Classical and minimal models of the moduli space of curves of genus two

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1 Introduction

This paper is an introduction to the minimal model program, as applied to the moduli space of curves. Our long-term goal is a geometric description of the canonical model of the moduli space when it is of general type. This entails proving that the canonical model exists and interpreting it as a parameter space in its own right.

Work of Eisenbud, Harris, and Mumford shows that M_g is of general type at least when $g \geq 24$ (see [HaMu] and subsequent papers). A standard conjecture of birational geometry–the finite generation of the canonical ring– would imply the canonical model is

$$\text{Proj} \ \oplus_{n \geq 0} \Gamma(\overline{M}_g, nK_{\overline{M}_g}).$$

Unfortunately, this has yet to be verified in a single genus!

There is some cause for optimism: Shepherd-Barron [SB] has recently shown that the canonical model of the moduli space of principally polarized abelian surfaces of dimension $g \geq 12$ is the first Voronoi compactification.

Another possible line of attack is to consider *log* canonical models of the moduli space. The moduli space is best regarded as a pair $(\overline{M}_g, \Delta)$, where $\overline{M}_g$ is Deligne-Mumford compactification by stable curves and Δ is its boundary. It is implicit in the work of Mumford [Mu1] that the moduli

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space of stable curves is its own log canonical model (see Theorem 4.7). Our basic strategy it to interpolate between the log canonical model and the (conjectural) canonical model by considering

$$\text{Proj }\oplus_{n\geq 0}\Gamma(\overline{M}_g, n(K_{\overline{M}_g}+\alpha\Delta)),$$

where $\alpha\in\mathbb{Q}\cap[0,1]$ is chosen so that $K+\alpha\Delta$ is effective.

This program is a subject of ongoing work, inspired by correspondence with S. Keel, in collaboration with D. Hyeon. Future papers will address the stable behavior of these spaces for successively smaller values of α. It is remarkable that their behavior is largely independent of the genus (see, for example, Remark 4.9.)

However, for small values of g special complexities arise. When $g=2$ or 3, the locus in $\overline{M}_g$ of curves with automorphism has codimension ≤ 1. To include these spaces under our general framework, we must take into account the properties of the moduli stack $\overline{\mathcal{M}}_g$. In particular, it is necessary to use the canonical divisor of the moduli stack rather than its coarse moduli space. These differ substantially, as the natural morphism $\overline{\mathcal{M}}_g \to \overline{M}_g$ is ramified at stable curves admitting automorphisms.

Luckily, we have inherited a tremendously rich literature on curves of small genus. The invariant-theoretic properties of M_2 were extensively studied by the 19th century German school [Cl], who realized it as an open subset of the weighted projective space $\mathbb{P}(1,2,3,5)$. Theorem 4.10 reinterprets this classical construction using the modern language of stacks and minimal models

We work over an algebraically closed field k of characteristic zero. We use the notation $\equiv$ for $\mathbb{Q}$-linear equivalence of divisors. Throughout, a *curve* is a connected, projective, reduced scheme of dimension one. The *genus* of a curve is its arithmetic genus.

The moduli stack of smooth (resp. stable) curves of genus g is denoted $\mathcal{M}_g$ (resp. $\overline{\mathcal{M}}_g$); the corresponding coarse moduli scheme is denoted M_g (resp. $\overline{M}_g$). The boundary divisors in $\overline{M}_g$ (resp. $\overline{\mathcal{M}}_g$) are denoted

$$\Delta_0, \Delta_1, \ldots, \Delta_{\lfloor g/2\rfloor}(\text{resp. }\delta_0, \delta_1, \ldots, \delta_{\lfloor g/2\rfloor}).$$

Let $Q : \overline{\mathcal{M}}_g \to \overline{M}_g$ denote the natural morphism from the moduli stack to the coarse moduli space, so that

$$Q^*\Delta_i = \delta_i, i \neq 1 \qquad Q^*\Delta_1 = 2\delta_1.$$

We write $\delta = \sum_{0 \le i \le g/2} \delta_i$; abusing notation, we also use δ for the corresponding divisor

$$\Delta_0 + 1/2\Delta_1 + \Delta_2 + \ldots + \Delta_{\lceil g/2 \rceil}$$

on $\overline{M}_g$.

2 Classical geometry

2.1 Elementary facts about curves of genus two

We recall results from standard textbooks, e.g., [Ha] IV Ex 2.2 and §5. Let C denote a smooth curve of genus two with sheaf of differentials ω_C. The global sections of ω_C give the canonical morphism

$$j : C \to \mathbb{P}(\Gamma(C, \omega_C)) \simeq \mathbb{P}^1$$

which is finite of degree two. The corresponding covering transformation $\iota : C \to C$ is called the hyperelliptic involution. By the Hurwitz formula, j is branched over six distinct points

$$\{b_1, \ldots, b_6\} \subset \mathbb{P}(\Gamma(C, \omega_C)).$$

On fixing an identification $\mathbb{P}(\Gamma(C, \omega_C)) \simeq \mathbb{P}^1$, we can write down a nontrivial binary sextic form vanishing at the branch points

$$F \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6)),$$

determined by $\{b_1, \ldots, b_6\} \in \mathbb{P}^1$ up to a scalar.

Conversely, suppose we have a binary sextic form F with six distinct zeros $b_1, \ldots, b_6 \in \mathbb{P}^1$. Then there is a unique degree-two cover of $\mathbb{P}^1$ branched over these points. This is a smooth curve of genus two and the map to $\mathbb{P}^1$ is the canonical morphism. Moreover, the isomorphism class of C depends only on the orbit of F under the action of GL_2.

To summarize: There is a one-to-one correspondence between isomorphism classes of curves of genus two and GL_2-orbits of binary sextic forms with distinct zeros.

We will need a relative version of this dictionary, following [Vi]. Let $\pi : \mathcal{C} \to S$ be a smooth morphism to a scheme of finite type over k, with fibers curves of genus two. Since the relative dualizing sheaf ω_π is globally generated, there is a relative double cover

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\;\;j\;\;} & \mathbb{P}^1(\pi_*\omega_\pi) := \mathbb{P} \\
\;\;{\scriptstyle \pi}\searrow & & \swarrow {\scriptstyle \psi} \\
& S &
\end{array}
$$

with associated involution ι. Using the trace we decompose $j_*\mathcal{O}_\mathcal{C} \simeq \mathcal{O}_\mathbb{P} \oplus \mathcal{L}$, where $\mathcal{L}$ has relative degree -3 on the fibers of ψ. The $\mathcal{O}_\mathbb{P}$-algebra structure on $\mathcal{O}_\mathcal{C}$ is thus determined by an isomorphism $\mathcal{L}^2 \to \mathcal{O}_\mathbb{P}$, i.e., by a nonvanishing section of $\mathcal{L}^{-2}$ with zeros along the branch locus of j. By relative duality

$$j_*\omega_\pi = \mathscr{H}om_{\mathcal{O}_\mathbb{P}}(j_*\mathcal{O}_\mathcal{C}, \omega_\psi) \simeq \omega_\psi \oplus (\omega_\psi \otimes \mathcal{L}^{-1})$$

which yields

$$\pi_*\omega_\pi \simeq \psi_*(\omega_\psi \otimes \mathcal{L}^{-1}).$$

Using the identifications

$$\psi_*\mathcal{O}_\mathbb{P}(+1) = \pi_*\omega_\pi, \quad \omega_\psi = (\psi^* \det \pi_*\omega_\pi)(-2), \quad \mathrm{Pic}(\mathbb{P}) = \mathrm{Pic}(S) \oplus \mathbb{Z}c_1(\mathcal{O}_\mathbb{P}(+1)),$$

we find

$$\mathcal{L}^{-1} \simeq \mathcal{O}_\mathbb{P}(+1) \otimes \omega_\psi^{-1} = \mathcal{O}_\mathbb{P}(3) \otimes (\psi^* \det \pi_*\omega_\pi)^{-1}.$$

The class of the branch divisor is thus

$$-2c_1(\mathcal{L}) = c_1(\mathcal{O}_\mathbb{P}(6)) - 2\psi^* c_1(\det \pi_*\omega_\pi). \tag{1}$$

Here are the practical implications of this analysis: An isomorphism of C induces a linear transformation on $\Gamma(C, \omega_C)$, which respects the binary sextic F up to a scalar. If we have an element $M \in \mathrm{GL}_2$, respecting F up to a scalar, when does it come from an automorphism of C? Specfically,

$$(x, y) \mapsto (x, y) \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

4

induces a natural left action on binary sextic forms

$$F \mapsto F(xm_{11} + ym_{21}, xm_{12} + ym_{22}).$$

Equation (1) indicates this should be renormalized

$$F \mapsto (M, F) := (\det M)^{-2} F(xm_{11} + ym_{21}, xm_{12} + ym_{22}), \qquad (2)$$

so that $(M, F) = F$ if and only M is induced from an automorphism of C.

A smooth curve is *bielliptic* if it admits a degree-two morphism $i : C \to E$ to an elliptic curve; the covering transformation is called a *bielliptic involution*. For curves of genus two any bielliptic involution commutes with the hyperelliptic involution, which yields a diagram

$$
\begin{array}{ccc}
C & \xrightarrow{i} & E \\
j \downarrow & & \downarrow \bar{j} \\
\mathbb{P}^1 & \xrightarrow{\bar{i}} & \mathbb{P}^1
\end{array}
$$

where $\bar{i}$ and $\bar{j}$ are the double covers induced on quotients. The branch locus of j is preserved by the covering transformation for $\bar{i}$, which is conjugate to $[x, y] \mapsto [y, x]$. The resulting involution of the branch locus will also be called a bielliptic involution. Thus C is isomorphic to a double cover branched over

$$\{[\alpha_1, 1], [1, \alpha_1], [\alpha_2, 1,][1, \alpha_2], [\alpha_3, 1], [1, \alpha_3]\}$$

for some $\alpha_1, \alpha_2, \alpha_3 \in k$. Conversely, each such curve admits a diagram as above and thus is bielliptic.

2.2   Invariant theory of binary sextics

We observe the classical convention for normalizing the coefficients of a bilinear sextic

$$F = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6$$

and consider the induced action of GL_2 on $k[a, b, c, d, e, f, g]$. Recall that a polynomial $P \in k[a, b, c, d, e, f, g]$ is SL_2-*invariant* if for each $M \in \mathrm{SL}_2$, we have

$$(M, P) = P.$$

We write
$$R := k[a, b, c, d, e, f, g]^{\mathrm{SL}_2}$$

for the ring of such invariants. If P is SL_2-invariant then each homogeneous component of P is as well, so R is a graded ring. Every homogeneous invariant satisfies the functional relation

$$M \cdot P = (\det M)^{\deg(P)} P, \quad M \in \mathrm{GL}_2; \tag{3}$$

here it is essential that the action (2) include the factor $(\det M)^2$. The transformation

$$(x, y) \rightarrow (y, x)$$

thus reverses the sign of invariants of odd degree. These are called *skew invariants* in the classical literature.

Explicit generators for R were first written down in the nineteenth century, e.g., [Cl], pp. 296, in symbolic notation, [Ca] and [Sa] as explicit polynomials–the second edition of Salmon's *Higher algebra* has the most detailed information, and also [El] pp. 322. A nice early twentieth-century discussion is [Sc] pp. 90 and a modern account invoking the representation theory of SL_2 is [Sp].

For our purposes, the symmetric function representation of the invariants in [Ig2] pp. 176 and 185 is the most useful. Let $\xi_1, \ldots, \xi_6$ denote the roots of the dehomogenized form $F(x, 1)$, and write (ij) as shorthand for $\xi_i - \xi_j$. We write

$$
\begin{aligned}
A &= a^2 \sum_{\text{fifteen}} (12)^2 (34)^2 (56)^2 \\
B &= a^4 \sum_{\text{ten}} (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2 \\
C &= a^6 \sum_{\text{sixty}} (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2 (14)^2 (25)^2 (36)^2 \\
D &= a^{10} \prod_{ij} (ij)^2 \\
E &= a^{15} \prod_{\text{fifteen}} \det \begin{pmatrix} 1 & \xi_1 + \xi_2 & \xi_1 \xi_2 \\ 1 & \xi_3 + \xi_4 & \xi_3 \xi_4 \\ 1 & \xi_5 + \xi_6 & \xi_5 \xi_6 \end{pmatrix} = a^{15} \prod_{\text{fifteen}} ((14)(36)(52) - (16)(32)(54))
\end{aligned}
$$

where the summations are chosen to make the expressions $\mathfrak{S}_6$-symmetric. Consequently, A, B, C, D, and E can all be expressed as polynomials in

6

$\mathbb{Q}[a, b, c, d, e, f, g]$, e.g.,

$$A \;=\; -240(ag - 6bf + 15ce - 10d^2)$$

$$B \;=\; -162000 \det \begin{pmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{pmatrix} + 1620(ag - 6bf + 15ce - 10d^2)^2.$$

In classical terminology, $(ag - 6bf + 15ce - 10d^2)$ is the *sixth transvectant of F over itself*; transvection is one of the main operations in Gordan's proof of finiteness for invariants of binary forms. The determinantal expression is the *catalecticant* of F: It vanishes precisely when F can be expressed as a sum of three sixth powers [El] pp. 276.

The following facts will be useful for subsequent analysis:

Proposition 2.1   *1. The expressions A, B, C, D, and E are invariant and generate R [Hi], pp. 100, [Cl], etc.*

   2. D is the discriminant and vanishes precisely when the binary form has a multiple root.

   3. B, C, D, and E vanish whenever the binary form has a triple root; A vanishes when the form has a quadruple root.

   4. E vanishes if and only if the form admits a bielliptic involution, as defined in §2.1 [El], pp. 327 and [Cl], pp. 457.

   5. The unique irreducible relation among the invariants is

   $$E^2 = G(A, B, C, D),$$

   where G is weighted-homogeneous of degree 30 [Cl], pp. 299.

The notation used for the generating invariants is not consistent among authors. Our notation is consistent with that of Igusa, but inconsistent with Clebsch's and Salmon's. Of course, the invariants of degree two and fifteen are unique up to scalar.

2.3 The projective invariant-theory quotient

We consider

$$X := \operatorname{Proj} R = \operatorname{Proj} \frac{k[A, B, C, D, E]}{\langle E^2 - G(A, B, C, D) \rangle}.$$

If $A = B = C = D = 0$ then $E = 0$ as well, so X is covered by the distinguished affine open subsets

$$\{A \neq 0\} \quad \{B \neq 0\} \quad \{C \neq 0\} \quad \{D \neq 0\}.$$

However, in each localization

$$(k[A, B, C, D, E][A^{-1}])_0 \quad (k[A, B, C, D, E][B^{-1}])_0$$

$$(k[A, B, C, D, E][C^{-1}])_0 \quad (k[A, B, C, D, E][D^{-1}])_0$$

only even powers of E appear, so all the functions over these distinguished open subsets can be expressed in terms of A, B, C, D. In light of Proposition 2.1, we find

Proposition 2.2   *1. $X \simeq \operatorname{Proj} k[A, B, C, D] \simeq \mathbb{P}(2, 4, 6, 10) \simeq \mathbb{P}(1, 2, 3, 5)$ [Ig2], pp. 177.*

 2. A binary sextic with a zero of multiplicity three, admitting a nonvanishing invariant of positive degree, is mapped to $p := [1, 0, 0, 0, 0] \in X$.

 3. All positive-degree invariants vanish at binary sextics with a zero of multiplicity four; they do not yield points of X.

 Geometric Invariant Theory gives an interpretation of the points of X:

Proposition 2.3   *1. a binary sextic is stable (resp. semistable) if and only if its zeros have multiplicity ≤ 2 (resp. ≤ 3) [GIT], ch. 4 §1;*

 2. $X - \{[1, 0, 0, 0, 0]\}$ is a geometric quotient for binary sextics with zeros of multiplicity ≤ 2 [GIT] 1.10.

The 'only if' part of the first assertion can be deduced from Proposition 2.2. As $X - \{D = 0\}$ is a geometric quotient for binary sextics with distinct zeros, our analysis of genus two curves in §2.1 yields

Proposition 2.4 *The moduli scheme M_2 can be identified with $X - \{D = 0\}$, where D is the discriminant.*

Remark 2.5 This construction definitely fails in characteristic two. If the double cover $j : C \to \mathbb{P}^1$ is wildly ramified, the branch divisor may have multiplicities > 3. These curves correspond to unstable points under the SL_2-action, and thus are not represented in the invariant-theory quotient. [Ig1] has a detailed account of what must be done in this case.

2.4 Invariant-theory quotient as a contraction

We sketch the relationship between the invariant-theory quotient and the moduli space of stable curves.

Definition 2.6 A birational map of normal projective varieties

$$\beta : Y \dashrightarrow X$$

is a *contraction* if β^{-1} has no exceptional divisors, i.e., the proper transform of each codimension-one subset in X has codimension one in Y.

Proposition 2.7 *There exists a birational contraction $\beta : \overline{M}_2 \dashrightarrow X$ restricting to the identity along the open subset M_2. β is an isomorphism over $\overline{M}_2 - \Delta_1$ and contracts Δ_1 to the point p.*

proof: To produce the birational contraction, we exhibit a morphism

$$\beta^{-1} : U \hookrightarrow \overline{M}_2$$

where $U \subset X$ is open with complement of codimension ≥ 2 and $\beta^{-1}|_{M_2 \cap U}$ is the identity. We shall take $U = X - p$, where p corresponds to the binary forms with a triple zero (cf. Proposition 2.2.)

Our argument uses the following generalization of §2.1

Lemma 2.8 There is a one-to-one correspondence between

1. binary sextic forms having zeros of multiplicity ≤ 2, up to projectivity;

2. stable curves of genus two *not* contained in the boundary divisor Δ_1, up to isomorphism.

9

The correspondence induces the desired morphism β^{-1}.

For any binary form F with zeros of multiplicity ≤ 2, we may consider the double cover branched along $F = 0$

$$j : C_F \to \mathbb{P}^1.$$

As we have seen, C_F is smooth away from the double zeros of F; at these points C_F has local equation $y^2 = x^2$, which defines a node. We have $j^* \mathcal{O}_{\mathbb{P}^1}(+1) = \omega_{C_F}$, which is ample on C_F, so C_F is stable. The normalization $\nu : C_F^\nu \to C_F$ is the double cover branched along $F' = 0$, where F' is the product of the factors of F with multiplicity one; C_F is obtained from C_F^ν by gluing the pairs of points over the each double root of F. There are three possibilities:

1. $\deg(F') = 4$, in which case C_F^ν is connected of genus one;

2. $\deg(F') = 2$, in which case C_F^ν is connected of genus zero;

3. $\deg(F') = 0$, in which case C_F^ν has two connected components of genus zero.

In any event, C_F cannot be expressed as the union of two subcurves of genus one meeting at a point, so the resulting curve is not in Δ_1.

Conversely, suppose we are given a stable curve C of genus two not contained in Δ_1. Quite generally, ω_C is globally generated for any stable curve without separating nodes; the only curves in $\overline{M}_2$ with separating nodes lie in Δ_1. Thus the sections of ω_C give a double cover

$$j : C \to \mathbb{P}^1$$

branched along a sextic, with zeros of multiplicity ≤ 2 because C is nodal. This completes the proof of the lemma.

The constructions given in the lemma yield a morphism

$$\beta^{-1} : U \to \overline{M}_2$$

taking U bijectively to $\overline{M}_2 - \Delta_1$. It follows that β contracts Δ_1 to p. $\square$

2.5 Blowing up the invariant-theory quotient

We recall the principal result of [Ig2]. Here is the necessary notation: Let A_g denote the moduli space of principally polarized abelian varieties of dimension g, $\overline{A}_g$ its Satake compactification. Recall that $\overline{A}_g = \mathrm{Proj}\, S$, where S is the ring of $\mathrm{Sp}(g, \mathbb{Z})$-modular forms; we use λ to denote the resulting polarization on $\overline{A}_g$. Let $t : M_g \hookrightarrow A_g$ denote the Torelli morphism, associating to each curve its Jacobian.

Now assume $g = 2$. Regarding M_2 as an open subset of X (see Proposition 2.4), t extends to a rational map

$$\tau : X \dashrightarrow \overline{A}_2.$$

The inclusion and the Torelli morphism induce

$$M_2 \hookrightarrow \widetilde{X} := \overline{\mathrm{Graph}(\tau)} \subset X \times \overline{A}_2. \qquad (4)$$

In particular, $\widetilde{X}$ compactifies M_2.

Theorem 2.9 *[Ig2] The indeterminacy of τ is the point $p = [1, 0, 0, 0, 0] \in X$ corresponding to binary sextic forms with a zero of multiplicity three. If we choose local coordinates at p*

$$x_1 = 2^4 3^2 B / A^2 \quad x_2 = 2^6 3^3 (3C - AB)/A^3 \quad x_3 = 2 \cdot 3^5 D/A^5$$

then τ is resolved by a weighted blow-up centered at p

$$b : \widetilde{X} \to X$$

with weights

$$weight(x_1) = 2 \quad weight(x_2) = 3 \quad weight(x_3) = 6.$$

The exceptional divisor of b is mapped isomorphically to the locus of principally polarized abelian surfaces that decompose as a product of two elliptic curves (with the induced product polarization).

Remark 2.10 Igusa's result is considerably more precise: He explicitly computes the correspondence between the ring of invariants R and the ring of modular forms S. In particular, S is a polynomial ring with generators in degrees $4, 6, 10, 12$ and the locus of products is given by the vanishing of a form of weight 10.

2.6 Comparing the blow-up with moduli space

Proposition 2.11 *The open imbedding $M_2 \hookrightarrow \widetilde{X}$ extends to a birational map*

$$\gamma : \overline{M}_2 \dashrightarrow \widetilde{X}$$

which is an isomorphism in codimension one.

In particular, γ and γ^{-1} are both birational contractions. In Proposition 4.2 we will prove that γ is an isomorphism.

proof The Torelli morphism admits an extension $\bar{t} : \overline{M}_g \to \overline{A}_g$ [Na1], Theorem 3. This is not an isomorphism for $g > 1$: The divisor $\Delta_0 \subset \overline{M}_g$ is mapped to a boundary stratum of $\overline{A}_g$, which has codimension ≥ 2. However, in genus two $\bar{t}$ is an isomorphism at the generic point of Δ_1. Indeed, the Jacobian of a curve $[E_1 \cup_q E_2] \in \Delta_1$, with E_1 and E_2 smooth of genus one, is the abelian surface $E_1 \times E_2$.

The following diagram summarizes the various birational maps and morphisms:

$$
\begin{array}{ccc}
\overline{A}_2 & \overset{\pi}{\leftarrow} & \widetilde{X} \\
\bar{t} \uparrow & & \downarrow b \\
\overline{M}_2 & \overset{\beta}{\dashrightarrow} & X
\end{array} .
$$

By Theorem 2.9, π_2 is also an isomorphism over the generic point of $\bar{t}(\Delta_1)$, so γ is an isomorphism at the generic point of Δ_1. β and b are both isomorphisms over the generic point of the divisor $\beta(\Delta_0)$ (see Proposition 2.7), so γ is an isomorphism at the generic point of Δ_0. Since γ is regular along $M_2 = \overline{M}_2 - \Delta_0 - \Delta_1$, the result follows. $\square$

The proper transforms of Δ_0 and Δ_1 in $\widetilde{X}$ are denoted $\widetilde{\Delta}_0$ and $\widetilde{\Delta}_1$. Thus $\widetilde{\Delta}_1$ is the exceptional divisor of $b : \widetilde{X} \to X$.

Remark 2.12 (Bibliographic note) There are a number of partial desingularizations of $\overline{A}_g$ through which $\bar{t}$ factors, e.g., the 'Igusa monoidal transform' [Ig3], [Na1] and the toroidal compactification associated to the 2nd Voronoi fan [Na2]. When $g = 2$, these approaches coincide [Na2] Remark 2.8 and yield a partial desingularization $\widetilde{A}_2 \to \overline{A}_2$. See [Ig3] Theorem 5 for a blow-up representation, expressed in terms of modular forms; the center of this blow-up is in the boundary $\overline{A}_2 - A_2$. Namikawa [Na1] §9 has shown that the factorization $\overline{M}_2 \overset{\sim}{\to} \widetilde{A}_2$ is an isomorphism. Notwithstanding Igusa's explicit formulas for $\tau : X \dashrightarrow \overline{A}_2$ and $\widetilde{A}_2 \to \overline{A}_2$, it is not entirely obvious how to extract an isomorphism $\widetilde{A}_2 \overset{\sim}{\to} \widetilde{X}$.

3 Stack geometry

3.1 A stack-theoretic quotient

Proposition 2.2 might suggest that the invariant E is irrelevant to the geometry of the quotient. However, we have so far ignored possible *stack structures* on the quotient, which are intertwined with the geometry of E. There are a number of natural stacks to consider, including the GL_2-quotient stack. Our choice is dictated by pedagogical imperatives, i.e., to exhibit concretely the nontrivial inertia along the bielliptic locus where E vanishes.

The ring of invariants R is graded by degree, so we have a natural $\mathbb{G}_m$-action on the affine variety $Y = \text{Spec } R$. Now $\mathbb{G}_m$ acts on the open subset $Y - (0,0,0,0,0)$ with finite stabilizers and closed orbits, so the quotient stack

$$\mathcal{X} := (Y - (0,0,0,0,0))/\mathbb{G}_m$$

is a separated Deligne-Mumford stack with coarse moduli space $q : \mathcal{X} \to X$ (see [LM] 10.13.2,7.6,8.1 for more information). The points of $Y - (0,0,0,0,0)$ with nontrivial stabilizer map to the points of $\mathcal{X}$ with nontrivial inertia groups; this is the ramification locus of q.

We collect some geometric properties of $\mathcal{X}$:

Proposition 3.1   *1. The closed imbedding* $\text{Spec } R \hookrightarrow \text{Spec } k[A, B, C, D, E]$ *induces a closed imbedding* $i : \mathcal{X} \hookrightarrow \mathcal{P}(2, 4, 6, 10, 15)$*, where*

$$\mathcal{P}(2, 4, 6, 10, 15) := (\text{Spec } k[A, B, C, D, E] - (0,0,0,0,0))/\mathbb{G}_m,$$

with $\mathbb{G}_m$ *acting with weights* $(2, 4, 6, 10, 15)$

$$t \cdot (A, B, C, D, E) \mapsto (t^2 A, t^4 B, t^6 C, t^{10} D, t^{15} E);$$

2. the dualizing sheaf of $\mathcal{X}$ *is given via adjunction*

$$\omega_{\mathcal{X}} = i^* \omega_{\mathcal{P}(2,4,6,10,15)}(\mathcal{X}) \simeq i^* \mathcal{O}_{\mathcal{P}(2,4,6,10,15)}(-7),$$

where $\mathcal{O}_{\mathcal{P}(2,4,6,10,15)}(+1)$ *is the invertible sheaf associated with the principal* $\mathbb{G}_m$*-bundle classified by the identity character of* $\mathbb{G}_m$*;*

3. the ramification divisor of q *is*

$$\mathcal{E} := \{E = 0\} \subset \mathcal{X}$$

and we have

$$q^* \omega_X \simeq \omega_{\mathcal{X}}(-\mathcal{E}) \simeq i^* \mathcal{O}_{\mathcal{P}(2,4,6,10,15)}(-22).$$

proof: The first two assertions do not require proof. As for the third, it follows from the classification of possible automorphisms of binary sextics [Bo], [Ig1] §8 The only automorphism type occuring in codimension one is the bielliptic involution (cf. Proposition 2.1). □

Remark 3.2 Notwithstanding Proposition 2.4, $\mathcal{M}_2$ is *not* contained in $\mathcal{X}$ as an open substack. Using the functional relation (3), the inertia group of $[F] \in \mathcal{X} - p$ is the quotient

$$\{M \in \mathrm{GL}_2 : (M, F) = F\}/\{M \in \mathrm{SL}_2 : (M, F) = F\},$$

which is trivial for generic bilinear forms.

The inertia group at $[C] \in \mathcal{M}_2$ is $\mathrm{Aut}(C)$, and the presence of the hyperelliptic involution ι means this is always nontrivial. Now $\mathrm{Aut}(C)$ has a natural representation on $\Gamma(C, \omega_C)$ and an induced representation on $\wedge^2 \Gamma(C, \omega_C)$ that is not faithful: We do not see elements of $\mathrm{Aut}(C)$ acting on $\Gamma(C, \omega_C)$ with determinant one, e.g., ι, which acts on $\Gamma(C, \omega_C)$ by $-I$. The corresponding quotient of $\mathrm{Aut}(C)$ is the inertia picked up by $\mathcal{X}$.

3.2 Geometry of the moduli stack

Lemma 3.3 Every stable curve of genus two admits a canonical hyperelliptic involution, which is central in its automorphism group.

proof: We claim that every stable curve of genus two is canonically an admissible double cover ([HaMu],§4 or [ACV]) of a nodal curve of genus zero

$$j : C \to R, \quad R = \begin{cases} \mathbb{P}^1 & \text{if } [C] \notin \Delta_1 \\ \mathbb{P}^1 \cup_r \mathbb{P}^1 & \text{if } [C] \in \Delta_1 \end{cases}$$

branched over six smooth points $b_1, \ldots, b_6 \in R$ and the node $r \in R$. The covering transformation ι therefore commutes with each automorphism of C.

The cover is induced by

$$C \to \mathbb{P}(\Gamma(C, \omega_C^2)) \simeq \mathbb{P}^2$$

14

which factors

$$C \xrightarrow{j} R \subset \mathbb{P}^2,$$

where R is a plane conic and j is finite of degree two. Indeed, for curves not in Δ_1 this is the double cover $C \to \mathbb{P}^1$ discussed in Lemma 2.8. For curves $C = E_1 \cup_q E_2 \in \Delta_1$ with $q = E_1 \cap E_2$ the disconnecting node joining the genus-one components E_1 and E_2, we have a double cover

$$j : E_1 \cup_q E_2 \longrightarrow \mathbb{P}^1 \cup_r \mathbb{P}^1,$$

with $j(q) = r$ and j mapping the genus one components two-to-one onto rational components, with ramification at q along each component. $\square$

Consider stable curves of genus two with automorphisms beyond ι. There are two possibilities: Either $(R, b_1, \ldots, b_6)$ admits automorphisms permuting the b_i or $j : C \to R$ admits covering transformations other than the canonical hyperelliptic involution. The classification of automorphism groups ([Bo] or [Ig1]§8) yields the following possibilities in codimension one:

1. the curves C in δ_1; here $j : C \to \mathbb{P}^1 \cup \mathbb{P}^1$ admits involutions fixing each component of C;

2. the closure of the locus of curves $j : C \to \mathbb{P}^1$ branched over six points admitting a bielliptic involution.

At each point $[C] \in \overline{M}_2$, the moduli space is étale-locally isomorphic to the quotient $T_{[C]}\overline{\mathcal{M}}_2/Aut(C)$ at the origin, where

$$T_{[C]}\overline{\mathcal{M}}_2 = \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C))$$

is the tangent space with the induced automorphism action. When C is smooth, Serre duality gives

$$T_{[C]}\overline{\mathcal{M}}_2 = \Gamma(C, \omega_C^2)^*;$$

equation (2) shows that the hyperelliptic involution acts trivially and a bielliptic involution acts by reflection

This local isomorphism can be chosen so the divisors Δ_0 and Δ_1, correspond to the images of unions of distinguished hyperplanes in $T_{[C]}\overline{\mathcal{M}}_2$. The local-global spectral sequence gives an exact sequence

$$0 \to H^1(\mathscr{H}om(\Omega_C^1, \mathcal{O}_C)) \to \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C) \to \Gamma(\mathscr{E}xt^1(\Omega_C^1, \mathcal{O}_C)) \to 0. \quad (5)$$

A local computation implies

$$\mathscr{E}xt^1(\Omega_C^1, \mathcal{O}_C) \simeq \oplus_{\text{nodes } p \in C} k.$$

If $C = E_1 \cup_q E_2$ is in Δ_1, let $\widehat{\delta}_1 \subset T_{[C]}\overline{\mathcal{M}}_2$ denote the hyperplane corresponding to the disconnecting node separating the genus-one components of C, i.e., the kernel of the projection onto the direct summand corresponding to q. The extra covering involutions of $j : C \to \mathbb{P}^1 \cup \mathbb{P}^1$ mentioned above act on $T_{[C]}\overline{\mathcal{M}}_2$, trivially on $\widehat{\delta}_1$ and by multiplication by (-1) on $\mathscr{E}xt^1(\Omega_C^1, \mathcal{O}_C)_q$. This corresponds to reflection across $\widehat{\delta}_1$. Let $\widehat{\delta}_0$ denote the union of the hyperplanes corresponding to each of the non-disconnecting nodes of C. The surjectivity of the last arrow in (5) means that $\widehat{\delta} = \widehat{\delta}_1 \cup \widehat{\delta}_0$ is normal crossings.

Definition 3.4 Let $\xi \subset \overline{\mathcal{M}}_2$ (resp. $\Xi \subset \overline{M}_2$) denote the closure of the smooth curves admitting a bielliptic involution.

These are irreducible of codimension one, e.g., by Proposition 2.1 and the characterization of bielliptic curves. This can also be seen infinitesimally: Under the local identification with $T_{[C]}\overline{\mathcal{M}}_2/\operatorname{Aut}(C)$, each branch of ξ is identified with the hyperplane $T_{[C]}\overline{\mathcal{M}}_2$ fixed by the corresponding bielliptic involution (acting by reflection on the tangent space). The union of these hyperplanes is denoted $\widehat{\xi}$.

The bielliptic divisor has more complicated local geometry, as Ξ may have quite a few local branches. For example,

$$F(x, y) = xy(x - \xi y)(x + \xi y)(x - \xi^{-1} y)(x + \xi^{-1} y)$$

has automorphism group isomorphic to the Klein four-group and admits two involutions

$$[x, y] \mapsto [y, x] \quad [x, y] \mapsto [-y, x].$$

The cyclotomic form

$$F(x, y) = x^6 + y^6 = (x - \zeta y)(x - \zeta^3 y)(x - \zeta^5 y)(x - \zeta^7 y)(x - \zeta^9 y)(x - \zeta^{11} y) \quad \zeta^{12} = 1$$

has automorphism group isomorphic to the dihedral group with 12 elements and admits *four* distinct involutions

$$[x, y] \mapsto [y, x] \quad [x, y] \mapsto [-x, y] \quad [x, y] \mapsto [\zeta^4 y, x] \quad [x, y] \mapsto [\zeta^8 y, x].$$

In particular, $\widehat{\xi}$ is *not* normal crossings.

Proposition 3.5 (Ramification formula for $Q : \overline{\mathcal{M}}_2 \to \overline{M}_2$)

$$K_{\overline{\mathcal{M}}_2} + \alpha\delta \equiv Q^*(K_{\overline{M}_2} + \alpha\Delta_0 + \frac{1+\alpha}{2}\Delta_1 + \frac{1}{2}\Xi)$$

proof: Lemma 3.3 allows us to consider the rigidification ([ACV] §5) of $\overline{\mathcal{M}}_2$ with respect to the group $\langle \iota \rangle$ generated by the canonical involution

$$r : \overline{\mathcal{M}}_2 \to \overline{\mathcal{M}}_2^{<\iota>}.$$

Given a scheme T, T-valued points of $\overline{\mathcal{M}}_2^{<\iota>}$ correspond to families of genus two stable curves over T, where two families are identified if they differ by a $\langle \iota \rangle$-valued cocycle over T. The inertia group of $\overline{\mathcal{M}}_2^{<\iota>}$ at $[C]$ is the quotient

$$\mathrm{Aut}(C)/\langle \iota \rangle.$$

We have a factorization

$$\overline{\mathcal{M}}_2 \xrightarrow{r} \overline{\mathcal{M}}_2^{<\iota>} \xrightarrow{\bar{q}} \overline{M}_2$$

so that r is étale of degree two and $\overline{M}_2$ is the coarse moduli space of $\overline{\mathcal{M}}_2^{<\iota>}$ [ACV] Theorem 5.1.5. We therefore obtain the following formula for dualizing sheaves

$$r^*\omega_{\overline{\mathcal{M}}_2^{<\iota>}} \simeq \omega_{\overline{\mathcal{M}}_2}.$$

Let $\delta_1^{<\iota>}$ and $\xi^{<\iota>}$ denote the corresponding Cartier divisors in $\overline{\mathcal{M}}_2^{<\iota>}$. As $\bar{q}$ has simple ramification along these divisors, we obtain

$$\bar{q}^*\Delta_1 = 2\delta_1^{<\iota>} \qquad \bar{q}^*\Xi = 2\xi^{<\iota>}$$
$$\bar{q}^*K_{\overline{M}_2} = K_{\overline{\mathcal{M}}_2^{<\iota>}} - \delta_1^{<\iota>} - \xi^{<\iota>},$$

which together imply the formula.$\square$

3.3 Analysis of $b : \widetilde{X} \to X$ along the exceptional divisor

By Theorem 2.9, the exceptional divisor $\widetilde{\Delta}_1$ is mapped isomorphically to the locus in $\overline{A}_2$ parametrizing abelian surfaces decomposing into products of elliptic curves (as a principally polarized abelian variety); this is isomorphic to $\mathbb{P}(2, 3, 6)$.

Proposition 3.6 *Let $\widetilde{\widetilde{\Xi}}$ denote the proper transform of Ξ in $\widetilde{X}$. We have the following formulas:*

$$
\begin{aligned}
K_{\widetilde{X}} &\equiv b^* K_X + 10\widetilde{\Delta}_1 \\
b^*\{G = 0\} &\equiv \widetilde{\widetilde{\Xi}} + 12\widetilde{\Delta}_1 \\
b^*\{D = 0\} &\equiv \widetilde{\Delta}_0 + 6\widetilde{\Delta}_1 \\
\widetilde{\widetilde{\Xi}} &\equiv 3\widetilde{\Delta}_0 + 12\widetilde{\Delta}_1.
\end{aligned}
$$

We take this an an excuse to explore the geometry of $\widetilde{\Delta}_1$ in detail: It is isomorphic to the symmetric square of the moduli space of elliptic curves. The standard theory of modular forms implies

$$
\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6) = \left(\mathrm{Spec}\ k[g_2, g_3] - (0,0)\right)/\mathbb{G}_m,
$$

with $\mathbb{G}_m$ acting with weights $(4,6)$

$$
t \cdot (g_2, g_3) \mapsto (t^4 g_2, t^6 g_3)
$$

with coarse moduli space

$$
\overline{M}_{1,1} = \overline{A}_1 \simeq \mathrm{Proj}\ k[g_2, g_3] \simeq \mathbb{P}^1.
$$

In taking the symmetric square we need to be mindful of the stack structures: The coarse moduli space of the symmetric square need not be isomorphic to the symmetric square of the coarse moduli space, i.e., $\mathrm{Sym}^2(\mathbb{P}^1) \simeq \mathbb{P}^2$. The symmetric square of the stack has the following quotient-stack presentation:

$$
\begin{aligned}
(\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1})/\mathfrak{S}_2 &= (\mathrm{Spec}\ k[g_2, g_3, h_2, h_3] - Z)/H \\
Z &= \{(g_2, g_3, h_2, h_3) : g_2 = g_3 = 0 \text{ or } h_2 = h_3 = 0\}.
\end{aligned}
$$

Here H is the group generated by the torus

$$
(t, u) \cdot (g_2, g_3, h_2, h_3) \mapsto (t^4 g_2, t^6 g_3, u^4 h_2, u^6 h_3)
$$

and the involution

$$
(g_2, g_3, h_2, h_3) \mapsto (h_2, h_3, g_2, g_3),
$$

i.e., $H = \mathfrak{S}_2 \ltimes \mathbb{G}_m^2$ where $\mathfrak{S}_2$ acts on $\mathbb{G}_m^2$ by interchanging the factors.

The coarse moduli space of the stack is the invariant-theory quotient for the action of H. Consider the elements $p \in k[g_2, g_3, h_2, h_3]$ with the following properties:

18

1. $p(g_2, g_3, h_2, h_3) = p(h_2, h_3, g_2, g_3)$;

2. $p(t^2 g_2, t^3 g_3, u^2 h_2, u^3 h_3) = (tu)^N p(g_2, g_3, h_2, h_3)$ for some N.

This ring is generated by $g_2 h_2, g_3 h_3$, and $g_2^3 h_3 + g_3^2 h_2^3$ and

$$\text{Proj } k[g_2 h_2, g_3 h_3, g_2^3 h_3^2 + g_3^2 h_2^3] \simeq \mathbb{P}(4, 6, 12) \simeq \mathbb{P}(2, 3, 6),$$

which explains why the weights of b are $(2, 3, 6)$. See [Ig2], Theorem 3, for a discussion in terms of the modular forms for $\mathrm{Sp}(2, \mathbb{Z})$ (see Remark 2.10).

proof of proposition: The first equation follows because $b : \widetilde{X} \to X$ has weights $(2, 3, 6)$. As for the second, $\widetilde{\Xi}$ is the proper transform of the divisor $\{G = 0\} \subset X$ parametrizing forms admitting an bielliptic involution. When a bielliptic curve of genus two specializes to a stable curve in Δ_1, the bielliptic involution specializes to a morphism exchanging the elliptic components. Therefore, $\widetilde{\Xi} \cap \widetilde{\Delta}_1 \subset \widetilde{\Delta}_1$ is the diagonal in the symmetric square, which is cut out by a form of weighted-degree twelve.

For the third equation, $\widetilde{\Delta}_0$ is the proper transform of $\{D = 0\}$. The intersection $\widetilde{\Delta}_0 \cap \widetilde{\Delta}_1 \subset \widetilde{\Delta}_1$ is the locus where the discriminant $\Delta = g_2^3 - 27 g_3^2$ vanishes, and thus has weighted-degree six. The last equation follows because G has weighted degree thirty and D has weighted degree ten.□

One consequence of this analysis is worth mentioning.

Proposition 3.7 *For sufficiently small $\epsilon > 0$, the divisor*

$$\widetilde{\Delta}_0 + (6 - \epsilon)\widetilde{\Delta}_1$$

is ample on $\widetilde{X}$.

proof: The divisor can be expressed

$$b^*(\text{ample divisor}) - \epsilon(b\text{-exceptional divisor}),$$

which yields a polarization of the blow-up $b : \widetilde{X} \to X$. □

4 Birational geometry of the moduli space

4.1 Divisor classes and birational contractions of $\overline{M}_2$

It is well known that the rational divisor class group of $\overline{M}_2$ is freely generated by the boundary divisors Δ_0 and Δ_1; see [HaMo] for a nice account of divisors

on $\overline{M}_g$ for arbitrary g. When $g = 2$, Proposition 2.7 gives an elementary proof of this fact: Since $X \simeq \mathbb{P}(1, 2, 3, 5)$ its divisor class group has rank one and is generated by the discriminant divisor $\{D = 0\}$; the same holds true for $X - p$. By Proposition 2.11 $\overline{M}_2$ is isomorphic to $\widetilde{X}$ up to codimension ≤ 1, so these have isomorphic class groups. It follows that the rational divisor class group of $\overline{M}_2$ is generated by Δ_1 and the proper transform of the discriminant, which is just Δ_0.

The nef cone of $\overline{M}_2$ is also well-known. We will not give a self-contained proof here, but rather rely on the general result of Cornalba-Harris [CH]:

Theorem 4.1 *The line bundle $a\lambda - b\delta$ is nef on $\overline{M}_g, g \geq 2$, if and only if $a \geq 11b \geq 0$.*

Here λ is the pull-back of the polarization on $\overline{A}_g$ via the extended Torelli map $\bar{t} : \overline{M}_g \to \overline{A}_g$ (see §2.6).

To apply this in our situation, we observe that

$$\lambda = \frac{1}{10}(\Delta_0 + \Delta_1)$$

over $\overline{M}_2$ (see [HaMo] pp. 175). The factor 10 can be explained by the fact that $\bar{t}(\Delta_1)$ is defined by the vanishing of a modular form of weight ten (see Remark 2.10 and [Ig2]). Substitution gives the first part of

Proposition 4.2 *The nef cone of $\overline{M}_2$ is generated by the divisors $\Delta_0 + \Delta_1$ and $\Delta_0 + 6\Delta_1$, respectively. These are both semiample, inducing the birational contractions*

$$\bar{t} : \overline{M}_2 \to \overline{A}_2 \quad \beta : \overline{M}_2 \to X.$$

The rational map $\gamma : \overline{M}_2 \dashrightarrow \widetilde{X}$ is an isomorphism.

remainder of proof: Of course, $\Delta_0 + \Delta_1$ is semiample and induces the birational contraction morphism $\bar{t} : \overline{M}_2 \to \overline{A}_2$. As $\widetilde{X}$ and $\overline{M}_2$ are isomorphic in codimension one, Proposition 3.7 says that $\widetilde{\Delta}_0 + (6 - \epsilon)\widetilde{\Delta}_1$ is ample on $\widetilde{X}$ and the corresponding divisor $\Delta_0 + (6 - \epsilon)\Delta_1$ is ample on $\overline{M}_2$. It follows that $\widetilde{X}$ and $\overline{M}_2$ are each isomorphic to Proj of

$$\oplus_{n \geq 0} \Gamma(\mathcal{O}_{\widetilde{X}}(n(\widetilde{\Delta}_0 + (6 - \epsilon)\widetilde{\Delta}_1))) \simeq \oplus_{n \geq 0} \Gamma(\mathcal{O}_{\overline{M}_2}(n(\Delta_0 + (6 - \epsilon)\Delta_1))).$$

In particular, the rational map γ is an isomorphism. Thus the contractions $b : \widetilde{X} \to X$ and $\beta : \overline{M}_2 \dashrightarrow X$ coincide. $\square$

4.2 Canonical class of $\overline{M}_2$

The canonical class $K_{\overline{M}_2}$ can also be computed by elementary methods. We know that

$$\omega_{\mathbb{P}(1,2,3,5)} \simeq \mathcal{O}_{\mathbb{P}(1,2,3,5)}(-11),$$

which also follows from the third part of Proposition 3.1. Since the discriminant has degree ten, we find

$$K_X = -\frac{11}{5}\{D = 0\}.$$

Applying the formulas of Proposition 3.6, we obtain

$$K_{\widetilde{X}} \equiv -\frac{11}{5}\widetilde{\Delta}_0 - \frac{16}{5}\widetilde{\Delta}_1$$

$$K_{\widetilde{X}} + \frac{1}{2}\widetilde{\Delta}_1 + \frac{1}{2}\widetilde{\Xi} \equiv -\frac{7}{10}\widetilde{\Delta}_0 + \frac{3}{10}\widetilde{\Delta}_1.$$

Since $\widetilde{X}$ and $\overline{M}_2$ agree in codimension one, they have the same canonical class

$$K_{\overline{M}_2} \equiv -\frac{11}{5}\Delta_0 - \frac{16}{5}\Delta_1.$$

In particular, we obtain

$$K_{\overline{M}_2} + \alpha\Delta_0 + \frac{1+\alpha}{2}\Delta_1 + \frac{1}{2}\Xi \equiv (-\frac{7}{10} + \alpha)\Delta_0 + (\frac{3}{10} + \alpha/2)\Delta_1. \quad (6)$$

Remark 4.3 The importance of this divisor stems from the ramification equation of Proposition 3.5. This divisor class pulls back to the class

$$K_{\overline{\mathcal{M}}_2} + \alpha\delta$$

on the moduli stack. We shall interpret log canonical models of the moduli stack using this divisor.

Using the computations of Proposition 3.6, we obtain a discrepancy equations for $\beta : \overline{M}_2 \to X$

$$K_{\overline{M}_2} + \alpha\Delta_0 + \frac{1}{2}\Xi = \beta^*(K_X + \alpha\{D = 0\} + \frac{1}{2}\{G = 0\}) + (4 - 6\alpha)\Delta_1 \quad (7)$$

$$K_{\overline{M}_2} + \alpha\Delta_0 + \frac{1+\alpha}{2}\Delta_1 + \frac{1}{2}\Xi = \beta^*(K_X + \alpha\{D = 0\} + \frac{1}{2}\{G = 0\}) + \frac{9 - 11\alpha}{2}\Delta_1. \quad (8)$$

4.3 Generalities on log canonical models

See [FA] §2 for definitions of the relevant terms and technical background. Let V be a normal projective variety, $D = \sum_{i=1}^{n} a_i D_i$ a $\mathbb{Q}$-divisor such that $0 \le a_i \le 1$ and $K_V + D$ is $\mathbb{Q}$-Cartier. Abusing notation, we write $V - D$ for $V - \cup_i D_i$.

Definition 4.4 (V, D) is a *strict log canonical model* if $K_V + D$ is ample, (V, D) has log canonical singularities, and $V - D$ has canonical singularities.

The idea here is to realize (V, D) as a log canonical model without introducing boundary divisors over $V - D$. This is natural if we want to respect the geometry of the open complement.

The following recognition criterion for strict log canonical models is based on [FA] §2:

Proposition 4.5 *Consider birational projective contractions* $\rho : \widetilde{V} \to V$ *where the exceptional locus of* ρ *is divisorial,* D_i' *denotes the proper transform of* D_i, *and* E_j *(resp.* F_k*) denotes the exceptional divisors of* ρ *with* $\rho(E_j) \subset D$ *(resp.* $\rho(F_k) \not\subset D$*).*

The following are equivalent:

1. (V, D) *is a strict log canonical model.*

2. *For some resolution of singularities* $\rho : \widetilde{V} \to V$, *with the union of the exceptional locus and* $\cup_i D_i'$ *normal crossings, there exist* $b_j \in \mathbb{Q} \cap [0, 1]$ *so that*
$$\widetilde{D} = \sum_i a_i D_i' + \sum_j b_j E_j$$

   satisfies the formula
   $$K_{\widetilde{V}} + \widetilde{D} \equiv \rho^*(K_V + D) + \sum_j d_j E_j + \sum_k e_k F_k, \quad d_j, e_k \ge 0. \qquad (9)$$

   For each such choice of b_j
   $$K_{\widetilde{V}} + \widetilde{D} - \sum_j d_j E_j - \sum_k e_k F_k$$

   is semiample and induces ρ. *Furthermore, we may take the* $b_j = 1$.

3. *For some contraction $\rho : \widetilde{V} \to V$, there exist $b_j \in \mathbb{Q} \cap [0, 1]$ so that*

$$\widetilde{D} = \sum_i a_i D'_i + \sum_j b_j E_j$$

satisfies the formula

$$K_{\widetilde{V}} + \widetilde{D} \equiv \rho^*(K_V + D) + \sum_j d_j E_j + \sum_k e_k F_k, \quad d_j, e_k \geq 0,$$

$(\widetilde{V}, \widetilde{D})$ is log canonical, and $\widetilde{V} - \widetilde{D}$ has canonical singularities. The divisor

$$K_{\widetilde{V}} + \widetilde{D} - \sum_j d_j E_j - \sum_k e_k F_k$$

is semiample and induces ρ.

Some general facts are worth mentioning before we indicate the proof. First, we can decide whether a pair is canonical or log canonical by computing discrepancies on *any* resolution. Second, discrepancies increase as the coefficients of the log divisor are decreased [FA] 2.17.3. Third, in situations (2) and (3) the pair (V, D) is the *log canonical model* of $(\widetilde{V}, \widetilde{D})$; such models are unique [FA] 2.22.1.

proof: It is trivial that the second statement implies the third. To see that the third implies the first, take a resolution for $(\widetilde{V}, \widetilde{D})$ so that the union of the exceptional locus and all the proper transforms of the D'_i, E_j, and F_k is normal crossings. Comparing discrepancies for $(\widetilde{V}, \widetilde{D})$ and (V, D), using the fact the coefficients of components in $\widetilde{D}$ are at least as large as the coefficents of the corresponding components appearing in $\rho^*(K_V + D)$, we find that (V, D) is log-canonical and has canonical singularities along $V - D$. Since $\rho^*(K_V + D)$ induces ρ, $K_V + D$ must be ample on V.

For the remaining implication, since (V, D) has log canonical singularities and canonical singularities away from D, the discrepancy equation (9) follows. Since $K_V + D$ is ample on V, its pull-back to $\widetilde{V}$ is semiample and induces ρ. $\square$

We shall also need the following basic fact, a special case of [FA] 20.2, 20.3:

Proposition 4.6 *Let W be a smooth variety and $h : W \to V$ be a finite dominant morphism to a normal variety. Let $D = \sum_i a_i D_i, 0 \leq a_i \leq 1$ be a*

$\mathbb{Q}$-*divisor on* V *containing all the divisorial components of the branch locus of* h. *Let* $\bar{D}$ *be a* $\mathbb{Q}$-*divisor on* W *so that* $\mathrm{supp}(h^{-1}(D)) = \mathrm{supp}(\bar{D})$ *and* $h^*(K_V + D) = K_W + \bar{D}$. *Then* (V, D) *has log canonical singularities along* D *iff* $(W, \bar{D})$ *has log canonical singularities along* $\bar{D}$.

If D *and* $\bar{D}$ *have multiplicity one at each component then* $h^*(K_V + D) = K_W + \bar{D}$ *follows from the other assumptions.*

4.4 Example of $\overline{M}_g$

The standpoint of this section owes a great deal to Mumford [Mu1] [Mu2]:

Theorem 4.7 *For* $g \geq 4$, *the pair* $(\overline{M}_g, \Delta)$ *is a strict log canonical model.*

Remark 4.8 This is also the natural log canonical model from the point of view of the moduli stack. Indeed, for $g \geq 4$ the locus in M_g of curves with automorphisms has codimension ≥ 2, so the branching divisor of $Q :$ $\overline{\mathcal{M}}_g \to \overline{M}_g$ is just Δ_1; over Δ_1, we have simple ramification. We therefore have ([HaMu] pp. 52) $Q^* K_{\overline{M}_g} = K_{\overline{\mathcal{M}}_g} - \delta_1$ and thus

$$Q^*(K_{\overline{M}_g} + \Delta) = K_{\overline{\mathcal{M}}_g} + \delta.$$

sketch proof: We first check that $K_{\overline{M}_g} + \Delta$ is ample. The formula from [HaMu] §2 (or [HaMo])

$$K_{\overline{M}_g} = 13\lambda - 2\Delta_0 - 3/2\Delta_1 - 2 \sum_{2 \leq i \leq g/2} \Delta_j$$

gives $K_{\overline{M}_g} + \delta = 13\lambda - \delta$, which is ample by Theorem 4.1 (see also [Mu1]).

The singularity analysis follows [HaMu]. $\overline{M}_g$ has canonical singularities by Theorem 1 of [HaMu]. To show that $(\overline{M}_g, \Delta)$ has log canonical singularities, we use the fact that $\overline{M}_g$ is étale-locally a quotient of a smooth variety by a finite group. At $[C]$ it has a local presentation

$$h : T_{[C]}\overline{\mathcal{M}}_g \twoheadrightarrow T_{[C]}\overline{\mathcal{M}}_g / \mathrm{Aut}(C)$$

in terms of its tangent space

$$T_{[C]}\overline{\mathcal{M}}_g = \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C).$$

We analyze the quotient morphism using Proposition 4.6. The preimage of the boundary divisor corresponds to a union of hyperplanes $\widehat{\delta} \subset T_{[C]}\overline{\mathcal{M}}_g$,

meeting in normal crossings. The pair $(T_{[C]}\overline{\mathcal{M}}_g, \bar{\Delta})$ then has log canonical singularities. An application of Proposition 4.6, utilizing the ramification discussion in Remark 4.8, implies $(K_{\overline{\mathcal{M}}_g}, \Delta)$ is log canonical. □

Remark 4.9 Using the full force of Theorem 4.1 we get a sharper statement. Consider the pair

$$(\overline{M}_g, \alpha\Delta_0 + \frac{1+\alpha}{2}\Delta_1 + \alpha(\Delta_2 + \ldots + \Delta_{\lfloor g/2 \rfloor})),$$

with log canonical divisor pulling back to $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ on the moduli stack. The $\mathbb{Q}$-divisor $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ is the pull-back of an ample line bundle if and only if $9/11 < \alpha \leq 1$. Since $\overline{M}_g$ is a locally a quotient of a smooth variety by a finite group, all divisors on $\overline{M}_g$ are $\mathbb{Q}$-Cartier. An easy computation with the discrepancy equation (9) then shows that the pair remains log canonical even as the coefficients are reduced.

4.5 Application to $\overline{\mathcal{M}}_2$

The source of the special difficulties in this case is the fact that $K_{\overline{M}_2} + \Delta$ is not effective. Indeed, in §4.1 we computed

$$K_{\overline{M}_2} \equiv -\frac{11\Delta_0 + 6\Delta_1}{5}$$

so $K_{\overline{M}_2} + \Delta = -$effective divisor.

In order to recover a result analogous to Theorem 4.7, we must take the 'log canonical model of the moduli stack', as interpreted on $\overline{M}_2$ via Proposition 3.5:

Theorem 4.10 *Consider the log canonical model of $\overline{\mathcal{M}}_2$ with respect to the $K_{\overline{\mathcal{M}}_2} + \alpha\delta$, i.e., the log canonical model of $\overline{M}_2$ with respect to*

$$K_{\overline{M}_2} + \alpha\Delta_0 + \frac{1+\alpha}{2}\Delta_1 + \frac{1}{2}\Xi.$$

1. *For $9/11 < \alpha \leq 1$, we recover $\overline{M}_2$.*

2. *For $7/10 < \alpha \leq 9/11$ we recover the invariant theory quotient $X \simeq \mathbb{P}(1,2,3,5)$.*

3. *For $\alpha = 7/10$ we get a point; the log canonical divisor fails to be effective for $\alpha < 7/10$.*

proof: The necessary ampleness results have already been stated. Proposition 4.2 and Equation (6) imply the log canonical divisor on $\overline{M}_2$ is ample if and only if $\alpha > 9/11$. Proposition 3.1 implies that

$$K_X + 1/2\{G = 0\} + \alpha\{D = 0\}$$

is positive on X if and only if $\alpha > 7/10$. When $\alpha = 7/10$ it is zero and when $\alpha < 7/10$ it is negative.

It remains to verify the singularity conditions: First, we check that $\overline{M}_2$ has canonical singularities away from Δ_0, Δ_1, and Ξ. Suppose that C is not in the boundary and does not admit admit a bielliptic involution. In Proposition 2.4 we saw

$$M_2 \simeq X - \{D = 0\} \subset \mathbb{P}(1, 2, 3, 5),$$

so we need to analyze the singularities of $\mathbb{P}(1, 2, 3, 5) - \{D = 0\}$. A point in weighted projective space is nonsingular when the weights corresponding to its non-vanishing coordinates are relatively prime, so the only possible singularity occurs when $A = B = C = 0$. The corresponding bilinear sextic form is

$$x(x^5 + y^5),$$

the unique form with an automorphism group of order five [Bo] pp. 51 [Ig1] pp. 645. At this point, $\mathbb{P}(1, 2, 3, 5)$ is locally isomorphic to the cyclic quotient singularity $\frac{1}{5}(1, 2, 3)$, i.e., the quotient of $\mathbb{A}^3$ under the action

$$(a, b, c) \mapsto (\zeta a, \zeta^2 b, \zeta^3 c) \quad \zeta \neq 1 \in \mu_5.$$

This is canonical by the Reid-Tai criterion; see [HaMu] pp. 28 for a general result.

Second, we address the singularities along the boundary. For $\alpha > 9/11$ we need that

$$K_{\overline{M}_2} + \alpha\Delta_0 + \frac{1 + \alpha}{2}\Delta_1 + 1/2\Xi$$

is log canonical. When $\alpha \leq 9/11$, Proposition 4.5 and the discrepancy computation (8) reduce us to showing that this is log canonical. Since $\overline{M}_2$ is

26

$\mathbb{Q}$-factorial and discrepancies increase as coefficients of log divisors decrease [FA] 2.17.3, it suffices to verify that

$$K_{\overline{M}_2} + \Delta_0 + \Delta_1 + \frac{1}{2}\Xi \tag{10}$$

is log canonical.

The proof relies on the description of the boundary divisors in terms of the local presentation

$$T_{[C]}\overline{\mathcal{M}}_2/\mathrm{Aut}(C),$$

as sketched in §3.2. The key observation is that Ξ does not play a rôle in the analysis. Each bielliptic involution acts on $T_{[C]}\overline{\mathcal{M}}_2$ by reflection across the corresponding hyperplane in $\widehat{\xi}$, so the quotient

$$h : T_{[C]}\overline{\mathcal{M}}_2 \to T_{[C]}\overline{\mathcal{M}}_2/H, \quad H = \mathrm{Aut}(C)/\langle \iota \rangle,$$

has simple ramification along $\widehat{\xi}$. Since Ξ has coefficient $1/2$ in (10), $\widehat{\xi}$ does not appear in the pull-back of the log canonical divisor to $T_{[C]}\overline{\mathcal{M}}_2$.

Thus (10) pulls back to

$$K_{T_{[C]}\overline{\mathcal{M}}_2} + \widehat{\delta},$$

and we have seen that $\widehat{\delta}$ is normal crossings. It follows that $(T_{[C]}\overline{\mathcal{M}}_2, \widehat{\delta})$ is log canonical and Proposition 4.6 gives the desired result.□

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