THE HASSE PRINCIPLE FOR COMPLETE INTERSECTIONS IN PROJECTIVE SPACE

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Abstract. Assuming the existence of a smooth geometrically integral complete intersection X of dimension ≥ 3 in $\mathbf{P}^n$ over a number field K, such that the Zariski closure of $X(K)$ is nonempty but of codimension ≥ 2 in X, we construct a 3-dimensional smooth geometrically integral complete intersection X' in $\mathbf{P}^n_K$ that violates the Hasse principle. Such a violation could not be explained by the Brauer-Manin obstruction or Skorobogatov's generalization thereof.

In [SW], Sarnak and Wang showed that the finiteness of $X(\mathbf{Q})$ for a certain nonsingular hypersurface X in $\mathbf{P}^5$ over $\mathbf{Q}$ would imply the existence of a nonsingular hypersurface Y in $\mathbf{P}^4$ that violated the Hasse principle. They chose X so that $X(\mathbf{C})$ was hyperbolic, so that the finiteness of $X(\mathbf{Q})$ would follow from a conjecture of Lang. Colliot-Thélène noted that the Brauer-Manin obstruction for a nonsingular complete intersection of dimension ≥ 3 in projective space vanishes, so Y then would be an example for which the Brauer-Manin obstruction to the Hasse principle was not the only one. Later, Skorobogatov [Sk] constructed a different sort of variety over $\mathbf{Q}$ (a bielliptic surface) for which the Brauer-Manin obstruction is not the only one, by applying the Brauer-Manin obstruction to unramified covers. A complete intersection Y as above is simply connected, however, so its lack of rational points would not be explainable by Skorobogatov's methods.

The construction of Sarnak and Wang involved slicing X with hyperplanes, and performing various calculations involving special properties of X, which had to be carefully chosen. The purpose of this note is to show that using a similarly elementary approach, but with *ramified coverings* instead of slices, one can show that the finiteness and non-emptiness of the rational points in *any* nonsingular hypersurface in $\mathbf{P}^n$ over a number field K implies the existence of a nonsingular hypersurface in $\mathbf{P}^n$ over K that violates the Hasse principle. In particular, it is no longer necessary to start with a variety of dimension one higher in order to obtain a Hasse principle counterexample of desired dimension. More generally, we prove the following.

Theorem 1. *Fix a number field K and integers $d \geq 2$, $n \geq 3$. If there exists a d-dimensional smooth geometrically integral complete intersection X in $\mathbf{P}^n$ over K such that the Zariski closure of $X(K)$ has dimension at most 1, but $X(K)$ is not empty, then there exists also a d-dimensional smooth geometrically integral complete intersection X' in $\mathbf{P}^n$ over K such that $X'(K_v)$ is nonempty for all completions K_v of K, but $X'(K)$ is empty.*

Proof. Suppose that X exists. It follows from Bertini's theorem (see Theorem II.8.18 of [Ha] and its proof) that after a linear change of variables on $\mathbf{P}^n$ over K, we may assume both of the following:

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1. Each coordinate subspace meets X transversely. (By "coordinate subspace," we mean a linear subspace of $\mathbf{P}^n$ of the form $x_{i_1} = x_{i_2} = \cdots = x_{i_q} = 0$.)
2. If Z is the Zariski closure of $X(K)$ in X, and R is any 1-dimensional irreducible component of $Z \times_K \overline{K}$, then each coordinate subspace meets R transversely (i.e., the hyperplane $x_i = 0$ meets R in r distinct points where r is the degree of R in $\mathbf{P}^n$, and the subspace $x_i = x_j = 0$ for $i < j$ does not meet R).

Let $X' = \phi^{-1}(X)$ where the morphism $\phi : \mathbf{P}^n \to \mathbf{P}^n$ is given in homogeneous coordinates by

$$(x_0 : x_1 : \cdots : x_n) \mapsto (c_0 x_0^p : c_1 x_1^p : \cdots : c_n x_n^p)$$

for some large prime p and for some $c_0, c_1, \ldots, c_n \in K^*$ to be specified later. Since the coordinate subspaces meet X transversely, X' is smooth. If X is given by the homogeneous equations

$$f_1(x_0, \ldots, x_n) = \cdots = f_{n-d}(x_0, \ldots, x_n) = 0,$$

then X' is a complete intersection of dimension d as well, given by the equations

$$f_1(c_0 x_0^p, \ldots, c_n x_n^p) = \cdots = f_{n-d}(c_0 x_0^p, \ldots, c_n x_n^p) = 0.$$

If one successively adjoins to the function field $\overline{K}(X)$ the p-th roots of the functions x_i/x_0 on X, each such extension is totally ramified along the poles and zeros of these functions, so X' (which is of pure dimension d) is geometrically integral.

Suppose that R is a 1-dimensional irreducible component of $Z \times_K \overline{K}$ as above. Let $S = \phi^{-1}(R)$, which is a curve over $\overline{K}$. The same proof that showed that X' was geometrically integral shows that S is irreducible. We will prove next that the genus g_S of S is at least 2 if $p > 2$. (By genus, we mean geometric genus, i.e., the genus of a smooth projective desingularization.) The Riemann-Hurwitz formula yields

$$2g_S - 2 = (\deg \iota)(2g_R - 2) + \sum (e_i - 1).$$

Let r be the degree of R in $\mathbf{P}^n$. Then each coordinate hyperplane meets R transversely in exactly r points, each belonging to only one coordinate hyperplane. These are the only points on R which ramify, and above each there are p^{n-1} points of S, each with ramification index p. Thus

$$\begin{aligned} 2g_S - 2 &= p^n(2g_R - 2) + (n+1) r p^{n-1}(p-1) \\ &\geq -2p^n + 4(p^n - p^{n-1}). \end{aligned}$$

If $p > 2$, then $g_S \geq 2$.

Thus $\phi^{-1}(Z) \times_K \overline{K}$ consists of isolated points and curves over $\overline{K}$ of genus ≥ 2. Applying Faltings' theorem to each curve (over a finite extension of K over which it is defined) shows that $\phi^{-1}(Z)(K)$ is finite. But $X'(K) \subseteq \phi^{-1}(Z)$, so $X'(K)$ is finite too. On the other hand, we are free to choose the c_i so that at least one point in $X(K)$ has a K-rational preimage. Thus by replacing X by X', we have reduced the problem to the case where $X(K)$ is finite (and nonempty).

The next step is to reduce to the case where $X(K)$ consists of a single point. Suppose instead that there exist distinct points $P_1, P_2 \in X(K)$. We construct a new covering $X' \to X$ exactly as before, choosing p large enough so that K has no nontrivial p-th roots of unity. Because we are allowed a generic change of variables on the projective space containing X, we may assume that P_1 and P_2 are outside all coordinate hyperplanes and that the values taken by the rational function x_1/x_0 at P_1 and P_2 are distinct in K^*/K^{*p}. Then we may pick the c_i so that P_1 has a K-rational preimage in X' but P_2 does not. Moreover, the choice of p guarantees that each K-rational point on X has at most one K-rational preimage on X', so we have succeeded in constructing a cover X' with fewer K-rational points than X, but still with at least one K-rational point. Iterating such covers eventually leads to a cover with exactly one K-rational point.

Therefore we now assume $X(K)$ consists of a single point P. Again perform a generic change of variables and define a covering X' as before, using $c_i = 1$ for $i \neq 0$ ($c_0 \in K^*$ will be chosen later). By Bertini's theorem, we may assume that the intersection Y of X' with the hyperplane $x_0 = 0$ is smooth and geometrically integral, and we may also assume that $P = (1 : 1 : \cdots : 1)$. Over all but finitely many completions of K, Y will have a point, and hence so will X'. Note that Y does not depend on c_0, so we are free to apply weak approximation to choose $c_0 \in K^*$ such that $c_0 \notin K^{*p}$ but such that $c_0 \in K_v^{*p}$ for each of the finitely many completions K_v of K for which $Y(K_v) = \emptyset$. This ensures that X' has a point (namely a preimage of P) over each of the finitely many completions remaining. On the other hand, $X'(K)$ is empty, since any K-rational point of X' would have to lie above P. $\square$

Corollary 2. *If there exists a smooth geometrically integral complete intersection X of dimension ≥ 3 in $\mathbf{P}^n$ over a number field K, such that the Zariski closure of $X(K)$ has codimension ≥ 2 in X, but $X(K)$ is not empty, then there exists also a 3-dimensional smooth geometrically integral complete intersection X' in projective space over K such that $X'(K)$ is empty, but such that the emptiness of $X'(K)$ cannot be explained by the Brauer-Manin obstruction or Skorobogatov's generalization thereof.*

Proof. Choose $P \in X(K)$. If $\dim X > 3$, then we may repeatedly use the Bertini-type theorems of [KA] to replace X by its intersection with a sufficiently general K-rational hypersurface of large degree m through P, in order to reduce to the case where $\dim X = 3$. Now apply Theorem 1. $\square$

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