Arithmetic Stratifications and Partial Eisenstein Series

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Abstract. Let $P\backslash G$ and $Q\backslash H$ be generalized flag varieties over a number field F. In this paper we study certain locally trivial fibre bundles Y_η over $P\backslash G$ having $Q\backslash H$ as general fibre, and determine the arithmetic stratification of Y_η with respect to a line bundle. The arithmetic stratification is defined in terms of height zeta functions and the height zeta function of a stratum is of the form

$$\sum_{\gamma\in P(F)\backslash G(F)} e^{\langle s\lambda, H_P(\gamma)\rangle} E_Q^{Qw^{-1}Q_0}(s\mu, \eta(p_\gamma)) \, ,$$

where $E_Q^{Qw^{-1}Q_0}$ is a "partial Eisenstein series" associated to the Schubert cell $Q\backslash Qw^{-1}Q_0$. The computation of the constant term of these gives estimates that allow one to determine the abcissa of convergence of the height zeta function of the stratum.

Résumé. Soient $P\backslash G$ et $H\backslash Q$ des variétés de drapeaux généralisées sur un corps de nombres F. Dans cet article nous étudions certaines fibrés localement triviales Y_η sur $P\backslash G$ ayant $Q\backslash H$ comme fibre générale, et nous déterminons la stratification arithmétique de Y_η relative à un faisceau inversible. La stratification arithmétique est définie par des fonctions zéta associées à un faisceau inversible métrisé et la fonction zéta d'une strate prend la forme

$$\sum_{\gamma\in P(F)\backslash G(F)} e^{\langle s\lambda, H_P(\gamma)\rangle} E_Q^{Qw^{-1}Q_0}(s\mu, \eta(p_\gamma)) \, ,$$

$E_Q^{Qw^{-1}Q_0}$ étant la série d'Eisenstein partielle associée à la cellule de Schubert $Q\backslash Qw^{-1}Q_0$. Le calcul du terme constant de celles-ci fournit des estimations permettent de déterminer l'abcisse de convergence de la fonction zéta d'une strate.

Contents

Introduction

Consider a projective variety Y over a number field F and a metrized line bundle $\mathbf{L} = (L, (\| \|_v)_v)$ on Y. For any subvariety $U \subset Y$ there is a counting function

$$N_U(\mathbf{L}, H) = \#\{y \in U(F) \mid H_{\mathbf{L}}(y) \leq H\}\,.$$

Suppose $N_U(\mathbf{L}, H)$ is finite for all H. Then one would like to understand the asymptotic behaviour of the counting function $N_U(\mathbf{L}, H)$ as H tends to infinity. A useful tool when studying this question is the (formal) Dirichlet series

$$Z_U(\mathbf{L}, s) = \sum_{y \in U(F)} \frac{1}{H_{\mathbf{L}}(y)^s}\,,$$

called a *height zeta function*. Whenever this series converges for all $s \gg 0$ let $\sigma_U(L)$ be its abcissa of convergence:

$$\sigma_U(L) = \inf\{s \in \mathbf{R} \mid Z_U(\mathbf{L}, s) \text{ converges}\}\,.$$

$\sigma_U(L)$ depends only on the line bundle L underlying $\mathbf{L}$ and not on the metric. $\sigma_U(L) = -\infty$ if and only if $U(F)$ is a finite; otherwise

$$\sigma_U(L) = \limsup \frac{\log N_U(\mathbf{L}, H)}{\log H}\,.$$

Sometimes one can go much further: namely, establish analytic continuation of the height zeta function. Then one even gets an asymptotic formula for the counting function, using a Tauberian Theorem (provided an assumption on the poles is satisfied). But before going into these questions, one encounters some features that should be considered beforehand.

A general phenomenon which may occur is this: there is a proper subvariety Y' in Y, such that $\sigma_Y(L) = \sigma_{Y'}(L) > \sigma_{Y-Y'}(L)$. Such a subvariety is called *accumulating*. In this case, counting rational points on Y' is asymptotically the same as counting rational points on Y, and therefore one should study Y' and $Y - Y'$ separately. It may happen that $Y - Y'$ contains again an accumulating subvariety. This leads us to the notion of the *arithmetic stratification* of Y with respect to L, introduced by V.V.Batyrev and Yu.I.Manin in [BM]. Suppose that $Y(F)$ is dense in Y and that there is an open subset Y' such that $Z_{Y'}(F)(\mathbf{L}, s)$ converges for all $s \gg 0$. The arithmetic stratification of Y with respect to L is the descending sequence of Zariski open subvarieties

$$Y \supset Y_0 \supset Y_1 \supset Y_2 \ldots$$

where Y_0 is the maximal Zariski open subset of Y for which the height zeta function $Z_{Y_0}(\mathbf{L}, s)$ converges for all $s \gg 0$, and $Y_{i+1} \subset Y_i$ is the maximal open subset of Y_i such that $\sigma_{Y_{i+1}}(L)$ is strictly smaller than $\sigma_{Y_i}(L)$. Note that the arithmetic stratification depends on the line bundle L. As an example, consider the variety obtained by blowing up the projective plane over F in a rational point. The projective line replacing this point will be an accumulating subvariety with respect to the anticanonical line bundle. For a further discussion cf. the introduction of [FMT] and [BM].

In this paper we study a class of smooth projective varieties, give a description of their Picard groups, and determine the arithmetic stratification with respect to an arbitrary line bundle. Here is a brief description of the varieties and the methods employed.

The varieties are locally trivial fibre bundles $Y_\eta \to W = P\backslash G$ over a generalized flag variety $P\backslash G$. The fibres are isomorphic to another flag variety $Q\backslash H$, and Y_η is constructed via a homomorphism $\eta : P \longrightarrow H$ which factors through a split torus. The Picard group is up to a finite cokernel given by the direct sum of character groups $X^*(Q) \oplus X^*(P)$. In $Pic(Y_\eta)_{\mathbf{R}}$ there is the closed cone $\Lambda(Y_\eta)_{\text{eff}}$ generated by the line bundles having non-zero global sections, and we denote the interior of this cone by $\Lambda(Y_\eta)^\circ_{\text{eff}}$. From now on we assume that η satisfies assumption 1.4 below. Let L be a line bundle on Y corresponding to the pair $(\mu, \lambda) \in X^*(Q) \oplus X^*(P)$. Suppose L lies in $\Lambda(Y_\eta)^\circ_{\text{eff}}$, and consider the arithmetic stratification $Y_\eta \supset Y_0 \supset Y_1 \supset Y_2! \ldots$ of Y_η with respect to L. It turns out that the strata $Y_i - Y_{i+1}$ are always unions of twisted products $Y^w = X^w \times^P G$ over $P\backslash G$. $X^w = Q\backslash Q w^{-1} Q_0$ is the Schubert cell in X corresponding to the Weyl group element w. The height zeta function of Y^w (with respect to a certain metric on L) is a double sum

$$\sum_{\gamma \in P(F)\backslash G(F)} e^{\langle s\lambda, H_P(\gamma)\rangle} E_Q^{Qw^{-1}Q_0}(s\mu, \eta(p_\gamma)) \,.$$

$E_Q^{Qw^{-1}Q_0}$ is what we call a "partial Eisenstein series", because it is a subseries of a Langlands-Eisenstein series: the summation runs only over $Q(F)\backslash Q(F)w^{-1}Q_0(F)$. The partial Eisenstein series can be bounded from above and from below by its constant term $E_{Q,Q_0}^{Qw^{-1}Q_0}$. The constant term in turn is equal to

$$e^{\langle s\check{\eta}(w\mu)-\check{\eta}(w\rho_0-\rho_0),H_P(\gamma)\rangle}E_{Q,Q_0}^{Qw^{-1}Q_0}(s\mu,1_H)\,.$$

Therefore, replacing the partial Eisenstein series by its constant term, one gets

$$E_{Q,Q_0}^{Qw^{-1}Q_0}(s\mu,1_H)\cdot E_P^G(s(\lambda+\check{\eta}(w\mu))-\check{\eta}(w\rho_0-\rho_0),1_G)\,.$$

This suffices to determine the abcissa of convergence of the height zeta functions of the strata. The arithmetic stratification with respect to L can be described purely in terms of a finite number of explicitely computable constants involving the parameters μ,λ and the Weyl group elements.

The results in this paper are part of the authors Ph.D. thesis, whose second part concerning asymptotic formulas shall appear in [St]. It is a pleasure to thank Jens Franke for introducing me to the methods used in this paper and for his constant support while working on these questions.

1 The fibre bundles: geometric-arithmetic preliminaries

(1.1) Let G and H be two semi-simple linear algebraic groups over the number field F, and fix minimal F-rational parabolic subgroups $P_0\subset G$ and $Q_0\subset H$ with (F-rational) Levi decompositions $P_0=L_0U_0$, $Q_0=M_0V_0$. The quotients of G, respectively H, by parabolic subgroups $P\supset P_0$, $Q\supset Q_0$ are (by definition) generalized flag varieties:

$$W=P\backslash G\quad\text{and}\quad X=Q\backslash H.$$

Consider a homomorphism $\eta:P\to T_0$, where T_0 is the maximal split torus in the center of M_0. P acts via η on $X\times G$ from the right: $(x,g)\cdot p=(x\eta(p),p^{-1}g)$, and there is a canonical morphism π from

$$Y=Y_\eta=(X\times G)/P$$

to W, giving Y the structure of a locally trivial fibre bundle over W with general fibre X, cf. [J], 5.14.

(1.2) Let $\mu\in X^*(Q)=Hom_{F\text{-}gps}(Q,\mathbf{G}_{m,F})$ be a character of Q and define an action of Q on $H\times\mathbf{A}^1$ by $q\cdot(h,a)=(qh,\mu(q)^{-1}a)$. The quotient of $H\times\mathbf{A}^1$ by Q is a line bundle over $X=Q\backslash H$, to be denoted by L_μ :

$$L_\mu = Q\backslash(H \times \mathbf{A}^1),$$

This gives an injection

$$X^*(Q) \longrightarrow Pic(X)$$

with finite cokernel, cf. for instance [Sa], Prop. 6.10, Lemma 6.9. The same applies to $X^*(P)$ and $Pic(W)$. The action of P on $H \times \mathbf{A}^1$, given by $(h,a)\cdot p = (h\eta(p), a)$, descends to an action of P on L_μ, and the quotient

$$L_\mu^Y = (L_\mu \times G)/P\,,$$

P acting on $L_\mu \times G$ by $(l,g)\cdot p = (l\cdot p, p^{-1}g)$, is a line bundle over Y. Using the exact sequence (6.10.2) in [Sa], it is not difficult to show that

$$(\mu,\lambda) \longmapsto \text{class of } L_\mu^Y \otimes \pi^* L_\lambda \tag{1.2.1}$$

is a monomorphism with finite cokernel

$$X^*(Q) \oplus X^*(P) \longrightarrow Pic(Y)\,.$$

(1.3) The choice of Q_0 determines a basis Δ_0^H of the root system of H with respect to T_0. The roots of T_0 occuring in the Lie algebra of the unipotent radical of Q_0 are defined to be positive and Δ_0^H is the basis of simple positive roots with respect to this set of positive roots. Let

$$\mathcal{W}_H = N_{H(F)}(T_0(F))/Z_{H(F)}(T_0(F))$$

be the Weyl group of H ($N_{H(F)}$ and $Z_{H(F)}$ denoting the normalizer, respectively centralizer in $H(F)$), and let $(.,.)$ be a $\mathcal{W}_H$-invariant positive definite form on $X^*(T_0)_{\mathbf{R}}$. Define the set

$$X^*(T_0)_{\mathbf{R}}^+ = \{x \in X^*(T_0)_{\mathbf{R}} \,|\, \text{for all}\, \beta \in \Delta_0^H : (x,\beta) > 0\,\}\,.$$

This is the positive Weyl chamber. Restriction of characters gives an injection and an isomorphism

$$X^*(Q)_{\mathbf{R}} \hookrightarrow X^*(Q_0)_{\mathbf{R}} \xrightarrow{\sim} X^*(T_0)_{\mathbf{R}}\,.$$

In the sequel we will always consider $X^*(Q)_{\mathbf{R}}$ as a subspace of $X^*(Q_0)_{\mathbf{R}}$ which we will identify with $X^*(T_0)$. Let M denote the unique Levi component of Q which contains M_0. There is a subset Δ_0^M of Δ_0^H such that

$$X^*(Q)_{\mathbf{R}} = \{x \in X^*(T_0)_{\mathbf{R}} \,|\, \text{for all}\, \beta \in \Delta_0^M : (x, \beta) = 0\,\}\,.$$

Denote by $X^*(Q)^+_{\mathbf{R}}$ the interior (in $X^*(Q)_{\mathbf{R}}$) of the intersection of $X^*(Q)_{\mathbf{R}}$ with the closure of $X^*(T_0)^+_{\mathbf{R}}$. Let Δ_Q^H be the complement of Δ_0^M in Δ_0^H. Then we have

$$X^*(Q)^+_{\mathbf{R}} = \{x \in X^*(Q)_{\mathbf{R}} \,|\, \text{for all}\, \beta \in \Delta_Q^H : (x, \beta) > 0\,\}\,.$$

It is well known, that under the isomorphism $X^*(Q)_{\mathbf{R}} \to Pic(X)_{\mathbf{R}}$ the closure of $X^*(Q)^+_{\mathbf{R}}$ is mapped onto the cone of effective divisors $\Lambda(X)_{\text{eff}}$, i.e., the closed cone that is generated by those line bundles having non-zero global sections (cf. [J], II, 2.6).

The definitions depend on the once and for all chosen minimal parabolic subgroup and the fixed Levi component M_0. Define $X^*(P)^+_{\mathbf{R}}$ by analogy (depending on P_0 and L_0).

The following assumption will be used to describe $\Lambda(Y)_{\text{eff}}$ and to determine the abcissa of convergence of the height zeta functions.

(1.4) Assumption. For all $\beta \in \Delta_0^H$ the element $-\beta \circ \eta$ of $X^*(P)$ lies in the closure of $X^*(P)^+_{\mathbf{R}}$.

Geometrically speaking, we consider those η which map the cone in $X^*(T_0)_{\mathbf{R}}$ generated by $-\Delta_0^H$ into the cone of effective divisors in $X^*(P)_{\mathbf{R}}$. This subset of $Hom_{F\text{-}gps}(P, T_0)$ generates a sublattice of finite index.
Moreover, it is easy to see that if one replaces η by a conjugate $w\eta w^{-1}$, w being an element of the Weyl group $\mathcal{W}_H$, the variety $Y_{w\eta w^{-1}}$ is isomorphic to Y_η. Sometimes such a conjugate of η satisfies (1.4). This is always the case if P is a maximal proper parabolic subgroup of G. Indeed, the group $X^*(P)$ is in this case of rank one, and if η is not trivial, the kernel of the induced homomorphism $\check{\eta} : X^*(T_0)_{\mathbf{R}} \to X^*(P)_{\mathbf{R}}$ is a hyperplane, consisting of elements which are orthogonal to some $\chi \in X^*(T_0)_{\mathbf{R}}$, orthogonal with respect to the $\mathcal{W}$-invariant positive definite form $(.,.)$ on $X^*(T_0)_{\mathbf{R}}$. Replacing χ by $-\chi$ if necessary, we can assume that the half-space of $\nu \in X^*(T_0)_{\mathbf{R}}$ with $(\nu, \chi) < 0$ is mapped to $X^*(P)^+_{\mathbf{R}}$ by $\check{\eta}$. Consider the (closure of the) chamber that contains χ and the base Δ of the root system that corresp! onds to that chamber. For $\beta \in \Delta$ one has $-\beta \circ \eta \in \overline{X^*(P)^+_{\mathbf{R}}}$, and this shows that a conjugate of η by an element of the Weyl group fulfills (1.4).

Identify $\mathcal{W}_H$ with its image in $Aut(X^*(T_0)_{\mathbf{R}})$. The reflections along the elements of Δ_0^H generate $\mathcal{W}_H$, and define the Bruhat ordering on $\mathcal{W}_H$ and a length function. Let w_H be the longest element of $\mathcal{W}$. Define

$$\Lambda_\eta = \{(\mu,\lambda) \in X^*(Q)_{\mathbf{R}} \oplus X^*(P)_{\mathbf{R}} \mid \mu \in \overline{X^*(Q)^+_{\mathbf{R}}} \text{ and } \check{\eta}(w_H\mu) + \lambda \in \overline{X^*(P)^+_{\mathbf{R}}}\ \}.$$

This is a closed cone in $X^*(Q)_{\mathbf{R}} \oplus X^*(P)_{\mathbf{R}}$.

(1.5) Proposition. *Suppose η fulfills assumption 1.4. Then the line bundle $L^Y_\mu \otimes \pi^* L_\lambda$ has non-zero global sections if and only if (μ,λ) lies in Λ_η.*

Proof. Suppose $H^0(Y, L^Y_\mu \otimes \pi^* L_\lambda) \neq 0$. The pull-back of $L^Y_\mu \otimes \pi^* L_\lambda$ by the quotient map $X \times G \longrightarrow Y$ is isomorphic to the pull-back of L_μ by the projection onto the first factor $X \times G \longrightarrow X$. It follows that L_μ has a non-zero global section, hence μ is an element of $\overline{X^*(Q)^+_{\mathbf{R}}}$ (cf. [J], II, 2.6). Let T_0 act on $L_\mu = Q\backslash(H \times \mathbf{A}^1)$ by $[h,a]t = [ht,a]$ and on $H^0(X, L_\mu)$ by $(t \cdot s)(x) = s(xt)t^{-1}$. Because T_0 is split, $H^0(X, L_\mu)$ decomposes as a direct sum of one-dimensional representations. For each $\chi \in X^*(T_0)$ let $m(\chi)$ be the multiplicity of the corresponding weight space. Then one has

$$\pi_*(L^Y_\mu \otimes \pi^* L_\lambda) \simeq \bigoplus_{\chi \in X^*(Q)} L^{\oplus m(\chi)}_{-\chi\circ\eta+\lambda}\,.$$

Hence, there is a χ with $m(\chi) > 0$ and $-\check{\eta}(\chi) + \lambda \in \overline{X^*(P)^+_{\mathbf{R}}}$. This implies

$$\check{\eta}(w_H\mu) + \lambda = -\check{\eta}(-w_H\mu - \chi) - \check{\eta}(\chi) + \lambda \in \overline{X^*(P)^+_{\mathbf{R}}}\,,$$

because $-w_H\mu - \chi \in \sum_{\beta\in\Delta_0^H} \mathbf{Z}_{\geq 0}$.

Now suppose that (μ,λ) lies in Λ_η. Then $-w_H\mu$ is a weight of the representation of T_0 on $H^0(X, L_\mu)$. Therefore,

$$\pi_*(L^Y_\mu \otimes \pi^* L_\lambda) \simeq L^{\oplus m(-w_H\mu)}_{(w_H\mu)\circ\eta+\lambda} \oplus \bigoplus_{\chi \neq -w_H\mu} L^{\oplus m(\chi)}_{-\chi\circ\eta+\lambda}$$

has a non-zero global section, and so does $L^Y_\mu \otimes \pi^* L_\lambda$. $\square$

By Proposition 1.5, under the isomorphism $X^*(Q)_{\mathbf{R}} \oplus X^*(P)_{\mathbf{R}} \longrightarrow Pic(Y)_{\mathbf{R}}$ induced by (1.2.1), the cone of effective divisors $\Lambda(Y_\eta)_{\text{eff}}$ is the image of Λ_η, and the interior Λ°_η of Λ_η is mapped onto $\Lambda(Y_\eta)^\circ_{\text{eff}}$.

(1.6) Next we discuss metrics on the line bundles introduced above. To do this, choose a maximal compact subgroup $\mathbf{K}_G = \prod_v \mathbf{K}_{G,v} \subset G(\mathbf{A})$, such that

$$G(\mathbf{A}) = P_0(\mathbf{A})\mathbf{K}_G \,,$$

where v runs over all places of F and $\mathbf{A}$ denotes the ring of adeles of F. For a place v and $w \in W(F_v)$ choose $k \in \mathbf{K}_{G,v}$ which is mapped to w. If $[g,a] \in L_\lambda(F_v)$ is an F_v-valued point of the line bundle L_λ over w, define its norm by

$$\|[g,a]\|_w = |\lambda(p)a|_v \,,$$

where $p \in P(F_v)$ is determined by $g = pk$ and $|\cdot|_v$ denotes the local absolute value that is the Haar multiplier. The family $\|\cdot\|_v = (\|\cdot\|_w)_{w \in W(F_v)}$ is a v-adic metric on L_λ and $\mathbf{L}_\lambda = (L_\lambda, (\|\cdot\|_v)_v)$ is a metrization of L_λ, cf. [P], Lemme 6.2.3. Define a map

$$H_P = H_{P,\mathbf{K}_G} : G(\mathbf{A}) \to Hom_{\mathbf{R}}(X^*(P) \otimes \mathbf{R}, \mathbf{R})$$

by $\langle \lambda, H_P(pk)\rangle = \log(\prod_v |\lambda(p_v)|_v)$, for $\lambda \in X^*(P)$, $p = (p_v)_v \in P(\mathbf{A})$, $k \in \mathbf{K}_G$ and extend linearly. An easy computation shows (cf. [FMT], p. 428) that for the (exponential) height function we have

$$H_{\mathbf{L}_\lambda}(w) = e^{-\langle \lambda, H_P(\gamma)\rangle} \,, \tag{1.6.1}$$

whenever $\gamma \in G(F)$ maps to $w \in W(F) = P(F)\backslash G(F)$.

Similarly, we fix a maximal compact subgroup $\mathbf{K}_H = \prod_v \mathbf{K}_{H,v} \subset H(\mathbf{A})$ such that $H(\mathbf{A}) = Q_0(\mathbf{A})\mathbf{K}_H$, and provide all line bundles L_μ, $\mu \in X^*(Q)$, with the metrics as defined above. Furthermore, $\mathbf{K}_H$ is supposed to induce maximal compact subgroups of all the standard Levi subgroups. More precisely, we assume that

for each parabolic subgroup $Q' \supset Q_0$ of H with Levi decomposition $Q' = M'V'$, $M' \supset M_0$, one has

- $Q'(\mathbf{A}) \cap \mathbf{K}_H = (M'(\mathbf{A}) \cap \mathbf{K}_H)(V'(\mathbf{A}) \cap \mathbf{K}_H)$,
- $M'(\mathbf{A}) \cap \mathbf{K}_H$ is a maximal compact subgroup of $M'(\mathbf{A})$.

By [MW], I.1.4., such $\mathbf{K}_H$ exist. This implies in particular that $\mathbf{K}_H$ contains the maximal compact subgroup of $T_0(\mathbf{A})$.

Finally, we define metrics on the line bundles $L_\mu^Y = (L_\mu \times G)/P$. Let $\|\cdot\|_v$ be the v-adic metric on L_μ defined by means of $\mathbf{K}_{H,v}$. Recall the action $L_\mu \times P \to L_\mu = Q\backslash(H \times \mathbf{A}^1)$, given by $l \cdot p = l \cdot \eta(p) = [h\eta(p), a]$ if $l = [h, a]$. For an element $[l, g] \in L_\mu(F_v)$ mapping to $y = [x, g] \in Y(F_v)$ define

$$\| \, [l, g] \, \|_y = \| \, l \cdot \eta(p) \, \|_{x\eta(p)} \, ,$$

where $g = pk$, $p \in P(F_v)$, $k \in \mathbf{K}_{H,v}$. It follows from the remark above, that this is well-defined (because $P(F_v) \cap \mathbf{K}_{G,v}$ is mapped into $\mathbf{K}_{H,v}$). The family $\| \cdot \|_v = (\| \cdot \|_y)_{y \in Y(F_v)}$ is a v-adic metric on L^Y_μ and $\mathbf{L}^Y_\mu = (L^Y_\mu, (\| \cdot \|_v)_v)$ is a metrization of L^Y_μ.
For $\gamma \in G(F)$ and $\delta \in H(F)$ mapping to $x \in X(F)$, the pair (x, γ) maps to an element $y \in Y(F)$, and

$$H_{\mathbf{L}^Y_\mu}(y) = e^{-\langle \mu, H_Q(\delta\eta(p)) \rangle} \quad , \tag{1.6.2}$$

with $g = pk$, $p \in P(\mathbf{A})$, $k \in \mathbf{K}_G$.

2 Height zeta functions

Denote by s a variable (taking values in $\mathbf{R}$). By (1.6.1) the height zeta function of the generalized flag variety $X = Q \backslash H$ corresponding to the metrized line bundle $\mathbf{L}_\mu$ is equal to (the value of) an *Langlands-Eisenstein series*, whenever the series converges:

$$Z_X(\mathbf{L}_\mu, s) = \sum_{x \in X(F)} H_{\mathbf{L}_\mu}(x)^{-s} = \sum_{\delta \in Q(F) \backslash H(F)} e^{\langle s\mu, H_Q(\delta) \rangle} = E^H_Q(s\mu, 1_H) \, .$$

This crucial observation is due to J.Franke. In [FMT] all analytic properties of these Eisenstein series that are needed for applications concerning rational points on X are proved. In this paper we only need results on the domain of convergence, and so we found it appropriate to prove these more accessible statements here again instead of deducing them from much deeper theorems. Hence our first task is to determine the domain of convergence of these Eisenstein series. Let ρ_Q be half the sum of the roots of T_0 that occur in the Lie algebra of V (= unipotent radical of Q), counted with multiplicity. For convenience, in the definition of E^H_Q, we shifted the coordinate on $X^*(Q) \otimes \mathbf{R}$ by $-\rho_Q$, compared with the usual convention.

The following elementary lemma will turn out to be very useful.

(2.1) Lemma. *Let $\mathcal{F} \subset H(\mathbf{A})$ and $\mathcal{C} \subset X^*(Q)_{\mathbf{R}}$ be relatively compact subsets. Then there is a constant $c > 1$ such that for all $v \in \mathcal{F}$, $\mu \in \mathcal{C}$ and $h \in H(\mathbf{A})$*

$$c^{-1}e^{\langle\mu,H_Q(h)\rangle} \leq e^{\langle\mu,H_Q(hv)\rangle} \leq ce^{\langle\mu,H_Q(h)\rangle}\,.$$

Proof. Put $h = qk$, $hv = q_1k_1$, $q, q_1 \in Q(\mathbf{A})$, $k, k_1 \in \mathbf{K}_H$. Note that $q^{-1}q_1 = kvk_1^{-1} \in \mathbf{K}_H\overline{\mathcal{F}}\mathbf{K}_H \cap Q(\mathbf{A})$ and the latter is a compact subset of $Q(\mathbf{A})$. Hence there is $c > 1$ such that for all $v_1 \in \mathbf{K}_H\mathcal{F}\mathbf{K}_H$ and $\mu \in \mathcal{C}$

$$c^{-1} \leq e^{\langle\mu,H_Q(v_1)\rangle} \leq c\,.$$

Hence we have for any triple $(\mu, v, h) \in \mathcal{C} \times \mathcal{F} \times H(\mathbf{A})$

$$\begin{array}{rcccl} c^{-1}e^{\langle\mu,H_Q(h)\rangle} & \leq & e^{\langle\mu,H_Q(q^{-1}q_1)\rangle}e^{\langle\mu,H_Q(q)\rangle} & = & e^{\langle\mu,H_Q(hv)\rangle}\,, \\ e^{\langle\mu,H_Q(hv)\rangle} & = & e^{\langle\mu,H_Q(q^{-1}q_1)\rangle}e^{\langle\mu,H_Q(q)\rangle} & \leq & ce^{\langle\mu,H_Q(hv)\rangle}\,. \end{array}$$

□

(2.2) The most important tool for us to investigate the Eisenstein series are Siegel domains. To introduce these, consider the homomorphism

$$\begin{array}{rrcl} \bar{\beta} = (\beta)_{\beta\in\Delta_0^H}: & T_0 & \to & \mathbf{G}_{m,F}^d\,, \\ & t & \mapsto & (\beta(t))_{\beta\in\Delta_0^H}\,. \end{array}$$

It is surjective with finite kernel ($d = dim(T)$). Embed $\mathbf{R}_{>0}$ into $\mathbf{A}^*$ by mapping $x \in \mathbf{R}$ to the idèle $(x_v)_v$ with archimedean components $x_v = x$ and non-archimedean components $x_v = 1$. This way we consider $\mathbf{R}_{>0}^d$ as a subgroup of $\mathbf{G}_{m,F}^d(\mathbf{A})$. Let A_{M_0} the connected component of the preimage of $\mathbf{R}_{>0}^d$ under $\bar{\beta}$, which contains the trivial element. For all $w \in \mathcal{W}$ we have $wA_{M_0}w^{-1} = A_{M_0}$. One has $M_0(\mathbf{A}) = A_{M_0}M_0(\mathbf{A})^1$ with

$$M_0(\mathbf{A})^1 = \bigcap_{\chi\in X^*(M_0)} ker(|\chi| : M_0(\mathbf{A}) \to \mathbf{R}_{>0})\,,$$

where $|\chi|((m_v)_v) := \prod_v |\chi(m_v)|_v$. For $t_0 \in A_{M_0}$ we define

$$A_{M_0}(t_0) = \{t \in A_{M_0} \,|\, \text{for all}\, \beta \in \Delta_0^H : \beta(t) > \beta(t_0)\,\}\,.$$

Recall the Levi decomposition $Q = MV$, M being the unique Levi component containing M_0. For a compact subset $\Omega \subset Q_0(\mathbf{A})$ put

$$\mathcal{S}_{Q,\Omega} = \Omega A_M A_{M_0}(t_0)\mathbf{K}_H\,,$$

where $A_M = A_{M_0} \cap Z(M(\mathbf{A}))$, $Z(M(\mathbf{A}))$ denoting the center of $M(\mathbf{A})$. If Ω is sufficiently big and t_0 is sufficiently small (i.e., $\beta(t_0)$ is sufficiently small for all $\beta \in \Delta_0^H$) then the following holds, c.f. [MW], I.2.1:

$$H(\mathbf{A}) = Q(F)\mathcal{S}_{Q,\Omega}\,.$$

In this case $\mathcal{S}_{Q,\Omega}$ is called a *Siegel domain.* With these preparations we can determine the abcissa of convergence of the Eisenstein series.

(2.3) Proposition. *a) The series*

$$E_Q^H(\mu+2\rho_Q,h) = \sum_{\delta\in Q(F)\backslash H(F)} e^{\langle \mu+2\rho_Q, H_Q(\delta h)\rangle}\,,$$

$\mu \in X^*(Q)\otimes \mathbf{C}$, $h\in H(\mathbf{A})$, converges uniformly for $(\Re(\mu),h)$ in any compact subset of $X^*(Q)^+_{\mathbf{R}}\times H(\mathbf{A})$.
b) For any $\mu\in X^*(Q)_{\mathbf{R}} - X^*(Q)^+_{\mathbf{R}}$ and any $h\in H(\mathbf{A})$ the series $E_Q^H(\mu+2\rho_Q,h)$ does not converge.

Proof. a) (This part is extracted from [Go].) For each $\beta\in\Delta_0^H$ there is a maximal proper parabolic subgroup Q_β containing Q_0 and a generator μ_β of $X^*(Q_\beta)$ lying in $X^*(Q_\beta)^+_{\mathbf{R}}$ such that

$$X^*(Q_0)^+_{\mathbf{R}} = \sum_{\beta\in\Delta_0^H} \mathbf{R}_{>0}\mu_\beta\,.$$

Attached to μ_β there is an ample line bundle L_{μ_β} on $Q_\beta\backslash H$. Hence, there exists $c_\beta>0$ such that for all $\delta\in H(F)$

$$e^{\langle \mu_\beta, H_{Q_\beta}(\delta)\rangle} = H_{\mathbf{L}_{\mu_\beta}}(Q_\beta(F)\delta)^{-1} \le c_\beta\,.$$

Now let $U\subset H(\mathbf{A})$ be an open and relatively compact subset. By Lemma (2.1) there is a constant $c>0$ such that for all $\beta\in\Delta_Q^H$, $\delta\in H(F)$ and $h\in U$

$$e^{\langle \mu_\beta, H_{Q_0}(\delta h)\rangle}\le c\,.$$

Therefore, $H(F)U$ is contained in $V_0(\mathbf{A})M_0(\mathbf{A})^1A^-_{M_0}(t_0^-)\mathbf{K}_H$ for some $t_0^-\in A_{M_0}$, where

$$A^-_{M_0}(t_0^-) = \{t\in A_{M_0}\,|\,\text{for all}\,\beta\in\Delta_0^H: \mu_\beta(t)<\mu_\beta(t_0^-)\}\,.$$

Next we use the fact that $H(\mathbf{A}) = Q(F)\mathcal{S}_{Q,\Omega}$ with some Siegel domain $\mathcal{S}_{Q,\Omega}$. By the definition of $\mathcal{S}_{Q,\Omega}$ we see that there is a fundamental domain for the action of $Q(F)$ on $H(F)U$ which is contained in

$$\Omega'(A_M\cap A^-_{M_0}(t_0^-))\mathbf{K}_H$$

for some compact subset $\Omega' \subset Q_0(\mathbf{A})$ which contains Ω. By Lemma 2.1, for h in a compact subset of $H(\mathbf{A})$, we can bound $E_Q^H(\mu + 2\rho_Q, h)$ by the integral

$$\int_{Q(F)\backslash H(F)U} e^{\langle \mu + 2\rho_Q, H_Q(h_1)\rangle} dh_1 \,,$$

with some Haar measure dh_1 on $H(\mathbf{A})$. Now, $dh_1 = e^{\langle -2\rho_Q, H_Q(m)\rangle} dv dm dk$ with Haar measures dv on $V(\mathbf{A})$, dm on $M(\mathbf{A})$, dk on $\mathbf{K}_H$ (recall $Q = MV$ and $H(\mathbf{A}) = Q(\mathbf{A})\mathbf{K}_H$). Therefore, the integral above can be bounded by

$$\int_{A_M \cap A_{M_0}^-(t_0^-)} e^{\langle \mu, H_Q(a)\rangle} da \,,$$

with some Haar measure da on A_M. Writing $\mu = \sum_{\beta \in \Delta_Q^H} s_\beta \mu_\beta$ with $s_\beta \in \mathbf{R}_{>0}$, we see that this integral is equal to

$$(\text{constant depending on } t_0^- \text{ and } da) \cdot \prod_{\beta \in \Delta_Q^H} \frac{1}{s_\beta} \,.$$

Hence we have shown that $E_Q^H(\mu + 2\rho_Q, h)$ converges uniformly for (μ, h) in compact subsets of $X^*(Q)_{\mathbf{R}}^+ \times H(\mathbf{A})$.

b) To prove the second assertion, it is enough to show that $E_Q^H(\mu + 2\rho_Q, 1_H)$ tends to infinity as μ approaches the boundary of $X^*(Q)_{\mathbf{R}}^+$.

Let $Q' \supset Q$ be a parabolic subgroup with Levi component $M' \supset M$ such that $Q \cap M'$ is a maximal proper parabolic subgroup of M'. Decompose $X^*(Q)_{\mathbf{R}}$ as follows:

$$X^*(Q)_{\mathbf{R}} = X^*(Q')_{\mathbf{R}} \oplus X^*(T_M/T_{M'})_{\mathbf{R}} \,,$$

where T_M (resp. $T_{M'}$) is the maximal split torus in the center of M (resp. M'). With respect to this decomposition we have $2\rho_Q = 2\rho_{Q'} + 2\rho_{Q\cap M'}^{M'}$, where $2\rho_{Q\cap M'}^{M'}$ is the sum of roots of T_0 (counted with multiplicity) which occur in the Lie algebra of the unipotent radical of $Q \cap M'$. $X^*(T_M/T_{M'})_{\mathbf{R}}$ is a one dimensional $\mathbf{R}$-vector space and the image of $X^*(Q)_{\mathbf{R}}^+$ under the projection onto this space is the open half-line generated by $\rho_{Q\cap M'}^{M'}$. Now let $\mu \in X^*(Q)_{\mathbf{R}}^+$ tend to the boundary of $X^*(Q)_{\mathbf{R}}^+$. Call μ' the image of μ under the projection onto $X^*(T_M/T_{M'})_{\mathbf{R}}$, and assume that μ' tends to zero. Next we note that

$$E_Q^H(\mu + 2\rho_Q, 1_H) = \sum_{\delta \in Q'(F)\backslash H(F)} e^{\langle \mu - \mu' + 2\rho_{Q'}, H_{Q'}(\delta)\rangle} E_Q^{Q'}(\mu' + 2\rho_{Q\cap M'}^{M'}, \delta) \,.$$

Hence it suffices to show that $E^{M'}_{Q\cap M'}(\mu' + 2\rho^{M'}_{Q\cap M'}, 1_H)$ tends to infinity as μ' tends to zero. In other words, it suffices to consider the case when Q is a maximal proper parabolic subgroup. Let Q^- be the opposite parabolic subgroup with Levi component V^-. The map from H to $Q\backslash H$ maps V^- isomorphically onto an open subset of $Q\backslash H$. On $V^-(\mathbf{A})$ the function $h \mapsto e^{\langle 2\rho_Q, H_Q(h)\rangle}$ is bounded from above but assumes every sufficiently small positive value. Let $\mathcal{F} \subset V^-(\mathbf{A})$ be an open and relatively compact subset such that $V^-(F)\mathcal{F} = V^-(\mathbf{A})$. Then, with the notations from part a) of the proof, there is some open and relatively compact subset $\omega \subset Q_0(\mathbf{A})$ such that a fundamental domain for the action of $Q(F)$ on $H(F)\mathcal{F}\mathbf{K}_H$ contains

$$\omega(A_M \cap A^-_{M_0}(t^-_0))\mathbf{K}_H .$$

Here we used the fact that A_M is one-dimensional. By Lemma 2.1, $E^H_Q(\mu + 2\rho_Q, 1_H)$ is bounded from below by the integral

$$\int_{Q(F)\backslash H(F)\mathcal{F}\mathbf{K}_H} e^{\langle \mu+2\rho_Q, H_Q(h)\rangle} dh\,,$$

with some Haar measure dh on $H(\mathbf{A})$, and this in turn can be bounded from below by

$$\int_{A_M\cap A^-_{M_0}(t^-_0)} e^{\langle \mu, H_Q(a)\rangle} da\,,$$

with some appropriate Haar measure da on A_M. This integral tends to infinity as μ tends to zero, and this completes the proof. □

(2.4) Of course, the Proposition 2.3 applies equally to the Eisenstein series

$$E^G_P(\lambda, G) = \sum_{\gamma\in P(F)\backslash G(F)} e^{\langle \lambda, H_P(\gamma g)\rangle} .$$

For $\mu \in X^*(Q)^+_{\mathbf{R}}$ denote by a_μ the abcissa of convergence of $E^H_Q(s\mu, h)$. By Prop. (2.3) we have

$$a_\mu = \inf\{a \mid a\mu \in 2\rho_Q + X^*(Q)^+_{\mathbf{R}}\,\} .$$

For $\mu \notin X^*(Q)^+_{\mathbf{R}}$ we put $a_\mu = \infty$. The height zeta function of the variety Y with respect to the metrized line bundle $\mathbf{L}^Y_\mu \otimes \pi^*\mathbf{L}_\lambda$ has, by (1.6.1) and (1.6.2), the following formal expression:

$$\begin{aligned}\sum_{y\in Y(F)} H_{\mathbf{L}_\mu^Y\otimes\pi^*\mathbf{L}_\lambda}(y)^{-s} &= \sum_{\gamma\in P(F)\backslash G(F)} e^{\langle s\lambda, H_P(\gamma)\rangle} \sum_{\delta\in Q(F)\backslash H(F)} e^{\langle s\mu, H_Q(\delta\eta(p_\gamma))\rangle} \\ &= \sum_{\gamma\in P(F)\backslash G(F)} e^{\langle s\lambda, H_P(\gamma)\rangle} E_Q^H(s\mu, \eta(p_\gamma))\end{aligned}$$

where $\gamma = p_\gamma k_\gamma$ with $p_\gamma \in P(\mathbf{A})$, $k_\gamma \in \mathbf{K}_G$. Even if $L_\mu^Y \otimes \pi^* L_\lambda$ lies in the interior of the cone of effective divisors $\Lambda(Y)_{\text{eff}}$, this series will in general not converge for any s. The reason for this is the occurrence of accumulating subvarieties. A posteriori it turns out that the right approach to find these is to decompose the fibre X into locally closed subvarieties, the images of the Bruhat cells:

$$X = \coprod_{w\in\mathcal{W}^M} X^w\,,$$

where $\mathcal{W}^M \subset \mathcal{W}$ consists of the elements of minimal length in the classes $w\mathcal{W}_M$, w running through $\mathcal{W}$, and $\mathcal{W}_M$ is the Weyl group of M with respect to T_0. X^w is the image of the generalized Bruhat cell $Qw^{-1}Q_0 \subset H$. Denote by w_H^M the element of minimal length in $w_H\mathcal{W}_M$. Then $X^{w_H^M}$ is the open stratum, which we denote by X°. The *partial Eisenstein series* are defined by summation over the rational points of X^w:

$$E_Q^{Qw^{-1}Q_0}(\mu, h) = \sum_{\delta\in Q(F)\backslash Q(F)w^{-1}Q_0(F)} e^{\langle\mu, H_Q(\delta h)\rangle}\,.$$

The behaviour of the partial Eisenstein series on Siegel domains is of importance for us. To understand how a partial Eisenstein series decreases when one goes to infinity on such a Siegel domain, we will study its *constant term.*

The constant term Θ_{Q_0} of a function $\Theta : H(\mathbf{A}) \to \mathbf{C}$, which is left-invariant under $V_0(F)$ and lies in $L_{1,\text{loc}}(H(\mathbf{A}))$, is defined by

$$\Theta_{Q_0}(h) = \int_{V_0(F)\backslash V_0(\mathbf{A})} \Theta(vh)dv\,,$$

where the measure dv on $V_0(F)\backslash V_0(\mathbf{A})$ is the quotient of the Haar measures and is normalized by $\int_{V_0(F)\backslash V_0(\mathbf{A})} dv = 1$.

The following statement is the key to determine the abcissa of convergence of the height zeta functions of the strata. We put $\rho_0 = \rho_{Q_0}$.

(2.5) Proposition. *Let $\mathcal{F} \subset H(\mathbf{A})$ and $\mathcal{C} \subset 2\rho_Q + X^*(Q)_{\mathbf{R}}^+$ be relatively compact subsets ($\mathcal{C}$ relatively compact in $X^*(Q)_{\mathbf{R}}$), and fix $t_0 \in A_{M_0}$.*

a) There is a constant $c > 1$, such that for all $\mu \in \mathcal{C}$, $w \in \mathcal{W}^M$, $t \in A_{M_0}(t_0)$ and $v \in \mathcal{F}$

$$c^{-1} E_{Q,Q_0}^{Qw^{-1}Q_0}(\mu, 1_H) \leq e^{-\langle w(\mu-\rho_0)+\rho_0, H_{Q_0}(t)\rangle} E_Q^{Qw^{-1}Q_0}(\mu, tv) \leq c E_{Q,Q_0}^{Qw^{-1}Q_0}(\mu, 1_H) .$$

b) Now let $w = w_H^M$. Then there is a constant $c > 1$, such that for all $\mu \in 2\rho_Q + X^*(Q)_{\mathbf{R}}^+$

$$E_Q^H(s\mu, 1_H) \leq c E_Q^{Qw^{-1}Q_0}(s\mu, 1_H) .$$

Proof. a) Let V_0 be the unipotent radical of Q_0, and let $\mathcal{F}_0 \subset V_0(\mathbf{A})$ be a compact subset such that $V_0(F)\mathcal{F}_0 = V_0(\mathbf{A})$. Because

$$\{t^{-1}vt \,|\, t \in A_{M_0}(t_0),\, v \in \mathcal{F}_0\}$$

is relatively compact, Lemma 2.1 ensures the existence of a $c > 1$ such that for all $h \in H(\mathbf{A})$, $t \in A_{M_0}$, $v \in \mathcal{F}$, $v_0 \in \mathcal{F}_0$ one has

$$c^{-1} e^{\langle \mu, H_Q(ht(t^{-1}v_0t))\rangle} \leq e^{\langle \mu, H_Q(htv)\rangle} \leq c e^{\langle \mu, H_Q(ht(t^{-1}v_0t))\rangle} .$$

Therefore

$$c^{-1} \int_{V_0(F)\backslash V_0(\mathbf{A})} E_Q^{Qw^{-1}Q_0}(\mu, v_0t) dv_0 \leq E_Q^{Qw^{-1}Q_0}(\mu, tv) \leq c \int_{V_0(F)\backslash V_0(\mathbf{A})} E^{Qw^{-1}Q_0}(\mu, v_0t) dv_0 .$$

Moreover,

$$\begin{aligned} \textstyle\int_{V_0(F)\backslash V_0(\mathbf{A})} E_Q^{Qw^{-1}Q_0}(\mu, v_0t) dv_0 &= \textstyle\int_{(V_0 \cap wV_0w^{-1})(\mathbf{A})\backslash V_0(\mathbf{A})} e^{\langle \mu, H_{Q_0}(w^{-1}v_0t))\rangle} dv_0 \\ &= e^{\langle w\mu+\rho_0-w\rho_0, H_Q(t)\rangle} E_{Q,Q_0}^{Qw^{-1}Q_0}(\mu, 1_H) . \end{aligned}$$

And this proves assertion a).

b) Because $H(F)$ is dense in H and $Qw^{-1}Q_0$ is open (and dense) in H, there are elements $\delta_1, \ldots, \delta_n \in H(F)$ such that

$$H(F) = \cup_{i=1}^n Q(F)w^{-1}Q_0(F)\delta_i .$$

Therefore,

$$E_Q^H(\mu, 1_H) \leq \textstyle\sum_{i=1}^n E_Q^{Qw^{-1}Q_0}(\mu, \delta_i) . \tag{2.5.1}$$

By Lemma 2.1, there exists $c_1 > 1$ such that for all $\mu \in \mathcal{C}$, $\delta \in H(F)$ and $i = 1, \ldots, n$

$$e^{\langle\mu,H_Q(\delta\delta_i)\rangle} \le c_1 e^{\langle\mu,H_Q(\delta)\rangle}\,.$$

The sum on the right of (2.5.1) can be bounded by $nc_1E_Q^{Qw^{-1}Q_0}(s\mu,1_H)$. □

(2.6) The following lemma explains the significance of assumption 1.4 : the geometric assertion that $\check\eta$ maps the cone in $X^*(T_0)_{\mathbf{R}}$ generated by $-\Delta_0^H$ into the cone $X^*(P)_{\mathbf{R}}^+ \subset X^*(P)_{\mathbf{R}}$, has as analytic consequence that there exists a fixed Siegel domain such that η maps the "P-part" of each rational element $\gamma \in G(F)$ into this fixed Siegel domain. To be more precise, put

$$T_0(\mathbf{A})^1 = M_0(\mathbf{A})^1 \cap T_0(\mathbf{A})\,,$$

cf. (2.2). By the product formula we have $T_0(F) \subset T_0(\mathbf{A})^1$, and the quotient $T_0(F)\backslash T_0(\mathbf{A})^1$ is compact. Let $\Omega_0 \subset T_0(\mathbf{A})^1$ be a compact subset such that $T_0(\mathbf{A})^1 = T(F)\Omega_0$.

(2.7) Lemma. *There is $t_0 \in A_{M_0}$, such that for all $\gamma \in G(F)$ one has*

$$\eta(p_\gamma) \in T_0(\mathbf{A})^1 A_{M_0}(t_0)\,,$$

where $\gamma = p_\gamma k_\gamma$, $p_\gamma \in P(\mathbf{A})$, $k_\gamma \in \mathbf{K}_G$.

Proof. For $\beta \in \Delta_0^H$ we have

$$e^{\langle\beta,H_{Q_0}(\eta(p_\gamma))\rangle} = e^{\langle\beta\circ\eta,H_P(\gamma)\rangle} = H_{\mathbf{L}_{-\beta\circ\eta}}(P(F)\gamma)\,.$$

By hypothesis, the line bundle $L_{-\beta\circ\eta}$ lies in the cone of effective divisors, hence the height function associated to $\mathbf{L}_{-\beta\circ\eta}$ is bounded from below. □

(2.8) Before proving our result about the abcissa of convergence we have to introduce the following constants. For $(\mu,\lambda) \in X^*(Q)_{\mathbf{R}} \oplus X^*(P)_{\mathbf{R}}$ and $w \in \mathcal{W}^M$ put

$$\begin{aligned}
\check a^w_{\mu,\lambda} &= \inf\{a \,|\, \text{for all } a' \ge a : a'(\lambda + \check\eta(w\mu)) - \check\eta(w\rho_0 - \rho_0) \in 2\rho_P + X^*(P)^+_{\mathbf{R}}\}\,,\\
a^w_{\mu,\lambda} &= \max\{a_\mu, \check a^w_{\mu,\lambda}\}\,.
\end{aligned}$$

In general, $\check a^w_{\mu,\lambda}$ may not be finite (by convention, the infimum of the empty set is ∞). For $w = w_H^M$ put $a^\circ_{\mu,\lambda} = a^w_{\mu,\lambda}$. The stratification of X gives rise to a stratification of Y because all strata $X^w = Q\backslash Qw^{-1}Q_0$ are stable under the action of T_0 from the right:

$$Y = \coprod_{w \in \mathcal{W}^M} Y^w \ \text{ with } Y^w = (X^w \times G)/P\,.$$

The open stratum, corresponding to $w = w_H^M$ will be denoted by Y°. The corresponding height zeta functions take the following form:

$$Z_{Y^w}(\mathbf{L}_\mu^Y \otimes \pi^* \mathbf{L}_\lambda, s) = \sum_{\gamma \in P(F)\backslash G(F)} e^{\langle s\lambda, H_P(\gamma)\rangle} E_Q^{Qw^{-1}Q_0}(s\mu, \eta(p_\gamma))\,,$$

with the notations from Lemma 2.7.

(2.9) Proposition. *a) If $\lambda + \check{\eta}(w\mu)$ is not contained in $X^+(P)_{\mathbf{R}}^+$, then $Z_{Y^w}(\mathbf{L}_\mu^Y \otimes \pi^* \mathbf{L}_\lambda, s)$ does not converge for any $s > 0$.*
b) If $\lambda + \check{\eta}(w\mu)$ is in $X^*(P)_{\mathbf{R}}^+$, then the abcissa of convergence of $Z_{Y^w}(\mathbf{L}_\mu^Y \otimes \pi^* \mathbf{L}_\lambda, s)$ is in the interval $[\check{a}_{\mu,\lambda}^w, a_{\mu,\lambda}^w]$.
c) Suppose that $L_\mu^Y \otimes \pi^* L_\lambda$ lies in $\Lambda(Y)_{\mathrm{eff}}^\circ$. Then the abcissa of convergence of $Z_{Y^\circ}(\mathbf{L}_\mu^Y \otimes \pi^* \mathbf{L}_\lambda, s)$ is $a_{\mu,\lambda}^\circ$.

Proof. a) Let us fix t_0 such that the assertion of Lemma 2.7 holds. Then we can write $\eta(p_\gamma) = \vartheta_\gamma o_\gamma a_\gamma$ with $\vartheta_\gamma \in T(F)$, $o_\gamma \in \Omega_0$ and $a_\gamma \in A_{M_0}(t_0)$. Since all partial Eisenstein series are left-invariant under $T(F)$ one has

$$E_Q^{Qw^{-1}Q_0}(s\mu, \eta(p_\gamma)) = E_Q^{Qw^{-1}Q_0}(s\mu, a_\gamma o_\gamma)\,.$$

For each real number $a > a_\mu$ there is by Proposition 2.5 a $c_1 > 0$ such that for all $w \in \mathcal{W}^M$ and all $s \in (a_\mu, a]$

$$c_1 E_{Q,Q_0}^{Qw^{-1}Q_0}(s\mu, 1_H) \le e^{-\langle w(s\mu - \rho_0) + \rho_0, H_{Q_0}(a_\gamma)\rangle} E_Q^{Qw^{-1}Q_0}(s\mu, \eta(p_\gamma)) \le c_1^{-1} E_{Q,Q_0}^{Qw^{-1}Q_0}(s\mu, 1_H)\,,$$

where $E_{Q,Q_0}^{Qw^{-1}Q_0}$ is the constant term of the partial Eisenstein series:

$$E_{Q,Q_0}^{Qw^{-1}Q_0}(s\mu, 1_H) = \int_{V_0(F)\backslash V_0(\mathbf{A})} E_Q^{Qw^{-1}Q_0}(s\mu, v_0)dv_0\,.$$

For $s \in (a_\mu, a]$ this function is bounded from below by a positive number c_2. For such s we have then

$$Z_{Y^w}(\mathbf{L}Y_\mu \otimes \pi^*\mathbf{L}_\lambda, s) \geq c_1 E_{Q,Q_0}^{Qw^{-1}Q_0}(s\mu, 1_H) \sum_{\gamma \in P(F)\backslash G(F)} e^{\langle s\lambda, H_P(a_\gamma)\rangle} e^{\langle sw\mu - w\rho_0 + \rho_0, H_{Q_0}(\eta(p_\gamma))\rangle}$$

$$\geq c_1 c_2 E_P^G(s(\lambda + \check{\eta}(w\mu)) - \check{\eta}(w\rho_0 - \rho_0), 1_G)\,.$$

It is well-known that $\rho_0 - w\rho_0$ lies in the closed cone generated by the positive simple roots, and by assumption 1.4 we conclude that $\check{\eta}(w\rho_0 - \rho_0)$ lies in the closure of $X^*(P)^+_{\mathbf{R}}$. Therefore, if $\lambda + \check{\eta}(w\mu)$ is not contained in $X^*(P)^+_{\mathbf{R}}$, the same is true for $s(\lambda + \check{\eta}(w\mu)) - \check{\eta}(w\rho_0 - \rho_0)$, for all $s > 0$. In this case, $Z_{Y^w}(\mathbf{L}_\lambda \otimes \pi^*\mathbf{L}_\mu, s)$ does not converge for any $s > 0$.
b) Suppose now that $\lambda + \check{\eta}(w\mu)$ is an element of $X^*(P)^+_{\mathbf{R}}$. The above estimate shows that the abcissa of convergence of $Z_{Y^w}(\mathbf{L}_\lambda \otimes \pi^*\mathbf{L}_\mu, s)$ is greater or equal to $\check{a}^w_{\mu,\lambda}$. On the other hand, using the estimate above, we get that

$$Z_{Y^w}(\mathbf{L}^Y_\mu \otimes \pi^*\mathbf{L}_\lambda, s) \leq c_1^{-1} E_{Q,Q_0}^{Qw^{-1}Q_0}(s\mu, 1_H) E_P^G(s(\lambda + \check{\eta}(w\mu)) - \check{\eta}(w\rho_0 - \rho_0), 1_G)\,.$$

Hence, this function converges for $s > \max\{a_\mu, \check{a}^w_{\mu,\lambda}\} = a^w_{\mu,\lambda}$, by Proposition 2.3 a).
c) Let $w = w^M_H$. By Proposition 2.5 b) and the estimate above, we have for $s \in (a_\mu, a]$

$$E_{Q,Q_0}^{Qw^{-1}Q_0}(s\mu, 1_H) \geq c_1 E_Q^{Qw^{-1}Q_0}(s\mu, 1_H) \geq c_3 E_Q^H(s\mu, 1_H)\,,$$

for some $c_3 > 0$. This shows that $Z_{Y^\circ}(\mathbf{L}^Y_\mu \otimes \pi^*\mathbf{L}_\lambda, s)$ converges only for $s > a^\circ_{\mu,\lambda}$. By part b) we conclude that the abcissa of convergence is exactly $a^\circ_{\mu,\lambda}$. □

3 Arithmetic stratification

In this section we determine the arithmetic stratification of $Y = (Q\backslash H \times G)/P$ with respect to line bundles $L^Y_\mu \otimes \pi^* L_\lambda$ which lie in the interior of the cone of effective divisors $\Lambda(Y)^\circ_{\text{eff}}$. Otherwise, there is no open non-empty subset for which the corresponding height zeta function converges for all $s \gg 0$. First we need a lemma.

(3.1) Lemma. *If $X^{w'} \subset \overline{X^w}$, then $a^{w'}_{\mu,\lambda} \geq a^w_{\mu,\lambda}$.*

Proof. The assertion follows if we can show that for all $a > a_\mu$ one has

$$a(\lambda + \check{\eta}(w\mu)) - \check{\eta}(w\rho_0 - \rho_0) - a(\lambda + \check{\eta}(w'\mu)) + \check{\eta}(w\rho_0 - \rho_0) \in \overline{X^*(P)^+_{\mathbf{R}}}\,.$$

Assumption 1.4 on η ensures us that this is true if we have

$$aw'\mu - w'\rho_0 - aw\mu + w\rho_0 \in \sum_{\beta \in \Delta_0^H} \mathbf{R}_{\geq 0}\beta$$

for all $a > a_\mu$. Note that

$$aw'\mu - w'\rho_0 - aw\mu + w\rho_0 = w'(a\mu - 2\rho_Q) - w(a\mu - 2\rho_Q) + w'(2\rho_Q - \rho_0) - w(2\rho_Q - \rho_0)\,.$$

Now let w_M be the longest element in $\mathcal{W}_M$. Then we have $w_H = w_H^M w_M$ and $2\rho_Q - \rho_0 = w_M\rho_0$. Therefore,

$$aw'\mu - w'\rho_0 - aw\mu + w\rho_0 = w'(a\mu - 2\rho_Q) - w(a\mu - 2\rho_Q) + w'w_M\rho_0 - ww_M\rho_0\,.$$

By [J], II, 13.8 (4), the assumption $X^{w'} \subset \overline{X^w}$ implies that $w' \leq w$ with respect to the Bruhat ordering on $\mathcal{W}_H$. It follows from the description of the Bruhat ordering given in [J], II, Proposition 13.7, together with

$$l(w'w_M) = l(w') + l(w_M),\ l(ww_M) = l(w) + l(w_M)$$

that moreover

$$w'w_M \leq ww_M\,.$$

Hence we are done, if we can show that whenever $w_1 \leq w_2$ and $x \in \overline{X^*(Q_0)^+_{\mathbf{R}}}$ one has

$$w_1x - w_2x \in \sum_{\beta \in \Delta_0^H} \mathbf{R}_{\geq 0}\beta\,.$$

For a root β of H with respect to T_0 we denote by s_β the reflection along β. By [BGG], Proposition 2.8, there exist positive roots $\beta_1, \ldots, \beta_k$ such that with $s_i = s_{\beta_i}$

$$\begin{gathered} w_2 = s_k \cdot s_{k-1} \cdot \ldots \cdot s_1 \cdot w_1\,, \\ \text{for all } i = 1, ..., k : l(s_i \cdot s_{i-1} \cdot \ldots \cdot s_1 \cdot w_1) = l(s_{i-1} \cdot \ldots \cdot s_1 \cdot w_1) + 1\,. \end{gathered}$$

Because

$$w_1x - w_2x = \sum_{i=1}^{k} s_{i-1} \cdot \ldots \cdot s_1 \cdot w_1x - s_i \cdot s_{i-1} \cdot \ldots \cdot s_1 \cdot w_1x\,,$$

we can assume that $w_2 = s_\beta w_1$ with a positive root β and $l(w_2) = l(w_1) + 1$. By [Bou], ch. VI, §1, Proposition 17, this implies $w_2^{-1}\beta < 0$, i.e. $w_1^{-1}\beta > 0$. Now suppose x lies in the closure of $X^*(Q_0)^+_{\mathbf{R}}$. Then one has

$$w_1x - w_2x = w_1x - (w_1x - 2\frac{(w_1x,\beta)}{(\beta,\beta)}\beta) = 2\frac{(x, w_1^{-1}\beta)}{(\beta,\beta)}\beta\,,$$

and this proves the assertion. □

To describe the arithmetic stratification, consider those $a^w_{\mu,\lambda}$ which are finite and order them:

$$\{a^w_{\mu,\lambda} |\, w \in \mathcal{W}^M, a^w_{\mu,\lambda} < \infty\} = \{a_0, a_1, \ldots, a_{r_{\mu,\lambda}}\}\,,$$

with $a_0 > a_1 > \ldots > a_{r_{\mu,\lambda}}$ and put formally $a_{-1} = \infty$.

(3.2) Theorem. *Let $Y = Y_\eta$ be the fibre bundle over $W = P\backslash G$ defined in (1.1), and suppose η fulfills the assumption of (1.4).*

a) If $L^Y_\mu \otimes \pi^* L_\lambda$ is not an element of $\Lambda(Y)^\circ_{\text{eff}}$, then there is no open non-empty subset such that the corresponding height zeta function converges for $s \gg 0$.

b) Suppose $L^Y_\mu \otimes \pi^* L_\lambda$ lies in $\Lambda(Y)^\circ_{\text{eff}}$. For $i = 0, 1, \ldots, r = r_{\mu,\lambda}$ put

$$Y_i = Y - \bigcup_{w\in\mathcal{W}^M,\, a^w_{\mu,\lambda} = a_{i-1}} \overline{Y^w}\,.$$

Then the arithmetic stratification of Y with respect to $L^Y_\mu \otimes \pi^* L_\lambda$ is

$$Y \supset Y_0 \supset Y_1 \supset \ldots \supset Y_r\,.$$

Proof. First we prove the second assertion. Lemma 3.1 implies that

$$Y_i = \bigcup_{w\in\mathcal{W}^M,\, a^w_{\mu,\lambda} \le a_i} Y^w\,.$$

Hence we conclude that Y_{i+1} is properly contained in Y_i $(i = 0, ..., r-1)$. Let $\sigma_{Y_i}(L)$ be the abcissa of convergence of $Z_{Y_i}(\mathbf{L}, s)$ with $\mathbf{L} = \mathbf{L}^Y_\mu \otimes \pi^* \mathbf{L}_\lambda$. We shall show that $\sigma_{Y_i}(L) = a_i$. Obviously,

$$Z_{Y_i}(\mathbf{L}, s) = \sum_{w\in\mathcal{W}^M,\, a^w_{\mu,\lambda} \le a_i} Z_{Y^w}(\mathbf{L}, s)\,.$$

By Proposition 2.9 b), $Z_{Y^w}(\mathbf{L},s)$ converges for $s > a^w_{\mu,\lambda}$. This shows that $Z_{Y_i}(\mathbf{L},s)$ converges for $s > a_i$. On the other hand, there is a $w \in \mathcal{W}^M$ such that $a^w_{\mu,\lambda} = a_i$ and in order for $Z_{Y^w}(\mathbf{L},s)$ to converge it is necessary that $s > \check{a}^w_{\mu,\lambda}$. Because the closure of X° is X, we have $a^\circ_{\mu,\lambda} = a_r$, by Lemma 3.1, and $Z_{Y^\circ}(\mathbf{L},s)$ is a summand of $Z_{Y_i}(\mathbf{L},s)$. By Proposition 2.5 c), we know that for $Z_{Y^\circ}(\mathbf{L},s)$ to converge it is necessary that $s > a_\mu$. Hence, for $Z_{Y_i}(\mathbf{L},s)$ to converge it is necessary that $s > \max\{\check{a}^w_{\mu,\lambda}, a_\mu\} = a^w_{\mu,\lambda} = a_i$ and we conclude that the abcissa of convergence of $Z_{Y_i}(\mathbf{L},s)$ is a_i.

Fix $i \in \{-1,\ldots,r\}$ and consider an open subset $Y' \subset Y_i$ which contains Y_{i+1} properly ($Y_{-1} = Y$, $Y_{r+1} = \emptyset$). The case $i = -1$ can only occur if $Z_Y(\mathbf{L},s)$ does not converge for any $s > 0$. We have to show that $Z_{Y'}(\mathbf{L},s)$ converges only for $s > a_i$ (and then this sum converges for all $s > a_i$). By hypothesis, Y' properly contains the open subset Y_{i+1}, hence there is a $w \in \mathcal{W}^M$ such that $a^w_{\mu,\lambda} = a_i$ und $Y' \cap Y^w \neq \emptyset$. $Y' \cap Y^w$ is an open and dense subset of Y^w. Therefore, there are open and dense subsets $X' \subset X^w$, $G' \subset G$, such that the image of $X' \times G'$ under the canonical projection $X \times G \to Y$ is contained in $Y' \cap Y^w$. Moreover, we may and we will assume that $PG' \subset G'$. Let $H' \subset H$ be the preimage of X' under! the projection $H \to X$. For $h \in H(\mathbf{A})$ put

$$E^{H'}_Q(s\mu,h) = \sum_{\delta \in Q(F)\backslash H'(F)} e^{\langle s\mu, H_Q(\delta t)\rangle} .$$

Then we have

$$Z_{Y'}(\mathbf{L},s) \geq \textstyle\sum_{\gamma \in P(F)\backslash G'(F)} e^{\langle s\lambda, H_P(\gamma)\rangle} E^{H'}_Q(s\mu, \eta(p_\gamma)) . \tag{3.2.1}$$

For each $\vartheta \in T_0(F)$ the set $H'\vartheta$ is open in $Qw^{-1}Q_0$, hence $\tilde{H} = \cup_{\vartheta \in T_0(F)} H'\vartheta$ is an open subset of $Qw^{-1}Q_0$ and there exist $\vartheta_1,\ldots,\vartheta_m \in T_0(F)$ such that $\tilde{H} = \cup_{1\leq j\leq m} H'\vartheta_j$. For $h \in H(\mathbf{A})$ we define as above

$$E^{\tilde{H}}_Q(s\mu,h) = \sum_{\delta \in Q(F)\backslash \tilde{H}(F)} e^{\langle s\mu, H_Q(\delta t)\rangle} .$$

Fix $a > a_\mu$. By Lemma 2.1 there is a $c_1 > 0$ such that for all $s \in (a_\mu, a]$ and $t \in T_0(\mathbf{A})$

$$E^{H'}_Q(s\mu,t) \geq c_1 E^{\tilde{H}}_Q(s\mu,t) . \tag{3.2.2}$$

Note that for all $\vartheta \in T_0(F)$ and $h \in H(\mathbf{A})$ one has $E^{\tilde{H}}_Q(s\mu,\vartheta h) = E^{\tilde{H}}_Q(s\mu,h)$. For each $\eta \in Q_0(F)$ the set $\tilde{H}\eta$ is open in $Qw^{-1}Q_0$. Therefore, there are $\eta_1,\ldots,\eta_n \in Q_0(F)$ such that $Qw^{-1}Q_0 = \cup_{1\leq k\leq n} \tilde{H}\eta_k$. Write $\eta(p_\gamma) = \vartheta_\gamma a_\gamma o_\gamma$ with $\vartheta_\gamma \in T_0(F)$, $o_\gamma \in \Omega_0$ and $a_\gamma \in A_{M_0}(t_0)$, where t_0 is as in Lemma 2.7. The set $\{a_\gamma^{-1}\eta_j a_\gamma o_\gamma \mid \gamma \in P(F)\backslash G'(F), j = 1,\ldots n\}$

is relatively compact in $H(\mathbf{A})$. So we can find $c_2 > 0$ such that for all $\gamma \in P(F)\backslash G'(F)$ and $j = 1, \dots, n$ we have

$$E_Q^{\tilde{H}}(s\mu, \eta_j a_\gamma o_\gamma) \geq c_2 E_Q^{\tilde{H}}(s\mu, a_\gamma o_\gamma) = c_2 E_Q^{\tilde{H}}(s\mu, \eta(p_\gamma)) \,.$$

Replacing nc_2 by c_2 we even get the following estimate

$$E_Q^{\tilde{H}}(s\mu, \eta(p_\gamma)) \geq c_2 E_Q^{Qw^{-1}Q_0}(s\mu, \eta(p_\gamma)) \ . \tag{3.2.3}$$

Putting (3.2.1), (3.2.2) and (3.2.3) together and using Proposition 2.5 we have

$$\begin{aligned} Z_{Y'}(\mathbf{L}, s) &\geq c_1 c_2 \textstyle\sum_{\gamma \in P(F)\backslash G'(F)} e^{\langle s\lambda, H_P(\gamma)\rangle} E_Q^{Qw^{-1}Q_0}(s\mu, \eta(p_\gamma)) \\ &\geq c_3 \textstyle\sum_{\gamma \in P(F)\backslash G'(F)} e^{\langle s(\lambda + \check{\eta}(w\mu)) - \check{\eta}(w\rho_0 - \rho_0), H_P(\gamma)\rangle} \,, \end{aligned}$$

with a suitable $c_3 > 0$. Because G' is dense in G there is $c_4 > 0$ such that

$$\sum_{\gamma \in P(F)\backslash G'(F)} e^{\langle s(\lambda + \check{\eta}(w\mu)) - \check{\eta}(w\rho_0 - \rho_0), H_P(\gamma)\rangle} \geq c_4 E_P^G(s(\lambda + \check{\eta}(w\mu)) - \check{\eta}(w\rho_0 - \rho_0)) \,.$$

(The argument is the same as in the proof of Prop. 2.5 b).) This shows that the abcissa of convergence of $Z_{Y'}(\mathbf{L}, s)$ is not less than $\check{a}_{\mu,\lambda}^w$. Because Y' is dense in Y, the intersection $Y' \cap Y^\circ$ is not empty. By the same reasoning as above we find $c_5 > 0$ such that

$$Z_{Y'}(\mathbf{L}, s) \geq c_5 \sum_{\gamma \in P(F)\backslash G'(F)} e^{\langle s\lambda, H_P(\gamma)\rangle} E_Q^{Q(w_H^M)^{-1}Q_0}(s\mu, \eta(p_\gamma)) \,.$$

Again by Proposition 2.5 b), $E_Q^{Q(w_H^M)^{-1}Q_0}(s\mu, \cdot)$ converges only for $s > a_\mu$, so we can conclude that $Z_{Y'}(\mathbf{L}, s)$ converges only for $s > \max\{\check{a}_{\mu,\lambda}^w, a_\mu\} = a_{\mu,\lambda}^w = a_i$, and this proves the second statement.

To prove the first assertion, observe that we have just seen that the height zeta function of any open non-empty subset can be bounded from below by the height zeta function of Y°. But this series does not converge for any $s > 0$ if $L_\mu^Y \otimes \pi^* L_\lambda$ is not in the interior of the positive cone, by proposition 2.9. $\square$

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