MAHLER MEASURE FOR DYNAMICAL SYSTEMS ON $\mathbb{P}^1$ AND INTERSECTION THEORY ON A SINGULAR ARITHMETIC SURFACE

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Abstract. The Mahler measure formula expresses the height of an algebraic number as the integral of the log of the absolute value of its minimal polynomial on the unit circle. The height is in fact the canonical height associated to the monomial maps x^n. We show in this work that for any rational map $\varphi(x)$ the canonical height of an algebraic number with respect to φ can be expressed as the integral of the log of its equation against the invariant Brolin-Lyubich measure associated to φ, with additional adelic terms at finite places of bad reduction. We give a complete proof of this theorem using integral models for each iterate of φ. In the last chapter, on equidistribution and Julia sets, we give a survey of results obtained by P. Autissier, M. Baker, R. Rumely, and ourselves. In particular, our results, when combined with technics of diophantine approximation, will allow us to compute the integrals in the generalized Mahler formula by averaging on periodic points.

1. Introduction

If F is the minimal polynomial over $\mathbb{Z}$ for an algebraic number x, the formula of Mahler [19] for the usual height $h(x)$ is

$$\deg(F)h(x) = \log \prod_{\text{all places } v} \sup(|x|_v, 1) = \int_0^1 \log |F(e^{2\pi i\theta})| d\theta.$$

One can notice the following facts:

(i) the height satisfies the functional equation $h(x^2) = 2h(x)$.

(ii) $d\theta$ is supported on the unit circle, which is the closure of the set of roots of unity, each root of unity having height 0. Along with

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the points 0 and ∞, the roots of unity are the only points that have finite forward orbits under iteration of the map $x \to x^2$.

We show in this article that these occurrences are general for any dynamical system on the Riemann sphere given by a rational function φ with coefficients in a number field. There is a canonical height h_φ that vanishes at precisely the points that have finite forward orbits under the iteration of φ. At each infinite place v, we have an integral $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi}$, where $d\mu_{v,\varphi}$ is the distribution associated to a canonical metric for φ on $\mathcal{O}_{\mathbb{P}}^1(1)$ (see Zhang [33]). At each finite v, we define an integral $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi}$ which is constructed via a limiting process that is analogous to Brolin's construction ([7]) of the φ-invariant measure at an infinite place, as we explain in Section 5. Our Theorem 6.1 asserts that

$$\deg(F) h_\varphi(\alpha) = \sum_{\text{places } v \text{ of } K} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi},$$

where α is an algebraic point and F is a minimal polynomial for α over K. We also show that for finite v we have $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi} = 0$ unless φ has bad reduction at v or all the coefficients of F have nonzero v-adic valuation. Moreover, we show that $\sum_{\text{finite}} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi}$ can be explicitly bounded in terms of F and polynomials P and Q for which $\varphi = P/Q$. In particular, Corollary 6.3 states that

$$\deg(F) h_\varphi(\alpha) \leq \sum_{v|\infty} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi}$$

if P is monic and F has coprime coefficients.

We use arithmetic intersection theory on a singular arithmetic surface. We introduce a blow-up associated to a model of φ^k to follow these iterates of φ in a coherent manner. We have found it convenient to work with Cohen-Macaulay surfaces instead of normal surfaces.

Dynamical systems have been studied by many authors; see, for example, C. T. McMullen [21] and J. Milnor [22]. Recently G. Everest and T. Ward, in their book [11], have studied algebraic dynamics on elliptic curves and on products of projective lines. They have pointed out particular cases of our main theorem (other cases have been studied by V. Maillot [20]).

In light of work of Szpiro, Ullmo, and Zhang ([27]), it seems natural to wonder if points with small canonical height h_φ on $\mathbb{P}^1$ are equidistributed with respect to $d\mu_{v,\varphi}$ for v an infinite place. P. Autissier [2] and M. Baker and R. Rumely [3] have recently shown that such an equidistribution result does indeed hold. One might also ask whether generalized Mahler measure can be computed via equidistribution. We will discuss conjectures and known results of this sort more precisely in Section 7.

The organization of the paper is as follows:

1-Introduction.

2-Canonical height, canonical metric, and canonical distribution associated to a dynamical system.

3-Three examples.

4-The blow-up associated to a model of a rational map from $\mathbb{P}^1$ to $\mathbb{P}^1$.

5-The integrals at finite places. 5.1-Existence of the integrals at finite places; 5.2-Invariance of the v-adic integral under change of variables; 5.3-Geometry of the v-adic integrals; 5.4-A remark on the use of integral notation at finite places.

6-The Mahler formula for dynamical systems.

7-Equidistribution and the Julia set.

A-Appendix: Schematic intersection theory on a Cohen-Macaulay surface. (This is needed to justify the use of intersections products in Section 4.)

B-Appendix: Arakelov intersection with the divisor associated to a rational function. (This is used in Section 6.)

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2. Canonical height, canonical metric, and canonical distribution associated to a dynamical system

The global theory of canonical heights was started by J. Silverman and G. Call [8]. Let X be a variety over a number field K. Suppose φ is a finite map of X to itself. Suppose that its degree d is greater than one and that there is an ample line bundle L on X satisfying $\varphi^*(L) \cong L^{\otimes d}$. Tate's recipe for the definition of the Néron-Tate height on an abelian variety carries over to this general case: one has a canonical height associated to φ defined by

$$h_\varphi(\alpha) = \lim_{k\to\infty} \frac{h_L(\varphi^k(\alpha))}{d^k}.$$

In this formula, h_L is associated with any set of smooth metrics at places at infinity (cf. [29] and Appendix B).

The canonical height h_φ satisfies the properties:

(i) h_φ satisfies Northcott's theorem: points over $\bar{K}$ with bounded degree and bounded height are finite in number.
(ii) $h_\varphi(\varphi(\alpha)) = dh_\varphi(\alpha)$.
(iii) h_φ is a non-negative function.
(iv) $h_\varphi(\alpha) = 0$ if and only if α has a finite forward orbit under iteration of φ.
(v) $|h_\varphi(\alpha) - h(\alpha)|$ is bounded on $\mathbb{P}^1(\overline{\mathbb{Q}})$ for h the usual height.

The canonical height h_φ is characterized as the the unique function on $\mathbb{P}^1(\bar{K})$ that satisfies (ii) and (v).

Definition 2.1. *A point is called periodic if it is a fixed point of φ^k for some integer k. A point is called preperiodic if its image under φ^m is periodic for some integer m. Equivalently, a point is preperiodic if and only if it has a finite forward orbit under iteration of φ.*

The periodic points are separated classically into three classes important for the dynamics:

Definition 2.2. *Let f be a differentiable map $f : \mathbb{P}^1 \to \mathbb{P}^1$. A fixed point x of f^k is called repelling (resp. attracting, resp. indifferent) if $|(f^k)'(x)| > 1$ (resp. $|(f^k)'(x)| < 1$, resp. $|(f^k)'(x)| = 1$). The closure in $\mathbb{P}^1(\mathbb{C})$ of the set of repelling periodic points is called the Julia set. The complement in $\mathbb{P}^1(\mathbb{C})$ of the Julia set is called the Fatou set of f.*

S. Zhang [31, 33] has shown the interest of the local theory of canonical heights. In [33, Section 2], he shows (following Tate) that if a line bundle L on a projective variety W has a metric $\|\cdot\|_v$ and there is an isomorphism $\tau : L^{\otimes d} \xrightarrow{\sim} \varphi^*(L)$ for some $d > 1$, then letting $\|\cdot\|_{v,0} = \|\cdot\|_v$ and $\|\cdot\|_{v,k+1} = (\tau^*\varphi^*\|\cdot\|_{v,k})^{1/d}$, one obtains a limit metric

$$\|\cdot\|_{v,\varphi} = \lim_{k\to\infty} \|\cdot\|_{v,k}.$$

The following proposition is due to S. Zhang ([33, Theorems 1.4 and 2.2]).

Proposition 2.3. *Let L and the metrics $\|\cdot\|_v$, $\|\cdot\|_{v,k}$, and $\|\cdot\|_{v,\varphi}$ on L be as above. Then:*

(i) *the $\|\cdot\|_{v,k}$ converge uniformly to $\|\cdot\|_{v,\varphi}$.*

(ii) *Suppose additionally that W is a curve, v is an infinite place, and that $\|\cdot\|_v$ is smooth and semipositive. Then the (normalized) curvatures $d\mu_{v,k} = -\frac{1}{(2\pi i)d^k} d\overline{d} \log \|\cdot\|_{v,k}$ have a limit distribution $d\mu_{v,\varphi}$ such that if s is a meromorphic section of a line bundle L' with metric $\|\cdot\|_v'$ at v, then $\lim_{k\to\infty} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log \|s\|_v' d\mu_{v,k}$ exists and is equal to $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log \|s\|_v' d\mu_{v,\varphi}$.*

Furthermore, neither $\|\cdot\|_{v,\varphi}$ *nor* $d\mu_{v,\varphi}$ *depends on our choice of smooth metric* $\|\cdot\|_v$.

By convention, all of our metrics at infinity are normalized; that is, on each stalk of L on $X(\mathbb{C}_v)$ the metric $\|\cdot\|_v$ behaves like $|\cdot|_v^{[K_v:\mathbb{R}]}$ where K_v is the completion of K at an infinite place v.

We note that canonical heights are the only Arakelov type heights which are non-negative naturally. The height is an intersection number in the sense of J. Arakelov [1] as extended by P. Deligne [10] and by S. Zhang [33] to this limit situation.

3. Examples

1-The squaring map on the multiplicative group and the naive height.

For the map $\varphi(t) = t^2$ on $\mathbb{P}^1$, the preperiodic points are zero, infinity, and the roots of unity. The unit circle is the Julia set (the closure of the repelling periodic points). The naive height satisfies the required functional equation to be the canonical height associated to φ. This can be verified via the usual definition

$$h([t_0 : t_1]) = \frac{1}{[K:\mathbb{Q}]} \log \prod_{\text{places } v \text{ of } K} \sup(|t_0|_v, |t_1|_v)^{\mathrm{N}_v},$$

where $\mathrm{N}_v = [K_v : \mathbb{Q}_v]$ and $[t_0 : t_1] \in \mathbb{P}^1(K)$. Thus, the naive height is the canonical height associated to φ.

The unit circle is the support of $d\mu_\varphi$ (in this case the Haar measure $d\theta$ on the unit circle). It is the curvature (in the sense of distributions) of the canonical metric

$$\|(\lambda T_0 + \mu T_1)([a:b])\| = \frac{|\lambda a + \mu b|}{\sup(|a|, |b|)}.$$

Note that the canonical height and curvature are the same for any map $\phi(t) = t^n$ with $n \geq 2$.

2- The Néron-Tate height is associated to multiplication by two on an elliptic curve.

Let E be an elliptic curve with Weierstrass equation $y^2 = G(t)$. We write $E = \mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau)$ with τ in the upper half plane. By passing to the quotient by $[-1]$, multiplication by 2 on E gives rise to the following rational map on $\mathbb{P}^1$:

$$\varphi(t) = \frac{G'^2(t) - 8tG(t)}{4G(t)}.$$

The preperiodic points of φ are the images of the torsion points in E. To see this note first that $2^n P = \pm 2^k P$ for $n \neq k$ implies P is a torsion point. Conversely, if $nP = 0$, then writing $2^k = qn + r_k$ with $r_k < n$, we see that there must be at least two different indices k and k' with $r_k = r_{k'}$ hence $2^k P = 2^{k'} P$. The fixed points of φ are the images of the inflection points of E, i.e. of the 3-torsion points in E.

The multiplication map $[2^n]$ on E is of course étale. The derivative of this map is 2^n everywhere, so the preperiodic points are all repelling. Hence the Julia set is the entire Riemann sphere. In [23] it is established that the Haar measure on E gives the curvature of the canonical metric associated to the Néron-Tate height. Its image on $\mathbb{P}^1$ is

$$d\mu_\varphi = \frac{i}{\mathrm{Im}(\tau)} \frac{dt \wedge \overline{dt}}{|G|}.$$

The curvature and canonical height will be the same for φ the map on $\mathbb{P}^1$ associated to any multiplication by n map on E for $n \geq 2$.

3-Parallel projection of a conic.

Consider the plane conic C over $\mathbb{Z}$ defined by the equation

$$X_0 X_1 + pX_2^2 = 0,$$

where p is an odd prime number. The reduction of C mod a prime ℓ is smooth and connected for $\ell \neq p$. The fiber over p is reduced and is the union of two lines. The arithmetic surface C is regular. The projection map from $\mathbb{P}^2$ to $\mathbb{P}^1$ defined by

$$\Phi([X_0 : X_1 : X_2]) = [X_0 + X_1 : X_2]$$

is well-defined as a map from C to $\mathbb{P}^1$. Projecting from $[0 : 1 : 0]$ yields an isomorphism between our conic and $\mathbb{P}^1$. Composing this with Φ gives rise to a map φ; the reader may check that this map is $\varphi(t) = \frac{t^2 - p}{t}$. This example is an illustration of a blowing-up allowing to define the map φ over $\mathbb{Z}$. This will be a systematic approach in the next section.

4. The Blow-up associated to a model of a rational map from $\mathbb{P}^1$ to $\mathbb{P}^1$

Let ψ be a rational map of degree d from $\mathbb{P}^1$ to $\mathbb{P}^1$ defined over the field of fractions K of a Dedekind domain B. For simplicity we will assume that ∞ is a fixed point of ψ. Note that we may also choose coordinates so that this is the case; hence, our assumption does not restrict our generality.

Definition 4.1. *A model over B of ψ is a map of polynomial rings*

$$\psi([T_0 : T_1]) = [G(T_0, T_1) : H(T_0, T_1)],$$

where the polynomials G and H are homogeneous of degree d with coefficients in B.

The models will allow us to work on arithmetic surfaces when K is a number field. These surfaces, obtained by blowing up non-regular centers, will be singular in general.

Let Y be the closed subscheme of $X = \mathbb{P}^1_B = \mathrm{Proj}(B[T_0, T_1])$ defined by the vanishing of G and H. We will call I the sheaf of ideals in $\mathcal{O}_X$ defining Y. The choice of G and H as "generators of I" gives rise to a surjection

$$\tilde{\psi}_1 : \mathcal{O}_X^2 \twoheadrightarrow I(d).$$

The support of the scheme Y is exactly where the map ψ cannot be extended to the fibers of the model $\mathbb{P}^1_B$.

Let N denote the projection from Y to $\mathrm{Spec}\, B$; we call this the *bad reduction* of the model. The scheme Y does not meet the generic fiber X_K.

Definition 4.2. *Let $\sigma : X_1 \to X$ be the blowing up of Y in X.*

By the universal property of the blow-up, the pull-back $\sigma^* I$ is locally generated by elements which are not zero divisors, i.e. there is a positive Cartier divisor E_1 on X_1 such that

$$\sigma^* I = \mathcal{O}_{X_1}(-E_1).$$

One then has a surjective map (which is simply σ^* of $\tilde{\psi}_1$)

$$\mathcal{O}_{X_1}^2 \twoheadrightarrow \sigma^*(\mathcal{O}_X(d)) \otimes \mathcal{O}_{X_1}(-E_1)$$

By the universal property characterizing the projective line this gives rises to a map:

$$\psi_1 : X_1 \to X$$

extending the original rational map ψ on the generic fibers.

Throughout this section we will use the scheme-theoretic intersection product $(\cdot.\cdot)$, defined in Appendix A.

Proposition 4.3. *The two dimensional scheme X_1 is reduced, irreducible and Cohen-Macaulay. The fiber of X_1 over $v \notin N$ is equal* *to* $\mathbb{P}^1_{k_v}$. *The fiber of* X_1 *over* $v \in N$ *has a finite number of components* $W_v, C_{v,1}, ..., C_{v,t_v}$, *where* W_v *is the strict transform of the fiber of* X *at* v *and the* $C_{v,i}$ *are components of the exceptional divisor of* σ. *Each* $C_{v,i}$ *is isomorphic to* $\mathbb{P}^1_{k_{v,i}}$ *where* $k_{v,i}$ *is the residual field of the closed point image of* $C_{v,i}$ *in* X. *The* $C_{v,i}$ *for* $i > 0$ *do not meet each other. The* $C_{v,i}$ *with* $i > 0$ *meet* W_v. *The exceptional divisor* E_1 *is a Cartier divisor equal to* $\mathbb{P}^1_Y$ *and, as a Weil divisor, it can be decomposed into* $\sum_{v\in N, i>0} r_{v,i} C_{v,i}$ *where the* $r_{v,i}$ *are positive integers equal to the local lengths of* $\mathcal{O}_Y$ *at the local rings of its support. The geometric self-intersection of* $C_{v,i}$ *is equal to* $-[k_{v,i} : k_v]/r_{v,i}$. *One has*

$$\psi_1^* \mathcal{O}_X(1) = \sigma^*(\mathcal{O}_X(d)) \otimes \mathcal{O}_{X_1}(-E_1).$$

Proof. Since the ideal I is generated by two elements, the scheme $X_1 = \mathrm{Proj}(\bigoplus I^n)$ is a closed subscheme of $\mathbb{P}^1_X = \mathrm{Proj}(\mathcal{O}_X[T_0, T_1])$. The exceptional fiber is then $\mathbb{P}^1_Y$. The surface X_1 embeds as a local complete intersection in the regular three-fold $\mathbb{P}^1_X$ and hence is Cohen-Macaulay. The $C_{v,i}$ are Weil divisors that are $\mathbb{Q}$-Cartier because E_1 and the total fibers F_v^* are Cartier divisors and the $C_{v,i}$ do not meet. Hence the intersection theory with the $C_{v,i}$ is as described in the Appendix A. One sees that the W_v are also $\mathbb{Q}$-Cartier since the total fiber of v is Cartier and the $C_{v,i}$ are $\mathbb{Q}$-Cartier. One has $\mathcal{O}_{X_1}(-E_1)|E_1 = \oplus_{i,v} \mathcal{O}_{\mathbb{P}^1_{Y_{i,v}}}(1)$. Hence

$$((-r_{v,i}C_{v,i}).(r_{v,i}C_{v,i})) = \deg(\mathcal{O}_{\mathbb{P}^1_{Y_{i,v}}}(1)) = [k_{v,i,k} : k_v] r_{v,i}.$$

This gives the value asserted for the self intersection of the components $C_{v,i}$. The formula for $\psi_1^* \mathcal{O}_X(1)$ comes from the universal property characterizing the projective line over X. □

Remark. This method of removing the indeterminacies of a map by blowing up is standard (see Hartshorne [14, p. 168] for example).

5. The integrals at finite places

Let v be a finite place of K. In this section we will: (1) define the v-adic integral and show that it exists, (2) show that the v-adic integral does not depend on our choice of polynomials P and Q defining a model for φ over $\mathcal{O}_K$, (3) relate the v-adic integral to the geometry of the blow-up maps associated to the φ^k, and (4) explain why it makes sense to think of this integral at the finite place v as a v-adic analogue of an integral at an archimedean place. We begin by developing some terminology.

As before, let $\varphi : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ be a rational map with a fixed point at ∞. We let $\mathcal{O}_K$ denote the ring of integers of a number field K. Let

$$\varphi([T_0 : T_1]) = [P(T_0, T_1) : Q(T_0, T_1)],$$

where P and Q are homogeneous polynomials of degree d in $\mathcal{O}_K[T_0, T_1]$, be a model for φ over $\mathcal{O}_K$, in the terminology of the Section 4. Letting $P_1 = P$ and $Q_1 = Q$, and recursively defining $P_{k+1} = P(P_k, Q_k)$ and $Q_{k+1} = Q(P_k, Q_k)$, we obtain models

$$\varphi^k([T_0 : T_1]) = [P_k(T_0, T_1) : Q_k(T_0, T_1)],$$

for iterates φ^k of φ. Recall that T_1 must be a factor of each Q_k since $\infty = [1 : 0]$ is a fixed point of φ_k.

Throughout this section, v will denote a finite valuation on K that has been extended to the algebraic closure $\overline{K}$ of K. We let $\mathcal{O}_v$ denote the set of all $z \in K$ for which $v(z) \geq 0$.

For $(a, b) \in \overline{K}^2 \setminus (0, 0)$, we define

(5.0.1)
$$S_{v,k}(P_k(a, b), Q_k(a, b)) := \min(v(P_k(a, b)), v(Q_k(a, b))) - \min(v(a^{d^k}), v(b^{d^k})).$$

Note that $S_{v,k}(P_k(a, b), Q_k(a, b))$ is a finite number, since P_k and Q_k have no common factor, and that

$$S_{v,k}(P_k(a, b), Q_k(a, b)) = S_{v,k}(P_k(za, zb), Q_k(za, zb)) \tag{5.0.2}$$

for any nonzero $z \in \overline{K}$. It follows that $S_{v,k}(P_k(a, b), Q_k(a, b))$ is nonnegative, since we may thus assume that $\min(v(a), v(b)) = 0$. We also define

$$R_v(P_k, Q_k) := \sup_{(a,b) \in \overline{K}^2 \setminus (0,0)} (S_{v,k}(P_k(a, b), Q_k(a, b))).$$

To see that $R_v(P_k, Q_k)$ exists and is finite, we apply the Euclidean algorithm to $P_k(T_0, 1)$ and $Q_k(T_0, 1)$ to obtain

$$x(T_0)P_k(T_0, 1) + y(T_0)Q_k(T_0, 1) = m$$

with $x, y \in \mathcal{O}_v[T_0]$ and nonzero $m \in \mathcal{O}_v$. Then, for any $(a, b) \in \overline{K}^2 \setminus (0, 0)$, we have

$$S_{v,k}(P_k(a, b), Q_k(a, b)) \leq \max\big(v(m), \min(v(P_k(1, 0)), v(Q_k(1, 0)))\big).$$

If $R_v(P_k, Q_k) > 0$, then P_k and Q_k have a common root modulo the prime ideal corresponding to v and thus our model for φ^k has bad reduction at v, as explained in Section 4.

5.1. **Existence of the v-adic integral.** Let F be a nonconstant polynomial in $K[t]$. Using the coordinates $[T_0 : T_1]$ and and letting $t = T_0/T_1$, we let $[a_1 : b_1], \dots, [a_{\deg F} : b_{\deg F}]$ be the points in $\mathbb{P}^1(\overline{K})$ at which F vanishes.

We will prove the following proposition.

Proposition 5.1. *The sequence* $\left(\sum_{\ell=1}^{\deg F} \frac{S_{v,k}(P_k(a_\ell,b_\ell),Q_k(a_\ell,b_\ell))}{d^k}\right)_k$ *is increasing and is bounded by* $(\deg D)(\frac{1}{d-1})R_v(P,Q)$.

This allows us to define the integrals at a finite place v as follows.

Definition 5.2. *For a finite place v, we define*

(5.2.1)

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi} := -\lim_{k\to\infty} \sum_{\ell=1}^{\deg F} \frac{S_{v,k}(P_k(a_\ell,b_\ell),Q_k(a_\ell,b_\ell))}{d^k} \log \mathrm{N}(v) - v(F)\log \mathrm{N}(v) + (\deg F)\frac{v(A_d)}{d-1}\log \mathrm{N}(v),$$

where $S_{v,k}$ is given by (5.0.1), $P = \sum_{i=1}^d A_i T_0^i T_1^{d-i}$, *and $v(F)$ is the v-adic valuation of the content of F, i.e.,* $v(F) = \min_i(v(m_i))$ *when* $F = \sum_{i=1}^{\deg F} m_i t^i$.

Proposition 5.1 will be a simple consequence of the following two lemmas.

Lemma 5.3. *Let* $(a,b) \in \overline{K}^2 \setminus (0,0)$. *For all integers $k \geq 1$, we have*

$$S_{v,k}(P_{k+1}(a,b),Q_{k+1}(a,b)) = dS_{v,k}(P_k(a,b),Q_k(a,b)) + S_{v,k}(P(P_k(a,b),Q_k(a,b)),Q(P_k(a,b),Q_k(a,b))).$$

Proof. By (5.0.2), we may assume that $\min(v(a),v(b)) = 0$. Then

$$\begin{aligned}
&S_{v,k}(P_{k+1}(a,b),Q_{k+1}(a,b))\\
&\quad= \min(v(P(P_k(a,b),Q_k(a,b))),v(Q(P_k(a,b),Q_k(a,b))))\\
&\quad= \min(v(P_k(a,b)^d),v(Q_k(a,b)^d)) - \min(v(P_k(a,b)^d),v(Q_k(a,b)^d))\\
&\quad+ \min(v(P(P_k(a,b),Q_k(a,b))),v(Q(P_k(a,b),Q_k(a,b))))\\
&\quad= dS_{v,k}(P_k(a,b),Q_k(a,b))\\
&\quad\ + S_{v,k}(P(P_k(a,b),Q_k(a,b)),Q(P_k(a,b),Q_k(a,b))).
\end{aligned}$$

□

Lemma 5.4. *For all integers $k \geq 1$, we have*

$$R_v(P_k,Q_k) \leq R_v(P,Q)\Big(\sum_{i=1}^k d^{i-1}\Big). \tag{5.4.1}$$

Proof. We proceed by induction. The case $k = 1$ is obvious. Now, let $a, b \in \overline{K}^2 \setminus (0,0)$; we may assume by (5.0.2) that $\min(w(a), w(b)) = 1$. If (5.4.1) holds for k, then applying Lemma 5.3 to the case of $k+1$ yields

$$\begin{aligned} S_{v,k}(P_{k+1}(a,b), Q_{k+1}(a,b)) &\leq dR_v(P_k, Q_k) + R_v(P,Q) \\ &= dR_v(P,Q)(\sum_{i=1}^{k} d^{i-1}) + R_v(P,Q) = R_v(P,Q)(\sum_{i=1}^{k+1} d^{i-1}). \end{aligned}$$

□

Now, we will prove Proposition 5.1.

Proof. (of Proposition 5.1). It follows from Lemma 5.3 that the sequence is increasing. By Lemma 5.4, we have

$$\begin{aligned} \frac{1}{d^k}(R_v(P_k, Q_k)) \leq \frac{1}{d^k} R_v(P,Q)\left(\sum_{i=1}^{k} d^{i-1}\right) &= R_v(P,Q)\left(\sum_{i=1}^{k} d^{i-k-1}\right) \\ \leq R_v(P,Q)\left(\sum_{i=1}^{\infty} \frac{1}{d^i}\right) &= R_v(P,Q)\left(\frac{1}{d-1}\right), \end{aligned}$$

which is precisely the bound given in the statement of Proposition 5.1. □

5.2. **Invariance of the v-adic integral under change of variables.** Our definition of $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi}$ in Definition 5.2 involves P and Q. We can show, however, that the definition depends only on our choice of the point at infinity. Let τ be a change of variable of the form $\tau(T_0) = mU_0 + nU_1$, $\tau(T_1) = zU_1$ (so that τ fixes $[1:0]$). To get a model from this change of variables, we let $\tau^*Q = zQ(\tau(T_0), U_1)$ and let

$$\tau^*P = \frac{zP(\tau(T_0), U_1)}{m} - \frac{znQ(\tau(T_0), U_1)}{m}$$

where $z \in \mathcal{O}_K$ is chosen so that τ^*Q and τ^*P are both in $\mathcal{O}_K[U_0, U_1]$. Note that P is written as it is since $\tau^{-1}(T_0) = U_0/m - nU_1/m$. We define τ^*P_k and τ^*Q_k recursively as we did with P_k and Q_k. We also write $\tau^*F(u) = F(\tau(T_0), U_1)/U_1^d$ where $u = U_0/U_1$. Note that $\deg \tau^*F = \deg F$.

With this notation we have the following proposition.

Proposition 5.5. *With τ as above, we have*

(5.5.1)

$$\lim_{k\to\infty}\sum_{\ell=1}^{\deg F}\frac{S_{v,k}(P_k(a_\ell,b_\ell),Q_k(a_\ell,b_\ell))}{d^k}-(\deg F)\frac{v(A_d)}{d-1}+v(F)$$

$$=\lim_{k\to\infty}\sum_{\ell=1}^{\deg F}\frac{S_{v,k}(\tau^*P_k(a'_\ell,b'_\ell),\tau^*Q_k(a'_\ell,b'_\ell))}{d^k}-(\deg F)\frac{v(\tau^*A_d)}{d-1}+v(\tau^*F),$$

where τ^*A_d is the leading coefficient of τ^*P and $\tau(a_\ell T_0+b_\ell T_1)=a'_\ell U_0+b'_\ell U_1$. Thus, Definition 5.2 does not depend on our choice of P and Q.

Proof. The proof is a simple computation. We compute

(5.5.2)

$$\lim_{k\to\infty}\sum_{\ell=1}^{\deg F}\frac{S_{v,k}(P_k(a_\ell,b_\ell),Q_k(a_\ell,b_\ell))}{d^k}=\lim_{k\to\infty}\sum_{\ell=1}^{\deg F}\frac{S_{v,k}(\tau^*P_k(a'_\ell,b'_\ell),\tau^*Q_k(a'_\ell,b'_\ell))}{d^k}$$

$$+\frac{(\deg F)v(z)}{d-1}+\sum_{\ell=1}^{\deg F}\min(v(a_\ell),v(b_\ell))-\sum_{\ell=1}^{\deg F}\min(v(a'_\ell),v(b'_\ell))$$

We also see that

(5.5.3) $$v(\tau^*A_d)=v(A_d)+v(z)-(d-1)v(m).$$

Now, we may choose our a_ℓ and b_ℓ so that $F(t)=\prod_{\ell=1}^{\deg F}(b_\ell t-a_\ell)$. Then $\tau^*F(u)=\prod_{\ell=1}^{\deg F}m(b'_\ell u-a'_\ell)$, so that

(5.5.4)

$$v(\tau^*F)=(\deg F)v(m)+\sum_{\ell=1}^{\deg F}\min(v(a'_\ell),v(b'_\ell))$$

$$=v(F)+(\deg F)v(m)+\sum_{\ell=1}^{\deg F}\min(v(a'_\ell),v(b'_\ell))-\sum_{\ell=1}^{\deg F}\min(v(a_\ell),v(b_\ell)).$$

Multiplying (5.5.3) by $\deg F/(d-1)$ and subtracting it from the sum of (5.5.2) and (5.5.4) gives (5.5.1). □

5.3. **Geometry of the v-adic integrals.** Let D be a horizontal divisor on X with support away from $[1:0]$. Let $[a_1:b_1],\dots,[a_{\deg D}:b_{\deg D}]$ denote the points in $X(\overline{K})$ corresponding to D.

Let F be a polynomial in the inhomogeneous variable $t=T_0/T_1$ obtained by taking a global section of $\mathcal{O}_{\mathbb{P}^1}(n)$ corresponding to D on the generic fiber (which will be a homogeneous polynomial of degree n) and dividing through by T_1^n. When D corresponds to a point $\alpha \in X(\overline{K})$, we will call such F a *minimal polynomial* for α over K.

Let $\sigma_k : X_k \longrightarrow X$ be the blow-up map associated to our model (P_k, Q_k) of φ^k described in Section 4. We will relate the v-adic integral of F with the geometry of $\sigma_k^* D$ for D the horizontal divisor on X corresponding to F. We write

$$\sigma_k^* D = D_k + \sum_v \sum_i x_{v,i,k} C_{v,i,k},$$

where $C_{v,i,k}$ are components of the exceptional fiber of σ_k and D_k is the horizontal divisor corresponding to D on X_k. We let $k_{v,i,k}$ denote the field of definition of the closed point on X_k corresponding to the component $C_{v,i,k}$.

We begin by treating the case where D corresponds to a single point defined over K. Let $v(P_K)$ and $v(Q_k)$ be the minimum of the v-adic valuations of the coefficients of P_K and Q_k respectively.

Lemma 5.6. *If D corresponds to a point $[a : b] \in \mathbb{P}^1(K)$, then, for any nonarchimedean v, we have*

$$\sum_i x_{v,i,k} = S_{v,k}(P_k(a,b), Q_k(a,b)) - \min(v(P_k), v(Q_k)).$$

Proof. We will work locally at a single nonarchimedean place v. Let $\sigma_{\mathcal{O}_v,k} : \left(X_k\right)_{\mathcal{O}_v} \longrightarrow X_{\mathcal{O}_v}$ denote σ_k with its base extended to $\operatorname{Spec} \mathcal{O}_v$. Since D corresponds to a single rational point, there is at most one nonzero $x_{v,i,k}$ for a fixed v and k. Thus, letting D_v be the localization of D at v, we have $\sigma_{\mathcal{O}_v,k}^*(D_v) = D_{v,k} + f_{v,k} C_{v,k}$ for some horizontal divisor $D_{v,k}$, some non-negative integer $f_{v,k}$, and some component $C_{v,k}$ of the exceptional fiber of $\sigma_{\mathcal{O}_v,k}$.

We may assume that $\min(v(a), v(b)) = 0$. Let I denote the ideal sheaf of D_v in $X_{\mathcal{O}_v}$; then $\sigma_{\mathcal{O}_v,k}^* I$ will be the ideal sheaf for a subscheme of $\left(X_k\right)_{\mathcal{O}_v}$ corresponding to $D_{v,k} + f_{v,k} C_{v,k}$. Note that $\sigma_{\mathcal{O}_v,k}^*(I \otimes \mathcal{O}_X(1))$ is generated by $(bT_0 - aT_1)$. Let U be an open subset of $X_{k,\mathcal{O}_v}$ containing $\operatorname{Supp}(D_{v,k} + C_{v,k})$. We may choose U to be the chart of $\left(X_k\right)_{\mathcal{O}_v}$ on which T_j doesn't vanish, where $j = 0$ if $v(a) = 0$ and $j = 1$ otherwise. Let i be the choice of $\{0, 1\}$ that is not equal to j. Let π be a generator for the maximal ideal in $\mathcal{O}_v$, let $\kappa = \min(v(P_k), v(Q_k))$, and let $G_k = P_k/\pi^\kappa$ and $H_k = Q_k/\pi^\kappa$. Then U is isomorphic to

$$\operatorname{Proj} \frac{\left(\mathcal{O}_v[T_i/T_j]\right)[t,u]}{\left(t \frac{G_k(T_0,T_1)}{T_j^{d^k}} - u \frac{H_k(T_0,T_1)}{T_j^{d^k}}\right)}.$$

Since we have $\mathcal{O}_v[T_i/T_j]/(b\,(T_0/T_j) - a\,(T_1/T_j)) \cong \mathcal{O}_v$, the subscheme of $\left(X_k\right)_{\mathcal{O}_v}$ determined by the vanishing of $\sigma^*_{\mathcal{O}_v,k} I$ will be isomorphic to

$$\begin{aligned}\operatorname{Proj}(\mathcal{O}_v[t,u]/(tP_k(a,b) - uQ_k(a,b))) &\cong \operatorname{Proj}(\mathcal{O}_v[t,u]/(\pi^r(\ell t - mu))) \\ &\cong \operatorname{Proj}((\mathcal{O}_v/\pi^r)[t,u]) \cup \operatorname{Spec}\mathcal{O}_v,\end{aligned}$$

where $r = S_{v,k}(P_k(a,b), Q_k(a,b)) - \kappa$, $G_k(a,b) = \ell\pi^r$, and $H_k(a,b) = m\pi^r$. Since the divisor corresponding to $\sigma^*_{\mathcal{O}_v,k} I$ is $D_{v,k} + f_{v,k} C_{v,i,k}$ where $D_{v,k}$ is horizontal and $C_{v,i,k}$ is reduced, this means that we must have

$$S_{v,k}(P_k(a,b), Q_k(a,b)) - \min(v(P_k), Q(P_k)) = r = f_{v,k} = \sum_i x_{v,i,k},$$

as desired. □

Since blowing up commutes with base extension from $\mathcal{O}_K$ to the ring of integers in a number field over which D splits into points, Lemma 5.6 generalizes easily to the following lemma.

Lemma 5.7. *We have*

$$\sum_i x_{v,i,k}[k_{v,i,k} : k] = \sum_{\ell=1}^{\deg F} S_{v,k}(P_k(a_\ell, b_\ell), Q_k(a_\ell, b_\ell)) - (\deg F)(\min(v(P_k, Q_k))).$$

Now, we will show that the contribution of $\min(v(P_k), v(Q_k))$ can be controlled.

Lemma 5.8. *The limit* $\lim_{k\to\infty} \frac{\min(v(P_k), v(Q_k))}{d^k}$ *exists.*

Proof. We will show that the sequence $\left(\frac{\min(v(P_k), v(Q_k))}{d^k}\right)$ is bounded and increasing. Since $\min(v(P_k), v(Q_k)) \le R_v(P_k, Q_k)$, boundedness follows immediately from Lemma 5.4. Now, for any $k \ge 1$, we have

$$\min(v(P_{k+1}), v(Q_{k+1})) \ge d \min(v(P_k), v(Q_k)))$$

since $P_{k+1} = P(P_k, Q_k)$ and $Q_{k+1} = Q(P_k, Q_k)$ so the sequence is increasing. □

Let ∞ denote the horizontal divisor on X corresponding to the point $[1:0]$ and let ∞_k denote the horizontal divisor on X_k corresponding to this point.

Lemma 5.9. *We have* $y_{v,i,k} = v(P_k(1,0)) - \min(v(P_k), v(Q_k))$ *where* $\sigma_k^*\infty = \infty_k + \sum_v \sum_i y_{v,i,k} C_{v,i,k}$. *Thus,*

$$\lim_{k\to\infty} \frac{y_{v,i,k}}{d^k} = \frac{v(A_d)}{d-1} - \lim_{k\to\infty} \frac{\min(v(P_k), v(Q_k))}{d^k}.$$

Proof. Since T_1 is a factor of Q, it is also a factor of Q_k for every k. Thus, for each k we have $S_{v,k}(P_k(1,0), Q_k(1,0)) = v(P_k(1,0))$, so by Lemma 5.6, we have $y_{v,i,k} = v(P_k(1,0)) - \min(v(P_k), v(Q_k))$. Since $P(1,0) = A_d$ and for all $k \geq 1$ we have

$$P_{k+1}(1,0) = P(P_k(1,0), 0) = A_d P_k(1,0)^d.$$

As in the proof of Lemma 5.4, it follows that

$$\lim_{k\to\infty} \frac{v(P_k(1,0))}{d^k} = v(A_d)\left(\sum_{i=1}^{\infty} \frac{1}{d^i}\right) = \frac{v(A_d)}{d-1}.$$

□

The following is now an immediate consequence of Definition 5.2 and Lemmas 5.7, 5.8, and 5.9.

Proposition 5.10. *With notation as above, we have*

$$\begin{aligned}\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi} &= -\lim_{k\to\infty} \frac{x_{v,i,k}[k_{v,i,k} : k_v]}{d^k} \log \mathrm{N}(v) \\ &\quad + (\deg F) \lim_{k\to\infty} \frac{y_{v,i,k}}{d^k} \log \mathrm{N}(v) - v(F) \log \mathrm{N}(v).\end{aligned} \tag{5.10.1}$$

5.4. **A remark on the use of integral notation at finite places.** Let us now add a few words about why it makes sense to think of our definition of $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v \, d\mu_{v,\varphi}$ as an integral when v is finite. H. Brolin ([7]) and M. Lyubich ([18]) have shown that if v is an infinite place and θ is a continuous, bounded function on $\mathbb{P}^1(\mathbb{C}_v)$, then for any $\xi \in \mathbb{C}$ with an infinite backward orbit under φ (i. e. for which the set $\cup_{k=1}^{\infty} (\varphi^k)^{-1}(\xi)$ is infinite), one has

$$\lim_{k\to\infty} \sum_{\varphi^{(z)}=\xi} \frac{\theta(z)}{d^k} = \int_{\mathbb{P}^1(\mathbb{C}_v)} \theta \, d\mu_{v,\varphi},$$

where $d\mu_{v,\varphi}$ is the unique φ-invariant measure (see [12]) on $\mathbb{P}^1$ (which is the same as our $d\mu_{v,\varphi}$, as we show in Proposition 7.2).

Our p-adic integrals can be written in a similar way. For example, suppose D is an irreducible divisor corresponding to a single point $[a : b]$ such that $P_k(a,b) \neq 0$. Then the polynomial $F(t) = bt - a$ defines D. Writing $P_k = \eta_k \prod_{j=1}^{d^k} (T_0 - u_j T_1)$, we then have

$$P_k(a,b) = \eta_k \prod_{j=1}^{d^k} (b - u_j a) = \eta_k \prod_{j=1}^{d^k} F(u_j).$$

Since $\varphi^k(z) = 0$ if and only if $[z : 1] \sim [u_j : 1]$ for some j, we thus have

$$\frac{\log|P_k(a,b)|_v}{d^k} = \frac{\log|\eta_k|_v}{d^k} + \sum_{\varphi^k(z)=0} \frac{\log|F(z)|_v}{d^k},$$

where the z with $\varphi^k(z) = 0$ are counted with multiplicities. Similarly, when Q is not a multiple of X_1^d, we have

$$\frac{\log|Q_k(a,b)|_v}{d^k} = \frac{\log|\gamma_k|_v}{d^k} + \sum_{\varphi^k(z)=\infty} \frac{\log|F(z)|_v}{d^k},$$

where γ_k is the leading coefficient of $Q_k(T_0, 1)$. Taking limits and subtracting off $\frac{\log|A_d|_v}{d-1}$, we see that $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi}$ is equal to

$$\lim_{k\to\infty} \max\left(\sum_{\varphi^k(z)=0} \frac{\log|F(z)|_v}{d^k}, \sum_{\varphi^k(z)=\infty} \frac{\log|F(z)|_v}{d^k}\right).$$

More generally, with a bit of diophantine geometry, we can show that for any point $\xi \in \mathbb{C}_v$ that has an infinite backwards orbit under φ, we have

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi} = \lim_{k\to\infty} \sum_{\substack{\varphi^k(z)=\xi \\ F(z)\neq 0}} \frac{\log|F(z)|_v}{d^k}.$$

We will prove this in a future paper. Thus, our v-adic integrals at finite places seem quite analogous to our integrals at the infinite places. We should note, however, that we do not know what classes of functions we can expect to be able to "integrate" in this way at finite places.

6. The Mahler formula for dynamical systems

The formula of Mahler for the naive height of a closed point $\alpha \neq \infty$,

$$\deg(F)h(\alpha) = \int_0^1 \log|F(\exp(2i\pi\theta))|d\theta,$$

where F is the minimal equation for the algebraic point α of the projective line over $\mathbb{Q}$ will be generalized to a dynamical system φ and its canonical height. The original Mahler formula is associated to the dynamics of the map $\varphi(t) = t^2$.

In general, the formula involves some adelic terms with support at places of bad reduction. Recall our definition of $d\mu_{v,\varphi}$ for v an infinite place from Section 2 and our definition of $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi}$ for v a finite place from Section 5. We give a proof of this theorem using integral models for each iterate of φ. It may be possible to give an independent proof using adelic metrics (*cf*. [35]). Recall our definition of $d\mu_{v,\varphi}$ for v an infinite place from Section 2 and our definition of $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi}$ for v a finite place from Section 5. Our main theorem is the following:

Theorem 6.1. *Let K be a number field, φ a rational map from $\mathbb{P}^1_K$ to $\mathbb{P}^1_K$ that has at least one K-rational fixed point, which we call ∞. For infinite v, (resp. finite v), let $d\mu_{v,\varphi}$ for v be defined as in Proposition 2.3 (resp. Definition 5.2). Then, given any point $\alpha \in \mathbb{P}^1(\bar{K})$ with $\alpha \neq \infty$ and any minimal polynomial F for α over K one has*

$$[K(\alpha):\mathbb{Q}]h_\varphi(\alpha) = \sum_{\text{places } v \text{ of } K} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi}. \tag{6.1.1}$$

For finite v, we have $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi} = 0$ unless φ has bad reduction at v or all the coefficients of F have nonzero v-adic valuation. In particular, $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi} = 0$ for all but finitely many v.

Proof. We will compute all our heights by using the Arakelov intersection product between Weil divisors and metrized line bundles, which is defined in Appendices A and B.

We can think of F as a rational function on X; we will then have $\operatorname{div} F = D - (\deg F)\infty + \sum_{\text{finite } v} v(F)X_v$, where D is the horizontal divisor on X corresponding to α and X_v is the fiber of X at v. We let $F_k = \sigma_k^* F$ and let $X_{k,v} = \sigma_k^* X_v$ denote the fiber of X_k at the finite place v. Let D_k be the horizontal divisor on X_k corresponding to D and let ∞_k be the horizontal divisor on X_k corresponding to ∞. We have

$$\begin{aligned}\operatorname{div} F_k = D_k + \sum_{\text{finite } v} \sum_i x_{v,i,k} C_{v,i,k} - (\deg F)\infty_k \\ - (\deg F) \sum_{\text{finite } v} \sum_i y_{v,i,k} C_{v,i,k} + \sum_{\text{finite } v} v(F) X_{k,v},\end{aligned}$$

where $\sigma_k^* D = D_k + \sum_v \sum_i x_{v,i,k} C_{v,i,k}$ and $\sigma_k^* \infty = \infty_k + \sum_v \sum_i y_{v,i,k} C_{v,i,k}$, as in Section 5.

We let L be the line bundle $\mathcal{O}_X(1)$. At each infinite place v, let $\|\cdot\|_v$ be the metric on L such that for any section $s = u_0 T_0 + u_1 T_1$, of $\mathcal{O}_{\mathbb{P}^1}(1)$, we have $\|s([T_0:T_1])\|_v = (|u_0T_0 + u_1T_1|_v^2/(|T_0|_v^2 + |T_1|_v^2))^{[K_v:\mathbb{R}]/2}$. This metric is smooth and semipositive (see [32, Section 6]). We denote as h_L the height function given by $[K(\alpha):\mathbb{Q}]h_L(\alpha) = (E_\alpha . L)_{\text{Ar}}$, where E_α is the horizontal divisor on X corresponding to α. Note that $h_L([1:0]) = 0$. We denote as L_k the line bundle $\varphi_k^* L$ endowed with the metric

$\varphi_k^* \| \cdot \|_v$. Then $\frac{1}{(\deg \varphi)^k} \frac{-1}{2\pi i} d\overline{d} \log \varphi_k^* \| \cdot \|_v$ converges to a distribution $d\mu_{v,\varphi}$ as $k \to \infty$ by Proposition 2.3.

Let $\alpha \in X(\bar{k})$ be a point on the generic fiber corresponding to D. By the definition of L_k, we have $(D_k.L_k)_{\mathrm{Ar}} = (\deg F) h_L(\varphi^k(\alpha))$ and $(\infty_k.L_k)_{\mathrm{Ar}} = 0$. Using Proposition B.3 and the fact that $(C_{v,i,k}.L_k) = [k_{v,i,k} : k_v]$ (by Proposition 4.3) and $(X_{k,v}.L_k) = 0$ (by Proposition A.8), we thus obtain

$$
\begin{aligned}
& d^k \sum_{v|\infty} \int_{X(\mathbb{C}_v)} \log |F|_v d\mu_{v,k} = (\operatorname{div} F_k.L_k)_{\mathrm{Ar}} \\
& = -((\deg F)\infty_k.L_k)_{\mathrm{Ar}} + (D_k.L_k)_{\mathrm{Ar}} + \sum_{\text{finite } v} (x_{v,i,k} C_{v,i,k}.L_k) \log \mathrm{N}(v) \\
& + \sum_{\text{finite } v} v(F)(X_{k,v}.L_k) \log \mathrm{N}(v) - (\deg F) \sum_{\text{finite } v} (y_{v,i,k} C_{v,i,k}.L_k) \log \mathrm{N}(v) \\
& = [K(\alpha) : \mathbb{Q}] h_L(\varphi^k(\alpha)) + \sum_{\text{finite } v} x_{v,i,k} [k_{v,i,k} : k_v] \log \mathrm{N}(v) \\
& + d^k \sum_{\text{finite } v} v(F) \log \mathrm{N}(v) - (\deg F) \sum_{\text{finite } v} y_{v,i,k} [k_{v,i,k} : k_v] \log \mathrm{N}(v).
\end{aligned}
$$

Now, we divide through by d^k and take limits. Since

$$
\lim_{k \to \infty} \sum_{v|\infty} \int_{X(\mathbb{C}_v)} \log |F|_v d\mu_{v,k} = \sum_{v|\infty} \int_{X(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi}
$$

and $\lim_{k \to \infty} h_L(\varphi^k(\alpha))/d^k = h_\varphi(\alpha)$, applying Proposition 5.10 yields

$$
\begin{aligned}
& [K(\alpha) : \mathbb{Q}] h_\varphi(\alpha) = \sum_{v|\infty} \int_{X(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi} - \sum_{\text{finite } v} v(F) \log \mathrm{N}(v) \\
& - \lim_{k \to \infty} \frac{1}{d^k} \sum_{\text{finite } v} x_{v,i,k} [k_{v,i,k} : k_v] \log \mathrm{N}(v) + (\deg F) \lim_{k \to \infty} \frac{y_{v,i,k}}{d^k} \log \mathrm{N}(v) \\
& = \sum_{v|\infty} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi} + \sum_{\text{finite } v} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi}.
\end{aligned}
$$

□

Remark 6.2. Letting $\varphi = P/Q$ be a model for φ and applying Lemma 5.1, we have the bound

$$\begin{aligned} -(\deg F)\Big(\sum_{\text{finite } v} \frac{R_v(P,Q)}{d-1}\Big) \log \mathrm{N}(v) - \sum_{\text{finite } v} v(F) \log \mathrm{N}(v) \\ \leq \sum_{\text{finite } v} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi} \\ \leq (\deg F) \sum_{\text{finite } v} \frac{v(A_d)}{d-1} \log \mathrm{N}(v) - \sum_{\text{finite } v} v(F) \log \mathrm{N}(v), \end{aligned}$$

where A_d is the coefficient of the T_0^d term of P, $v(F)$ is the minimum of the v-adic valuations of the coefficients of F, and $R_v(P,Q)$ is the supremum of $\min(v(P(a,b)), v(Q(a,b)))$ over all v-adic integers a and b in $\mathcal{O}_{\overline{K}}$ with $(a,b) \neq (0,0)$. Note that $\sum_{\text{finite } v} R_v(P,Q)$ is less than or equal to the resultant of $P_k(T_0,1)$ and $Q_k(T_0,1)$ as polynomials in T_0 (see Brieskorn and Knörrer [6, p. 279, Proposition 4], for example).

Corollary 6.3. *Suppose that φ can be written as $[P : Q]$ where $P(T_0,1)$ is monic in T_0 and that α has a minimal polynomial F over K with coprime coefficients in $\mathcal{O}_K$ (as is always the case when $K = \mathbb{Q}$, for example). Then*

$$[K(\alpha) : \mathbb{Q}] h_\varphi(\alpha) \leq \sum_{v|\infty} \int_{X(\mathbb{C})} \log |F|_v d\mu_{v,\varphi}, \tag{6.3.1}$$

with equality if φ has good reduction everywhere.

Example 6.4. In general, one cannot expect equality in (6.3.1). Suppose that $P(T_0,1)$ is monic, as in Corollary 6.3. Suppose furthermore that $\alpha \in \overline{K}$ has a minimal polynomial F with coprime coefficients in $\mathcal{O}_K$ and that $Q(\alpha,1) = 0$. Then, we have $\int_{X(\mathbb{C})} \log |F|_v d\mu_{v,\varphi} = -\frac{v(P(\alpha,1))}{d} \log \mathrm{N}(v)$ for any finite place v. Thus, if $v(P(\alpha,1)) > 0$ for some finite place v, then inequality (6.3.1) is strict.

7. Equidistribution and Julia sets

Let v be an archimedean place. As we have defined it, $d\mu_{v,\varphi}$ is only a distribution; that is to say, the integral $\int_{\mathbb{P}^1(\mathbb{C})} f \, d\mu_{v,\varphi}$ is only defined for smooth functions f. We will now show that $d\mu_{v,\varphi}$ extends to a linear functional on the space of continuous functions on $\mathbb{P}^1(\mathbb{C})$ and that this linear functional is the unique φ-invariant probability measure on $\mathbb{P}^1(\mathbb{C})$ with support on the Julia set of φ.

Let $\|\cdot\|_v$ be a metric on $\mathcal{O}^{\mathbb{P}^1}(1)$. Recall the definitions of $\|\cdot\|_{v,k}$ and $d\mu_{v,k}$ from Proposition 2.3.

Following [18], we define A to be the operator on the space of continuous functions of $\mathbb{P}^1(\mathbb{C})$ which sends a continuous function f on $\mathbb{P}^1(\mathbb{C})$ to

$$(Af)(z) := \frac{1}{d} \sum_{\varphi(w)=z} e_w f(w),$$

where $z \in \mathbb{P}^1(\mathbb{C})$ and e_w is the ramification index of φ at w.

Lemma 7.1. *Let U be an open set in $\mathbb{P}^1(\mathbb{C})$. Then*

$$\int_{\varphi^{-1}(U)} f \, d\mu_{v,k+1} = \int_U Af \, d\mu_{v,k} \tag{7.1.1}$$

for any $k \geq 1$.

Proof. Since the set of ramification points of φ is finite, it suffices to show that (7.1.1) holds when U contains no ramification points; since any open subset can be written as a union of simply connected open subsets, we may further assume that U is simply connected. We may then decompose $\varphi^{-1}(U)$ into d branches V_λ, $1 \leq \lambda \leq d$ such that φ is bijective on each V_λ with analytic inverse φ_λ^{-1}.

Choose a section s of $\mathcal{O}_{\mathbb{P}^1}(1)$ that does not vanish on U or $\varphi^{-1}(U)$. Let $\rho = \frac{\log \|s\|_{v,k}}{2\pi i}$. Then, on U, we have $d\overline{d}\rho = d\mu_{v,k}$ and on V_λ, we have $d\overline{d}(\rho \circ \varphi) = (\deg \varphi)(d\mu_{k+1,v})$. By change of variables, we then have

$$\begin{aligned} \int_{V_\lambda} f \, d\mu_{v,k+1} &= \frac{1}{d} \int_{V_\lambda} f(z) \, d\overline{d}(\rho(\varphi(z))) \\ &= \frac{1}{d} \int_U f(\varphi_\lambda^{-1}(u)) \, d\overline{d}(\rho(u)) \\ &= \frac{1}{d} \int_U f \circ \varphi_\lambda^{-1} \, d\mu_{v,k}. \end{aligned}$$

Since $(Af)(u) = \frac{1}{d} \sum_{\lambda=1}^d f \circ \varphi_\lambda^{-1}(u)$, we thus obtain

$$\begin{aligned} \int_{\varphi^{-1}(U)} f \, d\mu_{v,k+1} &= \sum_{\lambda=1}^d \int_{V_\lambda} f \, d\mu_{v,k+1} = \frac{1}{d} \sum_{\lambda=1}^d \int_U f \circ \varphi_\lambda^{-1} \, d\mu_{v,k} \\ &= \int_U \frac{1}{d} \sum_{\lambda=1}^d f \circ \varphi_\lambda^{-1} \, d\mu_{v,k} = \int_U Af \, d\mu_{v,k}. \end{aligned}$$

□

An *exceptional point* ξ for φ is a point such that $\varphi^2(\xi) = \xi$ and φ^2 ramifies completely at ξ. An exceptional point ξ is an super-attracting fixed point for φ^2 (see J. Milnor [22]).

Proposition 7.2. *The measures $d\mu_{v,k}$ converge to a measure $d\mu_{v,\varphi}$ that is supported on the Julia set. Furthermore, $d\mu_{v,\varphi}$ is the unique probability measure supported on the Julia set with the property that*

$$\int_{\varphi(U)} d\mu_{v,\varphi} = \int_U d\mu_{v,\varphi}$$

for any open subset $U \subset \mathbb{P}^1(\mathbb{C})$ such that φ is injective on U.

Proof. Let $\epsilon > 0$. We may choose an open set U_ϵ containing the exceptional points of φ for which

$$\int_{\mathcal{U}_\epsilon} d\mu_{v,1} \leq \frac{\epsilon}{2 \sup_{z \in \mathbb{P}^1(\mathbb{C})}(|f(z)|_v)}.$$

Such a set exists since $d\mu_{v,1}$ is a continous form. Let $W_\epsilon = \mathbb{P}^1(\mathbb{C}) \setminus U_\epsilon$. By Theorem 1 of [18], there is a constant C_f such that $(A^k f)(w)$ converges uniformly to C_f for $w \in W_\epsilon$. Thus, there is some M such that for any $k \geq M$, we have

$$|(A^k f)(w) - C_f|_v < \epsilon/2$$

for all $w \in W_\epsilon$. Using Lemma 7.1.1 and the fact that $\int_{W_\epsilon} d\mu_{v,1} \leq 1$, we then see that for all $k \geq M$ we have

$$\begin{aligned}
&\left| \int_{\mathbb{P}^1(\mathbb{C})} f \, d\mu_{v,k} - C_f \right|_v \\
&= \left| \int_{\mathbb{P}^1(\mathbb{C})} (A^k f) \, d\mu_{v,1} - C_f \right|_v \\
&\leq \left| \int_{W_\epsilon} (A^k f) \, d\mu_{v,1} - C_f \right|_v + \int_{U_\epsilon} |A^k f|_v \, d\mu_{v,1} \\
&\leq \int_{W_\epsilon} \frac{\epsilon}{2} \, d\mu_{v,1} + \int_{U_\epsilon} \big(\sup_{z \in \mathbb{P}^1(\mathbb{C})} (|f(z)|_v) \big) \, d\mu_{v,1} \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Thus, $d\mu_{v,\varphi}$ extends to a measure such that

$$\int_{\mathbb{P}^1(\mathbb{C})} f \, d\mu_{v,\varphi} = \lim_{k \to \infty} (A^k f)(z),$$

where z is any point in W_ϵ. Freir, Lopes, and Mane ([12]) have shown that the map sending a continuous function f to $\lim_{k \to \infty}(A^k f)(z)$, where z is a not an exceptional point of φ, is the unique φ-invariant probability measure on $\mathbb{P}^1(\mathbb{C})$ that is supported on the Julia set of φ. □

P. Autissier [2] has proved the following equidistribution theorem.

Theorem 7.3. *With the same hypothesis as Theorem 6.1, for any infinite v and any nonrepeating sequence of points (α_n) in $\mathbb{P}^1(K)$ such that $\lim_{n\to\infty} h(\alpha_n) = 0$, the sequence $\frac{1}{|\operatorname{Gal}(\alpha_n)|}\sum_{\sigma\in\operatorname{Gal}}\delta_{\sigma(\alpha_n)}$ of discrete measures converges weakly to $d\mu_\varphi$.*

M. Baker and R. Rumely ([3]) have given a new proof of this theorem, using capacity theory. Their proof also gives equidistribution results at finite places.

One might also ask whether the Mahler measure of this paper should also be computable by equidistribution; more precisely, we conjecture:

Conjecture 7.4. *With the same hypothesis as theorem 7.3, for F the minimal equation of point α not in the Galois orbit of any α_n and v an infinite place of K, one has*

$$\lim_{n\to\infty}\frac{1}{|\operatorname{Gal}(\alpha_n)|}\sum_{\sigma\in\operatorname{Gal}}\log|F(\sigma(\alpha_n)|_v = \int_{X(\mathbb{C}_v)}\log|F|_v d\mu_{v,\varphi}.$$

We can prove Conjecture 7.4 in the case that the points α are periodic points. This generalizes earlier work on "elliptic Mahler measure" by G. Everest and T. Ward in [11, Theorem 6.18].

Appendix A. Schematic Intersection theory on a Cohen-Macaulay surface.

Intersection theory has been developed by many authors. We are in this article very interested in signs and vanishing in an arithmetic situation. A positive local intersection number appears when it can be expressed as the length of a tensor product. This appendix uses material that exists in P. Deligne [10] and in W. Fulton [13, Chapter 20].

For us a surface will be a noetherian, irreducible, and reduced scheme X of dimension two. In this article, moreover, X will be Cohen-Macaulay and will be equipped with a flat and often projective, generically smooth, structural map $f : X \to \operatorname{Spec} B$, where B is a Dedekind domain with field of fractions K. Often K will be a number field or the field of fractions of a discrete valuation ring V. The geometric (or schematic) intersection numbers will be rational numbers (the need for denominators for an intersection theory on a singular scheme was noted previously in D. Mumford [24] and in C. Peskine and L. Szpiro [25]).

Definition A.1. *We define the schematic (or geometric) intersection number of a Cartier divisor D with a Weil cycle C when they have no common components as*

$$(D.C) = \operatorname{length}(\mathcal{O}_D\otimes\mathcal{O}_C) - \operatorname{length}(\operatorname{Tor}_1(\mathcal{O}_D,\mathcal{O}_C)).$$

Lemma A.2. *If C is of codimension 2, then $(D.C) = 0$.*

Proof. If f is locally the equation of D and I is the ideal of C in a local ring A of X, one has an exact sequence

$$0 \to \mathrm{Tor}_1(A/I, A/f) \to A/I \xrightarrow{f} A/I \to A/(I + (f)) \to 0.$$

The lemma follows since the four modules in this sequence are of finite length.

□

Proposition A.3. ***(bilinearity and symmetry)*** *The pairing $(D.C)$ we just defined is bilinear and symmetric when both sides are Cartier divisors.*

Proof. If C is a Weil divisor, it is a linear combination with integral coefficients of reduced and irreducible Weil divisors C_i (the coefficient of C_i is equal to length$((\mathcal{O}_C)_{\wp_i})$). The sheaf $\mathcal{O}_D$ being of Tor dimension 1 the pairing is linear on the right by devissage. To see the linearity on the left it is enough to look at the case where C is reduced and irreducible. The proof will be complete after the reader checks the following lemma:

Lemma A.4. *Let A be a commutative ring, I an ideal in A and f a non zero divisor in A. Then one has the following exact sequence:*

$$0 \to A/I \to A/fI \to A/fA \to 0$$

Proof. The symmetry is clear when D and C are Cartier divisors with no common components. □

If C' is a $\mathbb{Q}$-Cartier divisor, i.e. a Weil divisor with an integral multiple nC' which is Cartier, we will define

$$(C'.C) = \frac{1}{n}((nC').C)$$

In this article we use only intersection between two $\mathbb{Q}$-Cartier divisors. So, up to an integral multiple, the intersection number is locally a finite sum of length$(A/(f, g))$, where A is a local ring of dimension 2 and depth 2 and (f, g) is a regular sequence. The following propositions are classical:

Proposition A.5. ***(linear equivalence)*** *If C is a Cohen-Macaulay projective curve then*

$$(D.C) = \deg_C \mathcal{O}_X(D)_{|C}.$$

Proof. One needs only to check that the degree of a line bundle is a well behaved notion on Cohen-Macaulay projective curves. This is the Riemann-Roch theorem for curves.

If C is a projective curve and L is a line bundle on X a projective surface one can then speak of $(L.C)$ for L is the difference in $\mathrm{Pic}(X)$ between two very ample line bundles each of them having sections with no common components with C. ☐

Corollary A.6. *When C is a projective curve and D is a $\mathbb{Q}$-Cartier divisor, the intersection $(D.C)$ is well-defined by bilinearity and linear equivalence even when D and C have a common component.*

Corollary A.7. *Let F be a rational function on a reduced, irreducible surface X that is projective and generically smooth over B. Then for any Weil divisor C contained in a fiber over B, we have*

$$(\mathrm{div}(F).C) = 0.$$

Proof. This is clear for the line bundle $\mathcal{O}_X(\mathrm{div}(F))$ is equal to $\mathcal{O}_X$ and C is a projective curve. ☐

Proposition A.8. ***(projection formula)*** *Let $\varphi : Y \to X$ be a map between surfaces X and Y that are projective over B. If L is a line bundle on X and C closed subscheme of Y one has*

$$(\varphi^*(L).C) = (L.\varphi_*(C)).$$

In particular if C is contracted by φ to a subscheme of X of codimension 2 the intersection number $(\varphi^*(L).C)$ is zero.

Proof. By additivity we can suppose C is a reduced irreducible curve in X. There are two cases: $\varphi_*(C)$ is of dimension 1 and $\varphi_*(C)$ is of dimension zero. In the first case $C \to \varphi_*(C)$ is finite and by Lemma A.4 we have

$$\begin{aligned} \mathrm{length}(\mathcal{O}_C/(f\mathcal{O}_C)) &= \mathrm{length}(\mathcal{O}_X/(f\mathcal{O}_X) \otimes \mathcal{O}_C) \\ &= \mathrm{length}(\mathcal{O}_Y/(f\mathcal{O}_Y) \otimes \mathcal{O}_C). \end{aligned}$$

In the second case L can be realized as the line bundle associated to the difference of two very ample divisors on X each of them having no intersection with $\varphi_*(C)$. The reciprocal images of these divisors in Y do not meet C, so both side of the projection formula vanish as it is required. ☐

The following proposition shows that intersection theory for $\mathbb{Q}$-Cartier divisors does not change when we pass to the normalization.

Proposition A.9. ***(invariance under normalization)*** *Let A be a Cohen-Macaulay integral domain of dimension 2 and let $\tilde{A}$ be its integral closure. Let (f, g) a regular sequence in A then supposing $\tilde{A}$ is a finitely generated A-module (f, g) is a regular sequence in $\tilde{A}$ and*

$$\mathrm{length}(A/(f, g)) = \mathrm{length}(\tilde{A}/(f, g)).$$

Proof. We shall note that $\tilde{A}$ is finitely generated over A when we are in a geometric or arithmetic situation. When (f, g) is a regular sequence in a module M the only non vanishing Tor is the tensor product $M \otimes A/(f, g)$. One proves easily by induction that the alternating sum $\sum_i (-1)^i \,\mathrm{length}(\mathrm{Tor}_i(A/(f, g), M))$ is a non negative additive function on the set of finitely generated A-modules. This sum is is equal to zero on modules of the form A/h or $A/(h, k)$ for a system of parameters (h, k) in A and thus is zero on $A/\wp$ for any prime ideal $\wp$ containing h or (h, k) by devissage. Hence, it is equal to zero on any module of dimension less than or equal to one. Any non-zero prime ideal of A being of the previous form our assertion of vanishing is proved. The module $\tilde{A}/A$ being of dimension at most one and (f, g) remaining a regular sequence in $\tilde{A}$ the proposition follows by additivity. □

The following proposition is proved in P. Deligne [10] with the additional assumption that X is normal:

Proposition A.10. *Let X be a generically smooth, reduced, irreducible and locally Cohen-Macaulay surface. The geometric intersection product on a fiber of $X \to \mathrm{Spec}(B)$ when X is projective over B, is negative. Only combination of full fibers have zero self-intersection.*

By Proposition A.9, the assumption that X is normal may be replaced with the weaker assumption that X is Cohen-Macaulay.

Thus, Proposition A.10 applies to the surfaces used in this paper.

Appendix B. Arakelov intersection with the divisor associated to a rational function

Let X be as in Appendix A; in this section we specify that the Dedekind domain B we work with is the ring of integers in a number field K. Let F be a meromorphic function on X. We begin by working with "arithmetico-geometric" intersections; that is to say, if we intersect a Cartier divisor D and a Weil cycle C without common components, their arithmetico-geometric intersection $(D.C)_{\mathrm{fin}}$ is taken to be

$$(D.C)_{\mathrm{fin}} = \sum_{\text{finite places } v \text{ of } B} (D_v.C_v) \log \mathrm{N}(v),$$

where $(\cdot.\cdot)$ is the intersection product from Appendix A, the D_v and C_v are the pull-backs of D and C back to the fiber X_v, and $\mathrm{N}(v)$ is the cardinality of the residue field k_v of our base ring B at v. These intersections $(D.C)_{\mathrm{fin}}$ are thus sums of the geometric intersections of Appendix A "weighted" by the logarithm of the size of the residue fields k_v.

In practice, we will be computing our intersections between rational functions and reduced irreducible horizontal divisors $\operatorname{Spec} R$. Thus, for R an order in a number field K and f a nonzero element of the field of fractions of R, we define $\#(R/f)$ follows: Let S denote the primes $\mathcal{P}$ in R for which $f \in R_{\mathcal{P}}$ and let T denote the primes $\mathcal{Q}$ in R for which $f^{-1} \in R_{\mathcal{Q}}$. We let

$$\#(R/fR) = \frac{\prod_{\mathcal{P}\in S} \#(R_{\mathcal{P}}/fR_{\mathcal{P}})}{\prod_{\mathcal{Q}\in T} \#(R_{\mathcal{Q}}/f^{-1}R_{\mathcal{Q}})}.$$

Lemma B.1. *Let E_β be an irreducible horizontal divisor on X corresponding to the Galois orbit of the point $\beta \in X(\bar{\mathbb{Q}})$. Then*

$$(\operatorname{div} F.E_\beta)_{\mathrm{fin}} = \sum_{v|\infty} \sum_{i=1}^{\deg \beta} \mathrm{N}_v \log |F(\beta_v^{[i]})|_v,$$

where $\beta_v^{[i]}$ are the conjugates of β in $X(\mathbb{C}_v)$, the primes $v \mid \infty$ are the set of infinite places of K, each extended to an infinite place on the field of fractions of R, and N_v is the local degree $[K_v : \mathbb{R}]$.

Proof. The divisor E_β determines a closed immersion $i_\beta : \operatorname{Spec} R \longrightarrow X$ for an order R, so F pulls back to an element $i_\beta^* F$ of the field of fractions of R. By our definition of arithmetic intersection, we have $(\operatorname{div} F.E_\beta)_{\mathrm{fin}} = \log \#(R/i_\beta^* F)$. Since $\#(R/i_\beta^* F) = \mathrm{Norm}_{K/\mathbb{Q}}(i_\beta^* F)$ (see [28, Section III.4], for example), the product formula over $\mathbb{Q}$ and the definition of the norm gives

$$\log \#(R/i_\beta^* F) - \sum_{v|\infty} \sum_{i=1}^{\deg \beta} \mathrm{N}_v \log |F(\beta_v^{[i]})|_v = 0.$$

□

Now, let L be a line bundle on X endowed with a smooth metric $\|\cdot\|_v$ at each infinite place v of K. We will not require the metric on L to have any special properties, since we will not need the sort of adjunction formula that is used, for example, in P. Vojta [30] or S. Lang [17, Chapter IV]. Let s be a section of L such that $\operatorname{div} s$ and $\operatorname{div} F$ have no common horizontal components. Let v be an infinite place of K

and let F_v and s_v denote the pull-backs of F and s to $X_{\mathbb{C}_v}$. We write $\operatorname{div} F_v = \sum m_\alpha \alpha$ and $\operatorname{div} s_v = \sum n_\beta \beta$ where all of the α and β are in $X(\mathbb{C}_v)$.

Proposition B.2. *With notation as above, we have*

$$\frac{1}{2\pi i}\int_{X(\mathbb{C}_v)} \log|F|_v \, d\overline{d}\log\|s\|_v = \sum_\alpha m_\alpha \log\|s(\alpha)\|_v - \sum_\beta \mathrm{N}_v\, n_\beta \log|F(\beta)|_v.$$

Proof. We follow the proof of S. Lang [17, Lemma 2.1.1, pp. 22-23] closely. Since $\log|F|_v$ is harmonic away from the support of $\operatorname{div} F_v$, we have $d\overline{d}\log|F|_v = 0$ away from the points α; thus

$$\log|F|_v d\overline{d}\log\|s\|_v = \log|F|_v^2\, d\overline{d}\log\|s\|_v - \log\|s\|_v d\overline{d}\log|F|_v$$

away from the α and β. Since $d\eta\wedge\overline{d}\gamma = d\gamma\wedge\overline{d}\eta$ for any smooth functions η and γ (see [17, p. 22]), we therefore have

$$\log|F|_v d\overline{d}\log\|s\|_v - \log\|s\|_v d\overline{d}\log|F|_v = d(\log|F|_v\overline{d}\|s\|_v - \log\|s\|_v\overline{d}\log|F|_v)$$

away from the α and β.

Now, let let $Y(a)$ be the complement of the circles $C(\alpha,a)$ and $C(\beta,a)$ of radius a around all of the α and β and let

$$\omega = \log|F|_v\,\overline{d}\log\|s\|_v - \log\|s\|_v\,\overline{d}\log|F|_v.$$

By Stokes theorem, we have

$$\text{(B.2.1)}\qquad \int_{Y(a)} \log|F|_v\, d\overline{d}\|s\|_v = -\left(\sum_\alpha \int_{C(\alpha,a)} \omega + \sum_\beta \int_{C(\beta,a)} \omega\right)$$

(the minus sign here comes from the fact that applying Stokes theorem to the outside of a circle gives a negative orientation). Now, it is easy to see that when a is small we have

$$\int_{C(\alpha,a)} \log|F|_v\overline{d}\log\|s\|_v = O(a\log a),$$

so

$$\lim_{a\to 0}\int_{C(\alpha,a)} \log|F|_v\overline{d}\log\|s\|_v = 0.$$

Similarly, we have

$$\lim_{a\to 0}\int_{C(\beta,a)} \log\|s\|_v\,\overline{d}\log|F|_v = 0.$$

To evaluate $\lim_{a\to 0}\int_{C(\alpha,a)} \log|F|_v \overline{d}\log\|s\|$, we switch to polar coordinates r, θ; we then have

$$\overline{d}\log\|s\|_v = r\frac{\partial \log\|s\|_v}{\partial r}d\theta.$$

Since

$$\frac{\partial \log\|s\|_v}{\partial r} = \frac{\mathrm{N}_v\, n_\beta}{r} + C^\infty\text{-function}$$

for small r, we thus obtain

$$\lim_{a\to 0}\int_{C(\beta,a)} \log|F|_v\, \overline{d}\log\|s\|_v = (2\pi i)\,\mathrm{N}_v\, n_\beta \log|F(\beta)|_v.$$

A similar calculation shows that

$$\lim_{a\to 0}\int_{C(\alpha,a)} \log\|s\|_v \overline{d}\log|F|_v = (2\pi i) m_\alpha \log\|s(\alpha)\|_v.$$

Taking the limit of (B.2.1) as $a \to 0$, thus gives

$$\lim_{a\to 0}\frac{1}{2\pi i}\int_{Y(a)} \log|F|_v\; d\overline{d}\log\|s\|_v$$
$$= -\left(\sum_\beta \mathrm{N}_v\, n_\beta \log|F(\beta)|_v - \sum_\alpha m_\alpha \log\|s(\beta)\|_v\right).$$

□

Now, let us define the Arakelov intersection $(D.L)_{\mathrm{Ar}}$ of a metrized line bundle L with a Weil divisor. We begin by defining the Arakelov degree $\deg_{Ar} M$ of a metrized line bundle M over an order R as in J. Silverman [26]. A metrized line bundle M over an order R is a free R-module of rank one with a nonzero metric $\|\cdot\|_w$ on the completion M_w for each archimedean place $w \in R_\infty$ on the field of fractions of R. As with our metrized line bundles on X, the metrics at infinity are normalized so that they behave like $|\cdot|^{\mathrm{N}_w}$ where $|\cdot|$ is the usual absolute value on $\mathbb{C}$ and N_w is the local degree of R_w over $\mathbb{R}$. Let $m \neq 0$ be an element of M; then

(B.2.2) $$\deg_{\mathrm{Ar}} M = \log\#(M/Rm) - \sum_{w\in R_\infty}\log\|m\|_w.$$

Note that this definition does not depend on our choice of m by the product formula.

If L is a metrized line bundle and E_β is a horizontal divisor on X, then L pulls back to a metrized line bundle $i_\beta^* L$ over the order R where E_β is $i_\beta(\operatorname{Spec} R)$ and we define

$$(E_\beta.L)_{\mathrm{Ar}} = \deg_{\mathrm{Ar}} i_\beta^* L.$$

If s is a section of L such that $\operatorname{Supp} s$ doesn't meet E_β on the generic fiber, this gives

$$(E_\beta.L)_{\mathrm{Ar}} = \log \#(M/R(i_\beta^* s)) - \sum_{v|\infty} \sum_{i=1}^{\deg \beta} \log \|s(\beta_v^{[i]})\|_v,$$

where $\beta_v^{[i]}$ are the conjugates of β in $X(\mathbb{C}_v)$.

If D is a reduced irreducible divisor contained in a finite fiber X_v, then we define $(D.L)_{\mathrm{Ar}} = (D.L)_{\mathrm{fin}} = \deg(L|_D) \log \mathrm{N}(v)$.

Write $\operatorname{div} s = D_{\mathrm{ver}} + D_{\mathrm{hor}}$ where D_{ver} is vertical and D_{hor} is horizontal. For each Galois orbit of points β in $X(\mathbb{C}_v)$ as above, we pick a representative β'. Then we can write $D_{\mathrm{hor}} = \sum n_{\beta'} E_{\beta'}$, where E_β is an irreducible horizontal divisor on X corresponding to the Galois orbit of the point $\beta' \in X(\bar{\mathbb{Q}})$ and $n_{\beta'} = n_\beta$ for each $\beta \in X(\mathbb{C}_v)$ in the orbit of β'. We also pick representatives α' of each Galois orbit of points α and write the horizontal part of $\operatorname{div} F$ as $\sum m_{\alpha'} E_{\alpha'}$ where $E_{\alpha'}$ is the irreducible horizontal divisor corresponding to the Galois orbit of the point $\alpha' \in X(\bar{\mathbb{Q}})$.

We recall our definition of the curvature $d\mu_v$ of a smoothly metrized line bundle L. If s is a section of L, then away from the support of s, we have $d\mu_v = -\frac{1}{2\pi i} d\bar{d} \log \|s\|_v$.

Proposition B.3. *We have the formula*

$$(\operatorname{div} F.L)_{\mathrm{Ar}} = \sum_{v|\infty} \int_{X(\mathbb{C}_v)} \log |F|_v d\mu_v.$$

Proof. We have $(\operatorname{div} F.D_{\mathrm{ver}}) = 0$ by Corollary A.7 (since each component of D_{ver} is projective), so letting $\beta_v^{[i]}$ denote the conjugates of β' in $X(\mathbb{C}_v)$, we have

$$\begin{aligned}(\operatorname{div} F.L)_{\mathrm{Ar}} &= (\operatorname{div} F.E_\beta)_{\mathrm{fin}} - \sum_{v|\infty} \sum_{\alpha'} \sum_{i=1}^{\deg \alpha'} m_{\alpha'} \log \|s(\alpha_v^{[i]})\|_v \\ &= \sum_{v|\infty} \sum_{\beta'} \sum_{i=1}^{\deg \beta'} \mathrm{N}_v\, n_{\beta'} \log |F(\beta_v^{[i]})|_v - \sum_{v|\infty} \sum_{\alpha'} \sum_{i=1}^{\deg \alpha'} m_{\alpha'} \log \|s(\alpha_v^{[i]})\|_v\end{aligned}$$

by Lemma B.1. Applying Proposition B.2 at each $v \mid \infty$ and summing over v gives

$$(\operatorname{div} F.L)_{\mathrm{Ar}} = -\sum_{v\mid\infty} \frac{1}{2\pi i} \int_{X(\mathbb{C}_v)} \log|F|_v \, d\overline{d} \log\|s\|_v$$
$$= \sum_{v\mid\infty} \int_{X(\mathbb{C}_v)} \log|F|_v d\mu_v.$$

□

References

[1] S. Arakelov, *Intersection theory of divisors on an arithmetic surface*, Math. USSR Izvestija **8** (1974), 1167–1180.
[2] P. Autissier, *Points entiers sur les surfaces arithmétiques*, J. reine. angew. Math **531** (2001), 201–235.
[3] M. Baker and R. Rumely, *Equidistribution of small points on curves, rational dynamics, and potential theory*, in preparation.
[4] Y. Bilu, *Limit distribution of small points on algebraic tori*, Duke Math. **89** (1997), 465–476.
[5] J.-B. Bost, H. Gillet, C. Soulé, *Heights of projective varieties and positive Green forms*, J. Amer. Math. Soc. **7** (1994), 903–1027.
[6] E. Brieskorn and H. Knörrer, *Plane algebraic curves* (translated by J. Stillwell), Birkhäuser Verlag, Basel, 1986.
[7] H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965) 103–144.
[8] G. S. Call and J. Silverman, *Canonical heights on varieties with morphism*, Compositio Math. **89** (1993), 163–205.
[9] A. Chambert-Loir, *Points de petite hauteur sur les variété semi abéliennes*, Annales de l'Ecole Normale Supérieure **33-6** (2000), 789–821.
[10] P. Deligne, *Le déterminant de la cohomologie*, Contemporary Mathematics **67** (1987), 94–177.
[11] G. Everest and T. Ward, *Heights of Polynomials and Entropy in Algebraic Dynamics*, Springer Universitext (1999), 1–211.
[12] A. Freire, A. Lopes, R. Mañe, *An invariant measure for rational functions*, Boletim da Sociedade Brasileira de Matematica, **14**,(1983), 45–62.
[13] W. Fulton, *Intersection theory*, Springer Verlag, New York, 1975.
[14] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, Graduate Texts in Mathematics, vol. 52, New York, 1977.
[15] L. C. Hsia, *On the dynamical height zeta functions*, Journal of Number Theory **63** (1997),146–169.
[16] L .C. Hsia, *A weak Néron model with applications to p-adic dynamical systems*, Compositio Mathematica **100** (1996), 277–304.
[17] S. Lang, *Introduction to Arakelov theory*, Springer Verlag (1988).
[18] M. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dyman. Systems **3** (1983), 351–385.
[19] K. Mahler, *An application of Jensen's formula to Polynomials*, Mathematica **7** (1960), 98–100.

[20] V. Maillot, *Géométrie d'Arakelov des Variétés Toriques et fibrés en droites intégrables*, Mémoires de la S.M.F **80** (2000).

[21] C. T. McMullen, *Complex dynamics and renormalisation*, Annals of Mathematics Studies **135** (1994), 1–214.

[22] J. Milnor, *Dynamics in one complex variable. Introductory Lectures*, Vieweg, Braunschweig, 1999.

[23] L. Moret-Bailly, *Pinceaux de variété abéliennes*, Asterisque **129** (1985).

[24] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity.*, Inst. Hautes Études Sci. Publ. Math., No. 9, 1961, 5–22.

[25] C. Peskine and L. Szpiro, *Syzygies et multiplicités*, C. R. Acad. Paris Sci. Sér A **278** (1974), 1421–1424.

[26] J. Silverman, *The theory of height functions*, in *Arithmetic geometry* (edited by G. Cornell and J. Silverman), Springer-Verlag, New York, 1986, pp. 151–166.

[27] L. Szpiro, E. Ullmo, and S. Zhang, *Equirépartition des petits points*, Invent. Math. **127** (1997), 337–347.

[28] L. Szpiro, *Cours de géométrie arithmétique*, Orsay preprint.

[29] L. Szpiro, *Séminaire sur les pinceaux arithmétiques*, Astérisque **127** (1985), 1–287.

[30] P. Vojta, *A generalization of theorems of Faltings and Thue-Siegel-Roth-Wirsing*, J. Amer. Math. Soc. **5** (1992), 763–804.

[31] S. Zhang, *Positive line bundles on arithmetic surfaces*, Annals of Math. **136** (1992), 569–587.

[32] S. Zhang, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc. **8** (1995), 187–221.

[33] S. Zhang, *Small points and adelic metrics*, J. Algebraic Geometry **4** (1995), 281–300.

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