(2)(12)(23) & + & (4)(14)(34) & = & (13)(13,24) \\ \mathrm{VI} & -(1)(12)(1) & + & (4)(24)(34) & = & (23)(14,23) \\ \mathrm{VII} & (2)(4)(24)^2 & + & (1)(3)(13)^2 & = & (12,34)(14,23) \\ \mathrm{VIII} & -(1)(2)(12)^2 & + & (3)(4)(34)^2 & = & (13,24)(14,23) \\ \mathrm{IX} & (1)(4)(14)^2 & - & (2)(3)(23)^2 & = & (12,34)(13,24) \end{array}$$

The counting problem is to estimate the number of 13-tuples of *nonzero* integers, satisfying the equations above and subject to the inequality $\max_i\{|s'_i|\} \le \mathsf{B}$. Heath-Brown proved in [HB03] that there exists constants $0 < \mathsf{c} < \mathsf{c}'$ such that

$$\mathsf{c}\mathsf{B}\log(\mathsf{B})^6 \le \mathsf{N}(\mathsf{B}) \le \mathsf{c}'\mathsf{B}\log(\mathsf{B})^6.$$

Example 5.3.4 (The $2\mathsf{A}_2 + \mathsf{A}_3$ cubic surface). The equation

$$x_0x_1x_2 = x_3^2(x_1 + x_2)$$

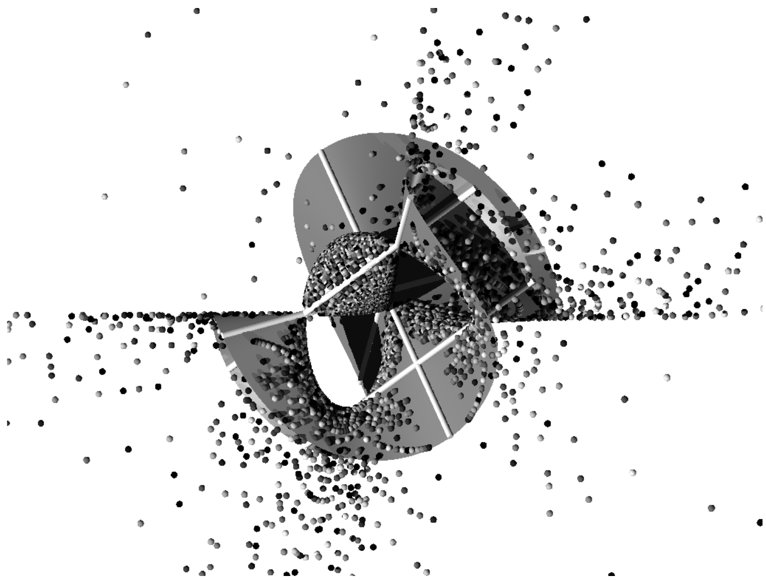

FIGURE 3. The 5332 rational points of height ≤ 100 on
$$x_0x_1x_2 = x_3^2(x_1 + x_2)$$

defines a cubic surface X with singularities of indicated type. It contains 5 lines. The Cox ring has the following presentation

$$\mathrm{Cox}(X) = F[\eta_1, \ldots, \eta_9]/(\eta_4\eta_6^2\eta_{10} + \eta_1\eta_2\eta_7^2 + \eta_8\eta_9)$$

The figure shows some rational points on this surface. [2] The expected asymptotic

$$\mathsf{N}(X^\circ(\mathbb{Q}), \mathsf{B}) \sim \mathsf{B}^1 \log(\mathsf{B})^6$$

on the complement of the 5 lines has not been proved, yet.

5.4. **Torsors over the Segre cubic threefold.** In this section we work over $\mathbb{Q}$. The threefold $X = \bar{\mathcal{M}}_{0,6}$ can be realized as the blow-up of $\mathbb{P}^3$ in the points

	x_0	x_1	x_2	x_3
q_1	1	0	0	0
q_2	0	1	0	0
q_3	0	0	1	0
q_4	0	0	0	1
q_5	1	1	1	1

[2] I am grateful to U. Derenthal for allowing me include it here.

and in the proper transforms of lines joining two of these points. The Segre cubic is given as the image of X in $\mathbb{P}^4$ under the linear system $2L - (E_1 + \cdots E_5)$ (quadrics passing through the 5 points):

$$\begin{aligned} s_1 &= (x_2 - x_3)x_1 \\ s_2 &= x_3(x_0 - x_1) \\ s_3 &= x_0(x_1 - x_3) \\ s_4 &= (x_0 - x_1)x_2 \\ s_5 &= (x_1 - x_2)(x_0 - x_3) \end{aligned}$$

It can be realized in $\mathbb{P}^5 = (y_0 : \ldots : y_5)$ as

$$\mathcal{S}_3 := \{\sum_{i=0}^{5} y_i^3 = \sum_{i=0}^{5} y_i = 0\}$$

(exhibiting the $\mathfrak{S}_6$-symmetry.) It contains 15 planes, given by the $\mathfrak{S}_6$-orbit of

$$y_0 + y_3 = y_1 + y_4 = y_3 + y_5 = 0,$$

and 10 singular double points, given by the $\mathfrak{S}_6$-orbit of

$$(1 : 1 : 1 : -1 : -1 : -1).$$

This is the maximal number of nodes on a cubic threefold and $\mathcal{S}_3$ is the unique cubic with this property. The hyperplane sections $\mathcal{S}_3 \cap \{y_i = 0\}$ are *Clebsch* diagonal cubic surfaces (unique cubic surfaces with $\mathfrak{S}_5$ as symmetry group. The hyperplane sections $\mathcal{S}_3 \cap \{y_i = 0\}$ are *Clebsch* cubics, a unique cubic surface with $\mathfrak{S}_5$-symmetry. The hyperplane sections $\mathcal{S}_3 \cap \{y_i - y_j = 0\}$ are *Cayley* cubic surfaces (see Example 5.3.3). The geometry and symmetry of these and similar varieties are described in detail in [Hun96].

The counting problem on $\mathcal{S}_3$ is: find the number $\mathsf{N}(\mathsf{B})$ of all 4-tupels of $(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4/\pm$ such that

- $\gcd(x_0, x_1, x_2, x_3) = 1$;
- $\max_{j=1,\ldots,5}(|s_j|)/|\gcd(s_1, \ldots, s_5) \leq \mathsf{B}$;
- $x_i \neq 0$ and $x_i - x_j \neq 0$ for all $i, j \neq i$.

The last condition is excluding rational points contained in accumulating subvarieties (there are B^3 rational points on planes $\mathbb{P}^2 \subset \mathbb{P}^4$, with respect to the $\mathcal{O}(1)$-height). The second condition is the bound on the *height*.

First we need to determine

$$a(L) = \inf\{a \,|\, aL + K_X \in \Lambda_{\rm eff}(X)\},$$

where L is the line bundle giving the map to $\mathbb{P}^4$. We claim that $a(L) = 2$. This follows from the fact that

$$\sum_{i,j} (ij)$$

is on the boundary of $\Lambda_{\text{eff}}(X)$ (where (ij) is the class in $\text{Pic}(X)$ of the preimage in X of the line $l_{ij} \subset \mathbb{P}^4$ through q_i, q_j.

Therefore, we expect

$$\mathsf{N}(\mathsf{B}) = O(\mathsf{B}^{2+\epsilon})$$

as $\mathsf{B} \to \infty$. In fact, it was shown in [BT98] that $b(L) = 6$. Consequently, one expects

$$\mathsf{N}(\mathsf{B}) \sim \mathsf{c}\mathsf{B}^2 \log(\mathsf{B})^5, \quad \text{as } \mathsf{B} \to \infty.$$

Remark 5.4.1. The difficult part is to keep track of $\gcd(s_1, \dots, s_5)$. Indeed, if we knew that this gcd $= 1$ we could easily prove the bound $O(\mathsf{B}^{2+\epsilon})$ by observing that there are $O(\mathsf{B}^{1+\epsilon})$ pairs of (positive) integers $(x_2 - x_3, x_1)$ (resp. $(x_0 - x_1, x_2)$) satisfying $(x_2 - x_3)x_1 \le \mathsf{B}$ (resp. $(x_0 - x_1)x_2 \le \mathsf{B}$). Then we could reconstruct the quadruple

$$(x_2 - x_3, x_1, x_0 - x_1, x_2)$$

and consequently

$$(x_0, x_1, x_2, x_3)$$

up to $O(\mathsf{B}^{2+\epsilon})$.

Thus it is necessary to introduce gcd between x_j, etc. Again, we use the symbols $(i), (ij), (ijk)$ for variables on the torsor for X corresponding to the classes of the preimages of points, lines, planes resp. Once we fix a point $(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$ (such that $\gcd(x_0, x_1, x_2, x_3) = 1$), the values of these coordinates over the corresponding point on X can be expressed as greatest common divisors. For example, we can write

$$x_3 = (123)(12)(13)(23)(1)(2)(3),$$

a product of integers (neglecting the sign of x_3; in the torsor language, we are looking at the orbit of $\mathsf{T}_{\text{NS}}(\mathbb{Z})$). Here is a self-explanatory list:

(123)	x_3	(12)	x_2, x_3
(124)	x_2	(13)	x_1, x_3
(125)	$x_2 - x_3$	(14)	x_1, x_2
(134)	x_1	(15)	$x_1 - x_3, x_1 - x_2$
(135)	$x_1 - x_3$	(23)	$x_3, x_0 - x_3$
(145)	$x_1 - x_2$	(24)	x_2, x_0
(234)	x_0	(25)	$x_3 - x_2, x_0 - x_3$
(235)	$x_0 - x_3$	(34)	x_1, x_0
(245)	$x_0 - x_2$	(35)	$x_1 - x_3, x_0 - x_1$
(345)	$x_0 - x_1$	(45)	$x_1 - x_2, x_0 - x_1$.

After dividing s_j by the gcd, we get

$$\begin{array}{rcl} s'_1 & = & (125)(134)(12)(15)(25)(13)(14)(34)(1) \\ s'_2 & = & (123)(245)(12)(13)(23)(24)(25)(45)(2) \\ s'_3 & = & (234)(135)(23)(24)(34)(13)(15)(35)(3) \\ s'_4 & = & (345)(124)(34)(35)(45)(12)(14)(24)(4) \\ s'_5 & = & (145)(235)(14)(15)(45)(23)(35)(25)(5) \end{array}$$

(observe the symmetry with respect to the permutation (12345)). We claim that $\gcd(s'_1, \ldots, s'_5) = 1$. One can check this directly using the definitions of $(i), (ij), (ijk)$'s as gcd's. For example, let us check that nontrivial divisors $d \neq 1$ of (1) cannot divide any other s'_j. Such a d must divide (123) or (12) or (13) (see s'_2). Assume it divides (12). Then it doesn't divide $(13), (14)$ and (15) (the corresponding divisors are disjoint). Therefore, d divides (135) (by s'_3) and (235) (by s'_5). Contradiction (indeed, (135) and (235) correspond to disjoint divisors). Assume that d divides (123). Then it has to divide either (13) or (15) (from s'_3) and either (12) or (14) (from s'_4). Contradiction.

The integers $(i), (ij), (ijk)$ satisfy a system of relations (these are equations for the torsor induced from fibrations of $\bar{\mathcal{M}}_{0,6}$ over $\mathbb{P}^1$):

I	x_0	x_1	$x_0 - x_1$
II	x_0	x_2	$x_0 - x_2$
III	x_0	x_3	$x_0 - x_3$
IV	x_1	x_2	$x_1 - x_2$
V	x_1	x_3	$x_1 - x_3$
VI	x_2	x_3	$x_2 - x_3$
VII	$x_0 - x_1$	$x_0 - x_2$	$x_1 - x_2$
VIII	$x_0 - x_1$	$x_0 - x_3$	$x_1 - x_3$
IX	$x_1 - x_2$	$x_1 - x_3$	$x_2 - x_3$
X	$x_2 - x_3$	$x_0 - x_3$	$x_0 - x_2$

which translates to

I	(234)(23)(24)(2) − (134)(13)(14)(1)	=	(345)(45)(35)(5)
II	(234)(23)(34)(3) − (124)(12)(14)(1)	=	(245)(25)(45)(5)
III	(234)(24)(34)(4) − (123)(12)(13)(1)	=	(235)(25)(35)(5)
IV	(134)(13)(34)(3) − (124)(12)(24)(2)	=	(145)(15)(45)(5)
V	(134)(14)(34)(4) − (123)(12)(23)(2)	=	(135)(15)(35)(5)
VI	(124)(14)(24)(4) − (123)(13)(23)(3)	=	(125)(15)(25)(5)
VII	(345)(34)(35)(3) − (245)(24)(25)(2)	=	−(145)(14)(15)(1)
VIII	(345)(34)(45)(4) − (235)(23)(25)(2)	=	−(135)(13)(15)(1)
IX	(145)(14)(45)(4) − (135)(13)(35)(3)	=	−(125)(12)(25)(2)
X	(125)(12)(15)(1) + (235)(23)(35)(3)	=	−(245)(24)(45)(4)

The counting problem now becomes: find all 25-tuples of *nonzero* integers satisfying the equations I−X and the inequality $\max(|s'_j|) \le \mathsf{B}$.

Remark 5.4.2. Note the analogy to the the case of $\bar{\mathcal{M}}_{0,5}$ (the unique split Del Pezzo surface of degree 5): the variety defined by the above equations is the Grassmannian $\mathrm{Gr}(2,6)$ (in its Plücker embedding into $\mathbb{P}^{24}$).

In [VW95] it is shown that there exist constants $\mathsf{c}, \mathsf{c}' > 0$ such that

$$\mathsf{c}\mathsf{B}^2 \log(\mathsf{B})^5 \le \mathsf{N}(\mathsf{B}) \le \mathsf{c}'\mathsf{B}^2 \log(\mathsf{B})^5.$$

This uses a different (an intermediate) torsor over X - the determinantal variety given by

$$\det(x_{ij})_{3\times 3} = 0.$$

Theorem 5.4.3. [dlB07]

$$\mathsf{N}(\mathsf{B}) = \frac{1}{24}\tau_\infty \prod_p \tau_p \cdot \mathsf{B}^2 \log(\mathsf{B})^5 \left(1 + O\Big(\frac{(\log\log B)^{1/3}}{(\log B)^{1/3}}\Big)\right),$$

where τ_∞ is the real density of points on X, and

$$\tau_p = \Big(1 - \frac{1}{p}\Big)^6 \Big(1 + \frac{6}{p} + \frac{6}{p^2} + \frac{1}{p^3}\Big)$$

is the p-adic density of points.

The proof of this result uses the Grassmannian $\mathrm{Gr}(2,6)$.

5.5. **Flag varieties and torsors.** We have seen that for a Del Pezzo surface of degree 5 and for the Segre cubic threefold the universal torsors are flag varieties; and that lifting the count of rational points to these flag variety yields the expected asymptotic results.

More generally, let G be a semi-simple algebraic group, $P \subset G$ a parabolic subgroup. The flag variety $P\backslash G$ admits an action by any subtorus of the maximal torus in G on the right. Choosing a linearization for this action and passing to the quotient we obtain a plethora of examples of *nonhomogeneous* varieties X whose torsors carry additional symmetries. These may be helpful in the counting rational points on X.

Example 5.5.1. A flag variety for the group G_2 is the quadric hypersurface

$$v_1u_1 + v_2u_2 + v_3u_3 + z^2 = 0,$$

where the torus $\mathbb{G}_m^2 \subset \mathsf{G}_2$ acts as

$$\begin{array}{lllll} v_j & \mapsto & \lambda_j v_j, & j = 1,2 & v_3 \mapsto (\lambda_1\lambda_2)^{-1} v_3 \\ u_j & \mapsto & \lambda_j^{-1} u_j, & j = 1,2 & u_3 \mapsto \lambda_1\lambda_2 u_3. \end{array}$$

The quotient by $\mathbb{G}_m^2$ is a subvariety in the weighted projective space $\mathbb{P}(1,2,2,2,3,3) = (z : x_1 : x_2 : x_3 : y_1 : y_2)$ with the equations

$$x_0 + x_1 + x_2 + z^2 = 0 \ \text{ and } \ x_1x_2x_3 = y_1y_2.$$

6. Height zeta functions

Consider the variety $X \subset \mathbb{P}^5$ over $\mathbb{Q}$ given by

$$x_0x_1 - x_2x_3 + x_4x_5 = 0.$$

It is visibly a quadric hypersurface and we could apply the circle method as in Section 4.6. It is also the Grassmannian variety $\mathrm{Gr}(2,4)$ and an equivariant compactification of $\mathbb{G}_a^4$. We could count points using any of the structures. In this section we explain counting strategies based on group actions and harmonic analysis.

6.1. **Tools from analysis.** Here we collect technical results from complex and harmonic analysis which will be used in the treatment of height zeta functions.

For $U \subset \mathbb{R}^n$ let

$$\mathsf{T}_U := \{s \in \mathbb{C} \,|\, \Re(s) \in U\}$$

be the tube domain over U.

Theorem 6.1.1 (Convexity principle). *Let $U \subset \mathbb{R}^n$ be a connected open subset and $\bar{U}$ the convex envelope of U, i.e., the smallest convex open set containing U. Let $\mathsf{Z}(s)$ be a function holomorphic in T_U. Then $\mathsf{Z}(s)$ is holomorphic in $\mathsf{T}_{\bar{U}}$.*

Theorem 6.1.2 (Phragmen-Lindelöf principle). *Let ϕ be a holomorphic function for $\Re(s) \in [\sigma_1, \sigma_2]$. Assume that in this domain ϕ satisfies the following bounds*

- $|\phi(s)| = O(e^{\epsilon|t|})$, *for all* $\epsilon > 0$;
- $|\phi(\sigma_1 + it)| = O(|t|^{k_1})$ *and* $|\phi(\sigma_s + it) = |O(|t|^{k_2})$.

Then, for all $\sigma \in [\sigma_1, \sigma_2]$ *one has*

$$|\phi(\sigma + it)| = O(|t|^k), \qquad \text{where } \frac{k - k_1}{\sigma - \sigma_1} = \frac{k_2 - k_1}{\sigma_2 - \sigma_1}.$$

Using the functional equation and known bounds for $\Gamma(s)$ in vertical strips one derives the *convexity bound*

$$|\zeta(\frac{1}{2} + it)| = O(|t|^{1/4+\epsilon}). \tag{6.1}$$

More generally, we have the following

Proposition 6.1.3. *Let* χ *be an unramified character of* $\mathbb{G}_m(\mathbb{A}_F)/\mathbb{G}_m(F)$, *i.e.,* χ_v *is trivial on* $\mathbb{G}(\mathfrak{o}_v)$, *for all* $v \nmid \infty$. *For all* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|L(s, \chi)| \ll (1 + |\Im(\chi)| + |\Im(s)|)^{\epsilon}, \qquad \text{for } \Re(s) > 1 - \delta. \tag{6.2}$$

Here $\Im(\chi) \in \oplus_{v|\infty}\mathbb{G}_m(F_v)/\mathbb{G}_m(\mathfrak{o}_v) \simeq \mathbb{R}^{r_1+r_2-1}$, *with* r_1, r_2 *the number of real, resp. pairs of complex embeddings of* F.

Theorem 6.1.4 (Tauberian theorem)**.** *Let* $\{\lambda_n\}$ *be an increasing sequences of positive real numbers, with* $\lim_{n\to\infty} \lambda_n = \infty$. *Let* $\{a_n\}$ *be another sequence of positive real numbers and put*

$$\mathsf{Z}(s) := \sum_{n\ge 1} \frac{a_n}{\lambda_n^s}.$$

Assume that this series converges absolutely and uniformly to a homomorphic function in the tube domain $\mathsf{T}_{>a} \subset \mathbb{C}$, *for some* $a > 0$, *and that it admits a representation*

$$\mathsf{Z}(s) = \frac{h(s)}{(s-a)^b},$$

where h *is holomorphic in* $\mathsf{T}_{>a-\epsilon}$, *for some* $\epsilon > 0$, *with* $h(a) = c > 0$, *and* $b \in \mathbb{N}$. *Then*

$$\mathsf{N}(B) := \sum_{\lambda_n \le B} a_n \sim \frac{c}{a(b-1)!} B^a \log(B)^{b-1}, \qquad \text{for } B \to \infty.$$

A frequently employed result in analytic number theory is

Theorem 6.1.5 (Poisson formula)**.** *Let* G *be a locally compact abelian group with Haar measure* dg *and* $H \subset G$ *a closed subgroup. Let* $\hat{G}$ *be* *the Pontryagin dual of G, i.e., the group of characters, i.e., continuous homomorphisms,*

$$\chi : G \to \mathbb{S}^1 \subset \mathbb{C}^*$$

into the unit circle. Let $f : G \to \mathbb{C}$ be a function, satisfying some mild assumptions (integrability, continuity) and let

$$\hat{f}(\chi) := \int_G f(g) \cdot \chi(g) dg$$

be its Fourier transform. Then there exist Haar measures dh on H and $dh^\perp$ on $H^\perp$, the subgroup of characters trivial on H, such that

$$\int_H f dh = \int_{H^\perp} \hat{f} dh^\perp. \tag{6.3}$$

A standard application is to $H = \mathbb{Z} \subset \mathbb{R} = G$. In this case $H^\perp = H = \mathbb{Z}$, and the formula reads

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

This is a powerful identity which is used, e.g., to prove the functional equation and meromorphic continuation of the Riemann zeta function. We will apply Equation (6.3) in the case when G is the group of adelic points of an algebraic torus or an additive group, and H is the subgroup of rational points. This will allow us to establish a meromorphic continuation of height zeta functions for equivariant compactifications of these groups.

Another application of the Poisson formula arises as follows: Let A be a lattice and Λ a convex cone in $A_{\mathbb{R}}$. Let $d\check{a}$ be the Lebesgue measure on the dual space $\check{A}_{\mathbb{R}}$ normalized by the dual lattice $\check{A}$). Let

$$\mathcal{X}_\Lambda(\mathbf{s}) := \int_{\check{\Lambda}} e^{-\langle \mathbf{s}, \check{a} \rangle} d\check{a}, \quad \Re(\mathbf{s}) \in \Lambda^\circ.$$

be the Laplace transform of the set-theoretic characteristic function of the dual cone $\check{\Lambda}$.

Let $\pi : (A, \Lambda) \to (\tilde{A}, \tilde{\Lambda})$ be a homomorphism, with finite cokernel A' and kernel $B \subset A$. Normalize $db : \mathrm{vol}(B_{\mathbb{R}}/B) = 1$. Then

$$\mathcal{X}_{\tilde{\Lambda}}(\pi(\mathbf{s})) = \frac{1}{(2\pi)^{k - \tilde{k}}} \frac{1}{|A'|} \int_{B_{\mathbb{R}}} \mathcal{X}_\Lambda(\mathbf{s} + ib) db.$$

In particular,

$$\mathcal{X}_\Lambda(\mathbf{s}) = \frac{1}{(2\pi)^d} \int_{M_{\mathbb{R}}} \prod_{j=1}^n \frac{1}{(s_j + im_j)} dm.$$

6.2. **Compactifications of groups and homogeneous spaces.** As already mentioned in Section 3, an easy way to generate examples of algebraic varieties with many rational points is to use actions of algebraic groups. Here we discuss the geometric properties of groups and their compactifications.

Let G be a linear algebraic group over a field F, and

$$\varrho \,:\, G \to \mathrm{PGL}_{n+1}$$

an algebraic representation over F. Let $x \in \mathbb{P}^n(F)$ be a point. The orbit $\varrho(G)\cdot x \subset \mathbb{P}^n$ inherits rational points from $G(F)$. Let $H \subset G$ be the stabilizer of x. In general, we have an exact sequence

$$1 \to H(F) \to G(F) \to G/H(F) \to \mathrm{H}^1(F, H) \to \cdots$$

We will only consider examples when $(G/H)(F) = G(F)/H(F)$.

By construction, the Zariski closure X of $\varrho(G)\cdot x$ is geometrically isomorphic to an *equivariant* compactification of the homogeneous space G/H. We have a dictionary

$$(\varrho, x \in \mathbb{P}^n) \Leftrightarrow \left\{ \begin{array}{l} \text{equivariant compactification } X \supset G/H, \\ G\text{-linearized very ample line bundle } L \text{ on } X. \end{array} \right.$$

Representations of semi-simple groups do not deform, and can be characterized by combinatorial data: lattices, polytopes etc. Note, however, that the choice of the initial point $x \in \mathbb{P}^n$ can still give rise to moduli. On the other hand, the classification of representations of unipotent groups is a *wild* problem, already for $G = \mathbb{G}_a^2$. In this case, understanding the moduli of representations of a fixed dimension is equivalent to classifying pairs of commuting matrices, up to conjugacy (see [GP69]).

6.3. **Basic principles.** Here we explain some common features in the study of height zeta functions of compactifications of groups and homogeneous spaces.

In all examples, we have $\mathrm{Pic}(X) = \mathrm{NS}(X)$, a torsion-free abelian group. Choose a basis $L_1, \ldots, L_r$ of $\mathrm{Pic}(X)$ and metrizations $\mathcal{L}_j = (L_j, \|\cdot\|_v)$. We obtain a *height system*:

$$\mathsf{H}_j \,:\, X(F) \to \mathbb{R}_{>0}, \quad \text{ for } \; j = 1, \ldots, r,$$

which can be extend to $\mathrm{Pic}(X)_{\mathbb{C}}$, by linearity:

$$\begin{array}{rcl} \mathsf{H} : X(F) \times \mathrm{Pic}(X)_{\mathbb{C}} & \to & \mathbb{R}_{>0} \\ (x, \mathbf{s}) & \mapsto & \prod_{j=1}^r \mathsf{H}_{\mathcal{L}_j}(x)^{s_j}, \end{array} \tag{6.4}$$

where $\mathsf{s} := \sum_{j=1}^r s_j L_j$. For each j, the 1-parameter zeta function

$$\mathsf{Z}_X(\mathcal{L}_j, s) = \sum_{x \in X(F)} H_{\mathcal{L}}(s)^{-s}$$

converges absolutely to a holomorphic function, for $\Re(s) \gg 0$ (see Remark **??**). It follows that

$$\mathsf{Z}_X(\mathbf{s}) := \sum_{x \in X(F)} \mathsf{H}(\mathbf{s}, x)^{-1}$$

converges absolutely to a holomorphic function for $\Re(\mathsf{s})$ contained some cone in $\mathrm{Pic}(X)_{\mathbb{R}}$.

Step 1. One introduces a generalized height height pairing

$$\mathsf{H} = \prod_v \mathsf{H}_v \;:\; G(\mathbb{A}) \times \mathrm{Pic}(X)_{\mathbb{C}} \to \mathbb{C}, \tag{6.5}$$

such that the restriction of H to $\mathrm{Pic}(X) \times G(F)$ coincides with the pairing in (6.4). Since X is projective, the height zeta function

$$\mathsf{Z}(g, \mathbf{s}) := \sum_{\gamma \in G(F)} \mathsf{H}(\gamma g, \mathbf{s})^{-1} \tag{6.6}$$

converges to a function which is continuous in g and holomorphic in $\mathbf{s}$ for $\Re(\mathbf{s})$ contained in some cone $\Lambda \subset \mathrm{Pic}(X)_{\mathbb{R}}$. The standard height zeta function is obtained by setting $g = e$, the identity in $G(\mathbb{A})$. Our goal is to obtain a meromorphic continuation to the tube domain T over an open neighborhood of $[-K_X] = \kappa \in \mathrm{Pic}(X)_{\mathbb{R}}$ and to identify the poles of Z in this domain.

Step 2. It turns out that

$$\mathsf{Z}(g, \mathbf{s}) \in \mathsf{L}^2(G(F)\backslash G(\mathbb{A})),$$

for $\Re(\mathbf{s}) \gg 0$. This is immediate in the cocompact case, e.g., for G unipotent or semi-simple anisotropic, and requires an argument in other cases. The L^2-space decomposes into unitary irreducible representations for the natural action of $G(\mathbb{A})$. We get a formal identity

$$\mathsf{Z}(g, \mathbf{s}) = \sum_{\varrho} \mathsf{Z}_{\varrho}(g, \mathbf{s}), \tag{6.7}$$

where the summation is over all irreducible unitary representations $(\varrho, \mathcal{H}_{\varrho})$ of $G(\mathbb{A})$ occuring in the right regular representation of $G(\mathbb{A})$ in $\mathsf{L}^2(G(F)\backslash G(\mathbb{A}))$.

Step 3. In many cases, the leading pole of $\mathsf{Z}(g,\mathbf{s})$ arises from the trivial representation, i.e., from the integral

$$\int_{G(\mathbb{A}_F)} \mathsf{H}(g,\mathbf{s})^{-1}dg = \prod_v \int_{G(F_v)} \mathsf{H}_v(g_v,\mathbf{s})^{-1}dg_v, \tag{6.8}$$

where dg_v is a Haar measure on $G(F_v)$. To simplify the exposition we assume that

$$X \setminus G = D = \cup_{i\in\mathcal{I}} D_i,$$

where D is a divisor with normal crossings whose components D_i are geometrically irreducible.

We choose integral models for X and D_i and observe

$$G(F_v) \subset X(F_v) \xrightarrow{\sim} X(\mathcal{O}_v) \to X(\mathbb{F}_q) = \cup_{I\subset\mathcal{I}} D_I^\circ(\mathbb{F}_q),$$

where

$$D_I := \cap_{i\in I} D_i, \quad D_I^\circ := D_I \setminus \cup_{I'\supsetneq I} D_{I'}.$$

Write $\mathbf{s} = \sum_i s_i D_i$ and For almost all v, we have:

$$\int_{G(F_v)} \mathsf{H}_v(g_v,\mathbf{s})^{-1}dg_v = \tau_v(G)^{-1}\left(\sum_{I\subset\mathcal{I}} \frac{\#D_I^\circ(\mathbb{F}_q)}{q^{\dim(X)}} \prod_{i\in I} \frac{q-1}{q^{s_i-\kappa_i+1}-1}\right), \tag{6.9}$$

where $\tau_v(G)$ is the local Tamagawa number of G and κ_i is the order of the pole of the (unique modulo constants) top-degree differential form on G along D_i. The height integrals are geometric versions of Igusa's integrals. They are closely related to "motivic" integrals of Batyrev, Kontsevich, Denef and Loeser (see [?], [?] and [?]).

This allows to regularize explicitely this adelic integral. For example, for unipotent G we have

$$\int_{G(\mathbb{A}_F)} \mathsf{H}(g,\mathbf{s})^{-1}dg = \prod_i \zeta_F(s_i-\kappa_i+1)\cdot\Phi(\mathbf{s}), \tag{6.10}$$

with $\Phi(s)$ holomorphic and absolutely bounded for $\Re(s_i) > \kappa_i - \delta$, for all i.

Step 4. Next, one has to identify the leading poles of $\mathsf{Z}_\varrho(g,\mathbf{s})$, and to obtain bounds which are sufficiently uniform in ϱ to yield a meromorphic continuation of the right side of (6.15). This is nontrivial already for abelian groups G (see Section 6.4 for the case when $G = \mathbb{G}_a^n$). Moreover, will need to show *pointwise* convergence of the series, as a function of $g \in G(\mathbb{A})$.

For G abelian, e.g., an algebraic torus, all unitary representation have dimension one, and equation (6.15) is nothing but the usual Fourier expansion of a "periodic" function. The adelic Fourier coefficient is an Euler product, and the local integrals can be evaluated explicitly.

For other groups, it is important to have some sort of parametrization of representations occurring on the right side of the spectral expansion. For example, for unipotent groups such a representation is provided by Kirillov's orbit method (see Section 6.6). For semi-simple groups one has to appeal to Langland's theory of automorphic representations.

6.4. **Additive groups.** Let X be an equivariant compactification of an additive group $G = \mathbb{G}_a^n$. For example, any blowup $X = \mathrm{Bl}_Y(\mathbb{P}^n)$, with $Y \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$, can be equipped with a structure of an equivariant compactification of $\mathbb{G}_a^n$. In particular, the Hilbert schemes of all algebraic subvarieties of $\mathbb{P}^{n-1}$ appear in the moduli of equivariant compactifications X as above. Some features of the geometry of such compactifications have been explored in [HT99]. The analysis of height zeta functions has to capture this geometric complexity. In this section we present an approach to height zeta functions developed in [CLT99], [CLT00], and [CLT02].

The Poisson formula yields

$$\begin{aligned}\mathsf{Z}(\mathbf{s}) &= \sum_{x \in G(F)} \mathsf{H}(x, \mathbf{s})^{-1} \\ &= \int_{G(\mathbb{A}_F)} \mathsf{H}(x, \mathbf{s})^{-1} dx \ + \sum_{\psi \neq \psi_0} \hat{\mathsf{H}}(\psi, \mathbf{s}),\end{aligned}$$

where the sum runs over all nontrivial characters $\psi \in (G(\mathbb{A}_F)/\mathbb{G}(F))^*$ and

$$\hat{\mathsf{H}}(\psi, \mathbf{s}) = \int_{G(\mathbb{A}_F)} \mathsf{H}(x, \mathbf{s})^{-1} \psi(x) dx$$

is the Fourier transform, with an appropriately normalized Haar measure dx.

Example 6.4.1. The simplest case is $G = \mathbb{G}_a = \mathbb{A}^1 \subset \mathbb{P}^1$, over $F = \mathbb{Q}$, with the standard height

$$\mathsf{H}_p(x) = \max(1, |x|_p), \quad \mathsf{H}_\infty(x) = \sqrt{1 + x^2}.$$

We have

$$\mathsf{Z}(s) = \sum_{x \in Q} \mathsf{H}(x)^{-s} = \int_{\mathbb{A}_\mathbb{Q}} \mathsf{H}(x)^{-s} dx + \sum_{\psi} \hat{\mathsf{H}}(\psi, s). \tag{6.11}$$

The local Haar measure dx_p is normalized by $\mathrm{vol}(\mathbb{Z}_p) = 1$ so that

$$\mathrm{vol}(|x|_p = p^j) = p^j(1 - \frac{1}{p}).$$

We have

$$\begin{aligned}\int_{\mathbb{Q}_p} \mathsf{H}_p(x)^{-s} dx_p &= \int_{\mathbb{Z}_p} \mathsf{H}_p(x)^{-s} dx_p + \sum_{j\geq 1} \int_{|x|_p = p^j} \mathsf{H}_p(x)^{-s} dx_p \\ &= 1 + \sum_{j \geq 1} p^{-js} \mathrm{vol}(|x|_p = p^j) = \frac{1 - p^{-s}}{1 - p^{-(s-1)}}\end{aligned}$$

$$\int_{\mathbb{R}} (1 + x^2)^{-s/2} dx = \frac{\Gamma((s-1)/2)}{\Gamma(s/2)}.$$

Now we analyze the contributions from nontrivial characters. Each such character ψ decomposes as a product of local characters, defined as follows:

$$\begin{aligned}\psi_p = \psi_{p,a_p} &: x_p \mapsto e^{2\pi i a_p \cdot x_p}, \quad a_p \in \mathbb{Q}_p, \\ \psi_\infty = \psi_{\infty, a_\infty} &: x \ \mapsto e^{2\pi i a_\infty \cdot x}, \qquad a \in \mathbb{R}.\end{aligned}$$

A character is *unramified* at p if it is trivial on $\mathbb{Z}_p$, i.e., $a_p \in \mathbb{Z}_p$. Then $\psi = \psi_a$, with $a \in \mathbb{A}_{\mathbb{Q}}$. A character $\psi = \psi_a$ is unramified for all p iff $a \in \mathbb{Z}$. Pontryagin duality identifies $\hat{\mathbb{Q}}_p = \mathbb{Q}_p$, $\hat{\mathbb{R}} = \mathbb{R}$, and $(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q})^* = \mathbb{Q}$.

Since H_p is invariant under the translation action by $\mathbb{Z}_p$, the local Fourier transform $\hat{\mathsf{H}}_p(\psi_{a_p})$ vanishes unless ψ_p is unramified at p. In particular, only unramified characters are present in the expansion (6.11), i.e., we may assume that $\psi = \psi_a$ with $a \in \mathbb{Z} \setminus 0$. For $p \nmid a$, we compute

$$\hat{\mathsf{H}}_p(s, \psi_a) = 1 + \sum_{j \geq 1} p^{-sj} \int_{|x|_p = p^j} \psi_a(x_p) dx_p = 1 - p^{-s}.$$

Putting together we obtain

$$\begin{aligned}\mathsf{Z}(s) &= \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{\Gamma((s-1)/2)}{\Gamma(s/2)} \\ &+ \sum_{a \in \mathbb{Z}} \prod_{p \nmid a} \frac{1}{\zeta_p(s)} \cdot \prod_{p | a} \hat{\mathsf{H}}_p(x_p)^{-s} dx_p \cdot \int_{\mathbb{R}} (1 + x^2)^{-s/2} \cdot e^{2\pi i a x} dx\end{aligned}$$

For $\Re(s) > 2-\delta$, we have the upper bounds

$$|\prod_{p|a} \hat{\mathsf{H}}_p(x_p)^{-s}dx_p| \ll |\prod_{p|a}\int_{\mathbb{Q}_p} \mathsf{H}_p(x_p)^{-s}dx_p \ll |a|^{\epsilon} \tag{6.12}$$

$$|\int_{\mathbb{R}}(1+x^2)^{-s/2}\cdot e^{2\pi i a x}dx| \ll_N \frac{1}{(1+|a|)^N}, \quad \text{for any } N\in\mathbb{N}, \tag{6.13}$$

where the second inequality is proved via repeated integration by parts.

This gives a meromorphic continuation of $\mathsf{Z}(s)$ and its pole at $s=2$ (corresponding to $-K_X = 2L \in \mathbb{Z} = \mathrm{Pic}(\mathbb{P}^1)$). The leading coefficient at this pole is the Tamagawa number defined by Peyre.

Now we turn to the general case.

- $\mathrm{Pic}(X) = \oplus_i \mathbb{Z}D_i$
- $-K_X = \sum_i \kappa_i D_i$, with $\kappa_i \geq 2$
- $\Lambda_{\mathrm{eff}}(X) = \oplus_i \mathbb{R}_{\geq 0}D_i$

Local and global heights are given by

$$\mathsf{H}_{D_i,v}(x) := \|\mathsf{f}_i(x)\|_v^{-1} \quad \text{and} \quad \mathsf{H}_{D_i}(x) = \prod_v \mathsf{H}_{D_i,v}(x),$$

where f_i is the unique G-invariant section of $\mathrm{H}^0(X, D_i)$. We get a *height pairing*:

$$\begin{array}{rcl} \mathsf{H} \ : G(\mathbb{A}_F)\times \mathrm{Pic}(X)_{\mathbb{C}} & \to & \mathbb{C} \\ (x, \sum_i s_i D_i) & \mapsto & \prod_i \mathsf{H}_{D_i}(x)^{s_i} \end{array}$$

Similarly, obtain characters of $\mathbb{G}_a^n(\mathbb{A}_F)$.

A character is determined by a "linear form" $\langle a,\cdot\rangle = f_{\mathbf{a}}$, on $\mathbb{G}_a^n$, which gives a rational function $f_{\mathbf{a}} \in F(X)^*$. We have

$$\mathrm{div}(f_{\mathbf{a}}) = E_{\mathbf{a}} - \sum_i d_i(f_{\mathbf{a}})D_i$$

with $d_i \geq 0$, for all i.

Define:

- $S(\mathbf{a}) \subset \mathrm{Val}(F)$
- $\mathcal{I}_0(a) := \{i \,|\, d_i(f_{\mathbf{a}}) = 0\} \subsetneq \mathcal{I}$

We have

$$\hat{\mathsf{H}}(\psi_{\mathbf{a}}, \mathbf{s}) = \prod_{i\in\mathcal{I}_0(\mathbf{a})} \zeta_F(s_i - \kappa_i + 1)\cdot \Phi_{\mathbf{a}}(\mathbf{s})\cdot \int_{G(\mathbb{A}_\infty)} H_\infty(x,\mathbf{s})^{-1}\psi_{\mathbf{a},\infty}(x)dx$$

with $\Phi_{\mathbf{a}}(\mathbf{s})$ holomorphic for $\Re(s_i) > \kappa_i - \delta$ and bounded by $(1+\|a\|)^{\epsilon}$.

$$\begin{aligned}\mathsf{Z}(\mathbf{s}) &= \int_{G(\mathbb{A}_F)} \mathsf{H}(x,\mathbf{s})^{-1}dx + \sum_{\mathcal{I}_0\subsetneq\mathcal{I}}\sum_{\psi_{\mathbf{a}}\,:\,\mathcal{I}_0(\mathbf{a})=\mathcal{I}} \hat{\mathsf{H}}(\psi_{\mathbf{a}},\mathbf{s}) \\ &= \prod_{i\in\mathcal{I}} \zeta_F(s_i-\kappa_i+1)\cdot\Phi(\mathbf{s}) \\ &\quad + \sum_{I\subset\mathcal{I}}\prod_{i\in I}\zeta_F(s_i-\kappa_i+1)\cdot\tilde{\Phi}_I(\mathbf{s})\end{aligned}$$

To be completed.

6.5. **Toric varieties.** Analytic properties of height zeta functions of toric varieties have been established in [BT95], [BT98], and [**?**].

An algebraic torus is a linear algebraic groups T over a field F such that

$$T_E \simeq \mathbb{G}_{m,E}^d$$

for some finite Galois extension E/F. Such an extension is called a *splitting field* of T. A torus is *split* if $T\simeq \mathbb{G}_{m,F}^d$. The group of algebraic characters

$$M := \mathfrak{X}^*(T) = \mathrm{Hom}(T,\bar{K}^*)$$

is a torsion free $\Gamma := \mathrm{Gal}(E/F)$-module. The standard notation for its dual, the cocharacters is $N := \mathfrak{X}^*(T)$. There is an equivalence of categories:

$$\left\{\begin{array}{l} d-\text{dimensional integral}\\ \Gamma-\text{representations,}\\ \text{up to equivalence}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} d-\text{dimensional}\\ \text{algebraic tori, split over } \;E,\\ \text{up to isomorphism}\end{array}\right\}$$

The local and global theory of tori can be summarized as follows: The local Galois groups $\Gamma_v := \mathrm{Gal}(E_w/F_v)\subset\Gamma$ act on $M_v := M^{\Gamma_v}$, the characters of $T(F_v)$. Let N_v be the lattice of local cocharacters. Write $T(\mathfrak{o}_v)\subset T(F_v)$ for a maximal compact subgroup (after choosing an integral model, it is indeed the group of $\mathfrak{o}_v$-valued points of T, for almost all v). Then

$$T(F_v)/T(\mathfrak{o}_v)\hookrightarrow N_v = N^{\Gamma_v},$$

an isomorphism for v unramified in E/F. Adelically, we have

$$T(\mathbb{A}_F)\supset T^1(\mathbb{A}_F) = \{t \mid \prod_v |m(t_v)|_v = 1 \;\; \forall m\in M^\Gamma\}$$

and

$$T(F)\hookrightarrow T^1(\mathbb{A}_F).$$

Let $\mathbf{K}_T := \prod_v T(\mathfrak{o}_v)$ be the maximal compact subgroup of $T(\mathbb{A}_F)$.

Theorem 6.5.1. *We have*

- *$T(\mathbb{A}_F)/T^1(\mathbb{A}_F) = N_{\mathbb{R}}^{\Gamma}$;*
- *$T^1(\mathbb{A}_F)/T(F)$ compact;*
- *$\mathbf{K}_T \cap T(F)$ finite;*
- *the map $(T(\mathbb{A}_F)/\mathbf{K}_T \cdot T(F))^* \to \oplus_{v|\infty} M_v \otimes \mathbb{R}$ has finite kernel (analogs of roots of 1) and maps the characters to a lattice $\oplus M_{\mathbb{R}}^{\Gamma}$.*

Over algebraically closed fields, complete *toric varieties*, i.e., equivariant compactifications of algebraic tori, are described and classified by a combinatorial structure (M, N, Σ), where

- M is the free abelian group of finite rank (the algebraic characters of the torus T),
- $N := \mathrm{Hom}(M, \mathbb{Z})$ is the dual group of cocharacters, and
- $\Sigma = \{\sigma\}$ is a *fan*, i.e., a collection of strictly convex cones in $N_{\mathbb{R}}$ such that
  1. $0 \in \sigma$ for all $\sigma \in \Sigma$,
  2. $N_{\mathbb{R}} = \cup_{\sigma\in\Sigma}\sigma$,
  3. every face $\tau \subset \sigma$ is in Σ
  4. $\sigma \cap \sigma' \in \Sigma$ and is face of σ, σ'.

A fan Σ is called *regular* if the generators of every $\sigma \in \Sigma$ form part of a basis of N. In this case, the corresponding toric variety X_Σ is smooth. The toric variety is constructed as follows:

$$X_\Sigma := \cup_\sigma U_\sigma \quad \text{where} \quad U_\sigma := \mathrm{Spec}(F[M \cap \check{\sigma}]),$$

and $\check{\sigma} \subset M_{\mathbb{R}}$ is the cone dual to $\sigma \subset N_{\mathbb{R}}$. The fan Σ encodes all geometric information about X_Σ. For example, 1-dimensional generators $e_1, \ldots, e_n$ of Σ correspond to boundary divisors $D_1, \ldots, D_n$, i.e., the irreducible components of $X_\Sigma \setminus T$. There is an explicit criterion for projectivity and a description of the cohomology ring, cellular structure etc.

Example 6.5.2. The simplest toric variety is $X = \mathbb{P}^1$. We have three distinguished Zariski open subsets:

- $\mathbb{P}^1 \supset \mathbb{G}_m = \mathrm{Spec}(F[x, x^{-1}]) = \mathrm{Spec}(F[x^{\mathbb{Z}}])$
- $\mathbb{P}^1 \supset \mathbb{A}^1 = \mathrm{Spec}(F[x]) = \mathrm{Spec}(F[x^{\mathbb{Z}_{\geq 0}}])$,
- $\mathbb{P}^1 \supset \mathbb{A}^1 = \mathrm{Spec}(F[x^{-1}]) = \mathrm{Spec}(F[x^{\mathbb{Z}_{\leq 0}}])$

They correspond to the semi-groups:

- $\mathbb{Z}$ dual to 0,
- $\mathbb{Z}_{\geq 0}$ dual to $\mathbb{Z}_{\geq 0}$,
- $\mathbb{Z}_{\leq 0}$ dual to $\mathbb{Z}_{\leq 0}$.

Over nonclosed ground fields F one has to account for the action of the Galois group of a splitting field E/F. The necessary modifications can be described as follows. The Galois group Γ acts on M, N, Σ. A fan Σ is called Γ-invariant if $\gamma \cdot \sigma \in \Sigma$, for all $\gamma \in \Gamma, \sigma \in \Sigma$. If Σ is a complete regular Γ-invariant fan such that, over the splitting field, the resulting toric variety $X_{\Sigma,E}$ is projective, then it can be descended to a complete algebraic variety $X_{\Sigma,F}$ over the groundfield F such that

$$X_{\Sigma,E} \simeq X_{\Sigma,F} \otimes_{\mathrm{Spec}(F)} \mathrm{Spec}(E),$$

as E-varieties with Γ-action.

Picard group

The split case

$PL(\Sigma)$ - piecewise linear $\mathbb{Z}$-valued functions φ on Σ determined by $M \supset \{m_{\sigma,\varphi}\}_{\sigma\in\Sigma}$, i.e., by its values $s_j := \varphi(e_j)$, $j = 1, \ldots, n$.

$$0 \to M \to PL(\Sigma) \xrightarrow{\pi} \mathrm{Pic}(X_\Sigma) \to 0$$

- every divisor is equivalent to a linear combination of boundary divisors $D_1, \ldots, D_n$, and φ is determined by its values on $e_1, \ldots, e_n$
- relations come from characters of T

The nonsplit case

$$0 \to M^\Gamma \to PL(\Sigma)^\Gamma \xrightarrow{\pi} \mathrm{Pic}(X_\Sigma)^\Gamma \to \mathrm{H}^1(\Gamma, M) \to 0$$

$$\begin{array}{rcl} \Lambda_{\mathrm{eff}}(X_\Sigma) & = & \pi(\mathbb{R}_{\geq 0}D_1 + \ldots + \mathbb{R}_{\geq 0}D_n) \\ -K_\Sigma & = & \pi(D_1 + \ldots + D_n) \end{array}$$

Example 6.5.3. $\mathbb{P}^1 = \{x = (x_0 : x_1)\} \supset \mathbb{G}_m$

$$\mathsf{H}_v(x) := \begin{cases} |\frac{x_0}{x_1}|_v & \text{if } |x_0|_v \geq |x_1|_v \\ |\frac{x_1}{x_0}|_v & \text{otherwise} \end{cases}$$

$$\mathsf{H}(x) := \prod_v \mathsf{H}_v(x)$$

In general, $T(F_v)/T(\mathcal{O}_v) \hookrightarrow N_v$. For $\varphi \in PL(\Sigma)$ put

$$\mathsf{H}_{\Sigma,v}(x, \varphi) := q_v^{\varphi(\bar{x}_v)} \quad \mathsf{H}_\Sigma(x, \varphi) := \prod_v \mathsf{H}_{\Sigma,v}(x, \varphi),$$

with $q_v = e$, for $v \mid \infty$. This height has the following properties:

- it gives a pairing $T(\mathbb{A}_F) \times PL(\Sigma)_{\mathbb{C}} \to \mathbb{C}$;
- the restriction to $T(F) \times PL(\Sigma)_{\mathbb{C}}$ descends to a well-defined pairing
$$T(F) \times \mathrm{Pic}(X_\Sigma)_{\mathbb{C}} \to \mathbb{C};$$
- $T(\mathcal{O}_v)$-invariance, for all v

Height zeta function:

$$\mathsf{Z}_\Sigma(\mathbf{s}) := \sum_{x \in T(F)} \mathsf{H}_\Sigma(x, \varphi_{\mathbf{s}})^{-1}$$

Poisson formula

$$\mathsf{Z}_\Sigma(\mathbf{s}) := \int_{(T(\mathbb{A}_F)/\mathbf{K}_T \cdot T(F))^*} \hat{\mathsf{H}}_\Sigma(\chi, \mathbf{s}) d\chi$$

$$\hat{\mathsf{H}}_\Sigma(\chi, \mathbf{s}) := \int_{T(\mathbb{A}_F)} \mathsf{H}_\Sigma(x, -\varphi_{\mathbf{s}})\chi(x) dx$$

- for χ nontrivial on $\mathbf{K}_T$ we have $\hat{\mathsf{H}}_\Sigma = 0$
- converges absolutely for $\Re(s_j) > 1$ (for all j)
- Haar measures on $T(F_v)$ normalized by $T(\mathcal{O}_v)$

Example 6.5.4. Consider the projective line $\mathbb{P}^1$ over $\mathbb{Q}$. We have

$$0 \to M \to PL(\Sigma) \to \mathrm{Pic}(\mathbb{P}^1) \to 0$$

with $M = \mathbb{Z}$ and $PL(\Sigma) = \mathbb{Z}^2$ The Fourier transforms of local heights can be computed as follows:

$$\hat{H}_p(\chi_0, \mathbf{s}) = 1 + \sum_{n \geq 1} p^{-s_1 - im} + \sum_{n \geq 1} p^{-s_1 + im} = \frac{\zeta_p(s_1 + im)\zeta_p(s_2 - im)}{\zeta_p(s_1 + s_2)},$$
$$\hat{H}_\infty(\chi_0, \mathbf{s}) = \int_0^\infty e^{(-s_1 - im)x} dx + \int_0^\infty e^{(-s_2 + im)x} dx = \frac{1}{s_1 + im} + \frac{1}{s_2 - im}.$$

We obtain

$$\mathsf{Z}_{\mathbb{P}^1}(s_1, s_2) = \int_{\mathbb{R}} \zeta(s_1 + im)\zeta(s_2 - im) \cdot \left(\frac{1}{s_1 + im} + \frac{1}{s_2 - im} \right) dm.$$

The integral converges for $\Re(s_1), \Re(s_2) > 1$, absolutely and uniformly on compact subsets. It remains to establish its meromorphically continuation. This can be achieved by shifting the contour of integration and computing the resulting residues.

It is helpful to compare this approach with the analysis of $\mathbb{P}^1$ as an additive variety in Example 6.4.1.

The Fourier transforms of local height functions in the case of $\mathbb{G}_m^d$ over $\mathbb{Q}$ are given by:

- $v \nmid \infty$:

$$\hat{\mathsf{H}}_{\Sigma,v}(\chi_v, -\mathbf{s}) = \sum_{k=1}^{d} \sum_{\sigma\in\Sigma(k)} (-1)^k \prod_{e_j\in\sigma} \frac{1}{1-q_v^{-(s_j+i\langle e_j,m\rangle)}}$$

- $v \mid \infty$

$$\hat{\mathsf{H}}_{\Sigma,v}(\chi_v, -\mathbf{s}) = \sum_{\sigma\in\Sigma(d)} \prod_{e_j\in\sigma} \frac{1}{(s_j+i\langle e_j,m\rangle)}$$

where $\chi = \chi_m$ is the character corresponding to $m \in M_{\mathbb{R}}$. The general case of nonsplit tori over number fields requires more care. We have an exact sequence of Γ-modules:

$$0 \to M \to PL(\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0,$$

with $PL(\Sigma)$ a permutation module. Duality gives a sequence of groups:

$$0 \to T_{\mathrm{Pic}}(\mathbb{A}_F) \to T_{PL}(\mathbb{A}_F) \to T(\mathbb{A}_F)$$

with

$$T_{PL}(\mathbb{A}_F) = \prod_{j=1}^{k} R_{F_j/F}\mathbb{G}_m(\mathbb{A}_F) \quad \text{(restriction of scalars)}$$

We get a map

$$\begin{array}{ccc} (T(\mathbb{A}_F)/\mathbf{K}_T\cdot T(F))^* & \to & \prod_{j=1}^{k} \left(\mathbb{G}_m(\mathbb{A}_{F_j})/\mathbb{G}_m(F_j)\right)^* \\ \chi & \mapsto & (\chi_1,\ldots,\chi_k) \end{array}$$

This map has finite kernel. Assembling local computations, we have

$$\hat{\mathsf{H}}_{\Sigma}(\chi, \mathbf{s}) = \frac{\prod_{j=1}^{k} \mathsf{L}(s_j,\chi_j)}{Q_\Sigma(\chi,\mathbf{s})}\zeta_{\Sigma,\infty}(\mathbf{s},\chi), \tag{6.14}$$

where $Q_\Sigma(\chi, \mathbf{s})$ bounded uniformly in χ, in compact subsets in $\Re(s_j) > 1/2+\delta$, $\delta > 0$, and

$$|\zeta_{\Sigma,\infty}(\mathbf{s},\chi)| \ll \frac{1}{(1+\|m\|)^{d+1}} \cdot \frac{1}{(1+\|\chi\|)^{d'+1}}.$$

This implies that

$$\mathsf{Z}_\Sigma(\mathbf{s}) = \int_{M_{\mathbb{R}}^{\Gamma}} f_\Sigma(\mathbf{s}+im)dm,$$

where

$$f_\Sigma(\mathbf{s}) := \sum_{\chi \in (T^1(\mathbb{A}_F)/\mathbf{K}_T \cdot T(F))^*} \hat{\mathsf{H}}_\Sigma(\chi, \mathbf{s})$$

We have

(1) $(s_1 - 1) \cdot \ldots \cdot (s_k - 1) f_\Sigma(\mathbf{s})$ is homolorphic for $\Re(s_j) > 1 - \delta$;
(2) f_Σ satisfies growth conditions in vertical strips (this follows by applying the Phragmen-Lindelöf principle 6.1.2 to bound L-functions appearing in equation (6.14));
(3) $\lim_{s_j \to 1} f_\Sigma(\mathbf{s}) = c(f_\Sigma) \neq 0$.

The Convexity principle 6.1.1 implies a meromorphic continuation of $\mathsf{Z}(\mathbf{s})$ to a tubular neighborhood of the shifted cone $\Lambda_{\mathrm{eff}}(X_\Sigma)$. The identification of the remaining factors β and τ requires another application of the Poisson formula. Other line bundles require a version of the technical theorem above.

6.6. **Unipotent groups.** Let $X \supset G$ be an equivariant compactification of a unipotent group over a number field F and

$$X \setminus G = D = \cup_{i \in \mathcal{I}} D_i.$$

Throughout, we will assume that G acts on X on both sides, i.e., that X is a compactification of $G \times G/G$, or a bi-equvariant compactification. We also assume that D is a divisor with normal crossings and its components D_i are geometrically irreducible. The main geometric invariants of X have been computed in Example 1.1.4: The Picard group is freely generated by the classes of D_i, the effective cone is simplicial, and the anticanonical class is sum of boundary components with nonnegative coefficients.

Local and global heights have been defined in Example 4.8.6:

$$\mathsf{H}_{D_i,v}(x) := \|\mathsf{f}_i(x)\|_v^{-1} \quad \text{and} \quad \mathsf{H}_{D_i}(x) = \prod_v \mathsf{H}_{D_i,v}(x),$$

where f_i is the unique G-invariant section of $\mathrm{H}^0(X, D_i)$. We get a *height pairing*:

$$\mathsf{H} \ : G(\mathbb{A}_F) \times \mathrm{Pic}(X)_{\mathbb{C}} \to \mathbb{C}$$

as in Section 6.3. The bi-equivariance of X implies that H is invariant under the action *on both sides* of a compact open subgroup $\mathbf{K}$ of the finite adeles. Moreover, we can arrange that H_v is smooth in g_v for archimedean v.

The height zeta function

$$\mathsf{Z}(\mathbf{s},g) := \sum_{\gamma\in G(F)} \mathsf{H}(\mathbf{s},g)^{-1}$$

is holomorphic in $\mathbf{s}$, for $\Re(\mathbf{s}) \gg 0$. As a function of g it is continuous and in $\mathsf{L}^2(G((F)\backslash G(\mathbb{A}_F))$, for these $\mathbf{s}$. We proceed to analyze its spectral decomposition. We get a formal identity

$$\mathsf{Z}(\mathbf{s};g) = \sum_{\varrho} \mathsf{Z}_\varrho(\mathbf{s};g), \tag{6.15}$$

where the sum is over all irreducible unitary representations $(\varrho, \mathcal{H}_\varrho)$ of $G(\mathbb{A}_F)$ occuring in the right regular representation of $G(\mathbb{A}_F)$ in $\mathsf{L}^2(G(F)\backslash G(\mathbb{A}_F))$. They are parametrized by F-rational orbits $\mathcal{O} = \mathcal{O}_\varrho$ under the coadjoint action of G on the dual of its Lie algebra $\mathfrak{g}^*$. The relevant orbits are *integral* - there exists a lattice in $\mathfrak{g}^*(F)$ such that $\mathsf{Z}_\varrho(\mathbf{s};g) = 0$ unless the intersection of $\mathcal{O}$ with this lattice is nonempty. The pole of highest order is contributed by the trivial representation and integrality insures that this representation is "isolated".

Let ϱ be an integral representation as above. It has the following explicit realization: There exists an F-rational subgroup $M \subset G$ such that

$$\varrho = \mathrm{Ind}_M^G(\psi),$$

where ψ is a certain character of $M(\mathbb{A}_F)$. In particular, for the trivial representation, $M = G$ and ψ is the trivial character. Further, there exists a finite set of valuations $S = S_\varrho$ such that $\dim(\varrho_v) = 1$ for $v \notin S$ and consequently

$$\mathsf{Z}_\varrho(\mathbf{s};g') = \mathsf{Z}^S(\mathbf{s};g') \cdot \mathsf{Z}_S(\mathbf{s};g'). \tag{6.16}$$

It turns out that

$$\mathsf{Z}^S(\mathbf{s};g') := \prod_{v\notin S} \int_{M(F_v)} \mathsf{H}_v(\mathbf{s};g_v g'_v)^{-1}\overline{\psi}(g_v)dg_v,$$

with an appropriately normalized Haar measure dg_v on $M(F_v)$. The function Z_S is the projection of Z to $\otimes_{v\in S}\varrho_v$.

The first key result is the explicit computation of *height integrals*:

$$\int_{M(F_v)} H_v(\mathbf{s};g_v g'_v)^{-1}\overline{\psi}(g_v)dg_v$$

for almost all v. This has been done in [**?**] for equivariant compactifications of additive groups $\mathbb{G}_a^n$ (see Section 6.4); the same approach works here too. The contribution from the trivial representation can be computed using the formula of Denef-Loeser, as in (6.10):

$$\int_{G(\mathbb{A}_F)} \mathsf{H}(\mathbf{s}; g)^{-1} dg = \prod_i \zeta_F(s_i - \kappa_i + 1) \cdot \Phi(\mathbf{s}),$$

where $\Phi(\mathbf{s})$ is holomorphic for $\Re(\mathbf{s}) \in \mathsf{T}_{-K_X - \epsilon}$, for some $\epsilon > 0$, and $-K_X = \sum_i \kappa_i D_i$. As in the case of additive groups in Section 6.4, this term gives the "correct" pole at $-K_X$. The analysis of 1-dimensional representations, with $M = G$, is similar to the additive case. New difficulties arise from infinite-dimensional ϱ on the right side of the expansion (6.15).

Next we need to estimate $\dim(\varrho_v)$ and the local integrals for nonarchimedean $v \in S_\varrho$. The key result here is that the contribition to the Euler product from these places is a holomorphic function which can be bounded from above by a *polynomial* in the coordinates of ϱ, for $\Re(\mathbf{s}) \in \mathsf{T}_{-K_X - \epsilon}$. The uniform convergence of the spectral expansion comes from estimates at the archimedean places: for every (left or right) G-invariant differential operator ∂ (and $\mathsf{s} \in \mathsf{T}$) there exists a constant $\mathsf{c}(\partial)$ such that

$$\int_{G(F_v)} |\partial \mathsf{H}_v(\mathbf{s}; g_v)^{-1} dg_v|_v \le \mathsf{c}(\partial). \tag{6.17}$$

Let v be real. It is known that ϱ_v can be modeled in $\mathsf{L}^2(\mathbb{R}^r)$, where $2r = \dim(\mathcal{O})$. More precisely, there exists an isometry

$$j \ : \ (\pi_v, \mathsf{L}^2(\mathbb{R}^r)) \to (\varrho_v, \mathcal{H}_v)$$

(an analog of the Θ-distribution). Moreover, the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ surjects onto the Weyl algebra of differential operators with polynomial coefficients acting on the smooth vectors $\mathsf{C}^\infty(\mathbb{R}^r) \subset \mathsf{L}^2(\mathbb{R}^r)$. In particular, we can find an operator Δ acting as the (r-dimensional) harmonic oscillator

$$\prod_{j=1}^{r} \left(\frac{\partial^2}{\partial x_j^2} - a_j x_j^2 \right),$$

with $a_j > 0$. We choose an orthonormal basis of $\mathsf{L}^2(\mathbb{R}^r)$ consisting of Δ-eigenfunctions $\{\tilde{\omega}_\lambda\}$ (which are well known) and analyze

$$\int_{G(F_v)} \mathsf{H}_v(\mathbf{s}; g_v)^{-1} \overline{\omega}_\lambda(g_v) dg_v,$$

where $\omega_\lambda = j^{-1}(\tilde{\omega}_\lambda)$. Using integration by parts we find that for $\mathbf{s} \in \mathsf{T}$ and any $N \in \mathbb{N}$ there is a constant $\mathsf{c}(N, \Delta)$ such that this integral is bounded by

$$(1 + |\lambda|)^{-N} \mathsf{c}(N, \Delta). \tag{6.18}$$

This estimate suffices to conclude that for *each* ϱ the function Z_{S_ϱ} is holomorphic in T.

Now the issue is to prove the convergence of the sum in (6.15). Using any element $\partial \in \mathfrak{U}(\mathfrak{g})$ acting in $\mathcal{H}_\varrho$ by a scalar $\lambda(\partial) \neq 0$ (for example, any element in the center of $\mathfrak{U}(\mathfrak{g})$) we can improve the bound (6.18) to

$$(1 + |\lambda|)^{-N_1} \lambda(\partial)^{-N_2} \mathsf{c}(N_1, N_2, \Delta, \partial)$$

(for any $N_1, N_2 \in \mathbb{N}$). However, we have to insure the uniformity of such estimates over the set of all ϱ. This relies on a parametrization of coadjoint orbits. There is a finite set $\{\Sigma_{\mathbf{d}}\}$ of "packets" of coadjoint orbits, each parametrized by a locally closed subvariety $Z_{\mathbf{d}} \subset \mathfrak{g}^*$, and for each $\mathbf{d}$ a finite set of F-rational polynomials $\{P_{\mathbf{d},r}\}$ on $\mathfrak{g}^*$ such that the restriction of each $P_{\mathbf{d},r}$ to $Z_{\mathbf{d}}$ is invariant under the coadjoint action. Consequenty, the corresponding derivatives

$$\partial_{\mathbf{d},r} \in \mathfrak{U}(\mathfrak{g})$$

act in $\mathcal{H}_\varrho$ by multiplication by the scalar

$$\lambda_{\varrho,r} = P_{\mathbf{d},r}(\mathcal{O}).$$

There is a similar uniform construction of the "harmonic oscillator" $\Delta_{\mathbf{d}}$ for each $\mathbf{d}$. Combining the resulting estimates we obtain the uniform convergence of the right hand side in (6.15).

The last technical point is to prove that both expressions (6.6) and (6.15) for $\mathsf{Z}(\mathbf{s}; g)$ define *continuous* functions on $G(F)\backslash G(\mathbb{A}_F)$. Then (6.15) gives the desired meromorphic continuation of $\mathsf{Z}(\mathbf{s}; e)$.

Background material on representation theory of unipotent groups can be found in the books [CG66], [Dix96] and the papers [Moo65], [Kir99].

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