

NESTED SETS AND JEFFREY KIRWAN CYCLES

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1. INTRODUCTION

In this paper we discuss some new notions in the theory of hyperplane arrangements. The paper grew out of our plan to give an improved and simplified version of some of the results of Szenes Vergne [6].

We start from a complex vector space U of finite dimension r and a finite central hyperplane arrangement in U^* , given by a finite set $\Delta \subset U$ of linear equations. From these data one constructs the ordered set of subspaces, obtained by intersection of the given hyperplanes, and the open set \mathcal{A}_Δ complement of the union of the hyperplanes of the arrangement.

This paper consists of 3 parts. Part 1 is a recollection of the results in [4]. In part 2 we present 3 new results. The first, of combinatorial nature, establishes a canonical bijective correspondence between the set of no broken circuit bases and maximal nested sets which satisfy a condition called properness.

Next we associate to each proper maximal nested set \mathbb{M} a geometric cycle $c_{\mathbb{M}}$ of dimension r in \mathcal{A}_Δ . We show that integration of a top degree differential form over this cycle is done, by a simple algorithm, taking a multiple residue with respect to a system of local coordinates. The last result is the proof that, under the duality given by integration, the basis of cohomology given by the forms associated to the no broken circuit bases is dual to the basis of homology determined by the cycles $c_{\mathbb{M}}$. Section 3 is dedicated to the application relevant for the computations of [6], that is to say the Jeffrey Kirwan cycles.

Finally we wish to thank M. Vergne for explaining to us some of the theory and for various discussions and suggestions.

1.1. Notations. With the notations of the introduction, let U be a complex vector space of dimension r , $\Delta \subset U$ a totally ordered finite set of vectors $\Delta = \{\alpha_1, \dots, \alpha_m\}$. These vectors are the linear equations of a hyperplane arrangement in U^* . For simplicity we also assume that Δ spans U and any two distinct elements in Δ are linearly independent.

An example is a (complete) set of positive roots in a root system, ordered by any total order which refines the reverse dominance order.

In the A_{n-1} case we could say that $x_i - x_j \geq x_h - x_k$ if $k - h \geq j - i$ and, if they are equal if $i \leq h$.

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We want to recall briefly the main points of the theory (cf. [5]). Let $\Omega_i(\mathcal{A}_\Delta)$ denote the space of algebraic differential forms of degree i on \mathcal{A}_Δ .

We shall use implicitly the *formality*, that is the fact that the \mathbb{Z} subalgebra of differential forms on \mathcal{A}_Δ generated by the linear forms $\frac{1}{2\pi i} d \log \alpha, \alpha \in \Delta$ is isomorphic (via De Rham theory) to the integral cohomology of \mathcal{A}_Δ .

Formality implies in particular that $\Omega_r(\mathcal{A}_\Delta) = H^r \oplus d\Omega_{r-1}(\mathcal{A}_\Delta)$, for top degree forms. $H^r \equiv H^r(\mathcal{A}_\Delta, \mathbb{C})$ is the \mathbb{C} span of the top degree forms $\omega_\sigma := d \log \gamma_1 \wedge \cdots \wedge d \log \gamma_r$ for all bases $\sigma := \{\gamma_1, \dots, \gamma_r\}$ extracted from Δ . The forms ω_σ satisfy a set of linear relations generated by the following ones. Given $r+1$ elements $\gamma_i \in \Delta$, spanning U , we have:

$$\sum_{i=1}^{r+1} (-1)^i d \log \gamma_1 \wedge \dots \wedge d \log \check{\gamma}_i \wedge \dots \wedge d \log \gamma_r = 0.$$

Recall that a *non broken circuit* in Δ (with respect to the given total ordering) is an ordered linearly independent subsequence $\{\alpha_{i_1}, \dots, \alpha_{i_t}\}$ such that, for each $1 \leq \ell \leq t$ there is no $j < i_\ell$ such that the the vectors $\alpha_j, \alpha_{i_\ell}, \dots, \alpha_{i_t}$ are linearly dependent. In other words α_{i_ℓ} is the minimum element of $\Delta \cap \langle \alpha_{i_\ell}, \dots, \alpha_{i_t} \rangle$. In [5] it is proved that the elements

$$\left(\frac{1}{2\pi i}\right)^r \omega_\sigma := \left(\frac{1}{2\pi i}\right)^r d \log \gamma_1 \wedge \dots \wedge d \log \gamma_r,$$

where $\sigma = \{\gamma_1, \dots, \gamma_r\}$ runs over all ordered bases of V which are non broken circuits, give a linear \mathbb{Z} -basis of the integral cohomology of \mathcal{A}_Δ .

1.2. Irreducibles. Let us now recall some notions from [4]. Given a subset $S \subset \Delta$ we shall denote by U_S the space spanned by S .

Definition 1. Given a subset $S \subset \Delta$ set $\overline{S} := U_S \cap \Delta$, the completion of S . S is called complete if $S = \overline{S}$.

A complete subset $S \subset \Delta$ is called reducible if we can find a partition $S = S_1 \dot{\cup} S_2$, called a **decomposition** such that $U_S = U_{S_1} \oplus U_{S_2}$, irreducible otherwise.

Equivalently we say that the space U_S is reducible. Notice that, in the reducible case $S = S_1 \dot{\cup} S_2$, also S_1 and S_2 are complete.

From this definition it is easy to see [4]:

Lemma 1.1. Given complete sets $A \subset S$ and a decomposition $S = S_1 \dot{\cup} S_2$ of S we have that $A = (A \cap S_1) \dot{\cup} (A \cap S_2)$ is a decomposition of A . Let $S \subset \Delta$ be complete. Then there is a sequence (unique up to reordering) S_1, \dots, S_m of irreducible subsets in S such that

- $S = S_1 \cup \dots \cup S_m$ as disjoint union.
- $U_S = U_{S_1} \oplus \dots \oplus U_{S_m}$.

The S_i 's are called the irreducible components of S and the decomposition $S = S_1 \cup \dots \cup S_m$, the irreducible decomposition of S .

In the example of root systems, a complete set S is irreducible if and only if $S \cup -S$ is an irreducible root system.

We shall denote by \mathcal{I} the family of all irreducible subsets in Δ .

1.3. A minimal model. In [4] we have constructed a minimal smooth variety X_Δ containing \mathcal{A}_Δ as an open set with complement a normal crossings divisor, plus a proper map $\pi : X_\Delta \rightarrow U^*$ extending the identity of \mathcal{A}_Δ . The smooth irreducible components of the boundary are indexed by the *irreducible subsets*. To describe the intersection pattern between these divisors, in [4] we developed the general theory of nested sets. Maximal nested sets correspond to special points at infinity, intersections of these boundary divisors. In the paper [6], implicitly the authors use the points at infinity coming from complete flags which correspond, in the philosophy of [4], to a *maximal* model with normal crossings. It is thus not a surprise that by passing from a maximal to a minimal model the combinatorics gets simplified and the constructions become more canonical.

Let us recall the main construction of [4]. For each $S \in \mathcal{I}$ we have a subspace $S^\perp \subset U^*$ where $S^\perp = \{a \in U^* \mid s(a) = 0, \forall s \in S\}$. We have the projective space $\mathbb{P}(U^*/S^\perp)$ of lines in U^*/S^\perp a map $i : \mathcal{A}_\Delta \rightarrow U^* \times_{S \in \mathcal{I}} \mathbb{P}(U^*/S^\perp)$. Set X_Δ equal to the closure of the image $i(\mathcal{A}_\Delta)$ in this product. In [4] we have seen that X_Δ is a smooth variety containing a copy of \mathcal{A}_Δ and the complement of \mathcal{A}_Δ in X_Δ is a union of smooth irreducible divisors D_S , having transversal intersection, indexed by the elements $S \in \mathcal{I}$.

1.4. Nested sets. Still in [4] we showed that a family D_{S_i} of divisors indexed by irreducibles S_i has non empty intersection (which is then smooth irreducible) if and only if the family is *nested* according to:

Definition 2. A subfamily $\mathbb{M} \subset \mathcal{I}$ is called *nested* if, given any subfamily $\{S_1, \dots, S_m\} \subset \mathbb{M}$ with the property that for no $i \neq j$, $S_i \subset S_j$, then $S := S_1 \cup \dots \cup S_m$ is complete and the S_i 's are the irreducible components of S .

Lemma 1.2. 1) Let $\mathbb{M} = \{S_1, \dots, S_m\}$ be a nested set. Then $S := \cup_{i=1}^m S_i$ is complete. The irreducible components of S are the maximal elements of \mathbb{M} .

2) Any nested set is the set of irreducible components of the elements of a flag $A_1 \supset A_2 \supset \dots \supset A_k$, where each A_i is complete.

Proof. 1) By definition of nested set, the maximal elements of \mathbb{M} decompose their union which is complete.

2) It is clear that, if $A \subset B$, the irreducible components of A are contained each in an irreducible component of B . From this follows that the irreducible components of the sets of a flag form a nested set. Conversely let $\mathbb{M} = \{S_1, \dots, S_m\}$ be a nested set. Set $A_1 = \cup_{i=1}^m S_i$. Next remove from \mathbb{M} the irreducible components of A_1 (in \mathbb{M} by part 1)). We have a new nested set to which we can apply the same procedure. Working inductively we construct a flag of which \mathbb{M} is the decomposition. \square

One way of using the previous result is the following, given a basis $\sigma : \{\gamma_1, \dots, \gamma_r\} \subset \Delta$ one can associate to σ a maximal flag $F(\sigma)$ by setting $A_i(\sigma) := \Delta \cap \langle \gamma_{i+1}, \dots, \gamma_r \rangle$. Clearly the map from bases to flags is surjective and from flags to maximal nested sets is also surjective. We thus obtain a surjective map from bases to maximal nested sets. We will see that it induces a bijective map between no broken circuit bases and proper maximal nested sets.

Proposition 1.3. *1) Let $A_1 \supsetneq A_2 \cdots \supsetneq A_k$, be a maximal flag of complete non empty sets. Then $k = r$ and for each i , A_i spans a subspace of codimension $i - 1$.*

2) Let $\Delta = S_1 \cup \dots \cup S_t$ be the irreducible decomposition of Δ .

i) Then the S_i 's are the maximal elements in \mathcal{I} .

ii) Every maximal nested set contains each of the elements S_i ,

$i = 1, \dots, t$ and is a union of maximal nested sets in the sets S_i .

3) Let \mathbb{M} be a maximal nested set, $A \in \mathbb{M}$ and $B_1, \dots, B_r \in \mathbb{M}$ maximal among the elements in \mathbb{M} properly contained in A .

Then the subspaces U_{B_i} form a direct sum and

$$\dim(\oplus_{i=1}^k U_{B_i}) + 1 = \dim U_A.$$

4) A maximal nested set always has r elements.

Proof. 1) By definition $A_1 = \Delta$ spans U . If $\alpha \in A_i - A_{i+1}$ the completion of $A_{i+1} \cup \{\alpha\}$ must be A_i by the maximality of the flag. On the other hand by definition α is not in the subspace spanned by A_{i+1} hence we have that $\dim U_{A_i} = \dim U_{A_{i+1}} + 1$ which implies 1). 2) is immediate from the definitions. As for 3) by definition the subspaces U_{B_i} form a direct sum and since A is irreducible $\oplus_{i=1}^k U_{B_i} \subsetneq U_A$. Let $\alpha \in A - \cup_{i=1}^k B_i$ and B be the completion of $\{\alpha\} \cup \cup_{i=1}^k B_i$. We must have $B = A$ otherwise we can add the irreducible components of B to \mathbb{M} which remains nested, contradicting the maximality. Thus $\dim(\oplus_{i=1}^k U_{B_i}) + 1 = \dim U_A$. 4) follows from 3) and an easy induction. \square

A maximal nested set \mathbb{M} corresponds thus to a set of r divisors in X_Δ which by [4], intersect transversally in a single point $P_{\mathbb{M}}$. Let us explicit the example of the positive roots of type A_{n-1} . We think of such a root as a pair (i, j) with $1 \leq i < j \leq n$. The irreducible subsets are indexed by subsets (which we display as sequences) (i_1, \dots, i_s) , $s > 1$, with $1 \leq i_1 < \dots < i_s \leq n$. To such a sequence corresponds the set S of pairs supported in the sequence. A family of subsets is nested if any two of them are either disjoint or one is contained in the other.

In this case a maximal nested set \mathbb{M} has the following property, if $A \in \mathbb{M}$ has k elements and $k > 2$ we have two possibilities. Either the maximal elements of \mathbb{M} reduce to one subset with $k - 1$ elements or to two disjoint subsets A_1, A_2 with $A = A_1 \cup A_2$.

We define a map $\phi : \mathcal{I} \rightarrow \Delta$ by associating to each $S \in \mathcal{I}$ its minimum $\phi(S) := \min(a \in S)$ with respect to the given ordering.

For example, in the root system case, with the ordering given before, we have that $\phi(S)$ is the highest root in S .

We come to the main new definition:

Definition 3. *A maximal nested set \mathbb{M} is called proper if the set $\phi(\mathbb{M}) \subset \Delta$ is a basis of V .*

Example In the A_n case with the previous ordering $\phi(i_1, \dots, i_s) = x_{i_1} - x_{i_s} = (i_1, i_s)$.

A proper maximal nested set \mathbb{M} is thus encoded by a sequence of $n - 1$ subsets each having at least two elements, with the property that, taking the minimum and maximum for each set, these pairs are all distinct.

It is easy to see how to inductively define a bijection between proper maximal nested sets and permutations of $1, \dots, n$ fixing n . To see this consider \mathbb{M} as a sequence $\{S_1, \dots, S_{n-1}\}$ of subsets of $\{1, \dots, n\}$ with the above properties. We can assume that $S_1 = (1, 2, \dots, n)$ and have seen that $\mathbb{M}' := \mathbb{M} - \{S_1\}$ has either one or two maximal elements. If S_2 is the unique maximal element and $1 \notin S_2$, by induction we get a permutation $p(\mathbb{M}')$ of $2, \dots, n$. We then set $p(\mathbb{M})$ equal to the permutation which fixes 1 and is equal to $p(\mathbb{M}')$ on $2, \dots, n$. If S_2 is the unique maximal element and $n \in S_2$, we get a permutation $p(\mathbb{M}')$ of $1, \dots, n - 1$. We then set $p(\mathbb{M})$ equal to the permutation which fixes n and is equal to $\tau p(\mathbb{M}') \tau$ on $S_2 = \{1, \dots, n - 1\}$, τ being the permutation which reverses the order in S_2 . If S_2 and S_3 are the two maximal elements so that $\{1, \dots, n\}$ is their disjoint union, and $1 \in S_2$, $n \in S_3$ then by induction we get two permutations p_2 and p_3 of S_2 and S_3 respectively. We then set $p(\mathbb{M})$ equal to p_3 on S_3 and equal to $\tau p_2 \tau$ on S_2 , τ being the permutation which reverses the order in S_2 . In particular this shows that there are $(n - 1)!$ proper maximal nested sets, which can be recursively constructed. This is the rank of the top cohomology of the complement of the corresponding hyperplane arrangement. We will see presently that this is a general phenomenon.

Remark Notice that a proper maximal nested set inherits a total ordering from the total ordering of $\phi(\mathbb{M})$, and that this ordering is clearly a refinement of the partial ordering by reverse inclusion.

Now fix a maximal nested set \mathbb{M} . We clearly have:

Lemma 1.4. *Given $\alpha \in \Delta$ there exists a unique minimal irreducible $S \in \mathbb{M}$ such that $\alpha \in S$.*

This allows us to define a map $p_{\mathbb{M}} : \Delta \rightarrow \mathbb{M}$ by setting $p_{\mathbb{M}}(\alpha) := S$.

Definition 4. *If $\sigma \subset \Delta$ is a basis of V , we say that σ is adapted to \mathbb{M} if the restriction of $p_{\mathbb{M}}$ to σ is a bijection.*

Notice that, if \mathbb{M} is proper, then the basis $\phi(\mathbb{M})$ is clearly adapted to \mathbb{M} .

2. A BASIS FOR HOMOLOGY

We have seen in section 1) that, given a basis $\sigma = \{\gamma_1, \dots, \gamma_r\}$ we can associate to σ a maximal nested set, which we now denote by $\eta(\sigma)$. $\eta(\sigma)$ is

the decomposition of the flag $A_i = \Delta \cap \langle \gamma_i, \dots, \gamma_r \rangle$. Let us denote by \mathcal{C} the set of non broken circuit bases of V , by \mathcal{M} denote the set of proper maximal nested set.

Lemma 2.1. *If a no broken circuit basis σ is adapted to a proper nested set $\mathbb{M} = \{S_1, \dots, S_r\}$, then $\sigma = \phi(\mathbb{M})$.*

Proof. Let $\sigma = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$. Clearly $i_1 = 1$, and α_{i_1} is the minimum element of Δ . Let A be the irreducible component of Δ containing α_1 . We have that $A \in \mathbb{M}$, $\phi(A) = \alpha_1$ so $A = S_1$. We claim that $p_{\mathbb{M}}(\alpha_1) = A$. This follows from the fact that \mathbb{M} is proper so α_1 cannot be contained in two distinct elements A, B of \mathbb{M} , otherwise $\phi(A) = \phi(B)$. By lemma 1.2 $\Delta' := S_2 \cup \dots \cup S_r$ is complete. $A \not\subset \Delta'$ otherwise, still by 1.2 we would have that A is one of the S_i , $i \geq 2$. Since $\sigma' := \{\alpha_{i_2}, \dots, \alpha_{i_r}\}$ is adapted to $\mathbb{M}' := \{S_2, \dots, S_r\}$ we must have that the space $U_{\Delta'}$ spanned by Δ' , is $r-1$ dimensional, $\{\alpha_{i_2}, \dots, \alpha_{i_r}\}$ is a no broken circuit basis for $U_{\Delta'}$ relative to Δ' ordered by the total order induced from that of Δ and adapted to the proper nested set \mathbb{M}' . We can thus finish by induction. \square

Theorem 2.2. *We have that η maps \mathcal{C} to \mathcal{M} and ϕ maps \mathcal{M} to \mathcal{C} . Furthermore η and ϕ are bijections which are one the inverse of the other.*

Proof. Let $\sigma = \{\gamma_1, \dots, \gamma_r\} \in \mathcal{C}$. By definition, for each i we have that γ_i is the minimum element in $A_i = \Delta \cap \langle \gamma_i, \dots, \gamma_r \rangle$. Thus it is also the minimum element in one of the irreducibles decomposing A_i . It follows that $\eta(\sigma)$ is proper and that $\phi\eta(\sigma) = \sigma$.

Conversely let $\mathbb{M} = \{S_1, \dots, S_r\} \in \mathcal{M}$, and let $\gamma_i = \phi(S_i)$. By definition the γ_i 's are linearly independent, $\gamma_i < \gamma_{i+1}$ and \mathbb{M} is the decomposition of the flag $A_i := \cup_{j \geq i} S_j$. We thus have by the definition of ϕ that γ_i is the minimum element in A_i . Since A_i is complete we deduce that $\sigma = \{\gamma_1, \dots, \gamma_r\} \in \mathcal{C}$. Clearly $\eta(\phi(\mathbb{M})) = \mathbb{M}$. \square

Corollary 2.3. *A no broken circuit basis σ is adapted to a unique maximal proper nested set \mathbb{M} and $\sigma = \phi(\mathbb{M})$.*

Let us now fix a basis $\sigma \subset \Delta$. Write $\sigma = \{\gamma_1, \dots, \gamma_r\}$ and consider the r -form

$$\omega_\sigma := d \log \gamma_1 \wedge \dots \wedge d \log \gamma_r.$$

This is a holomorphic form on the open set \mathcal{A}_Δ of U^* which is the complement of the arrangement formed by the hyperplanes whose equation is in Δ . In particular if $\mathbb{M} \in \mathcal{M}$, we shall set $\omega_{\mathbb{M}} := \omega_{\phi(\mathbb{M})}$.

Also if $\mathbb{M} \in \mathcal{M}$, we can define a homology class in $H_r(\mathcal{A}_\Delta, \mathbb{Z})$ as follows.

Identify U^* with \mathbb{A}^r using the coordinates $\phi(S)$, $S \in \mathbb{M}$. Consider another complex affine space \mathbb{A}^r with coordinates z_S , $S \in \mathbb{M}$. In \mathbb{A}^r take the small torus T of equation $|z_S| = \varepsilon$ for each $S \in \mathbb{M}$. Define a map

$$f : \mathbb{A}^r \rightarrow U^*, \quad \text{by } \phi(S) := \prod_{S' \supset S} z_{S'}.$$

In [4] we have proved that this map lifts, in a neighborhood of 0, to a local system of coordinates of the model X_Δ . To be precise for a vector $\alpha \in \Delta$, set $B = p_{\mathbb{M}}(\alpha)$. In the coordinates z_S , we have that

$$(1) \quad \alpha = \sum_{B' \subset B} a_{B'} \prod_{S \supseteq B'} z_S = \prod_{S \supseteq B} z_S (a_B + \sum_{B' \subset B} a_{B'} \prod_{B \supsetneq S \supseteq B'} z_S)$$

with $a_{B'} \in \mathbb{R}$ and $a_B \neq 0$. Set $f_{M,\alpha}(z_S) := a_B + \sum_{B' \subset B} a_{B'} \prod_{B \supsetneq S \supseteq B'} z_S$ and $A_{\mathbb{M}}$ be the complement in the affine space \mathbb{A}^r of coordinates z_S of the hypersurfaces of equations $f_{M,\alpha}(z_S) = 0$. The main point is that, $A_{\mathbb{M}}$ is an open set of X_Δ . The point 0 in $A_{\mathbb{M}}$ is the *point at infinity* $P_{\mathbb{M}}$. The open set \mathcal{A}_Δ is contained in $A_{\mathbb{M}}$ as the complement of the divisor with normal crossings given by the equations $z_S = 0$. From this one sees immediately that, if ε is sufficiently small f maps T homeomorphically into \mathcal{A}_Δ . Let us give to T the obvious orientation coming from the total ordering of \mathbb{M} , so that $H_r(T, \mathbb{Z})$ is identified with \mathbb{Z} and set $c_{\mathbb{M}} = f_*(1) \in H_r(\mathcal{A}_\Delta, \mathbb{Z})$.

Proposition 2.4. *Let $\sigma = \{\gamma_1, \dots, \gamma_r\} \subset \Delta$ be a basis of V . Let $\mathbb{M} \in \mathcal{M}$. Then*

1) *If σ is not adapted to \mathbb{M} ,*

$$\int_{c_{\mathbb{M}}} \omega_\sigma = 0.$$

2) *If σ is adapted to \mathbb{M} , consider the sequence $p_{\mathbb{M}}(\gamma_1), \dots, p_{\mathbb{M}}(\gamma_r)$. This is a permutation π of the totally ordered set \mathbb{M} and we denote by $s(\mathbb{M}, \sigma)$ its sign. Then*

$$\frac{1}{(2\pi i)^r} \int_{c_{\mathbb{M}}} \omega_\sigma = s(\mathbb{M}, \sigma).$$

Proof. Given $\alpha \in \Delta$, from equation (1) we deduce that, in the neighborhood $A_{\mathbb{M}}$, the 1-form $d \log \alpha$ equals the sum of the 1-form $\sum_{S \supseteq B} d \log z_S$ and of a 1-form $\psi_B := d \log(a_B + \sum_{B' \subset B} a_{B'} \prod_{B \supsetneq S \supseteq B'} z_S)$ which is exact and holomorphic on the solid torus in \mathbb{A}^r defined by $|z_S| \leq \varepsilon$.

When we substitute these expressions in the linear forms $d \log \gamma_i$ and expand the product ω_σ we obtain various terms. Some terms vanish since we repeat twice a factor $d \log z_S$, some terms contain a factor ψ_B hence they are exact. The only possible contribution which gives a non exact form is when σ is adapted to \mathbb{M} , and then it is given by the term $s(\mathbb{M}, \sigma) \omega_{\mathbb{M}}$. From this observation both 1) and 2) easily follow. \square

Given the class $c_{\mathbb{M}}$ and an r -dimensional differential form ψ we can compute $\int_{c_{\mathbb{M}}} \psi$. Denoting by $P_{\mathbb{M}}$ the point at infinity corresponding to 0 in the previously constructed coordinates $z_i := z_{S_i}$ we shall say:

Definition 5. *The integral $\frac{1}{(2\pi i)^r} \int_{c_{\mathbb{M}}} \psi$ is called the **residue** of ψ at the point at infinity $P_{\mathbb{M}}$. We will also denote it by $\text{res}_{\mathbb{M}}(\psi)$.*

NOTE The algebraic forms, in a neighborhood of the point $P_{\mathbb{M}}$ and in the coordinates z_i have the form $\psi = f(z_1, \dots, z_r) dz_1 \wedge \dots \wedge dz_r$ with

$f(z_1, \dots, z_r)$ a Laurent series which can be explicitly computed. Then the residue $\text{res}_{\mathbb{M}}(\psi)$ equals the coefficient of $(z_1 \dots z_r)^{-1}$, in this series.

We can summarize this section with the main Theorem.

Theorem 2.5. *The set of elements $c_{\mathbb{M}}$, $\mathbb{M} \in \mathcal{M}$ is the basis of $H_r(\mathcal{A}, \mathbb{Z})$, dual, under the residue pairing, to the basis given by the forms $\omega_{\phi(\mathbb{M})}$: the forms associated to the no broken circuit bases relative to the given ordering.*

Proof. This is a consequence of 2.2, 2.3, 2.4. \square

Remarks. 1) The formulas found give us an explicit formula for the projection π of $\Omega_r(\mathcal{A}) = H^r \oplus d\Omega_{r-1}(\mathcal{A}_{\Delta})$ to H^r with kernel $d\Omega_{r-1}(\mathcal{A}_{\Delta})$. We have:

$$(2) \quad \pi(\psi) = \sum_{\mathbb{M} \in \mathcal{M}} \text{res}_{\mathbb{M}}(\psi) \omega_{\mathbb{M}}.$$

2) Using the projection π any linear map on H^r , in particular the Jeffrey Kirwan residue (see below), can be thought of as a linear map on $\Omega_r(\mathcal{A})$ vanishing on $d\Omega_{r-1}(\mathcal{A}_{\Delta})$. Our geometric description of homology allows us to describe any such map as integration on a cycle, linear combination of the cycles $c_{\mathbb{M}}$.

3) There are several possible applications of these formulas to combinatorics and counting integer points in polytopes. The reader is referred to [2],[1].

4) We have treated only top homology but all homology can be described in a similar way due to the fact that for each k the k^{th} cohomology decomposes into the contributions relative to the subspaces of codimension k and the corresponding transversal configuration.

3. THE JEFFREY-KIRWAN RESIDUE

In this section V is a real r -dimensional vector space and $U := V \otimes_{\mathbb{R}} \mathbb{C}$, $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset V$. We further restrict to the case in which there exists a linear function on V which is positive on Δ .

Now let us assume that we have fixed once and for all an orientation of V^* by choosing an ordered basis $\xi = (x_1, \dots, x_r)$ of V and taking the orientation form $d\underline{x} = dx_1 \wedge dx_2 \wedge \dots \wedge dx_r$, for example if Γ is the lattice spanned by the vectors in Δ we can take an ordered basis of Γ .

This gives a canonical way of identifying the r -forms with functions on \mathcal{A}_{Δ} . The form $\omega_{\sigma} = d \log \gamma_1 \wedge \dots \wedge d \log \gamma_r$ is identified with the function $\epsilon_{\sigma} n_{\sigma}^{-1} \prod_i \gamma_i^{-1}$ where n_{σ} is the index in Γ of the lattice spanned by the elements γ_i (a volume element), and ϵ_{σ} a sign expressing the orientation. Let \tilde{H}^r denote this space of functions. Finally, when all the elements in Δ are on the same side of some hyperplane there is another interesting way of representing H^r in which the Jeffrey Kirwan residue appears in a very natural way. This is done via the Laplace transform $\int e^{-(x,y)} f(y) dy$. Which in our setting has to be understood as a transform from functions on V^* with a

prescribed invariant Lebesgue measure (the one induced by Γ) to functions on V (more intrinsically to r differential forms).

Precisely consider the cone C spanned by the vectors in Δ . For each basis $\sigma := \{\gamma_1, \dots, \gamma_r\}$ extracted from Δ let $C(\sigma)$ be the positive cone that it generates and χ_σ its characteristic function. Let finally K^r be the vector space spanned by the functions χ_σ . From the basic formula $\int_0^\infty \dots \int_0^\infty e^{-\sum_{i=1}^r x_i y_i} dy_1 \dots dy_r = \frac{1}{x_1 \dots x_r}$ and linear coordinate changes, it is easy to verify that, the Laplace transform of χ_σ is $n_\sigma^{-1} \prod_i \gamma_i^{-1}$, as function on the dual positive cone, consisting of all x such that $(x, y) > 0$, $\forall y \in C$. Therefore combining with the isomorphism of \tilde{H}^r with H^r we have a Laplace transform $L : K^r \rightarrow H^r$ with $L(\chi_\sigma) = \epsilon_\sigma \omega_\sigma$.

L is a linear isomorphism in which it is easy to reinterpret geometrically the linear relations previously described. Finally the Jeffrey Kirwan residue is a linear function $\psi \rightarrow J\langle c | \psi \rangle$ on H^r depending on a regular vector c . It corresponds to the linear function defined on K^r which just consists in evaluating the functions f in c . In other words $J\langle c | \psi \rangle = L^{-1}(\psi)(c)$. By the definition of K^r it is clear that this linear function depends only on the chamber C in which c lies. Our final result is the description of a geometric cycle $\delta(C)$ such that $J\langle c | \psi \rangle = \frac{1}{(2\pi i)^r} \int_{\delta(C)} \psi$. Given an ordered basis τ of V , we set ν_τ equal to 1 if the ordered basis τ has the same orientation as ξ , -1 otherwise. For a proper maximal nested set \mathbb{M} we denote $\nu_{\phi(\mathbb{M})}$ by $\nu_{\mathbb{M}}$.

For each basis $\tau \subset \Delta$, set $C(\tau) = \{x \in V | x = \sum_{\alpha \in \tau} a_\alpha \alpha, a_\alpha > 0\}$. Set for simplicity, for a proper maximal nested set \mathbb{M} , $C(\mathbb{M}) := C(\phi(\mathbb{M}))$.

At this point we can recall some facts from [6]. Assume that in V we have a lattice Γ which we interpret as the character group of an r -dimensional torus T_r . Assume that $\Delta \subset \Gamma$ is a set of characters.

We have the following sequence of ideas. First of all we use the elements α_i , $i = 1, \dots, n$ to construct an n -dimensional representation Z of T direct sum of the 1-dimensional representations with character α_i^{-1} . Call $R = \mathbb{C}[x_1, \dots, x_n]$ the ring of polynomial functions on Z . On Z we have an action of the n -dimensional torus D_n of diagonal matrices and T acts via a homomorphism into D_n . Hence the torus T acts on R and x_i has weight α_i . If $\gamma \in \Gamma$ is a character, define $R(\gamma)$ to be the subspace of R of weight γ with respect to T .

We shall denote by \mathcal{R} the set of regular vectors i.e. the set of vectors which cannot be written as a linear combination of the elements in a subset S of Δ of cardinality smaller than r .

Consider a regular vector $\xi \in \Gamma$ and let $R_\xi = \bigoplus_{k=0}^\infty R(k\xi)$.

The following facts are well known [3]. R_ξ is a finitely generated subalgebra stable under the torus D_n . So, if we grade R_ξ so that $R(k\xi)$ has degree k we can consider the projective variety $T_\xi := \text{Proj}(R_\xi)$ with a line bundle \mathcal{L} such that $H^0(T_\xi, k\mathcal{L}) = R(k\xi)$. T_ξ is an embedding of the $n - r$ dimensional torus D_n/T and the regularity of ξ implies that T_ξ is an orbifold. The elements α_i index the boundary divisors of this torus embedding. Thus to

each α_i we can associate a degree 2 cohomology class, the Chern class of the corresponding divisor, which is still expressed with the same symbol α_i . According to the theory developed by Jeffrey Kirwan, discussed in [3] one can compute the intersection numbers $\int_{T_\xi} P(\alpha_1, \dots, \alpha_n)$ using the notion of Jeffrey Kirwan residue. Denoting by C the chamber in which ξ lies, one has:

$$(3) \quad \int_{T_\xi} P(\alpha_1, \dots, \alpha_n) = J\langle c \mid \frac{P(\alpha_1, \dots, \alpha_n)}{\alpha_1 \alpha_2 \dots \alpha_n} d\mu \rangle.$$

We want to represent this residue by integration over a cycle:

$$(4) \quad J\langle c \mid \psi \rangle = \frac{1}{(2\pi i)^r} \int_{\delta(C)} \psi.$$

From the formula $L(\chi_\sigma) = \epsilon_\sigma \omega_\sigma$, and definition of the Jeffrey Kirwan cycle discussed in the introduction, we see that $\delta(C)$ is the r -cycle whose value on the r -form ω_σ , for every basis $\sigma = \{\gamma_1, \dots, \gamma_r\} \subset \Delta$ of V , is given by

$$(5) \quad \frac{1}{(2\pi i)} \int_{\delta(C)} \omega_\sigma = \begin{cases} 0 & \text{if } C \cap C(\sigma) = \emptyset \\ \nu_\sigma & \text{if } C \subset C(\sigma) \end{cases}$$

Using this description of $\delta(C)$ and the fact that our homology basis $c_{\mathbb{M}}$ is dual to the cohomology basis $\omega_{\phi(\mathbb{M})}$, one immediately has:

Theorem 3.1.

$$\delta(C) = \sum_{\mathbb{M} \in \mathcal{M} \mid C \subset C(\mathbb{M})} \nu_{\mathbb{M}} c_{\mathbb{M}}.$$

REFERENCES

- [1] W. Baldoni-Silva, J.A. De Loera, M. Vergne, *Counting Integer flows in Networks*, preprint math.CO/0303228
- [2] W. Baldoni-Silva, M. Vergne, *Residues formulae for volumes and Ehrhart polynomials of convex polytopes*, preprint math.CO/0103097
- [3] M. Brion, M. Vergne *Arrangement of hyperplanes. I. Rational functions and Jeffrey-Kirwan residue*. Ann. Sci. cole Norm. Sup. (4) 32 (1999), no. 5, 715–741.
- [4] C. DeConcini, C. Procesi *Wonderful Models of subspace arrangements*. Selecta Math. (N.S.) 1 (1995), no. 3, 459–494.
- [5] P. Orlik, H. Terao, *Arrangements of hyperplanes*. Grundlehren der Mathematischen Wissenschaften 300. Springer-Verlag, Berlin, 1992.
- [6] A. Szenes, M. Vergne *Toric reduction and a conjecture of Batyrev and Materov* preprint, AT/0306311.

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