Hecke orbits on Siegel modular varieties

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Abstract We sketch a proof of the Hecke orbit conjecture. We also explain several techniques developed for the Hecke orbit problem, including a generalization of the Serre-Tate coordinates.

§1. Introduction

In this article we give an overview of the proof of a conjecture of F. Oort that every prime-to-p Hecke orbit in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties over $\overline{\mathbb{F}_p}$ is dense in the *leaf* containing it. See 4.1 for a precise statement, 2.4 for the definition of Hecke orbits, and 3.1 for the definition of a leaf. Roughly speaking, a *leaf* is the locus in $\mathcal{A}_g$ consisting of all points x such that the quasi-polarized Barsotti-Tate group attached to x belongs to a fixed isomorphism class, while the prime-to-p Hecke orbit of a point x consists of all points y such that there exists a prime-to-p quasi-isogeny from A_x to A_y which preserves the polarizations. Here A_x and A_y denote the principally polarized abelian varieties attached to x, y respectively; a prime-to-p quasi-isogeny is the composition of a prime-to-p isogeny with the inverse of a prime-to-p isogeny.

For clarity in logic, it is convenient to separate the Hecke orbit conjecture into two parts; the continuous part asserts that the Zariski closure of a prime-to-p Hecke orbit has the same dimension as the dimension of the leaf containing it, the discrete part asserts that the prime-to-p Hecke correspondences operate transitively on the set of irreducible components of every leaf; see 4.1.

The prime-to-p Hecke correspondences on $\mathcal{A}_g$ form a pretty large family of symmetries on $\mathcal{A}_g$; this fact leads to the expectation that every prime-to-p Hecke orbit should be "as large as possible". The decomposition of $\mathcal{A}_g$ into the disjoint union of leaves constitutes a "fine" geometric structure of $\mathcal{A}_g$, existing only in characteristic p and called *foliation* in [22]. The Hecke orbit conjecture says, in particular, that the foliation structure on $\mathcal{A}_g$ over $\overline{\mathbb{F}_p}$ is determined by the Hecke symmetries.

The prime-to-p Hecke orbit $\mathcal{H}^{(p)}(x)$ of a point x is a countable subset of $\mathcal{A}_g$. Experience indicates that determining the Zariski closure of a countable subset of an algebraic variety in positive characteristic is often difficult. We developed a number of techniques to deal with the Hecke orbit problem. They include

(M) the ℓ-adic monodromy of leaves,

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(C) the theory of canonical coordinates on leaves,

(R) a rigidity result for p-divisible formal groups,

(S) splitting at supersingular point,

(H) hypersymmetric points,

and will be described in §5, §7, §8, §11, and §10 respectively. We hope that the above techniques will also be useful in other situations. Among them, the most significant is perhaps the theory of canonical coordinates on leaves, which generalizes the Serre-Tate coordinates for the local moduli space of ordinary abelian varieties. At a non-ordinary closed point x of $\mathcal{A}_g$, there is no description of the formal completion $\mathcal{A}_g^{/x}$ of $\mathcal{A}_g$ at x comparable to what the Serre-Tate theory provides. But if we restrict to the leaf $\mathcal{C}$ passing through x, then there is a "good" structural theory of the formal completion $\mathcal{C}^{/x}$. To get an idea, the simplest situation is when the Barsotti-Tate group $A_x[p^\infty]$ is isomorphic to a direct product $X \times Y$, where X, Y are isoclinic Barsotti-Tate group of Frobenius slopes λ_X, λ_Y respectively, and $\lambda_X < \lambda_Y = 1 - \lambda_X$. In this case, $\mathcal{C}^{/x}$ has a natural structure as an isoclinic p-divisible formal group of height $\frac{g(g+1)}{2}$, Frobenius slope $\lambda_Y - \lambda_X$, and $\dim(\mathcal{C}^{/x}) = (\lambda_Y - \lambda_X) \cdot \frac{g(g+1)}{2}$. Moreover, there is a natural isomorphism of F-isocrystals

$$\mathrm{M}(\mathcal{C}^{/x}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{sym}}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y)) \otimes_{\mathbb{Z}} \mathbb{Q}\,,$$

where $\mathrm{M}(\mathcal{C}^{/x}), \mathrm{M}(X), \mathrm{M}(Y)$ denote the Cartier-Dieudonné modules of $\mathcal{C}^{/x}$, X, Y respectively, and the right-hand side of the formula denotes the symmetric part of the internal Hom, with respect to the involution induced by the principal polarization on A_x. In the general case, $\mathcal{C}^{/x}$ is build up by a successive system of fibrations, each fibration has a natural structure as a torsor for a suitable p-divisible formal group.

The fundamental idea underlying our method is to exploit the action of the local stabilizer subgroups. Recall that the prime-to-p Hecke correspondences come from the action of the action of the group $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ on the prime-to-p tower of the moduli space $\mathcal{A}_g$, where $\mathbb{A}_f^{(p)}$ denotes the restricted product of $\mathbb{Q}_\ell$'s, where ℓ runs through all primes not equal to p. Suppose $Z \subset \mathcal{A}_g$ is a closed subscheme of $\mathcal{A}_g$ which is stable under all prime-to-p Hecke correspondences. It is clear that for any closed point $x \in Z(k)$, the subscheme Z is stable under the set $\mathrm{Stab}(x)$ consisting of all prime-to-p Hecke correspondences having x as a fixed point. This is an elementary fact, referred to as the *local stabilizer principle*, to be transformed into a more usable form below.

The stabilizer $\mathrm{Stab}(x)$ comes from the unitary group $G_x = \mathrm{U}(\mathrm{End}_k(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}, *_x)$ attached to the pair $(\mathrm{End}_k(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}, *_x)$, where $*_x$ denotes the Rosati involution on the semisimple algebra $\mathrm{End}_k(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$. The group $G_x(\mathbb{Z}_p)$ of $\mathbb{Z}_p$-points of G_x is a subgroup of $\mathrm{U}(\mathrm{End}_k(A_x[p^\infty]), *_x)$; the latter operates naturally on the formal completion $\mathcal{A}_g^{/x}$ by deformation theory. Notice that G_x has a natural $\mathbb{Z}$-model attached to the $\mathbb{Z}$-lattice $\mathrm{End}_k(A_x) \subset$

$\mathrm{End}_k(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $G_x(\mathbb{Z}_p)$ is defined using that $\mathbb{Z}$-model. With the help of the weak approximation theorem, applied to G_x, the local stabilizer principle then says that the formal completion $Z^{/x}$ of Z at x, as a closed formal subscheme of $\mathcal{A}_g^{/x}$, is stable under the action of $G_x(\mathbb{Z}_p)$. See §6 for details.

The tools (C), (R), (H) mentioned above allows us to use the local stabilizer principle effectively. A useful consequence is that, if Z is a closed subscheme of $\mathcal{A}_g$ stable under all prime-to-p Hecke correspondences, and x is a *split hypersymmetric* point in Z, then Z contains an irreducible component of the leaf passing through x; see Thm. 10.2. Here a *split* point in $\mathcal{A}_{g,n}$ is a point y such that A_y is isogenous to a product of abelian varieties where each factor has at most two slopes, while a *hypersymmetric* point is a point y in $\mathcal{A}_{g,n}$ such that $\mathrm{End}_k(A_y) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}_k(A_y[p^\infty])$. It should not come as a surprise that the local stabilizer principle gives us a lot of information at a hypersymmetric point, where the local stabilizer subgroup is quite large.

Let $x \in \mathcal{A}_{g,n}(k)$ be a point of $\mathcal{A}_{g,n}$. Let $\overline{\mathcal{H}^{(p)}(x)}$ be the Zariski closure of the prime-to-p Hecke orbit $\mathcal{H}^{(p)}(x)$ of x, and let $\overline{\mathcal{H}^{(p)}(x)}^0 := \overline{\mathcal{H}^{(p)}(x)} \cap \mathcal{C}(x)$.[3] The conclusion of the last paragraph tells us that, to show that $\overline{\mathcal{H}^{(p)}(x)}$ is irreducible, it suffices to show that $\overline{\mathcal{H}^{(p)}(x)}^0$ contains a split hypersymmetric point. The result that $\overline{\mathcal{H}^{(p)}(x)}^0$ contains a split hypersymmetric point is accomplished through what we call the *Hilbert trick* and the *splitting at supersingular points*.

The Hilbert trick refers to a special property of $\mathcal{A}_g$: Up to an isogeny correspondence, there exists a Hilbert modular subvariety of maximal dimension passing through any given $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_g$; see §9. To elaborate a bit, let x be a given point of $\mathcal{A}_g(\overline{\mathbb{F}_p})$. The Hilbert trick tells us that there exists an isogeny correspondence f, from a g-dimensional Hilbert modular subvariety $\mathcal{M}_E \subset \mathcal{A}_g$ to $\mathcal{A}_g$, whose image contains x. The Hilbert modular variety above is attached to a commutative semisimple subalgebra E of $\mathrm{End}_{\overline{\mathbb{F}_p}}(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$, such that $[E : \mathbb{Q}] = g$ and E is fixed by the Rosati involution. There are Hecke correspondences on $\mathcal{M}_E$ coming from $\mathrm{SL}(2, E)$, and $\mathrm{SL}(2, E)$ can be regarded as a subgroup of Sp_{2g}. The isogeny correspondence f above respects the prime-to-p Hecke correspondences. So, among other things, the Hilbert trick tells us that, for an $\overline{\mathbb{F}_p}$-point x of $\mathcal{A}_{g,n}$ as above, the Hecke orbit $\mathcal{H}^{(p)}(x)$ contains the f-image of a prime-to-p Hecke orbit $\mathcal{H}_E^{(p)}(\tilde{x})$ on the Hilbert modular variety $\mathcal{M}_E$, where $\tilde{x}$ is a pre-image of x under the isogeny correspondence f.

A consequence of the Hilbert trick and the local stabilizer principle, is the following trick of "splitting at supersingular points"; see 11.1. This "splitting trick" says that, in the interior of the Zariski closure of a given Hecke orbit, there exists a point y such that A_y is a *split* abelian variety. The last clause means that A_y is isogenous to a product of abelian varieties, each of which has at most two slopes.

One can formulate the notion of leaves and the Hecke orbit conjecture for Hilbert modular

[3]In fact $\overline{\mathcal{H}^{(p)}(x)}^0$ is equal to the subset of $\overline{\mathcal{H}^{(p)}(x)}$ consisting of all points y in $\overline{\mathcal{H}^{(p)}(x)}$ such that the Newton polygon of A_y is equal to the Newton polygon of A_x.

varieties. It turns out that the Hecke orbit problem for Hilbert modular varieties is easier to solve than the Siegel modular varieties, reflecting the fact that the a Hilbert modular variety comes from a reductive group G over $\mathbb{Q}$ such that every $\mathbb{Q}$-simple factor of the adjoint group G^{ad} has $\mathbb{Q}$-rank one. The trick "splitting at infinity" and a standard technique in algebraic geometry implies that, when one tries to prove the Hecke orbit conjecture, one may assume that the point x of $\mathcal{A}_g$ is defined over $\overline{\mathbb{F}_p}$ and the abelian variety A_x is split. Now we apply the Hilbert trick to x. To simplify the exposition, we will assume, for simplicity, that we have a Hilbert modular variety $\mathcal{M}_E$ in $\mathcal{A}_g$ passing through the point x, suppressing the isogeny correspondence f. We will also assume (or "pretend") that the leaf $\mathcal{C}_E(x)$ on $\mathcal{M}_E$ passing through x is the intersection of $\mathcal{C}(x)$ with $\mathcal{M}_E$. (The last assumption is not far from the truth, if we interpret "intersection" as a suitable fiber product.)

It is easy to see that every leaf in $\mathcal{M}_E$ contains a hypersymmetric point y of $\mathcal{A}_g$. Moreover A_y is split because A_x is split. So if we can prove the Hecke orbit conjecture for $\mathcal{M}_E$, then we will know that the Zariski closure of the Hecke orbit $\mathcal{H}^{(p)}(x)$ in $\mathcal{C}(x)$ contains a split hypersymmetric point y. Therefore the Hecke orbit conjecture for Hilbert modular varieties implies the continuous part of the Hecke orbit conjecture for $\mathcal{A}_g$. See also 12.6.1 for a sketch.

The general methods we developed, when applied to $\mathcal{M}_E$, produce a proof of the continuous part of the Hecke orbit conjecture for $\mathcal{M}_E$. Thus we are left with the discrete part of the Hecke orbit conjecture for $\mathcal{A}_g$ and the Hilbert modular varieties.

The discrete Hecke orbit problem is equivalent to the statement that every non-supersingular leaf is irreducible, see Thm. 5.1; the same holds for Hilbert modular varieties. Generally such irreducibility statements do not come by easily; so far there is no unified approach which works for all modular varieties of PEL-type. Using the techniques (H) and (M), one can reduce the discrete Hecke orbit conjecture for $\mathcal{A}_g$ to the statement that the prime-to-p Hecke correspondences operates transitively on the set of irreducible components of every non-supersingular Newton polygon strata in $\mathcal{A}_g$. Happily the results of Oort in [20], [21] can be applied to settle the latter irreducibility statement; see 13.1.1, [24], and references cited in 13.1.1. The discrete Hecke orbit problem for the Hilbert modular varieties, however, requires a different approach, based on the Lie-alpha stratification of Hilbert modular varieties, and the following property of Hilbert modular varieties: For each slope data ξ for $\mathcal{M}_E$, there exists a Lie-alpha stratum $\mathcal{N}_{\underline{e},\underline{a}} \subset \mathcal{M}_E$, contained in the Newton stratum in $\mathcal{M}_E$ attached to the given slope data ξ, and a dense open subset $\mathcal{U}_{\underline{e},\underline{a}}$ of $\mathcal{N}_{\underline{e},\underline{a}}$ such that $\mathcal{U}_{\underline{e},\underline{a}}$ is a leaf in $\mathcal{M}_E$. A critical step in the proof of the discrete Hecke orbit problem, due to C.-F. Yu, is constructing "enough" deformations to analyze the incidence relation of the Lie-alpha stratification; see 13.3.

Details of the proof of the Hecke orbit conjecture will appear in a manuscript with F. Oort. All unattributed results are due to suitable subsets of {Oort, Yu, CLC}. The author is responsible for all errors and imprecision.

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§2. Hecke orbits

(2.1) Let p be a prime number, fixed throughout this article. Let $\mathbb{Z}_f^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$, where ℓ runs through all prime numbers different from p. Let $\mathbb{A}_f^{(p)} := \prod'_{\ell \neq p} \mathbb{Q}_\ell \cong \mathbb{Z}_f^{(p)} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the ring of prime-to-p finite ideles attached to $\mathbb{Q}$.

Let k be an algebraically closed field of characteristic p. Choose and fix an isomorphism $\zeta : \mathbb{Z}_f^{(p)} \xrightarrow{\sim} \mathbb{Z}_f^{(p)}(1)$ over k, i.e. a compatible system of isomorphisms $\zeta_m : \mathbb{Z}/m\mathbb{Z} \simeq \mu_m(k)$, where m runs through all positive integers which are not divisible by p. For any natural number g and any integer $n \geq 3$ with $(n,p) = 1$, denote by $\mathcal{A}_{g,n}$ the moduli space over k classifying g-dimensional principally polarized abelian varieties with a symplectic level-n structure with respect to ζ.

(2.2) For any two integers $n_1, n_2 \geq 3$, such that $(p, n_1 n_2) = 1$ and $n_1 | n_2$, there is a canonical map $\mathcal{A}_{g,n_2} \to \mathcal{A}_{g,n_1}$. Denote by $\mathcal{A}_{g,(p)}$ the resulting projective system of the moduli spaces $\mathcal{A}_{g,n}$, where n runs through all integers $n \geq 3$ with $(p,n) = 1$. By definition, a geometric point in $\mathcal{A}_{g,(p)}(k)$ corresponds to a triple (A, λ, η), where A is a g-dimensional principally polarized abelian variety over k, λ is a principal polarization on A, and η is a level-$\mathbb{Z}_f^{(p)}$ structure on A, i.e. η is a symplectic isomorphism from $\prod_{\ell \neq p} A[\ell^\infty]$ to $(\mathbb{Z}_f^{(p)})^{2g}$, where the free $\mathbb{Z}_f^{(p)}$-module $(\mathbb{Z}_f^{(p)})^{2g}$ is endowed with the standard symplectic pairing.

(2.3) From the definition of $\mathcal{A}_{g,(p)}$ we see that there is a natural action of $\mathrm{Sp}_{2g}(\mathbb{Z}_f^{(p)})$ on $\mathcal{A}_{g,(p)}$, operating as covering transformations over the moduli stack $\mathcal{A}_g$. Moreover there is a natural action of the group $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ on $\mathcal{A}_{g,(p)}$, extending the action of $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ and gives a much larger collection of symmetries on the tower $\mathcal{A}_{g,(p)}$. The automorphism h_γ of $\mathcal{A}_{g,(p)}$ attached to an element $\gamma \in \mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ is characterized by the property that there is a prime-to-p isogeny α_γ from the universal abelian variety A to $h_\gamma^* A$ such that

$$\eta \circ \alpha_\gamma[(p)] = \gamma \circ \eta \,,$$

where $\alpha_\gamma[(p)]$ denotes the prime-to-p quasi-isogeny induced by α_γ, between the prime-to-p-divisible groups attached to A and $h_\gamma^* A$ respectively. On each individual moduli space $\mathcal{A}_{g,n}$, the action of $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ induces algebraic correspondences to itself; they are the classical Hecke correspondences on the Siegel moduli spaces.

(2.4) Definition Let $n \geq 3$ be an integer, $(n,p)=1$. Let $x \in \mathcal{A}_{g,n}(k)$ be a geometric point of $\mathcal{A}_{g,n}$, and let $\tilde{x} \in \mathcal{A}_{g,(p)}(k)$ be a geometric point of the tower $\mathcal{A}_{g,(p)}$ above x.

(i) The *prime-to-p Hecke orbit* of x in $\mathcal{A}_{g,n}$, denoted by $\mathcal{H}^{(p)}(x)$, or $\mathcal{H}(x)$ for short, is the image of the subset $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)}) \cdot \tilde{x}$ of $\mathcal{A}_{g,(p)}$ under the projection map $\pi_n : \mathcal{A}_{g,(p)} \to \mathcal{A}_{g,n}$.

(ii) Let ℓ be a prime number, $\ell \neq p$. The *ℓ-adic Hecke orbit* of x in $\mathcal{A}_{g,n}$, denoted by $\mathcal{H}_\ell(x)$, is the image of $\mathrm{Sp}_{2g}(\mathbb{Q}_\ell) \cdot \tilde{x}$ under $\pi : \mathcal{A}_{g,(p)} \to \mathcal{A}_{g,n}$.

(2.4.1) Remark (i) It is easy to see that the definition of $\mathcal{H}_\ell(x)$ does not depend on the choice of $\tilde{x}$. One can also use the ℓ-adic tower above $\mathcal{A}_{g,n}$ to define the ℓ-adic Hecke orbits.

(ii) Explicitly, the countable set $\mathcal{H}^{(p)}(x)$ (resp. $\mathcal{H}_\ell(x)$) consists of all points $y \in \mathcal{A}_{g,n}(k)$ such that there exists an abelian variety B over k and two prime-to-p isogenies (resp. ℓ-power isogenies) $\alpha : B \to A_x$, $\beta : B \to A_y$ such that $\alpha^*(\lambda_x) = \beta^*(\lambda_y)$.

(iii) The moduli stack $\mathcal{A}_g$ over k has a natural pro-étale $\mathrm{GSp}_{2g}(\mathbb{Z}_f^{(p)}$ cover; and the group $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)}$ operate on the projective limit. Then for any geometric point $x \in \mathcal{A}_{g,n}(k)$, we can define the $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})$-orbit of x and the $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$-orbit of x) as in Def. 2.4 using the pro-étale $\mathrm{GSp}_{2g}(\mathbb{Z}_f^{(p)}$-tower. Explicitly, the $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})$-orbit of x (resp. the $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$-orbit of x) on $\mathcal{A}_{g,n}$ for a geometric point $x \in \mathcal{A}_{g,n}(k)$ can be explicitly described as follows. It consists of all points $y \in \mathcal{A}_{g,n}(k)$ such that there exists a prime-to-p isogeny (resp. an ℓ-power isogeny) $\beta : A_x \to A_y$ such that $\beta^*(\lambda_y) = m(\lambda_x)$, where m is a prime-to-p positive integer (resp. a non-negative integer power of ℓ.)

(2.4.2) Remark In 2.4 we used the group $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ to define the prim-to-p Hecke orbits of a closed point x in $\mathcal{A}_{g,n} \to \mathrm{Spec}(k)$. Geometrically that means consider the orbit of x under all prime-to-p symplectic quasi-isogenies. One can also consider the orbit of x under all symplectic quasi-isogenies, or, as a slight variation, the orbit of x under all quasi-isogenies which preserve the polarization up to a multiple. The latter was used in [19, 15.A]. We considered only the prime-to-p Hecke correspondences in this article, since they are finite étale correspondences on $\mathcal{A}_{g,n}$, and reflect well the underlying group-theoretic properties.

(2.5) For any totally real number field F and any integer $n \geq 3$, $(n,p)=1$, denote by $\mathcal{M}_{F,n}$ the Hilbert modular variety over k attached to F as defined in [10]. Just as in the case of the Siegel modular varieties, the varieties $\mathcal{M}_{F,n}$ form a projective system, with a natural action by the group $\mathrm{SL}_2(F \otimes_{\mathbb{Q}} \mathbb{A}_f^{(p)})$, The prime-to-$p$ Hecke orbit $\mathcal{H}_F^{(p)}(x)$ and the ℓ-adic Hecke orbit $\mathcal{H}_{F,\ell}(x)$ of a geometric point $x \in \mathcal{M}_{F,n}(k)$ are, by definition, the image in $\mathcal{M}_{F,n}(k)$ of $\mathrm{SL}_2(F \otimes_{\mathbb{Q}} \mathbb{A}_f^{(p)}) \cdot \tilde{x}$ and $\mathrm{SL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \cdot \tilde{x}$ respectively, where $\tilde{x}$ is a k-valued point, lying above x, of the projective system $\mathcal{M}_{F,(p)} := \{\mathcal{M}_{F,m} \,:\, (m,p)=1\}$.

(2.5.1) More generally, if $E = F_1 \times \cdots \times F_r$ is a product of totally real number fields, we can define the Hilbert modular variety $\mathcal{M}_E$ over k attached to E, in the same fashion as in [10], with $\mathcal{O}_E := \mathcal{O}_{F_1} \times \cdots \times \mathcal{O}_{F_r}$. Then we have a canonical isomorphism $\mathcal{M}_E = \mathcal{M}_{F_1} \times \cdots \times \mathcal{M}_{F_r}$. The notion of Hecke orbits generalizes in the obvious way to the present situation.

(2.5.2) Remark The notion of prime-to-p Hecke orbits can be generalized to other modular varieties over k of PEL-type in a natural way. Furthermore, one expects that the notion of prime-to-p Hecke orbits can be generalized to the reduction over k of a Shimura variety X, with satisfactory properties.

§3. Leaves

In this section we work over an algebraically closed field k of characteristic $p > 0$. The modular varieties $A_{g,n}$ and $\mathcal{M}_{E,n}$ are considered over the fixed based field k.

(3.1) Theorem (Oort) *Let $n \geq 3$ be an integer, $(n,p) = 1$. Let $x \in \mathcal{A}_{g,n}(k)$ be a geometric point of $\mathcal{A}_{g,n}$.*

(i) *There exists a unique reduced constructible subscheme $\mathcal{C}(x)$ of $\mathcal{A}_{g,n}$, called the* leaf *passing through x, characterized by the following property. For every algebraically closed field $K \supseteq k$, $\mathcal{C}(x)(K)$ consists of all elements $y \in \mathcal{A}_{g,n}(K)$ such that*

$$(A_x[p^\infty], \lambda_x[p^\infty]) \times_{\operatorname{Spec} k} \operatorname{Spec} K \simeq (A_y[p^\infty], \lambda_y[p^\infty]) ,$$

where $\lambda_x[p^\infty], \lambda_y[p^\infty]$ are the principal quasi-polarizations induced by the principal polarizations λ_x, λ_y on the abelian varieties $A_x[p^\infty], A_y[p^\infty]$ respectively.

(ii) *The leaf $\mathcal{C}(x)$ is a locally closed subscheme of $\mathcal{A}_{g,n}$. Moreover it is smooth over k.*

(3.1.1) Remark (i) Thm. 3.1 is proved in [22, 3.3, 3.14]. The statement that the subset of $\mathcal{A}_{g,n}(k)$ consisting of all geometric points y such that $(A_y[p^\infty], \lambda_y[p^\infty])$ is isomorphic to $(A_x[p^\infty], \lambda_x[p^\infty])$ is the set of geometric points of a constructible subset of $\mathcal{A}_{g,n}$, follows from the following fact, proved in Manin's thesis [15]: A Barsotti-Tate group over k of a given height h is determined, up to non-unique isomorphism, by its truncation modulo a sufficiently high level $N \geq N(h)$.

(ii) T. Zink showed, in a letter to C.-L. Chai dated May 1, 1999, the following generalization of Manin's result: A *crystal* M over k is determined, up to non-unique isomorphisms, by its quotient modulo p^N, for some suitable $N > 0$ depending only on the height of M.

(iii) In [22], $\mathcal{C}(x)$ is called the *central leaf* passing through x.

(iv) It is clear from the definition that each leaf in $\mathcal{A}_{g,n}$ is stable under all prime-to-p Hecke correspondences. In particular, the Hecke orbit $\mathcal{H}^{(p)}(x)$ is contained in the leaf $\mathcal{C}(x)$ passing through x.

(v) Every leaf is contained in an *open Newton stratum* of $\mathcal{A}_{g,n}$, and every open Newton stratum is a disjoint union of leaves. Recall that an *open Newton stratum* $W^0_\xi(\mathcal{A}_{g,n})$ in $\mathcal{A}_{g,n}$

over k is, by definition, the subset of $\mathcal{A}_{g,n}$ such that $W^0_\xi(\mathcal{A}_{g,n})(K)$ consists of all K-points y of $\mathcal{A}_{g,n}$ such that the Newton polygon of $A_y[p^\infty]$ is equal to ξ, for all fields $K \supset k$. By Grothendieck-Katz, $W^0_\xi(\mathcal{A}_{g,n})$ is a locally closed subset of $\mathcal{A}_{g,n}$; see [13] for a proof. There are infinitely many leaves in $\mathcal{A}_{g,n}$ if $g \geq 2$. In particular the decomposition of $\mathcal{A}_{g,n}$ into a disjoint union of leaves is not a stratification in the usual sense: There are infinitely many leaves, and the closure of some leaves contain infinitely many leaves.

(3.1.2) Examples.

(i) The ordinary locus of $\mathcal{A}_{g,n}$, that is the largest open subscheme of $\mathcal{A}_{g,n}$ over which each geometric fiber of the universal abelian scheme is an ordinary abelian variety, is a leaf.

(ii) The "almost ordinary" locus of $\mathcal{A}_{g,n}$, or, the locus consisting of all geometric points x such that the maximal étale quotient of the attached Barsotti-Tate group $A_x[p^\infty]$ has height $g-1$, is a leaf.

(iii) Every supersingular leaf in $\mathcal{A}_{g,n}$ is finite over k. Hence there are infinitely many supersingular leaves in $\mathcal{A}_{g,n}$ if $g \geq 2$.

(iv) Consider the open Newton polygon stratum $\mathcal{W}^0_\xi(\mathcal{A}_{3,n})$ in $\mathcal{A}_{3,n}$, where the Newton polygon ξ has slopes $(\frac{1}{3}, \frac{2}{3})$. Every leaf $\mathcal{C}$ contained in $\mathcal{W}^0_\xi(\mathcal{A}_{3,n})$ is two-dimensional, while $\dim(\mathcal{W}^0_\xi(\mathcal{A}_{3,n})) = 3$.

(3.2) Proposition *Let $\mathcal{C}$ be a leaf in $\mathcal{A}_{g,n}$. For each integer $N \geq 1$, denote by $A[p^N] \to \mathcal{C}$ p^N-torsion subgroup scheme of the restriction to $\mathcal{C}$ of the universal abelian scheme. Then there exists a finite surjective morphism $f : S \to \mathcal{C}$ such that $(A[p^N], \lambda[p^N]) \times_{\mathcal{C}} S$ is a constant polarized truncated Barsotti-Tate group over S.*

See [22, 1.3] for a proof of 3.2.

(3.2.1) Using Prop. 3.2, one can show that there exist finite surjective isogeny correspondences between any two leaves lying in the same open Newton stratum; see [22, Lemma 3.14]. In particular, any two leaves in the same open Newton stratum have the same dimension.

(3.2.2) Remark In this article we have focused our attention on leaves in $\mathcal{A}_{g,n}$ over k. The notion of leaves can be extended to other modular varieties of PEL-type in a similar way, and the basic properties of leaves, including 3.1, 3.2, 3.3, can all be generalized; some of the generalized statements become a little stronger. It is expected that the notion of leaves can be defined on reduction over k of a Shimura variety X, with nice properties.

(3.3) Proposition *Let $\mathcal{C}$ be a leaf in $\mathcal{A}_{g,n}$. Denote by $A[p^\infty] \to \mathcal{C}$ the Barsotti-Tate group attached to the restriction to $\mathcal{C}$ of the universal abelian scheme. Then there exists a slope filtration on $A[p^\infty] \to \mathcal{C}$. More precisely, there exist Barsotti-Tate subgroups*

$$0 = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_m = A[p^\infty]$$

of $A[p^\infty] \to \mathcal{C}$ over the leaf $\mathcal{C}$ such that G_i/G_{i-1} is a Barsotti-Tate group over $\mathcal{C}$ with a single Frobenius slope μ_i, $i = 1, \dots, m$, and $\mu_1 > \mu_2 > \cdots > \mu_m$.

Remark (i) The statement that $H_i := G_i/G_{i-1}$ has Frobenius slope μ_i means that there exists constants $c, d > 0$ such that

$$\mathrm{Ker}([p^{\lfloor N\mu_i - c\rfloor}]) \subseteq \mathrm{Ker}(\mathrm{Fr}_{H_i}^{(p^N)}) \subseteq \mathrm{Ker}([p^{\lfloor N\mu_i + d\rfloor}])$$

for all $N \gg 0$.

(ii) The Frobenius slopes measures divisibility property of the Frobenius: A Barsotti-Tate group X has slope μ if $(\mathrm{Fr}_X)^N/p^{\mu N}$ and $p^{\mu N}/(\mathrm{Fr}_X)^N$ are both bounded as $N \to \infty$. In the literature the terminology "slope" is sometimes also used to measure the divisibility of the Verschiebung, hence we use "Frobenius slope" to avoid possible confusion.

(iii) When all fibers of $\mathcal{A}[p^\infty]$ at points of $\mathcal{C}$ are *completely slope divisible*, the existence of the slope filtration was proved by in [31]; see also [26]. The statement of Prop. 3.3 has not appeared in the literature, but the following stronger statement can be deduced from the main results of [31] and [26]: If $S \to \mathrm{Spec}(\mathbb{F}_p)$ is an integral noetherian normal scheme of characteristic p, and G is a Barsotti-Tate group over S which is geometrically fiber-wise constant, then $G \to S$ admits a slope filtration.

(iv) The slope filtration on a leaf holds the key to the theory of canonical coordinates on a leaf; see §7.

(v) It is clear that on a Barsotti-Tate group over a reduced base scheme S over k, there exists at most one slope filtration.

(vi) One can construct a Barsotti-Tate group over G a smooth base scheme S over k, for instance $\mathbb{P}^1$, such that G does not have a slope filtration.

(3.4) Denote by $\Pi_0(\mathcal{C}(x))$ the scheme of geometrically *irreducible* components of $\mathcal{C}(x)$, or equivalently, the set of geometrically *connected* components of $\mathcal{C}(x)$, since $\mathcal{C}(x)$ is smooth over k. The scheme $\Pi_0(\mathcal{C}(x))$ is finite and étale over k; this assertion holds even if the base field k is not assumed to be algebraically closed.

(3.5) Let $E = F_1 \times \cdots \times F_r$ be the product of totally real fields $F_1, \dots, F_r$, and let $n \geq 3$ be an integer with $(n, p) = 1$. The notion of *leaves* can be extended to the Hilbert modular variety $\mathcal{M}_{E,n}$ over k, as follows. Let $x \in \mathcal{M}_{E,n}(k)$ be a geometric point of the Hilbert modular variety $\mathcal{M}_{E,n}(k)$. The *leaf* in $\mathcal{M}_{E,n}$ passing through x is the smooth locally closed subscheme $\mathcal{C}_E(x)$, characterized by the property that $\mathcal{C}_E(x)(K)$ consists of all geometric points $y \in \mathcal{M}_{E,n}(K)$ such that there exists an $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$-linear isomorphism from $A_y[p^\infty]$ to $A_x[p^\infty]$ compatible with the $\mathcal{O}_E$-polarizations, for every algebraically closed field $K \supset k$.

(3.5.1) Just as in the case of Siegel modular varieties, each leaf in $\mathcal{M}_{E,n}$ is stable under all prime-to-p Hecke correspondences on $\mathcal{M}_{E,n}$.

(3.5.2) The slope filtration on the Barsotti-Tate group over a leaf in $\mathcal{M}_{E,n}$ takes the following form. Let $\mathcal{C}_E$ be a leaf in $\mathcal{M}_{E,n}$, and denote by G the Barsotti-Tate group attached to the restriction to $\mathcal{C}_E$ of the universal abelian scheme over $\mathcal{C}_E$. Write $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{j=1}^{s} \mathcal{O}_{E_{\wp_j}}$, where each $\mathcal{O}_{E_{\wp_j}}$ is a complete discrete valuation ring. The natural action of $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$ on G gives a decomposition

$$G = G_1 \times \cdots \times G_s \,,$$

where each G_j is a Barsotti-Tate group over $\mathcal{C}_E$, with action by $\mathcal{O}_{E_{\wp_j}}$, and the height of G_j is equal to $2\,[\mathcal{O}_{E_{\wp_j}} : \mathbb{Z}_p]$. Moreover, if G_j is not isoclinic of slope $\frac{1}{2}$, then there exists a Barsotti-Tate subgroup $H_j \subset G_j$ over $\mathcal{C}_E$, stable under the action of $\mathcal{O}_{E_{\wp_j}}$, such that

- the height of H_j is equal to $[\mathcal{O}_{E_{\wp_j}} : \mathbb{Z}_p]$,
- both H_j and G_j/H_j are isoclinic, of Frobenius slopes μ_j, μ'_j respectively, and
- $\mu_j > \mu'_j$ and $\mu_j + \mu'_j = 1$

for $j = 1, \ldots, s$.

§4. The Hecke orbit conjecture

Let k be an algebraically closed field of characteristic p, and let $n \geq 3$ be an integer, $(n,p) = 1$.

(4.1) Conjecture *Denote by $\mathcal{A}_{g,n}$ the moduli space of g-dimensional principally polarized abelian varieties over k with symplectic level-n structures as before.*

(i) **(HO)** *For any geometric point x of $\mathcal{A}_{g,n}$, the Hecke orbit $\mathcal{H}^{(p)}(x)$ is dense in $\mathcal{C}(x)$.*

(ii) $(\mathbf{HO})_{\mathrm{ct}}$ *For any geometric point x of $\mathcal{A}_{g,n}$, we have $\dim(\overline{\mathcal{H}^{(p)}(x)}) = \dim(\mathcal{C}(x))$, where $\overline{\mathcal{H}^{(p)}(x)}$ denotes the Zariski closure of the countable subset $\mathcal{H}^{(p)}(x)$ in $\mathcal{A}_{g,n}$. Equivalently, $\overline{\mathcal{H}^{(p)}(x)}$ contains the irreducible component of $\mathcal{C}(x)$ passing through x.*

(iii) $(\mathbf{HO})_{\mathrm{dc}}$ *For any geometric point x of $\mathcal{A}_{g,n}$, the canonical map*

$$\Pi_0(\overline{\mathcal{H}^{(p)}(x)}^{\circ}) \to \Pi_0(\mathcal{C}(x))$$

is surjective, where $\overline{\mathcal{H}^{(p)}(x)}^{\circ} := \overline{\mathcal{H}^{(p)}(x)} \cap \mathcal{C}(x)$ denotes the Zariski closure of the Hecke orbit $\mathcal{H}^{(p)}(x)$ in the leaf $\mathcal{C}(x)$. In other words, the prime-to-p Hecke correspondences operate transitively on the set $\Pi_0(\mathcal{C}(x))$ of geometrically irreducible components of $\mathcal{C}(x)$.

(4.1.1) Remark (i) The conjecture (HO) is due to Oort, see [22, 6.2]. It implies Conj. 15.A in [19], which asserts that the orbit of a point x in $\mathcal{A}_{g,n}(k)$ under all Hecke correspondences, including all purely inseparable ones, is Zariski dense in the Newton polygon stratum containing x.

(ii) It is clear that the conjecture (HO) is equivalent to the conjunction of $(\mathrm{HO})_{\mathrm{ct}}$ and $(\mathrm{HO})_{\mathrm{dc}}$. We call $(\mathrm{HO})_{\mathrm{ct}}$ (resp. $(\mathrm{HO})_{\mathrm{dc}}$) the continuous (resp. discrete) part of the Hecke orbit conjecture (HO).

(iii) The conjecture $(\mathrm{HO})_{\mathrm{dc}}$ is essentially an irreducibility statement; see Thm. 5.1.

(iv) We can also formulate the ℓ-adic version of the Hecke orbit conjecture, $(\mathrm{HO})_\ell$, for any prime number $\ell \neq p$. It asserts that $\mathcal{H}_\ell(x)$ is dense in $\mathcal{C}(x)$. One can define the continuous part $(\mathrm{HO})_{\ell,\mathrm{ct}}$, and the discrete part $(\mathrm{HO})_{\ell,\mathrm{dc}}$ of $(\mathrm{HO})_\ell$ as in 4.1. Clearly, $(\mathrm{HO})_\ell \Longleftrightarrow (\mathrm{HO})_{\ell,\mathrm{ct}} + (\mathrm{HO})_{\ell,\mathrm{dc}}$.

(v) Thm. 5.1 tells us that $(\mathrm{HO})_{\ell,\mathrm{dc}} \Longleftrightarrow (\mathrm{HO})_{\mathrm{dc}}$, and $(\mathrm{HO})_\ell \Longleftrightarrow (\mathrm{HO})$. Strictly speaking, Thm. 5.1 gives the implications when the Hecke orbit in question is not supersingular, however the supersingular case can be dealt with directly, using the weak approximation theorem.

(4.1.2) Let E be a finite product of totally real number fields, and let $\mathcal{M}_E$ be the Hilbert modular variety over k attached to E. Then we can formulate the Hecke orbit conjectures for $\mathcal{M}_n$ as in 4.1, and will use $(\mathrm{HO})_E$, $(\mathrm{HO})_{E,\mathrm{ct}}$, and $(\mathrm{HO})_{E,\mathrm{dc}}$ to denote the Hecke orbit conjecture for $\mathcal{M}_n$ and its two parts. Remark 4.1.1 (ii), (iii), (iv) hold in the present context.

(4.1.3) Remark The Hecke orbit conjecture(s) can be formulated for other modular varieties of PEL-type, and the reduction over k of any Shimura variety X if one is optimistic. It should be noted, however, that the statement in 4.1.1 (iii) needs to be modified, using the $G^{\mathrm{der}}(\mathbb{A}_f^{(p)})$-orbit instead of the $G(\mathbb{A}_f^{(p)}$-orbit, where G is the connected reductive group over $\mathbb{Q}$ in the input data of the Shimura variety X.

(4.2) Theorem *The Hecke orbit conjecture* (HO) *holds for the Siegel modular varieties. In other words, every prime-to-p Hecke orbit is Zariski dense in the leaf containing it.*

(4.2.1) Remark The Hecke orbit conjecture $(\mathrm{HO})_\ell$ also holds for $\mathcal{A}_{g,n}$, for any prime number $\ell \neq p$. Although $(\mathrm{HO})_\ell$ appears to be a stronger statement than (HO), it is essentially equivalent to it, by Thm. 5.1.

(4.3) In the rest of this note we present an outline of the proof of Thm. 4.2. We have already seen that Thm. 5.1 on ℓ-adic monodromy groups is helpful in clarifying the discrete Hecke orbit conjecture, and for the equivalence between $(\mathrm{HO})_\ell$ and (HO). The foundation underlying our approach is the *local stabilizer principle*, explained in §6; this principle is quite general and can be applied to all PEL-type modular varieties. We will also use a special property of the Siegel modular varieties, called the *Hilbert trick*, explained in §9. That property holds for modular varieties of PEL-type C, but not for PEL-type A or D. Both the local stabilizer principle and the Hilbert trick were used in [2]; the former was used not only for points in the ordinary locus, but also the zero-dimensional cusps and supersingular points.

There are several techniques which make the local stabilizer principle more potent. They include:

(C) the theory of canonical coordinates on leaves, generalizing Serre-Tate parameters for local moduli space of ordinary abelian varieties,

(R) a rigidity result for p-divisible formal groups,

(S) a trick "splitting at supersingular points", exploiting the action of the local stabilizer subgroup at supersingular points,

(H) a trick of using "hypersymmetric points" on a leaf.

They are explained in §7, §8, §11, §10 respectively. Among them, the methods (C), (R), (H) can be generalized to all modular varieties of PEL-type, while (S) depends on the Hilbert trick, therefore applies only to modular varieties of PEL-type C.

(4.4) The Hecke orbit conjecture for the Hilbert modular varieties enters the proof of $(\mathrm{HO})_{\mathrm{ct}}$ for $\mathcal{A}_g$ at an critical point, through the Hilbert trick.

(4.4.1) Theorem *The Hecke orbit conjecture holds for Hilbert modular varieties. In other words, every prime-to-p Hecke orbit in a Hilbert modular variety is Zariski dense in the leaf containing it.*

See 13.2 and 13.3 for a description of the proof of Thm. 4.4.1.

§5. ℓ-adic monodromy of leaves

Theorem 5.1 below explores the relation between the Hecke symmetries and the ℓ-adic monodromy. It asserts that the ℓ-adic monodromy of any non-supersingular leaf on $\mathcal{A}_g$ is maximal. A byproduct of 5.1, from group theoretic consideration, is an irreducibility statement. The irreducibility statement implies that for a non-supersingular leaf $\mathcal{C}$ in $\mathcal{A}_g$, the discrete part $(\mathrm{HO})_{\mathrm{dc}}$ of the Hecke orbit conjecture holds for $\mathcal{C}$ if and only if $\mathcal{C}$ is irreducible.

(5.1) Theorem *Let k be an algebraically closed field of characteristic p. Let $n \geq 3$ be a natural number which is prime to p. Let ℓ be a prime number $\ell \not| pn$. Let Z be a smooth locally closed subvariety of $\mathcal{A}_{g,n}$ over k. Assume that Z is stable under all ℓ-adic Hecke correspondences coming from $\mathrm{Sp}_{2g}(\mathbb{Q}_\ell)$, and that the ℓ-adic Hecke correspondences operate transitively on the set of irreducible components of Z. Let $A \to Z$ be the restriction to Z of the universal abelian scheme. Let Z_0 be an irreducible component of Z, and let $\bar{\eta}$ be a geometric generic point of Z_0. Assume that $A_{\bar{\eta}}$ is not supersingular. Then the image $\rho_{A,\ell}(\pi_1(Z_0,\bar{\eta}))$ of the ℓ-adic monodromy representation of $A \to Z_0$ is equal to $\mathrm{Sp}(\mathrm{T}_\ell, \langle\ ,\ \rangle_\ell) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$, where $\mathrm{T}_\ell = \mathrm{T}_\ell(A_{\bar{\eta}}) = \varprojlim_n A[\ell^n](\bar{\eta})$ denotes the ℓ-adic Tate module of $A_{\bar{\eta}}$. Moreover $Z = Z_0$, i.e. Z is irreducible, and Z is stable under all prime-to-p Hecke correspondences on $\mathcal{A}_{g,n}$.*

(5.1.1) Remark (i) Theorem 5.1 is handy when one tries to prove the irreducibility of certain subvarieties of $\mathcal{A}_g$.

(ii) The proof of 5.1 can be generalized to other modular varieties of PEL-type, but one has to make suitable modification of the statement if the derived group of G is not simply connected.

(iii) The proof of Thm. 5.1 is mostly group-theoretic; the algebro-geometric input is the semisimplicity of the ℓ-adic monodromy group.

§6. The action of the local stabilizer subgroup

(6.1) Let k be an algebraically closed field of characteristic p. Let $n \geq 3$ be an integer, $(n,p) = 1$. Let ℓ be a prime number, $\ell \neq p$. Let $Z \subset \mathcal{A}_{g,n}$ be a reduced closed subscheme stable under all ℓ-adic Hecke correspondences. In other words, Z is a union of ℓ-adic Hecke orbits. Let $x = ([A_x, \lambda_x]) \in Z(k)$ be a closed point of Z. Let $E = \mathrm{End}_k(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}_p$, and let $*$ be the Rosati involution of E induced by the principal polarization λ_x. Let

$$H = \{u \in E^\times | u \cdot u^* = u^* \cdot u = 1\}$$

be the unitary group attached to the pair $(E \otimes_{\mathbb{Q}} \mathbb{Q}_p, *)$. Let $U_x := H \cap \mathrm{End}_k(A_x[p^\infty])^\times$, called the local stabilizer subgroup at $x \in \mathcal{A}_g(k)$.

Similarly, let $\tilde{E} := \mathrm{End}_k(A_x[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and let $\tilde{*}$ be the involution on $\tilde{E}$ induced by λ_x. Denote by $\tilde{H}$ the unitary group attached to the pair $(\tilde{E}, \tilde{x})$, and let $\tilde{U}_x = \tilde{H} \cap \mathrm{End}_k(A_x[p^\infty])^\times$. The group $\tilde{U}_x$ operates naturally on $\mathcal{A}_{g,n}^{/x}$ by deformation theory. Since there is a natural inclusion $U_x \hookrightarrow \tilde{U}_x$, the subgroup U_x inherits an action on $\mathcal{A}_{g,n}^{/x}$.

(6.2) Proposition (local stabilizer principle) *Notation as above. Then the closed formal subscheme $Z^{/x}$ of $\mathcal{A}_{g,n}^{/x}$ is stable under the action of the local stabilizer subgroup U_x on $\mathcal{A}_{g,n}^{/x}$*

Sketch of Proof. Let $\underline{U}$ be the unitary group attached to the pair $(E, *)$, a reductive linear algebraic group over $\mathbb{Q}$. In particular the weak approximation theorem holds for $\underline{U}$. Choose and fix a "standard embedding" $\underline{U}(\mathbb{A}_f^{(p)}) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ coming from a choice of symplectic level-$\mathbb{Z}_f^{(p)}$ structure of A_x. Then every element of the subgroup $\underline{U}(\mathbb{A}_f^{(p)})$ of $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ gives rise to a prime-to-p Hecke correspondence having x as a fixed point. For any given element $\gamma_p \in U_x$, choose an element $\gamma \in \underline{U}(\mathbb{Q})$ close to γ_p in $\underline{U}(\mathbb{Q}_p)$. Note that the image of γ in $\underline{U}(\mathbb{A}_f^{(p)})$ gives rise to a prime-to-p Hecke correspondence, which has x as a fixed point and sends the formal subscheme $Z^{/x}$ of $\mathcal{A}_{g,n}^{/x}$ to $Z^{/x}$ itself. Interpreted in terms of deformation theory, the last assertion implies that a formal neighborhood $\mathrm{Spec}\left(\mathcal{O}_{Z^{/x}}/\mathfrak{m}_x^N\right)$ of x in $Z^{/x}$, as a formal subscheme of $\mathcal{A}_{g,n}^{/x}$, is stable under the natural action of γ_p, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{Z^{/x}}$, and $N = N(\gamma_p, \gamma)$ depends on how close γ is to γ_p, $N(\gamma_p, \gamma) \to \infty$ as $\gamma \to \gamma_p$. Taking the limit as γ goes to γ_p, we see that $Z^{/x}$ is stable under the action of γ_p. ∎

(6.2.1) Remark (i) The action of the local stabilizer subgroup on the deformation space goes back to Lubin and Tate in [14].

(ii) In [2], the local stabilizer principle was applied to the zero-dimensional cusps of $\mathcal{A}_{g,n}$, and also to points of $\mathcal{A}_{g,n}$ defined over finite fields. The calculation in [2] at the zero-dimensional cusps is a bit complicated, and can be avoided, using "Larsen's example" instead.

(iii) The bigger the local stabilizer subgroup U_x, the more information the action of U_x on $\mathcal{A}_{g,n}^{/x}$ contains. The size of $\underline{U}$ is maximal when the abelian variety A_x is supersingular. If x is supersingular point, then $\underline{U}$ is an inner twist of Sp_{2g}, so in some sense almost all information about the prime-to-p Hecke correspondences on $\mathcal{A}_{g,n}$ are encoded in the action of U_x on $\mathcal{A}_{g,n}^{/x}$. The challenge, however, is to dig the buried information out of this action.

§7. Canonical coordinates for leaves

(7.1) Let k be an algebraically closed field of characteristic p. Let $\mathcal{C}$ be a leaf on $\mathcal{A}_{g,n}$, where $n \geq 3$ is a natural number relatively prime to p. Let $x \in \mathcal{C}(k)$ be a closed point of $\mathcal{C}$. Recall that the leaf $\mathcal{C}$ is defined by a point-wise property, namely, a point $y \in \mathcal{C}(k)$ is in $\mathcal{C} = \mathcal{C}(x)$ if and only if the quasi-polarized Barsotti-Tate groups $(A_y[p^\infty], \lambda_y[p^\infty])$ and $(A_x[p^\infty], \lambda_x[p^\infty])$ are isomorphic. One can also use the same point-wise property to define leaves (on the base scheme) for a (quasi-polarized) Barsotti-Tate group over a Noetherian integral base scheme over k; see [22].

From the definition it is not immediately clear how to "compute" the formal completion $\mathcal{C}^{/x}$ of the leaf $\mathcal{C}$ at x. However this turns out to be possible, and the resulting theory is a generalization of the classical Serre-Tate theory for the local moduli of ordinary abelian varieties. Some highlights of the description of $\mathcal{C}^{/x}$ will be explained in this section. More details can be found in [5], [6].

(7.2) Recall that the deformation theory of (A_x, λ_x) is the same as that of the associated quasi-polarized Barsotti-Tate group $(A_x[p^\infty], \lambda_x[p^\infty])$. Let

$$0 = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_m = A_{\mathcal{C}}[p^\infty]$$

be the slope filtration of the restriction to $\mathcal{C}$ of the Barsotti-Tate group attached to the universal abelian scheme, so that each G_i/G_{i-1} is a Barsotti-Tate group over $\mathcal{C}$ with slope μ_i, $i = 1, 2, \ldots, m$, and $\mu_1 > \mu_2 > \cdots > \mu_m$. Moreover, each subquotient G_i/G_{i-1} is constant over the formal completion $\mathcal{C}^{/x}$ of $\mathcal{C}$ at x.

Let $\mathfrak{Def}(A_x) = \mathfrak{Def}(A_x[p^\infty])$ be the local deformation space of A_x over k, or equivalently the local deformation space of $A_x[p^\infty]$ over k; it is a g^2-dimensional smooth formal scheme over k. A basic phenomenon here is that $\mathcal{C}^{/x}$ is determined by the slope filtration on $A[p^\infty] \to \mathcal{C}^{/x}$. More precisely, the formal subscheme $\mathcal{C}^{/x} \subset \mathcal{A}_{g,n}^{/x} \subset \mathfrak{Def}(A_x)$ is contained in the "extension part" $\mathfrak{MDE}(A_x[p^\infty])$ of $\mathfrak{Def}(A_x)$, where $\mathfrak{MDE}(A_x[p^\infty])$ is the maximal closed formal subscheme of the local deformation space $\mathfrak{Def}(A_x) = \mathfrak{Def}(A_x[p^\infty])$ such that the restriction to $\mathfrak{MDE}(A_x[p^\infty])$ of the universal Barsotti-Tate group is a successive extension of constant Barsotti-Tate groups G_i/G_{i-1}, extending the slope filtration of $A_x[p^\infty]$. For each Artinian local k-algebra R, $\mathfrak{MDE}(R)$ is the set of isomorphism classes of tuples

$$\left(0 = \tilde{G}_0 \subset \tilde{G}_1 \subset \cdots \subset \tilde{G}_m; \alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_m\right),$$

such that

- $\tilde{G}_i$ is a Barsotti-Tate group over R for each i,
- each quotient $\tilde{G}_i/\tilde{G}_{i-1}$ is a Barsotti-Tate group over R, $i = 1, \dots, m$,
- α_i is an isomorphism from $\tilde{G}_i \times_{\mathrm{Spec}(R)} \mathrm{Spec}\, k$ to G_i, for $i = 1, \dots, m$,
- β_i is an isomorphism from $\tilde{G}_i/\tilde{G}_{i-1}$ to G_i/G_{i-1}, for $i = 1, \dots, m$,
- the isomorphisms $\alpha_1, \dots, \alpha_m$ are compatible with the inclusion maps $G_i \hookrightarrow G_{i+1}$ and $\tilde{G}_i \hookrightarrow \tilde{G}_{i+1}$, $i = 1, \dots, m-1$, and
- the isomorphisms $\beta_1, \dots, \beta_m$ are compatible with $\alpha_1, \dots, \alpha_m$.

Our theory of canonical coordinates provides description of the closed formal subscheme $\mathcal{C}^{/x}$ of $\mathfrak{MDE}(A_x[p^\infty])$ in terms of the structure of $\mathfrak{MDE}(A_x[p^\infty])$, independent of the notion of leaves. If the abelian variety A_x is ordinary, then $m = 1$, G_1 is toric, G_1/G_2 is étale, and the theory reduces to the classical Serre-Tate coordinates.

(7.3) The computation of $\mathcal{C}^{/x}$ can be reduced to the following two "essential cases". In both cases we have two p-Barsotti-Tate groups X and Y over k; X has slope μ_X, while Y has slope μ_Y. We assume that $\mu_X < \mu_Y$. Let $\mathrm{Spf}(R)$ be the equi-characteristic deformation space of $X \times Y$. Let $G \to \mathrm{Spf}(R)$ be the universal deformation of $X \times Y$. For each $n \geq 1$, since $G[p^n]$ is a finite locally free group scheme over $\mathrm{Spf}(R)$, it is the formal completion of a unique finite locally free group scheme over $\mathrm{Spec}(R)$, denoted by $G_n \to \mathrm{Spec}(R)$. The inductive system of finite locally free group schemes $G_n \to \mathrm{Spec}(R)$ form a Barsotti-Tate group over $\mathrm{Spec}(R)$, denoted by $G \to \mathrm{Spec}(R)$, abusing the notation.

- (unpolarized case) In this case, our goal is to compute the leaf passing through the closed point for the Barsotti-Tate group $G \to \mathrm{Spec}(R)$. This leaf will be denoted by $\mathcal{C}^\wedge_{\mathrm{up}}$.
- (polarized case) Suppose that λ is a principal quasi-polarization on $X \times Y$. This assumption implies that $\mu_X + \mu_Y = 1$. The equi-characteristic deformation space of $(X \times Y, \lambda)$ is a closed formal subscheme $\mathrm{Spf}(R/I)$ of $\mathrm{Spf}(R)$. We would like to compute the leaf passing through the closed point for the polarized Barsotti-Tate group $G \to \mathrm{Spec}(R/I)$; denote this leaf by $\mathcal{C}^\wedge$.

(7.3.1) Our starting point in the computation of $\mathcal{C}_{\rm up}^{/x}$ and $\mathcal{C}^{/x}$ is the following observation. There is a closed formal subscheme $\mathfrak{DE}(X,Y)$ of the deformation space $\mathrm{Spf}(R)$, maximal with respect to the property that the restriction to $\mathfrak{DE}(X,Y)$ of the universal deformation of $X\times Y$ is an extension of the constant Barsotti-Tate group X by the constant Barsotti-Tate group Y. It is not difficult to see that $\mathfrak{DE}(X,Y)$ is formally smooth over k. The existence of the canonical filtration of the restriction of G to the leaves implies that both $\mathcal{C}_{\rm up}^{\wedge}$ and $\mathcal{C}^{\wedge}$ are closed formal subschemes of $\mathfrak{DE}(X,Y)$. On the other hand, the Baer sum for extensions produced a group law on $\mathfrak{DE}(X,Y)$, so that $\mathfrak{DE}(X,Y)$ has a natural structure as a smooth formal group over k.

(7.4) Theorem *Notation as in 7.3.*

(i) *In the unpolarized case, the leaf $\mathcal{C}_{\rm up}^{\wedge}$ is naturally isomorphic to the maximal p-divisible formal subgroup $\mathfrak{DE}(X,Y)_{p\text{-div}}$ of $\mathfrak{DE}(X,Y)$. The p-divisible group $\mathfrak{DE}(X,Y)_{p\text{-div}}$ has slope $\mu_Y-\mu_X$.*

(ii) *In the polarized case, the principal quasi-polarization λ on $X\times Y$ induces an involution on $\mathfrak{DE}(X,Y)_{p\text{-div}}$, and $\mathcal{C}^{\wedge}$ is equal to the fixer subgroup $\mathfrak{DE}(X,Y)_{p\text{-div}}^{\rm sym}$ under the involution. Again, $\mathfrak{DE}(X,Y)_{p\text{-div}}^{\rm sym}$ is a p-divisible formal group with slope $\mu_Y-\mu_X$.*

(7.4.1) Remark Thm. 7.4 gives a structural characterization of the leaves $\mathcal{C}_{\rm up}^{\wedge}$ and $\mathcal{C}^{\wedge}$ in the formal subscheme $\mathfrak{DE}(X,Y)$ of the deformation space $\mathrm{Spf}(R)$ of $X\times Y$. In Thm. 7.6.3 and Prop. 7.6.4, we will see a structural characterization of a leaf $\mathcal{C}(\mathfrak{Def}(G))$ in the equi-characteristic deformation space $\mathfrak{Def}(G)$ of a general Barsotti-Tate group G over k, in a similar spirit. The above characterization deals with the differential property of leaves, and complements the global point-wise definition of leaves.

(7.5) Theorem *Let $\mathrm{M}(X),\mathrm{M}(Y)$ be the covariant Dieudonné module of X,Y respectively. Let $B(k)$ be the fraction field of $W(k)$. The $B(k)$-vector space*

$$\mathrm{Hom}_{W(k)}(\mathrm{M}(X),\mathrm{M}(Y))\otimes_{W(k)}B(k)$$

has a natural structure as a V-isocrystal.

(i) *Let $\mathrm{M}(\mathfrak{DE}(X,Y)_{p\text{-div}})$ be the covariant Diedonné module of $\mathcal{C}_{\rm up}^{\wedge}=\mathfrak{DE}(X,Y)_{p\text{-div}}$. Then there exists a natural isomorphism of V-isocrystals*

$$\mathrm{M}(\mathfrak{DE}(X,Y)_{p\text{-div}})\otimes_{W(k)}B(k)\xrightarrow{\sim}\mathrm{Hom}_{W(k)}(\mathrm{M}(X),\mathrm{M}(Y))\otimes_{W(k)}B(k)\,.$$

(ii) *Suppose that λ is a principal quasi-polarization λ on $X\times Y$. Let ι be the involution on $\mathrm{Hom}_{W(k)}(\mathrm{M}(X),\mathrm{M}(Y))\otimes_{W(k)}B(k)$ induced by λ. Let $\mathrm{M}(\mathfrak{DE}(X,Y)_{p\text{-div}}^{\rm sym})$ be the covariant Diedonné module of $\mathcal{C}^{\wedge}=\mathfrak{DE}(X,Y)_{p\text{-div}}^{\rm sym}$. Then there exists a natural isomorphism of V-isocrystals*

$$\mathrm{M}(\mathfrak{DE}(X,Y)_{p\text{-div}}^{\rm sym})\otimes_{W(k)}B(k)\xrightarrow{\sim}\mathrm{Hom}_{W(k)}^{\rm sym}(\mathrm{M}(X),\mathrm{M}(Y))\otimes_{W(k)}B(k)\,,$$

where the right-hand side is the subspace of $\mathrm{Hom}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y)) \otimes_{W(k)} B(k)$ *fixed under the involution* ι*.*

(7.5.1) Remark (i) Thm. 7.5 can be regarded as a generalization of the appendix of [17]. The method is a generalization of Mumford's seminal paper [16]. It also gives an explicit description of the Cartier-Dieudonné module of $\mathfrak{MDE}(X, Y)$. See [6] for details.

(ii) A key ingredient of [6] is the set $\mathrm{Cart}_p(k[[t]])$ of all formal curves in the functor of reduced Cartier ring for algebras over $\mathbb{Z}_{(p)}$. It has a natural $(\mathrm{Cart}_p(k), \mathrm{Cart}_p(k))$-bimodule structure because $\mathrm{Cart}_p(k)$ is a subring of $\mathrm{Cart}_p(k[[t]])$. Moreover $\mathrm{Cart}_p(k[[t]])$ has another $\mathrm{Cart}_p(k)$-module structure, compatible with the above bimodule structure.

(iii) We do not know a convenient characterization of the the p-divisible formal group $\mathfrak{DE}(X, Y)_{p\text{-div}}$ inside its isogeny class, in terms of the Dieudonneé modules $\mathrm{M}(X), \mathrm{M}(Y)$. When both X and Y are *minimal* in the sense of [23], i.e. the endomorphism algebra of X, Y are maximal orders, we expect that $\mathfrak{DE}(X, Y)_{p\text{-div}}$ is also maximal. It is easy to check that this conjectural statement holds when the denominators of the Brauer invariant of X and Y are relatively prime.

(7.5.2) Corollary *Let* $h(X), h(Y)$ *be the height of* X, Y *respectively.*

(i) *In the unpolarized case, the height of* $\mathfrak{DE}(X, Y)_{p\text{-div}}$ *is equal to* $h(X) \cdot h(Y)$*, and* $\dim(\mathfrak{DE}(X, Y)_{p\text{-div}}) = (\mu_Y - \mu_X) \cdot h(X) \cdot h(Y)$*.*

(ii) *In the polarized case, we have* $h(X) = h(Y)$*, the height of* $\mathfrak{DE}(X, Y)^{\mathrm{sym}}_{p\text{-div}}$ *is equal to* $\frac{h(X) \cdot (h(X)+1)}{2}$*, and* $\dim(\mathfrak{DE}(X, Y)^{\mathrm{sym}}_{p\text{-div}}) = \frac{1}{2}(\mu_Y - \mu_X) \cdot h(X) \cdot (h(X) + 1)$*.*

(7.5.3) Remark (1) Except for the factor $\mu_Y - \mu_X$, the formulae (i), (ii) in Cor. 7.5.2 is quite similar to the formulae for the dimension of the deformation space of an h-dimensional abelian variety and the dimension of $\mathcal{A}_h$ respectively.

(2) The theory of canonical coordinates inspires a conjectural group-theoretic formula for the dimension of leaves in the reduction over k of a Shimura variety. That formula will be explained in a future article with C.-F. Yu, and verified for modular varieties of PEL-type.

(7.6) We go back to the general case and reset the notation similar to 7.2. Denote by $\mathcal{C}(\mathfrak{Def}(A_x[p^\infty]))$ the leaf in the deformation space $\mathfrak{Def}(A_x[p^\infty])$ of the Barsotti-Tate group $A_x[p^\infty]$. Just as in Prop. 3.3, there exists a slope filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_m = A_{\mathcal{C}(\mathfrak{Def}(A_x[p^\infty]))}[p^\infty]$$

on the universal Barsotti-Tate group over $\mathcal{C}(\mathfrak{Def}(A_x[p^\infty]))$, where each G_i/G_{i-1} is an isoclinic Barsotti-Tate group over $\mathcal{C}(\mathfrak{Def}(A_x[p^\infty]))$ with slope μ_i, $\mu_1 > \cdots > \mu_m$. Therefore the leaf $\mathcal{C}(\mathfrak{Def}(A_x[p^\infty]))$ is contained in $\mathfrak{MDE}(A_x[p^\infty])$, the maximal closed formal subscheme of $\mathfrak{Def}(A_x[p^\infty])$ such that the restriction to $\mathfrak{MDE}(A_x[p^\infty])$ of the universal Barsotti-Tate group has a slope filtration extending the slope filtration of $A_x[p^\infty]$. We would like to have a structural description of the leaf $\mathcal{C}(\mathfrak{Def}(A_x[p^\infty]))$ as a closed formal subscheme of $\mathfrak{MDE}(A_x[p^\infty])$, independent of the "point-wise" definition of the leaf. This will be achieved in an inductive way, allowing us to understand how $\mathcal{C}(\mathfrak{Def}(A_x[p^\infty]))$ is "built up" from the p-divisible formal groups $\mathfrak{DE}(G_i/G_{i-1}, G_j/G_{j-1})_{p\text{-div}}$, $1 \le j < i \le m$.

(7.6.1) For each Barsotti-Tate group G over k, we can consider the leaf $\mathcal{C}(\mathfrak{Def}(G))$ in the deformation space $\mathfrak{Def}(G)$ over k, and we know that $\mathcal{C}(\mathfrak{Def}(G))$ is contained in $\mathfrak{MDE}(G)$, the maximal closed formal subscheme of $\mathfrak{Def}(G)$ such that the restriction to $\mathfrak{MDE}(G)$ of the universal Barsotti-Tate group has a slope filtration extending the slope filtration of G.

(7.6.2) Let $0 = G_0 \subset G_1 \subset \cdots \subset G_m$ be the slope filtration of a Barsotti-Tate group G over k. Suppose that $0 \le j_1 \le j_2 < i_2 \le i_1 \le m$. Then there exists a natural formally smooth morphism

$$\pi_{[j_2,i_2],[j_1,i_1]} : \mathfrak{MDE}(G_{i_1}/G_{j_1}) \to \mathfrak{MDE}(G_{i_2}/G_{j_2})\,.$$

These morphisms form a finite projective system, that is

$$\pi_{[j_3,i_3],[j_2,i_2]} \circ \pi_{[j_2,i_2],[j_1,i_1]} = \pi_{[j_3,i_3],[j_1,i_1]}$$

if $0 \le j_1 \le j_2 \le j_3 < i_3 \le i_2 \le i_1 \le m$. Moreover, using the theory of biextensions of Mumford and Grothendieck in [16] and [12], one can show that the morphism

$$\mathfrak{MDE}(G_i/G_j) \longrightarrow \mathfrak{MDE}(G_{i-1}/G_j) \times_{\mathfrak{MDE}(G_{i-1}/G_{j+1})} \mathfrak{MDE}(G_i/G_{j+1})$$

attached to the pair of morphisms $(\pi_{[j,i-1],[j,i]}, \pi_{[j+1,i],[j,i]})$ has a natural structure as a torsor for the formal group $\mathfrak{DE}(G_i/G_{i-1}, G_j/G_{j-1})$.

(7.6.3) Theorem *Notation as in 7.6.2.*

(i) *Suppose that $1 \le i \le m-1$. Then $\mathcal{C}(\mathfrak{Def}(G_{i+1}/G_{i-1}))$ is a torsor for the p-divisible formal group $\mathfrak{DE}(G_{i+1}/G_i, G_i/G_{i-1})_{p\text{-div}}$.*

(ii) *Suppose that $0 \le j_1 \le j_2 < i_2 \le i_1 \le m$. Then the restriction of $\pi_{[j_2,i_2],[j_1,i_1]}$ to the closed formal subscheme $\mathcal{C}(\mathfrak{Def}(G_{i_1}/G_{j_1}))$ of $\mathfrak{MDE}(G_{i_1}/G_{j_1})$ factors through $\mathcal{C}(\mathfrak{Def}(G_{i_2}/G_{j_2})) \hookrightarrow \mathfrak{MDE}(G_{i_2}/G_{j_2})$, and induces a formally smooth morphism*

$$\pi_{[j_2,i_2],[j_1,i_1]} : \mathcal{C}(\mathfrak{Def}(G_{i_1}/G_{j_1})) \to \mathcal{C}(\mathfrak{Def}(G_{i_2}/G_{j_2}))\,.$$

(iii) *Suppose that $1 \le i, j \le m$, $i \ge j+2$. Then the morphism*

$$\mathcal{C}(\mathfrak{Def}(G_i/G_j)) \longrightarrow \mathcal{C}(\mathfrak{Def}(G_{i-1}/G_j)) \times_{\mathcal{C}(\mathfrak{Def}(G_{i-1}/G_{j+1}))} \mathcal{C}(\mathfrak{Def}(G_i/G_{j+1}))$$

attached to the pair of morphisms $(\pi_{[j,i-1],[j,i]}, \pi_{[j+1,i],[j,i]})$ is a torsor for the p-divisible formal group $\mathfrak{DE}(G_i/G_{i-1}, G_j/G_{j-1})_{p\text{-div}}$, respecting the $\mathfrak{DE}(G_i/G_{i-1}, G_j/G_{j-1})$-torsor structure of

$$\mathfrak{MDE}(G_i/G_j) \longrightarrow \mathfrak{MDE}(G_{i-1}/G_j) \times_{\mathfrak{MDE}(G_{i-1}/G_{j+1})} \mathfrak{MDE}(G_i/G_{j+1})$$

at the end of 7.6.2.

(7.6.4) Proposition *The properties* (i), (ii), (iii) *in Thm. 7.6.3 determine uniquely the family of formal schemes* $\{\mathcal{C}(\mathfrak{Def}(G_i/G_j)) \,:\, 0 \le j < i \le m\}$, *where each member* $\mathcal{C}(\mathfrak{Def}(G_i/G_j))$ *of the family is considered as a closed formal subscheme of* $\mathfrak{Def}(G_i/G_j)$.

(7.6.5) Remark It is actually possible to do better than what was stated in Prop. 7.6.4. Namely, one can actually *construct* closed subschemes $\mathfrak{MDE}(G_i/G_j)_{p\text{-div}}$ of $\mathfrak{MDE}(G_i/G_j)$, satisfying the properties (i), (ii), (iii) in Thm. 7.6.3, using structural properties of the formal schemes $\mathfrak{MDE}(G_i/G_j)$, without the concept of leaves, in an inductive way. An important ingredient of the construction uses the theory of biextensions due to Mumford [16] and Grothendieck [12]. Of course, $\mathfrak{MDE}(G_i/G_j)_{p\text{-div}}$ is canonically isomorphic to $\mathcal{C}(\mathfrak{Def}(G_i/G_j))$ by Prop. 7.6.4. However that construction is a bit complicated, so we do not give further indication here.

(7.6.6) Corollary *Notation as in Thm. 7.6.3. Then*

$$\dim(\mathcal{C}(\mathfrak{Def}(G))) = \sum_{1\le j<i\le m} (\mu_i - \mu_j)\cdot h_i \cdot h_j \,,$$

where μ_i *is the slope of* G_i/G_{i-1} *and* h_i *is the height of* G_i/G_{i-1}, *for* $i = 1, \ldots, m$.

(7.7) Proposition *Let* G *be a Barsotti-Tate group over* k, *with a principal quasi-polarization* λ. *Then* λ *induces an involution on* $\mathfrak{MDE}(G)_{p\text{-div}}$. *Denote by* $\mathfrak{MDE}(G)^{\rm sym}_{p\text{-div}}$ *the maximal closed subscheme of* $\mathfrak{MDE}(G)_{p\text{-div}}$ *which is fixed by the involution. Then* $\mathfrak{MDE}(G)^{\rm sym}_{p\text{-div}}$ *is the largest closed formal subscheme of* $\mathfrak{MDE}(G)_{p\text{-div}}$ *such that* λ *extends to a quasi-polarization on the restriction to* $\mathfrak{MDE}(G)^{\rm sym}_{p\text{-div}}$ *of the universal Barsotti-Tate group over* $\mathfrak{MDE}(G)_{p\text{-div}} \subset \mathfrak{Def}(G)$. *If* $(G,\lambda) = (A_x[p^\infty], \lambda_x[p^\infty])$ *for some point* $x \in \mathcal{A}_{g,n}(k)$, *then there is a natural isomorphism of formal schemes from* $\mathfrak{MDE}(G)^{\rm sym}_{p\text{-div}}$ *to* $\mathcal{C}^{/x}$, *where* $\mathcal{C}$ *is the leaf in* $\mathcal{A}_{g,n}$ *passing through* x.

(7.7.1) Proposition *Let* A_x *be a* g*-dimensional principally polarized abelian variety over* k. *Suppose that* $A_x[p^\infty]$ *has Frobenius slopes* $\mu_1 < \mu_2 < \cdots < \mu_m$, *so that* $\mu_i + \mu_{m-i+1} = 1$ *for* $i = 1, \ldots, m$. *Let* h_i *be the multiplicity of* μ_i, *so that* $h_i = h_{m-i+1}$ *for all* i, $\sum_{i=1}^m h_i = 2g$, $\sum_{i=1}^m h_i\mu_i = g$. *Then*

$$\dim(\mathcal{C}(x)) = \frac{1}{2}\sum_{i<j,\, i+j\neq 1} (\mu_j - \mu_i)\cdot h_i\cdot h_j + \frac{1}{2}\sum_{2i\le m} (1-2\mu_i)\cdot h_i(h_i+1)\,.$$

Remark Prop. 7.7.1 follows from Prop. 7.7 and Cor. 7.5.2; see [6].

(7.7.2) Remark Historically, the formula for the dimension of a leaf $\mathcal{C}(x)$ in $\mathcal{A}_{g,n}$ and for the dimension of the leaf $\mathcal{C}(\mathfrak{Def}(G))$ in the deformation space of a Barsotti-Tate group G were first conjectured by Oort, in terms of the number of lattice points inside suitable regions attached to the Newton polygon of A_x and G, after suggestions by B. Poonen. See [5] for the original proofs of 7.6.6 and 7.7.1, which depend on the following fact, proved in [23]: If G_1, G_2 are Barsotti-Tate group over k, G_1 is minimal, and $G_1[p]$ is isomorphic to $G_2[p]$, then G_2 is isomorphic to G_1.

§8. A rigidity result for p-divisible formal groups

(8.1) Let k be an algebraically closed field of characteristic p. Let X be a p-divisible formal group over k. Then $\mathrm{End}_k(X)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is a semisimple algebra of finite dimension over $\mathbb{Q}_p$, and $\mathrm{End}_k(X)$ is an order in $\mathrm{End}_k(X)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Let H be a connected reductive linear algebraic group over $\mathbb{Q}_p$. Let $\rho : H(\mathbb{Q}_p)\to(\mathrm{End}_k(X)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^\times$ be a rational representation of H, i.e. ρ comes from a $\mathbb{Q}_p$-homomorphism of linear algebraic groups. Let $U\subset H(\mathbb{Q}_p)$ be an open subgroup of $H(\mathbb{Q}_p)$ such that $\rho(U)\subseteq \mathrm{End}_k(X)^\times$, so that U operates on X via ρ.

(8.2) Theorem *Notation as above. Let Z be an irreducible closed formal subscheme of X which is stable under the action of U. We assume that the composition $\mathbf{r}_X\circ\rho$ of ρ with the left regular representation $\mathbf{r}_X$ of $(\mathrm{End}_k(X)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^\times$ on $\mathrm{End}_k(X)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ does not contain the trivial representation of H as a subquotient. Then Z is a p-divisible formal subgroup of X.*

(8.2.1) Remark Thm. 8.2 is a considerable strengthening of [2, §4, Prop. 4], in several aspects. There, the p-divisible formal group is a formal torus, and the formal subvariety is assumed to be formally smooth. The most significant part is that, in [2, §4, Prop. 4], the symmetry group $\mathcal{O}_{\wp_1}^\times\times\cdots\times\mathcal{O}_{\wp_r}^\times$ has about the same size as the formal torus

$$\left(Y_1\otimes_{\mathbb{Z}_p}\widehat{\mathbb{G}}_m\right)\times\cdots\times\left(Y_r\otimes_{\mathbb{Z}_p}\widehat{\mathbb{G}}_m\right)$$

in some sense, while the symmetry group H in Thm. 8.2 can be quite small compared with the p-divisible formal group X. A typical special case is to take $H=\mathbb{G}_m$, $U=\mathbb{Z}_p^\times$, and each $u\in\mathbb{Z}_p^\times$ operates as $[u]_X$ on X, the map "multiplication by u" on X.

§9. The Hilbert trick

(9.1) Let $n\geq 3$ be an integer prime to p. Let $x\in\mathcal{A}_{g,n}(\overline{\mathbb{F}_p})$ be an $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_{g,n}$. Let $B=\mathrm{End}_{\overline{\mathbb{F}_p}}(A_x)\otimes_{\mathbb{Z}}\mathbb{Q}$, and let $*$ be the involution of B induced by λ_x. Let $E=F_1\times\cdots\times F_m$ be a product of totally real fields contained in B, fixed under the involution $*$, such that $\dim_{\mathbb{Q}}(E)=g$. Let $\mathcal{O}_E=\mathcal{O}_{F_1}\times\cdots\times\mathcal{O}_{F_m}$. Let G_j be the linear algebraic group over $\mathbb{Q}$ such that $G_j(\mathbb{Q})=\mathrm{SL}_2(F_j)$, $j=1,\ldots,m$, and denote by G the product group $G=G_1\times\cdots\times G_m$. There exists a "standard embedding" $h: G\hookrightarrow \mathrm{Sp}_{2g}$, well-defined up to conjugation.

(9.1.1) We will use the following variant of the definition of Hilbert modular varieties in [30], slightly different from the definition in [10]. Denote by $\mathcal{M}_{E,n}$ the Hilbert modular scheme attached to $\mathcal{O}_E$, such that for every $\overline{\mathbb{F}_p}$-scheme S, $\mathcal{M}_{E,n}(S)$ is the set of isomorphism classes of $(A\to S,\lambda,\iota,\eta)$, where $A\to S$ is an abelian scheme, $\iota:\mathcal{O}_E\to\mathrm{End}_S(A)$ is a injective ring homomorphism, λ is an $\mathcal{O}_E$-linear principal polarization of $A\to S$ of degree prime to p, and η is a level-n structure on $A\to S$. See [30, §5]. The modular scheme $\mathcal{M}_{E,n}$ is locally of finite type over k, and every irreducible component of $\mathcal{M}_{E,n}$ is of finite type over $\overline{\mathbb{F}_p}$. There is a set of algebraic correspondences on $\mathcal{M}_{E,n}$, coming from the adelic group $G(\mathbb{A}_f^{(p)})$, called the prime-to-p Hecke correspondences on the Hilbert modular scheme $\mathcal{M}_{E,n}$.

(9.2) Proposition *Notation as above. Then there exists*

- *a non-empty open-and-closed subscheme $\mathcal{M}_0$ of $\mathcal{M}_{E,m}$ for some m,*
- *a finite morphism $\mathcal{M}_0 \to \mathcal{M}_{E,n}$,*
- *a point $y_0 \in \mathcal{M}_0(\overline{\mathbb{F}_p})$, and*
- *a finite morphism $f : \mathcal{M}_0 \to \mathcal{A}_{g,n}$*

such that

(i) $f(y) = x$,

(ii) *f is compatible with the prime-to-p Hecke correspondences on $\mathcal{M}_0$ and $\mathcal{A}_{g,n}$, coming from the embedding $h : G \hookrightarrow \mathrm{Sp}_{2g}$, and*

(iii) *the pull-back by f of the universal abelian scheme over $\mathcal{A}_{g,n}$ is isogenous to the universal abelian scheme over $\mathcal{M}_0$.*

(9.2.1) The idea of the proof of Prop. 9.2 is as follows. It is well-known that every abelian variety defined over a finite field has "sufficiently many complex multiplication". Hence every maximal commutative semisimple subalgebra L of B stable under the Rosati involution $*$ is a product of CM-fields, and the subalgebra of L fixed under $*$ is a product of totally real fields. In particular this shows the existence of subalgebras E with the required properties in 9.1. If $\mathrm{End}_{\overline{\mathbb{F}_p}}(A_x)$ contains $\mathcal{O}_E$, then we obtain a natural morphism $\mathcal{M}_{E,n} \to \mathcal{A}_{g,n}$ passing through $x = [(A_x, \lambda_x, \eta_x)] \in \mathcal{A}_{g,n}(\overline{\mathbb{F}_p})$. In general $E \cap \mathrm{End}_{\overline{\mathbb{F}_p}}(A_x)$ is an order of $\mathcal{O}_E$, and we have to use an isogeny correspondence to conclude the proof of 9.2.

(9.2.2) Remark The local stabilizer principle and Thm. 8.2, applied to a point y of a Hilbert modular variety $\mathcal{M}_{E,n}$ over $\overline{\mathbb{F}_p}$, implies that there are only a finite number of possibilities of $\overline{\mathcal{H}_{E,n}(y)}^{/y}$, as a closed formal subscheme of $\mathcal{M}_{E,n}$ over k, where $\mathcal{H}_{E,n}(y)$ denotes the prime-to-p Hecke orbit of y in $\mathcal{M}_{E,n}$. The possibilities are parametrized by non-empty subsets of the finite set of maximal ideals of $\mathcal{O}_E$ containing p. The above phenomenon makes it relatively easy to verify the Hecke orbit conjecture for the Hilbert modular varieties. It also makes the Hilbert trick an effective tool for the Hecke orbit problem for the Siegel modular varieties $\mathcal{A}_{g,n}$, as well as other modular varieties of PEL-type.

(9.3) In this section the base field is $\overline{\mathbb{F}_p}$, because every abelian variety over $\overline{\mathbb{F}_p}$ has sufficiently many complex multiplications. So it seems that if we use the Hilbert trick, one can would be able to deal with the Hecke orbit conjecture (HO) "only" in the case when the algebraically closed base field k is equal to $\overline{\mathbb{F}_p}$. However every closed subvariety of $\mathcal{A}_{g,n}$ over k is finitely presented over k, and a standard argument in algebraic geometry shows that the validity of (HO) over $\overline{\mathbb{F}_p}$ implies the validity of (HO) over every algebraically closed field k. See the beginning of §3 of [2] for details.

§10. Hypersymmetric points

Let k be an algebraically closed field of characteristic p as before.

(10.1) Definition An abelian variety A over k is *hypersymmetric* if the natural map

$$\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}_k(A[p^\infty])$$

is an isomorphism. An equivalent condition is that the canonical map

$$\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p \to \mathrm{End}_k(A[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

(10.1.1) Remark It is clear from the definition that the abelian variety A_x has sufficiently many complex multiplication for any hypersymmetric point x. Therefore a theorem of Grothendieck tells us that A_x is isogenous to an abelian variety defined over $\overline{\mathbb{F}_p}$; see [18] for a proof of Grothendieck's theorem.

(10.1.2) EXAMPLES. (i) A g-dimensional ordinary abelian variety over k is hypersymmetric if and only if it is isogenous to a g-fold self-product $E \times \cdots \times E$, where E is an ordinary elliptic curve defined over $\overline{\mathbb{F}_p}$.

(ii) Let A be a abelian variety over k such that $A[p^\infty]$ has exactly two slopes, $g = \dim(A)$. Then A is hypersymmetric if and only if $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a central simple algebra over an imaginary quadratic field, and $\dim_{\mathbb{Q}}(\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}) = 2g^2$.

The assertions in the two examples can be verified using Honda-Tate theory for abelian varieties over finite fields. See [28] and [29] for the Honda-Tate theory.

(10.1.3) In every given Newton polygon stratum W^0_ξ in $\mathcal{A}_{g,n}$ over k, there exists a hypersymmetric point $x \in W^0_\xi(k)$. This statement follows easily from the Honda-Tate theory; see [25] for a proof.

(10.1.4) Let $E = F_1 \times \cdots \times F_r$ be a totally real number field such that there is only one place of F_i above p for $i = 1, \ldots, r$. Let $\mathcal{M}_E$ be the Hilbert modular variety over k attached to $\mathcal{M}_E$. Then there exists a hypersymmetric point in every given Newton stratum of $\mathcal{M}_E$. Similarly, there exists a hypersymmetric point in every given leaf of $\mathcal{M}_E$. This statement can be derived from the Honda-Tate theory and the "folation structure" on $\mathcal{M}_E$.

(10.2) Theorem *Let $[(A_x, \lambda_x)]$ be a point of $\mathcal{A}_g(k)$ such that*

- *A_x is hypersymmetric, and*
- *A_x is split, i.e. A_x is isomorphic to a product $B_1 \times \cdots \times B_m$, where each B_i is an abelian variety over k, and each B_i has at most two slopes.*

Then Zariski closure in $\mathcal{A}_g$ of the the prime-to-p Hecke orbit $\mathcal{H}^{(p)}(x)$ contains the irreducible component of the leaf $\mathcal{C}(x)$ passing through x.

(10.2.1) Remark A special case of Thm. 10.2 is an example of M. Larsen; see [2, §1].

(10.2.2) The proof of Thm. 10.2 uses Prop. 6.2, Thm. 8.2 and the theory of canonical coordinates. Here we sketch a proof of the special case when $A_x[p^\infty]$ is isomorphic to a product $X \times Y$, where X, Y are isoclinic Barsotti-Tate group of height g, with slopes $\mu_X < \mu_Y$, $\mu_X + \mu_Y = 1$. The principal polarization λ_x induces an isomorphism between X and the Serre dual of Y. The theory of canonical coordinates tells us that $\mathcal{A}_{g,n}^{/x}$ is isomorphic to the maximal subgroup $\mathfrak{DE}(X,Y)_{p\text{-div}}^{\text{sym}}$ of the Barsotti-Tate group $\mathfrak{DE}(X,Y)_{p\text{-div}}$ fixed under the involution induced by the principal polarization λ_x. Let $Z(x)$ be the Zariski closure of the Hecke orbit $\mathcal{H}(x)$, and let $Z(x)^{/x}$ be the formal completion of $Z(x)$ at x. The local stabilizer principle says that the subgroup $Z(x)^{/x}$ is stable under the natural action of the local stabilizer U_x. By Thm. 8.2, $Z(x)^{/x}$ is a Barsotti-Tate subgroup of the Barsotti-Tate group $\mathfrak{DE}(X,Y)_{p\text{-div}}^{\text{sym}}$.

Now we are ready to use Dieudonné theory and translate the last assertion into a statement in linear algebra. Let $V_X = \mathrm{M}(X) \otimes_{W(k)} B(k)$, $V_Y = \mathrm{M}(Y) \otimes_{W(k)} B(k)$. The principal polarization λ_x induces a duality pairing between V_X and V_Y. Thm. 7.5 tells us that $\mathrm{M}(\mathfrak{DE}(X,Y)_{p\text{-div}}) \otimes_{W(k)} B(k)$ is naturally isomorphic to $\mathrm{Hom}_{B(k)}^{\text{sym}}(V_X, V_Y)$, the symmetric part of the internal Hom. The group U_x operates naturally on $\mathrm{M}(X) \otimes_{W(k)} B(k)$ and $\mathrm{M}(Y) \otimes_{W(k)} B(k)$. One checks that, after passing to the algebraic closure $\overline{B(k)}$ of $B(k)$, the Zariski closure of U_x operating on $V_Y \otimes_{B(k)} \overline{B(k)}$ is isomorphic to the standard representation of GL_g, and the Zariski closure of U_x operating on $V_X \otimes_{B(k)} \overline{B(k)}$ is isomorphic to the dual of the standard representation of GL_g. So the action of the Zariski closure of U_x on $\mathrm{Hom}_{B(k)}^{\text{sym}}(V_X, V_Y) \otimes_{B(k)} \overline{B(k)}$ is isomorphic to the second symmetric product of the standard representation of GL_g. The last representation is absolutely irreducible; in fact it is one of the fundamental representations. Since $\mathrm{M}(Z(x)^{/x}) \otimes_{W(k)} B(k)$ is a non-trivial subrepresentation of the absolutely irreducible representation $\mathrm{Hom}_{B(k)}^{\text{sym}}(V_X, V_Y)$ of U_x, we conclude that $\mathrm{M}(Z(x)^{/x}) \otimes_{W(k)} B(k)$ is equal to $\mathrm{Hom}_{B(k)}^{\text{sym}}(V_X, V_Y)$, therefore $Z(x)^{/x} = \mathcal{A}_{g,n}^{/x}$. ∎

(10.3) Proposition *Let $\mathcal{C}^+$ be an irreducible component of a leaf $\mathcal{C}$ in $\mathcal{A}_{g,n}$, and let W^0_ξ be the open Newton stratum in $\mathcal{A}_{g,n}$ containing $\mathcal{C}^+$. Assume that W^0_ξ is irreducible. Then for every point $y \in W^0_\xi(k)$, there exists a point $x \in \mathcal{C}^+(k)$ such that there exists an isogeny from A_x to A_y, which respects the polarizations.*

Idea of Proof. Prop. 10.3 is an immediate consequence of the "almost product structure" on each irreducible component of a Newton polygon stratum W^0_ξ; see [22, Thm. 5.3]. We sketch the proof below.

Using Prop. 3.2, one constructs a finite surjective morphism $f : S \to \mathcal{C}^+$, a scheme T over k, and a morphism $g : S \times_{\operatorname{Spec} k} T \to W^0_\xi$ such that

(i) For any $s_1, s_2 \in S(k)$, $t_1, t_2 \in T(k)$, if $f(s_1) = f(s_2)$, then there exists an isogeny from $A_{g(s_1,t_1)}$ to $A_{g(s_2,t_2)}$, which respects the polarizations up to a multiple.

(ii) The image of g, in the naive sense, is a union of irreducible components of W^0_ξ.

So far we have not used the assumption that W^0_ξ is irreducible. The irreducibility of W^0_ξ implies that f is surjective. Prop. 10.3 follows.

(10.4) Proposition *Let $\mathcal{C}$ be a leaf in $\mathcal{A}_{g,n}$, and let W^0_ξ be the open Newton stratum in $\mathcal{A}_{g,n}$ containing $\mathcal{C}$. Assume that W^0_ξ is irreducible. Then the prime-to-p Hecke correspondences operate transitively on $\pi_0(\mathcal{C})$. Consequently $\mathcal{C}$ is irreducible if W^0_ξ is not the supersingular locus of $\mathcal{A}_{g,n}$.*

Idea of Proof. Let y be a hypersymmetric point in W^0_ξ; such a point exists by 10.1.3. By Prop. 10.3, in each irreducible component $\mathcal{C}^+_j$ of $\mathcal{C}$, there exists a hypersymmetric point x_j in $\mathcal{C}^+_j$, related to y by a (possibly inseparable) isogeny which preserves the polarizations up to a multiple. Using the weak approximation theorem, one sees that the x_j's are related by suitable prime-to-p Hecke correspondences. This shows that the prime-to-p Hecke correspondences operate transitively on the irreducible components of the leaf $\mathcal{C}$. The last statement follows from 5.1.

(10.5) We would like to discuss an emerging picture about the leaves and the hypersymmetric points. In many ways each non-supersingular leaf in $\mathcal{A}_{g,n}$ has properties similar to the modular variety $\mathcal{A}_{g,n}$ in characteristic 0. The Hecke orbit conjecture (HO) is an example of this phenomenon, so is Thm. 5.1. Borrowing an idea from Hindu mythology, one might want to think of the decomposition of $\mathcal{A}_{g,n}$ into leaves as Indra-inspired.

(10.5.1) For a leaf $\mathcal{C}$ in $\mathcal{A}_{g,n}$, the hypersymmetric points on $\mathcal{C}$ serve as an analogue of the notion of *special points* (or CM points) on a Shimura variety in characteristic 0. The following is an analogue of the André-Oort conjecture in characteristic p. Let $\mathcal{C}$ be a leaf of $\mathcal{A}_{g,n}$ over k, and let Z be a closed irreducible subvariety in $\mathcal{C}$. Assume that there is a subset $S \subset Z(k)$ such that S is dense in Z, and every point of S is hypersymmetric. Then there is a closed subvariety $X \subset \mathcal{A}_{g,n}$ which is the reduction over k of a Shimura subvariety such that Z is an irreducible component of $\mathcal{C} \cap X$. This conjecture seems to be more difficult than the André-Oort conjecture.

(10.5.2) In another direction, one expects that the p-adic monodromy of a subvariety Z in a leaf $\mathcal{C} \subset \mathcal{A}_{g,n}$ can be described in terms of the canonical coordinates and the naive p-adic monodromy of Z. The case when $\mathcal{C}$ is the ordinary locus of $\mathcal{A}_{g,n}$ has been considered in [4], and one expects that the general phenomenon is similar. In particular, there should be a more global theory of canonical coordinates on a leaf, and we hope to carry out such a project in the recent future.

§11. Splitting at supersingular points

(11.1) Proposition *Let k be an algebraically closed field of characteristic p. Let x be a point of $\mathcal{A}_{g,n}$ over k, and let $\mathcal{H}^{(p)}(x)$ be the prime-to-p Hecke orbit of x. Then there exists a point z_0 in the Zariski closure of $\mathcal{H}^{(p)}(x)$ such that A_{z_0} is a supersingular abelian variety over k.*

(11.1.1) Remark (i) Similarly, every prime-to-p Hecke orbit in a Hilbert modular variety has a supersingular point in its closure.

(ii) One can replace "prime-to-p" by "ℓ-adic" in 11.1, and also in (i) above.

(iii) See [2, Prop. 6] for a proof of 11.1 and (i), (ii) above. A key ingredient is the fact that every Ekedahl-Oort stratum in $\mathcal{A}_{g,n}$ is quasi-affine; see [21].

(11.2) Theorem *Let $x \in \mathcal{A}_{g,n}(\overline{\mathbb{F}_p})$ be an $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_{g,n}$. Let Z be the Zariski closure in $\mathcal{A}_{g,n}$ of the prime-to-p Hecke orbit $\mathcal{H}^{(p)}(x)$ of x, and let Z^0 be the intersection of Z with the leaf $\mathcal{C}(x)$ passing through x. Then there exists*

- *a point $y \in Z^0(\overline{\mathbb{F}_p})$,*
- *totally real fields $L_1, \ldots, L_s$, and*
- *an injective ring homomorphism $\beta : L_1 \times \cdots \times L_s \longrightarrow \mathrm{End}_k(A_y) \otimes_{\mathbb{Z}} \mathbb{Q}$*

such that

(i) $[L_1 : \mathbb{Q}] + \cdots + [L_s : \mathbb{Q}] = g$,

(ii) *$\beta(L_1 \times \cdots \times L_s)$ is fixed by the Rosati involution on $\mathrm{End}_{\overline{\mathbb{F}_p}}(A_y) \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by λ_y, and*

(iii) *there is only one maximal ideal in $\mathcal{O}_{L_j}$ which contains p, for $j = 1, \ldots, s$.*

In particular, there exists a point $y \in Z^0(\overline{\mathbb{F}_p})$ and abelian varieties $B_1, \ldots, B_s$ over $\overline{\mathbb{F}_p}$ such that A_y is isogenous to $B_1 \times \cdots \times B_s$, and each B_j has at most two slopes, $j = 1, \ldots, s$.

(11.2.1) Remark Thm. 11.2 depends crucially on the fact that x is an $\overline{\mathbb{F}_p}$-rational point. However we have seen in Rem. 9.3 that we may assume that the base field k is $\overline{\mathbb{F}_p}$ when considering the Hecke orbit conjecture (HO).

(11.3) We sketch a proof of 11.2, which uses the action of the local stabilizer subgroup at a supersingular point in the closure of $\mathcal{C}$ and the Hilbert trick.

We may and do assume that there exists a product $E = F_1 \times \cdots \times F_r$ of totally real fields, $[E : \mathbb{Q}] = g$, such that there exists an embedding $\iota : \mathcal{O}_E \hookrightarrow \mathrm{End}_{\overline{\mathbb{F}_p}}(A_x)$ of rings, and $\iota(\mathcal{O}_E)$ is fixed under the Rosati involution. This means that we have a natural morphism $f : \mathcal{M}_{E,m} \longrightarrow \mathcal{A}_{g,n}$ passing through x, compatible with the Hecke correspondences, for some m prime to p, such that for every geometric point $u \in \mathcal{M}_{E,m}(\overline{\mathbb{F}_p})$, the map induced by f on the strict henselizations

$$f^{(u)} : \mathcal{M}_{E,m}^{(u)} \to \mathcal{A}_{g,n}^{(f(u))}$$

is a closed embedding. Here $\mathcal{M}_{E,m}^{(u)}$ denotes the strict henselization of $\mathcal{M}_{E,m}$ at u, and $\mathcal{A}_{g,n}^{(f(u))}$ denotes the strict henselization of $\mathcal{A}_{g,n}$ at $f(u)$. Let W be the Zariski closure of the prime-to-p Hecke orbit $\mathcal{H}_E^{(p)}(x)$ in $\mathcal{M}_{E,n}$.

By 11.1.1 (i), there exists a supersingular point $z \in W(k)$. The local stabilizer principle tells us that the formal subscheme $Z^{/z} \subset \mathcal{A}_{g,n}$ is stable under the natural action of the local stabilizer subgroup U_z attached to z. Recall that U_z is a subgroup of $\mathrm{End}_{\overline{\mathbb{F}_p}}(A_z[p^\infty])^\times$ by definition.

One checks that there exists an element $\gamma \in U_z$ such that the subring

$$\mathrm{Ad}(\gamma)(E \otimes_{\mathbb{Q}} \mathbb{Q}_p) = \gamma \cdot (E \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cdot \gamma^{-1}$$

of $\mathrm{End}_{\overline{\mathbb{F}_p}}(A_z) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is equal to the $\mathbb{Q}_p$-linear span of

$$E' := (\mathrm{Ad}(\gamma)(E) \otimes_{\mathbb{Q}} \mathbb{Q}_p)) \cap \Big(\mathrm{End}_{\overline{\mathbb{F}_p}}(A_z) \otimes_{\mathbb{Z}} \mathbb{Q}\Big) ,$$

and E' is a product of totally real fields $L_1 \times \cdots \times L_s$, such that there is only one maximal ideal in $\mathcal{O}_{L_j}$ above p for $j = 1, \ldots, s$.

Denote by $\gamma^{/z}$ the automorphism of $\mathcal{A}_{g,n}^{/z}$ attached to γ. The fact that $\gamma^{/z}(W^{/z}) \subset Z^{/z}$ tells us, in the case when $\mathrm{Ad}(\gamma)(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p) \subset \mathrm{End}_{\overline{\mathbb{F}_p}}(A_z[p^\infty]) = \mathrm{End}_{\overline{\mathbb{F}_p}}(A_z) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, that there is a natural finite morphism

$$f_1 : \mathcal{M}_{E',m} \longrightarrow \mathcal{A}_{g,n}$$

with the following properties:

(1) There exists a point $z_1 \in \mathcal{M}_{E',m}(\overline{\mathbb{F}_p})$ such that $f_1(z_1) = z$.

(2) For every point $u \in \mathcal{M}_{E',m}(\overline{\mathbb{F}_p})$, the morphism f_1 induces a closed embedding, from the henselization $\mathcal{M}_{E',m}^{(u)}$ of $\mathcal{M}_{E',m}$ at u, to the henselization $\mathcal{A}_{g,n}^{(f_1(u))}$ of $\mathcal{A}_{g,n}$ at $f_1(u)$.

(3) $\gamma^{/z}(W^{/z}) \subset f_1^{/z_1}\left(\mathcal{M}_{E',n}^{/z_1}\right) \cap Z^{/z}$.

Hence the fiber product $\mathcal{M}_{E',m} \times_{\mathcal{A}_{g,n}} Z^0$ is not empty. Pick an $\overline{\mathbb{F}_p}$-point $\tilde{y}$ of $\mathcal{M}_{E',m} \times_{\mathcal{A}_{g,n}} Z^0$, and let y be the image of $\tilde{y}$ in $Z^0(\overline{\mathbb{F}_p})$. It is easy to see that y has the property stated in Prop. 11.2, and we are done. In general $(\mathrm{Ad}(\gamma)(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p)) \cap \left(\mathrm{End}_{\overline{\mathbb{F}_p}}(A_z) \otimes_{\mathbb{Z}} \mathbb{Z}_p\right)$ is of finite index in $\mathrm{Ad}(\gamma)(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Q}_p)$ and may not be equal to $\mathrm{Ad}(\gamma)(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, and we have to use an isogeny correspondence to conclude the proof.

(11.3.1) Remark (i) The last sentence in the statement of Thm. 11.2 follows from the properties (i), (ii), (iii) of A_y stated in the 11.2.

(ii) The local stabilizer subgroup U_x at a supersingular point x of $\mathcal{A}_{g,n}$ is a compact open subgroup of the group of $\mathbb{Q}_p$-points of an inner form of Sp_{2g}, and the prime-to-p Hecke orbit $\mathcal{H}^{(p)}(x)$ on $\mathcal{A}_{g,n}$ is *finite*. Hence the action of U_x on $\mathcal{A}_{g,n}^{/x}$ for such a supersingular point x contains a tremendous amount of information about the prime-to-p Hecke correspondences. However it is not always easy to mine this source of information; the success stories include Thm. 11.2, and [2, §5, Prop. 7].

§12. Logical interdependencies

Let k be an algebraically closed field of characteristic p as before. We summarize the logical interdependencies of various statements.

(12.1) We have seen that

$$(\mathrm{HO}) \iff (\mathrm{HO})_{\mathrm{ct}} + (\mathrm{HO})_{\mathrm{dc}}$$

(12.2) Suppose that $x \in \mathcal{A}_g(k)$ is not supersingular. Then Thm. 5.1 shows that

$$(\mathrm{HO})_{\mathrm{dc}} \text{ for x} \iff \mathcal{C}(\mathrm{x}) \text{ is irreducible}$$

(12.3) Suppose that $x, y \in \mathcal{A}_g(k)$, and there is an isogeny from A_x to A_y which preserves the polarizations up to multiples. Then

$$(\mathrm{HO})_{\mathrm{ct}} \text{ for x} \iff (\mathrm{HO})_{\mathrm{ct}} \text{ for y}$$

This is a consequence of 3.2.1, which depends on Prop. 3.2.

(12.4) Suppose that $x, y \in \mathcal{A}_g(k)$, and there is an isogeny from A_x to A_y which preserves the polarizations up to multiples. Then

$$(\mathrm{HO})_{\mathrm{dc}} \ \text{ for x} \iff (\mathrm{HO})_{\mathrm{dc}} \ \text{ for y}$$

The proof of the above statement is similar to the argument of Prop. 10.4, using hypersymmetric points.

(12.5) Let W_ξ be a non-supersingular Newton polygon stratum on $\mathcal{A}_g$, and let $\mathcal{C}$ be a leaf in W_ξ. Then

$$W_\xi \text{ is irreducible} \implies \mathcal{C} \text{ is irreducible}\,.$$

See Prop. 10.4.

(12.6) The implication

$$(\mathrm{HO}) \text{ for Hilbert modular varieties} \implies (\mathrm{HO})_{\mathrm{ct}}$$

holds.

(12.6.1) Here is a sketch of the proof of 12.6. Assume the Hecke orbit conjecture for Hilbert modular varieties. As remarked in 9.3, we may and do assume that the base field is $\overline{\mathbb{F}_p}$. Apply the trick "splitting at supersingular points" to get a point y in $\mathcal{A}_{g,n}(\overline{\mathbb{F}_p})$ contained in $\overline{\mathcal{H}^{(p)}(x)} \cap \mathcal{C}(x)$ as in Thm. 11.2. The Hilbert trick and the Hecke orbit conjecture for Hilbert modular varieties show that there exists a point $y_2 \in \left(\overline{\mathcal{H}^{(p)}(x)} \cap \mathcal{C}(x)\right)(\overline{\mathbb{F}_p})$ such that A_{y_2} is hypersymmetric and split. Here we used 10.1.4 on the existence of hypersymmetric points on every leaf of the Hilbert modular subvariety in $\mathcal{A}_{g,n}$ passing through the point y. Apply Thm. 10.2; the continuous part of the Hecke orbit conjecture for a Siegel modular variety $\mathcal{A}_{g,n}$ follows.

§13. Outline of the proof of the Hecke orbit conjecture

(13.1) PROOF OF $(\mathrm{HO})_{\mathrm{dc}}$.

(13.1.1) Theorem *Every non-supersingular Newton polygon stratum in $\mathcal{A}_{g,n}$ is irreducible.*

See [24] for a proof of Thm. 13.1.1. The proof of uses Thm. 5.1 and the results in [20], [8], [21].

(13.1.2) We have seen in Prop. 10.4 and 12.5 that $(\mathrm{HO})_{\mathrm{dc}}$ follows from Thm. 13.1.1. We are left with the continuous part $(\mathrm{HO})_{\mathrm{ct}}$ of the Hecke orbit conjecture.

(13.2) $(\mathrm{HO})_{\mathrm{ct}}$ FOR HILBERT MODULAR VARIETIES.

The continuous part $(\mathrm{HO})_{\mathrm{ct}}$ of the Hecke orbit conjecture for Hilbert modular varieties uses Thm. 8.2 and the argument in [4, §8]; the latter depends on the main result of [7] by de Jong. It is also possible to avoid de Jong's theorem in [7], using instead the local stabilizer principle at a supersingular point, similar to the argument of [2, §5, Prop. 7]. But the argument will not be as clean.

By 12.6, to complete the proof of the Hecke orbit conjecture for the Siegel modular varieties $\mathcal{A}_{g,n}$, it suffices to prove the discrete part of the Hecke orbit conjecture for Hilbert modular varieties.

(13.3) $(\mathrm{HO})_{\mathrm{dc}}$ FOR HILBERT MODULAR VARIETIES.

The proof of the discrete part of the Hecke orbit conjecture for Hilbert modular varieties uses the Lie-alpha stratification on Hilbert modular varieties. See [30] for some properties of the Lie-alpha stratification; see also [11] for the case when p is unramified in the totally real field, and [1] for the case when p is totally ramified in the totally real field. The starting point is the fact that for each given Newton polygon stratum W_ξ on a given Hilbert modular variety $\mathcal{M}_F$, there exists a leaf $\mathcal{C}$ contained in W_ξ which is an open subset of some Lie-alpha stratum of $\mathcal{M}_F$. A standard degeneration argument shows that it suffices to prove that the closure of every Lie-alpha stratum contains a superspecial point of a specific type. This observation allows us to bring in deformation theory. The last and the most crucial step was done by C.-F. Yu, who constructed enough deformations to facilitate an induction on the partially ordering on the family of irreducible components of Lie-alpha strata induced by the incidence relation.

§14. p-adic monodromy of leaves

In the last section we mention a maximality property of the naive p-adic monodromy group. The notion of hypersymmetric points plays an important role in the proof of 14.1.

(14.1) Theorem *Let x be a hypersymmetric point such that $\mathrm{End}_k(A_x[p^\infty])$ is a maximal order of $\mathrm{End}_k(A_x[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then the naive p-adic monodromy group of the leaf $\mathcal{C}(x)$ is maximal. In other words, if we use x as the base point, then the image of the naive p-adic monodromy group is equal to the intersection of $\mathrm{Aut}(A_x[p^\infty])$ with the unitary group attached to the pair $(\mathrm{End}_k(A_x[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, *)$, where $*$ denotes the involution on the semisimple algebra $\mathrm{End}_k(A_x[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ over $\mathbb{Q}_p$ induced by the principal polarization λ_x on A_x.*

(14.1.1) Remark A Barsotti-Tate group G over an algebraically closed field k of characteristic p such that $\mathrm{End}_k(G)$ is a maximal order in $\mathrm{End}_k(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is called a *minimal* Barsotti-Tate group in [23].

(14.2) Corollary *Let $x \in \mathcal{A}_{g,n}(k)$ be a closed point of $\mathcal{A}_{g,n}$ such that $\mathrm{End}_k(A_x[p^\infty])$ is a maximal order of $\mathrm{End}_k(A_x[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then the naive p-adic monodromy group of the leaf $\mathcal{C}(x)$ is maximal.*

(14.3) The idea of the proof of Thm. 14.1 is the following. First we prove an analogous statement for the naive p-adic monodromy group using Ribet's method in [27], [9]. Use a hypersymmetric point x with the properties in the statement of Thm. 14.1 as the base point for computing the p-adic monodromy group. This allows us to overcome the usual sticky issues related to different choices of base points, and reduce Thm. 14.1 to showing that the conjugates of the p-adic monodromy group of a leaf in a Hilbert modular subvariety already generates the target group of the naive p-adic monodromy representation. The last group-theoretic statement is elementary and can be verified directly.

References

[1] F. Andreatta and E. Z. Goren. Hilbert modular varieties of low dimension. To appear in *Geometric Aspects of Dwork's Theory, A Volume in memory of Bernard Dwork*. A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz, and F. Loeser (Eds.), 62 pages.

[2] C.-L. Chai. Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli. *Invent. Math.*, 121:439–479, 1995.

[3] C.-L. Chai. Monodromy of Hecke-invariant subvarieties. Prinprint, 10 pp., April, 2003. Available from `http://www.math.upenn.edu/~chai/`.

[4] C.-L. Chai. Families of ordinary abelian varieties: canonical coordinates, p-adic monodromy, Tate-linear subvarieties and Hecke orbits. Preprint, 55 pp., July 2003. Available from `http://www.math.upenn.edu/~chai/`.

[5] C.-L. Chai and F. Oort. Canonical coordinates on leaves of p-divisible groups. Part I. General properties. Preprint, 24 pp., August 2003.

[6] C.-L. Chai. Canonical coordinates on leaves of p-divisible groups. Part II. Cartier theory. Preprint, 28 pp., September 2003.

[7] A. J. de Jong. Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic. *Invent. Math.*, 134:301–333, 1998.

[8] A. J. de Jong and F. Oort. Purity of the stratification by Newton polygons. *Journ. A. M. S.*, 13:209–241, 2000.

[9] P. Deligne and K. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Inv. Math.*, 59:227–286, 1980.

[10] P. Deligne and G. Pappas. Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant. *Compos. Math.*, 90:59–79, 1994.

[11] E. Z. Goren and F. Oort. Sratification of Hilbert modular varieties. *J. Alg. Geom.*, 9:111–154, 2000.

[12] A. Grothendieck. *Groupes de Monodromie en Géométrie Algébrique (SGA 7) I.* Lecture Notes in Math. 288, Springer Verlag, 1972.

[13] N. M. Katz. Slope filtration of F-crystals. *Journ. Géom. Alg. Rennes,I, Astérisque*, vol. 63, pp. 113-164, 1979.

[14] J. Lubin and J. Tate. Formal complex multiplication in local fields. *Ann. Math.*, 81:380–387, 1965.

[15] Yu. I. Manin. The theory of commutative formal groups over fields of finite characteristic. *Usp. Math.*, 18:3–90, 1963. Russian Math. Surveys 18 (1963), 1–80.

[16] D. Mumford. Biextensions of formal groups. In *Algebraic Geometry*, number 307–322. Oxford Univ. Press, 1969. Proceedings of Internat. Coll. Bombay, 1968.

[17] T. Oda and F. Oort. Supersingular abelian varieties. pages 595–621, 1978.

[18] F. Oort. The isogeny class of a CM abelian variety is defined over a finite field. *J. Pure. Appl. Algebra*, 3:399–408, 1973.

[19] F. Oort. Some questions in algebraic geometry. Preliminary version, June 1995. Available from `http://www.math.uu.nl/people/oort/`

[20] F. Oort. Newton polygons and formal groups: conjectures by Manin and Grothendieck. *Ann. of Math.*, 152:183–206, 2000.

[21] F. Oort. A stratification of a moduli space of polarized abelian varieties. In *Moduli of Abelian Varieties*, Progress in Math. 195, Birkhäuser, pp. 345–416, 2001.

[22] F. Oort. Foliations in moduli spaces of abelian varieties. *J. A.M.S.*, 17:267–296, 2004.

[23] F. Oort. Minimal p-divisible groups. To appear in *Ann. Math.* Available from `http://www.math.uu.nl/people/oort/`

[24] F. Oort. Monodromy, Hecke orbits and Newton polygon strata. Seminar at MPI, Bonn, Feb. 14, 2004. Available from `http://www.math.uu.nl/people/oort/`

[25] F. Oort. Hypersymmetric abelian varieties. Preliminary version, Feb. 12, 2004. Available from `http://www.math.uu.nl/people/oort/`

[26] F. Oort and T. Zink. Families of p-divisible groups with constant Newton polygon.

[27] K. Ribet. p-adic interpolation via Hilbert modular forms. *Proc. Symp. Pure Math.*, vol. 29, *Algebraic Geometry–Arcata 1974*, pp. 581–592, Amer. Math. Soc., 1975.

[28] J. Tate. Endomorphisms of abelian varieties over finite fields. *Invent. Math.*, 2:134–144, 1966.

[29] J. Tate. Classes d'isogeny de variétés abéliennes sur un corps fini (d'apès T. Honda. *Séminaire Bourbaki 1968/69*, Exposé 352, Lecture Notes in Math. 179, pp. 95–110, 1971.

[30] C.-F. Yu. On reduction of Hilbert-Blumenthal varieties,. 33 pp. To appear in Ann. Inst. Fourier (Grenoble).

[31] T. Zink. On the slope filtration. *Duke Math. J.*, 109:79–95, 2001.

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