THE COX RING OF A DEL PEZZO SURFACE

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Abstract. Let X_r be a smooth Del Pezzo surface obtained from $\mathbb{P}^2$ by blow-up of $r \leq 8$ points in general position. It is well known that for $r \in \{3,4,5,6,7,8\}$ the Picard group $\mathrm{Pic}(X_r)$ contains a canonical root system $R_r \in \{A_2 \times A_1, A_4, D_5, E_6, E_7, E_8\}$. In this paper, we prove some general properties of the Cox ring of X_r ($r \geq 4$) and show its similarity to the homogeneous coordinate ring of the orbit of the highest weight vector in some irreducible representation of the algebraic group G associated with the root system R_r.

1. Introduction

Let X be a projective algebraic variety over a field $\Bbbk$. Assume that the Picard group $\mathrm{Pic}(X)$ is a finitely generated abelian group. Consider the vector space

$$\Gamma(X) := \bigoplus_{[D] \in \mathrm{Pic}(X)} H^0(X, \mathcal{O}(D)).$$

One wants to make it an $\Bbbk$-algebra which is graded by the monoid of effective classes in $\mathrm{Pic}(X)$ such that the algebra structure will be compatible with the natural bilinear map

$$b_{D_1,D_2} \; : \; H^0(X, \mathcal{O}(D_1)) \times H^0(X, \mathcal{O}(D_2)) \to H^0(X, \mathcal{O}(D_1 + D_2)).$$

However, there exist some problems in the realization of this idea. We remark that first of all there is no any natural isomorphism between $H^0(X, \mathcal{O}(D))$ and $H^0(X, \mathcal{O}(D'))$ if $[D] = [D']$. There exists only a canonical bijection between the linear systems $|D| \cong |D'|$ ($|D|$ denotes a projectivization of of the $\Bbbk$-vector space $H^0(X, \mathcal{O}(D))$). As a consequence, the bilinear map b_{D_1,D_2} depends not only on the classes $[D_1], [D_2], [D_1 + D_2] \in \mathrm{Pic}(X)$, but also on their particular representatives. One can easily see that only the morphism

$$s_{[D_1],[D_2]} \; : \; |D_1| \times |D_2| \to |D_1 + D_2|$$

of the product of two projective spaces $|D_1| \times |D_2|$ to another projective space $|D_1 + D_2|$ is well-defined. For this reason, it is much more natural to consider the graded set of projective spaces

$$\mathbb{P}(X) := \bigsqcup_{[D] \in \mathrm{Pic}(X)} |D|$$

together with the all possible morphisms $s_{[D_1],[D_2]}$ ($[D_1], [D_2] \in \mathrm{Pic}(X)$).

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Inspired by the paper of Cox on the homogeneous ring of a toric variety [Cox], Hu and Keel [H-K] suggested a definition of a **Cox ring**

$$Cox(X) = R(X, L_1, \dots, L_r) := \bigoplus_{(m_1,\dots,m_r)\in\mathbb{Z}^r} H^0(X, \mathcal{O}(m_1 L_1 + \cdots + m_r L_r)$$

which uses a choice of some $\mathbb{Z}$-basis $L_1, \dots, L_r$ in $\mathrm{Pic}(X)$ (e.g. if $\mathrm{Pic}(X) \cong \mathbb{Z}^r$ is a free abelian group). Using such a $\mathbb{Z}$-basis, one obtains a particular representative for each class in $\mathrm{Pic}(X)$ together with a well-defined multiplication so that $R(X, L_1, \dots, L_r)$ becomes a well-defined $\Bbbk$-algebra. If $L_1', \dots, L_r'$ is another $\mathbb{Z}$-basis of $\mathrm{Pic}(X)$, then the corresponding Cox algebra $R(X, L_1', \dots, L_r')$ is isomorphic to $R(X, L_1, \dots, L_r)$. Unfortunately, we can not expect to choose a $\mathbb{Z}$-basis of $\mathrm{Pic}(X)$ in a natural canonical way. More often one can choose in a natural way some effective divisors $D_1, \dots, D_n$ on X such that $\mathrm{Pic}(X)$ is generated by $[D_1], \dots, [D_n]$. If we set

$$U := X \setminus (D_1 \cup \cdots \cup D_n)$$

and assume that X is smooth, then $Pic(U) = 0$ and we obtain the exact sequence

$$1 \to \Bbbk^* \to \Bbbk[U]^* \to \bigoplus_{i=1}^{n} \mathbb{Z}[D_i] \to \mathrm{Pic}(X) \to 0.$$

Choosing a $\Bbbk$-rational point p in U, we can split the monomorphism $\Bbbk^* \to \Bbbk[U]^*$, so that one has an isomorphism

$$\Bbbk[U]^* \cong \Bbbk^* \oplus G,$$

where $G \subset \Bbbk[U]^*$ is a free abelian group of rank $n - r$. The choice of a $\Bbbk$-rational point $p \in U$ allows to give another approach to the graded space $\Gamma(X)$ and to the Cox algebra:

Definition 1.1. Let $X, U, p, D_1, \dots, D_n$ be as above. We consider the graded $\Bbbk$-algebra

$$\Gamma(X, U, p) := \bigoplus_{(m_1,\dots,m_n)\in\mathbb{Z}^n} H^0(X, \mathcal{O}(m_1 D_1 + \cdots + m_n D_n)$$

and define

$$Cox(X, U, p) := \Gamma(X, U, p)^G$$

as the $\Bbbk$-subalgebra of all G-invariant elements in $\Gamma(X, U, p)$.

Since $Pic(X) \cong \mathbb{Z}^n/G$, we obtain a natural $\mathrm{Pic}(X)$-grading on $Cox(X, U, p)$. We expect that the algebra $Cox(X, U, p)$ can de applied to some arithmetic questions about $\Bbbk$-rational points in $U \subset X$ (e.g. see [S]).

Remark 1.2. If X is a smooth projective toric variety and $U \subset X$ is the open dense torus orbit, then the choice of a point $p \in U$ defines an isomorphism of U with the algebraic torus T, so that the subgroup $G \subset \Bbbk[U]^*$ can be identified with the character group of T. In this way, one can show that $Cox(X, U, p)$ is isomorphic to a polynomial ring in n variables (n is the number of irreducible components of $X \setminus U$, cf. [Cox]).

Let X_r be a smooth Del Pezzo surface obtained from $\mathbb{P}^2$ by blow-up of $r \leq 8$ points in general position. It is well known that for $r \in \{3, 4, 5, 6, 7, 8\}$ the Picard group $\mathrm{Pic}(X_r)$ contains a canonical root system $R_r \in \{A_2 \times A_1, A_4, D_5, E_6, E_7, E_8\}$. Moreover, the natural embedding $\mathrm{Pic}(X_{r-1}) \hookrightarrow \mathrm{Pic}(X_r)$ induces induces the inclusion of root systems $R_{r-1} \hookrightarrow R_r$. If $G(R_r)$ is a connected algebraic group corresponding to the root system R_r, then the embedding $R_{r-1} \hookrightarrow R_r$ defines a maximal parabolic subgroup $P(R_{r-1}) \subset G(R_r)$. We expect that for $r \geq 4$ there should be some relation between a Del Pezzo surface X_r and the GIT-quotient of the homogeneous space $G(R_r)/P(R_{r-1})$ modulo the action of a maximal torus T_r of $G(R_r)$.

Our starting observation is the well-known isomorphism $X_4 \cong Gr(2,5)//T_4$ which follows from an isomorphism between the homogeneous coordinate ring of the Grassmaniann $Gr(3,5) = G(A_4)/P(A_2 \times A_1) \subset \mathbb{P}^9$ and the Cox ring of X_4. Another proof of this fact follows form the identification of X_4 with the moduli space $\overline{M_{0,5}}$ of stable rational curves with 5 marked points [K].

In this paper, we start an investigation of the Cox ring of Del Pezzo surfaces X_r ($r \geq 4$). It is natural to choose the classes of all exceptional curves $E_1, \dots, E_{N_r} \subset X_r$ as a generating set for the Picard group $\mathrm{Pic}(X_r)$. There is a natural $\mathbb{Z}_{\geq 0}$-grading on $\mathrm{Pic}(X_r)$ defined by the intersection with the anticanonical divisor $-K$.

We prove some general properties of Cox rings of Del Pezzo surfaces X_r ($r \geq 4$) and show their similarity to the homogeneous coordinate ring of $G(R_r)/P(R_{r-1})$. We remark that the homogeneous space $G(R_r)/P(R_{r-1})$ can be interpreted as the orbit of the highest weight vector in some natural irreducible representation of $G(R_r)$.

Remark 1.3. Some other connections between Del Pezzo surfaces and the corresponding algebraic groups were considered also by Friedman and Morgan in [F-M]. A similar topic was considered by Leung in [Le].

In this paper, we show that the Cox ring of a Del Pezzo surface X_r is generated by elements of degree 1. This implies that the homogeneous coordinate ring of $G(R_r)/P(R_{r-1})$ is naturally graded by the monoid of effective divisor classes on the surface X_r (the same monoid defines the multigrading of the Cox ring of X_r). Moreover, we obtains some results of the quadratic relations between the generators of the Cox ring of X_r.

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2. Del Pezzo Surfaces

Let us summarize briefly some well-known classical results on Del Pezzo surfaces which can be found in [Ma, Dem, Na].

One says that r ($r \leq 8$) points $p_1, \dots, p_r$ in $\mathbb{P}^2$ are in *general position* if there are no 3 points on a line, no 6 points on a conic ($r \geq 6$) and a cubic having seven points and one of them double does not have the eighth one ($r = 8$).

Denote by X_r ($r \geq 3$) the Del Pezzo surfaces obtained from $\mathbb{P}^2$ by blowing up of r points $p_1, \dots, p_r$ in general position. If $\pi : X_r \to \mathbb{P}^2$ the corresponding projective morphism, then the Picard group $Pic(X_r) \cong \mathbb{Z}^{r+1}$ contains a $\mathbb{Z}$-basis $l_i, (0 \leq i \leq r)$, $l_0 = [\pi^*\mathcal{O}(1)]$ and $l_i := [\pi^{-1}(p_i)]$, $i = 1, \dots, r$. The intersection form $(*,*)$ on $Pic(X_r)$ is determined in the chosen basis by the diagonal matrix: $(l_0, l_0) = 1, (l_i, l_i) = -1$ for $i \geq 1$, $(l_i, l_j) = 0$ for $i \neq j$,. The anticanonical class of X_r equals $-K = 3l_0 - l_1 - \cdots - l_r$. The number

$d := (K, K) = 9 - r$ is called the *degree* of X_r. The anticanonical system $| - K|$ of a Del Pezzo surface X_r is very ample if $r \leq 6$, it determines a two-fold covering of $\mathbb{P}^2$ if $r = 7$, and it has one base point, determining a rational map to $\mathbb{P}^1$ if $r = 8$. Smooth rational curves $E \subset X_r$ such that $(E, E) = -1$ and $(E, -K) = 1$ are called *exceptional curves*.

Theorem 2.1. [Ma] *The exceptional curves on X_r are the following:*

(1) *blown-up points $p_1, \dots, p_r$;*
(2) *lines through pairs of points p_i, p_j;*
(3) *conics through 5 points from $\{p_1, \dots, p_r\}$ $(r \geq 5)$;*
(4) *cubics, containing 7 points and 1 of them double $(r \geq 7)$;*
(5) *quartics, containing 8 points and 3 of them double $(r = 8)$;*
(6) *quintics, containing 8 of point and 6 of them double $(r = 8)$;*
(7) *sextics, containing 8 of those points, 7 of them double and 1 triple $(r = 8)$.*

The number N_r of exceptional curves on X_r is given by the following table:

r	3	4	5	6	7	8
N_r	6	10	16	27	56	240

The root system $R_r \subset \mathrm{Pic}(X_r)$ is defined as

$$R_r := \{\alpha \in \mathrm{Pic}(X_r) \ : \ (\alpha, \alpha) = -2, \ (\alpha, -K) = 0\}.$$

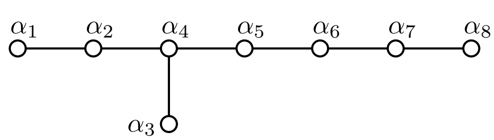
It is easy to show that R_r is exactly the set of all classes $\alpha = [E_i] - [E_j]$ where E_i and E_j are two exceptional curves on X_r such that $E_i \cap E_j = \emptyset$.

The corresponding Weyl group W_r is generated by the reflections $\sigma \ : \ x \mapsto x + (x, \alpha)\alpha$ for $\alpha \in R_r$. There are so called *simple roots* $\alpha_1, \dots, \alpha_r$ such that the corresponding reflexions $\sigma_1, \dots, \sigma_r$ form a minimal generating subset of W_r. The set of simple roots can be chosen as

$$\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = l_0 - l_1 - l_2 - l_3,$$

$$\alpha_i = l_{i-1} - l_i, \ \ i \geq 4.$$

The blow up morphism $X_r \to X_{r-1}$ determines an isometric embedding of the Picard lattices $\mathrm{Pic}(X_{r-1}) \hookrightarrow \mathrm{Pic}(X_r)$. This induces the embeddings for root systems, simple roots and Weyl groups W_r. For $r \geq 3$, the Dynkin diagram of R_r can be considered as the subgraph on the vertices α_i $(i \leq r)$ of the following graph:



In particular, we obtain $R_3 = A_2 \times A_1, R_4 = A_4, R_5 = D_5, R_6 = E_6, R_7 = E_7, R_8 = E_8$.

Denote by $\varpi_1, \dots, \varpi_r$ the dual basis to the $\mathbb{Z}$-basis $-\alpha_1, \dots, -\alpha_r$. Each ϖ_i is the highest weight of an irreducible representation of $G(R_r)$ which is called a *fundamental representation*. We will denote by $V(\varpi)$ the representation space of $G(R_r)$ with the highest weight ϖ.

Definition 2.2. A dominant weight ϖ is called *minuscule* if all weights of $V(\varpi)$ are nonzero and the W_r-orbit of the highest weight vector is a $\Bbbk$-basis of $V(\varpi)$ [G/P-I]). A dominant weight ϖ is called *quasiminuscule* [G/P-III], if all nonzero weights of $V(\varpi)$ have multiplicity 1 and form an W_r-orbit of ϖ (the zero weight of $V(\varpi)$ may have some positive multiplicity).

One can see from the explicit description of the root systems R_r that ϖ_r is minuscule for $3 \le r \le 7$, and ϖ_8 is quasiminuscule.

The dimension d_r of of the irreducible representation $V(\varpi_r)$ of $G(R_r)$ is given by the following table:

r	4	5	6	7	8
d_r	10	16	27	56	248

We will need the following statement:

Proposition 2.3. *Let D be a divisor on a Del Pezzo surface X_r $(2 \le r \le 8)$ such that $(D,E) \ge 0$ for every exceptional curve $E \subset X_r$. Then the following statements hold:*
(i) the linear system $|D|$ has no base points on any exceptional curve $E \subset X_r$;
(ii) if $r \le 7$, then the linear system $|D|$ has no base points on X_r at all.

Proof. Induction on r. If $r = 2$, then there exists exactly 3 exceptional curves E_0, E_1, E_2, whose classes in the standard basis are $l_0 - l_1 - l_2, l_1, l_2$. Moreover $[E_0]$, $[E_1]$ and $[E_3]$ form a basis of the Picard lattice $Pic(X_2)$. The dual basis w.r.t. the intersection form is $l_0, l_0 - l_1, l_0 - l_2$. Therefore the conditions on D imply that

$$[D] = n_0 l_0 + n_1(l_0 - l_1) + n_2(l_0 - l_2),\ n_0, n_1, n_2 \in \mathbb{Z}_{\ge 0}$$

So it is sufficient to check that the linear systems with the classes $l_0, l_0 - l_1, l_0 - l_2$ have no base points. The latter immediately follows from the fact that the first system defines the birational morphism $X_2 \to \mathbb{P}^2$ contracting E_1 and E_2, the second and third linear systems define conic bundle fibrations over $\mathbb{P}^1$.

For $r > 2$, we consider a second induction on $\deg D = (D, -K)$.

If there is an exceptional curve $E \subset X_r$ with $(D,E) = 0$, then the invetible sheaf $\mathcal{O}(D)$ is the inverse image of an invertible sheaf $\mathcal{O}(D')$ on the Del Pezzo surface X_{r-1} obtained by the contraction of E. Since the pull back of any exceptional curve on X_{r-1} under the birational morphism $\pi_E : X_r \to X_{r-1}$ is again an exceptional curve on X_r, we obtain that D' satisfy all conditions of the proposition on X_{r-1}. By the induction assumption $(r-1 \le 7)$, $|D'|$ has no base points on X_{r-1}. Therefore, $|D| = |\pi_E^* D'|$ has no base points on X_r.

If there is no exceptional curve $E \subset X_r$ with $(D,E) = 0$, then we denote by m the minimal intersection number (D,E) where E runs over all exceptional curves. Since we have $(E, -K) = 1$ for all exceptional curves, the divisor $D' := D + mK$ has nonnegative intersections with all exceptional curves and there exists an exceptional curve $E \subset X_r$ with $(D', E) = 0$. Since $\deg D' = (D', -K) = (D, -K) - m(K,K) < (D, -K) = \deg D$, by the induction assumption, we obtain that $|D'|$ is base point free. If $r \le 7$, then the anticanonical linear system $|-K|$ has no base points. Therefore, $|D| = |D' + m(-K)|$ is also base point free. In the case $r = 8$, $|-K|$ does have a base point $p \in X_8$. However, p cannot lie on an exceptional curve E, because the short exact sequence

$$0 \to H^0(X_8, \mathcal{O}(-K-E)) \to H^0(X_8, \mathcal{O}(-K)) \to H^0(E, \mathcal{O}(-K)|_E) \to 0$$

induces an isomorphism $H^0(X_8, \mathcal{O}(-K)) \cong H^0(E, \mathcal{O}(-K)|_E)$ (since $\deg(-K-E) = 0$ and $H^0(X_8, \mathcal{O}(-K-E)) = 0$). □

3. Generators of $Cox(X_r)$

Let $\{E_1, \ldots, E_{N_r}\}$ be the set of all exceptional curves on a Del Pezzo surface X_r. We choose a $\Bbbk$-rational point $p \in U := X_r \setminus (\bigcup_{i=1}^{N_r} E_i)$ and denote the ring $Cox(X_r, U, p)$ (see 1.1) simply by $Cox(X_r)$.

The ring $Cox(X_r)$ is graded by the semigroup $M_{\text{eff}}(X_r) \subset \text{Pic}(X_r)$ of classes of effective divisors on X_r. There is a coarser grading on $Cox(X_r)$ given by

$$Cox(X_r)^d \cong \bigoplus_{(D,-K)=d} H^0(X_r, \mathcal{O}(D)),$$

with respect to which we shall speak about the degree $d = (D, -K)$ of a divisor D.

Proposition 3.1. *The ring $Cox(X_3)$ is isomorphic to a polynomial ring in 6 variables $\Bbbk[x_1, \ldots, x_6]$, where x_i are sections defining all 6 exceptional curves on X_3.*

Proof. The Del Pezzo surface X_3 is a toric variety which can be descrobed as the blown-up of 3 torus invariant points (1:0:0), (0:1:0) and (0:0:1) on $\mathbb{P}^2$. So we can apply a general result of Cox on toric varieties [Cox] (see also 1.2). □

Theorem 3.2. *For $3 \leq r \leq 8$, the ring $Cox(X_d)$ is generated by elements of degree 1. If $r \leq 7$, then the generators of $Cox(X_d)$ are global sections of invertible sheaves defining the exceptional curves. If $r = 8$, then we should add to the above set of generators two linearly independent global sections of the anticanonical sheaf on X_8.*

Proof. Induction on r. The case $r = 3$ is settled by the previous proposition.

For $r > 3$ we choose an effective divisor D on X_r. We call a section $s \in H^0(X_r, \mathcal{O}(D))$ a **distinguished global section** if its support is contained in the union of exceptional curves of X_r ($r \leq 7$), or if its support is contained in the union of exceptional curves of X_8 and some anticanonical curves on X_8. Our purpose is to show that the vector space $H^0(X_r, \mathcal{O}(D))$ is spanned by all distinguished global sections.

This will be proved by induction on $\deg D := (D, -K) > 0$.

We consider several cases:

- If there exists an exceptional curve E such that $(D, E) < 0$, then $H^0(X_r, \mathcal{O}(D)|_E) = 0$ and it follows from the exact sequence

  $$H^1(X_r, \mathcal{O}(D)|_E) \to H^0(X_r, \mathcal{O}(D-E)) \to H^0(X_r, \mathcal{O}(D)) \to 0$$

  that the multiplication by a non-zero distinguished global section of $\mathcal{O}(E)$ induces an epimorphism $H^0(X_r, \mathcal{O}(D-E)) \to H^0(X_r, \mathcal{O}(D))$. Since $\deg(D-E) = \deg D - 1$, using the induction assumption for $D' = D - E$, we obtain the required statement for D.

- If there exists an exceptional curve E such that $(D, E) = 0$, then $\mathcal{O}(D)$ is the inverse image of a sheaf $\mathcal{O}(D')$ on the Del Pezzo surface X_{r-1} obtained by the contraction of E. Therefore we have an isomorphism $H^0(X_r, \mathcal{O}(D)) \cong H^0(X_{r-1}, \mathcal{O}(D'))$ and, by the induction assumption for $r-1$, we obtain the required statement for D, because distinguished global sections of $\mathcal{O}(D')$ lift to distinguished global sections of $\mathcal{O}(D)$.
- If $D = -K$, (or, equivalently, if $(D, E) = 1$ for every exceptional curve E), then $\mathcal{O}(D)|_E$ is isomorphic to $\mathcal{O}_E(1)$ and we have $H^1(X_r, \mathcal{O}(D)|_E) = 0$ together with the exact sequence

$$0 \to H^0(X_r, \mathcal{O}(D-E)) \to H^0(X_r, \mathcal{O}(D)) \to H^0(X_r, \mathcal{O}(D)|_E) \to 0,$$

where $H^0(X_r, \mathcal{O}(D)|_E)$ is 2-dimensional. Since one has $\deg(D-E) = \deg D - 1$. Using the induction assumption for $D' = D - E$, it remains show that there exists two linearly independent distinguished global sections of $\mathcal{O}(D)$ such that their restriction to E are two linearly independent global sections of $\mathcal{O}(D)|_E$. We describe these two distinguisched sections explicitly for each value of $r \in \{4, 5, 6, 7, 8\}$. Without loss of generality we can assume that $[E] = l_1$.

If $r = 4$, then we write the anticanonical class $-K = 3l_0 - l_1 - \cdots - l_4$ in the following two ways:

$$\begin{aligned} -K &= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_2 - l_3) + l_2 + l_3 \\ &= (l_0 - l_1 - l_3) + (l_0 - l_2 - l_4) + (l_0 - l_2 - l_3) + l_2 + l_3. \end{aligned}$$

These two decompositions of $-K$ determine two distinguished global sections of $\mathcal{O}(-K)$ with support on 5 exceptional curves. The projections of these sections under the morphism $X_4 \to \mathbb{P}^2$ are shown below in Figure 1.

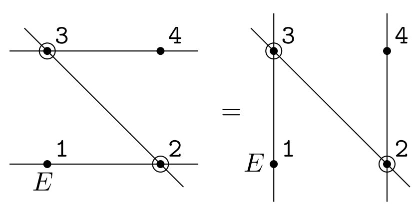


FIGURE 1. Two distinguished anticanonical classes for $r = 4$.

The restriction of the first section to E vanishes at the intersection point q_1 of E with the exceptional curve with the class $l_0 - l_1 - l_2$. The restriction of the second section to E vanishes at the intersection point q_2 of E with the exceptional curve with the class $l_0 - l_1 - l_3$. It is clear that $q_1 \neq q_2$. So the distinguished anticanonical sections are linearly independent.

If $r = 5$, then we write the anticanonical class as

$$\begin{aligned} -K &= 3l_0 - l_1 - \cdots - l_5 \\ &= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_4 - l_5) + l_4 \\ &= (l_0 - l_1 - l_5) + (l_0 - l_2 - l_3) + (l_0 - l_3 - l_4) + l_3. \end{aligned}$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of E with the exceptional curves belonging to the classes $l_0 - l_1 - l_2$ and $l_0 - l_1 - l_5$.

If $r = 6$, then we write the anticanonical class as

$$\begin{aligned} -K &= 3l_0 - l_1 - \cdots - l_6 \\ &= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_5 - l_6) \\ &= (l_0 - l_1 - l_6) + (l_0 - l_5 - l_4) + (l_0 - l_3 - l_2). \end{aligned}$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of E with the exceptional curves belonging to the classes $l_0 - l_1 - l_2$ and $l_0 - l_1 - l_6$.

If $r = 7$, then we write the anticanonical class as

$$\begin{aligned} -K &= 3l_0 - l_1 - \cdots - l_7 \\ &= (2l_0 - l_1 - l_2 - l_3 - l_4 - l_5) + (l_0 - l_6 - l_7) \\ &= (2l_0 - l_7 - l_6 - l_5 - l_4 - l_3) + (l_0 - l_2 - l_1). \end{aligned}$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of E with the exceptional curves belonging to the classes $2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$ and $l_0 - l_2 - l_1$.

If $r = 8$, then $\deg D - E = 0$. Therefore, $H^0(X_8, \mathcal{O}(D - E)) = 0$ (see the proof of 2.3) and we have an isomorphism

$$H^0(X_8, \mathcal{O}(D)) \cong H^0(X_8, \mathcal{O}(D)|_E).$$

So $H^0(X_8, \mathcal{O}(D)|_E)$ is generated by the restrictions of the anticanonical sections and we're done.

- If $(D, E) \geq 1$ for all exceptional curves E and $D \neq -K$, then we denote by m the minimum of the numbers (D, E) for all exceptional curves. Let E_0 be an exceptional curve such that $(D, E_0) = m \geq 1$. We define $D' = D - E_0$ and $D'' := D + mK$. By 2.3, $|D'|$ and $|D''|$ have no base points (if $r \leq 7$). Moreover, D'' can be seen as zero of a distinguished global section $s \in H^0(X_r, \mathcal{O}(D + mK))$ whose support does not contain the exceptional curve E_0 (if $r \leq 8$). We have the short exact sequence

$$0 \to H^0(X_r, \mathcal{O}(D')) \to H^0(X_r, \mathcal{O}(D)) \to H^0(X_r, \mathcal{O}(D)|_{E_0}) \to 0.$$

By the induction assumption, the space $H^0(X_r, \mathcal{O}(D'))$ is generated by distinguished global sections. It remains to show that there exist distinguished global sections of $\mathcal{O}(D)$ such that their restriction to E_0 generate the space $H^0(X_r, \mathcal{O}(D)|_{E_0})$. Since $(-mK, E_0) = (D, E_0) = m$, the space $H^0(X_r, \mathcal{O}(D)|_{E_0})$ is isomorphic to $H^0(X_r, \mathcal{O}(-mK)|_{E_0})$. Since $(D'', E_0) = 0$ the distinguished global section $s \in H^0(X_r, \mathcal{O}(D + mK))$ nowhere vanish on E_0. Therefore the multiplication by the distinguished global section s defines a homomorphism

$$H^0(X_r, \mathcal{O}(-mK)) \to H^0(X_r, \mathcal{O}(D))$$

whose restriction to E_0 is an isomorphism

$$H^0(X_r, \mathcal{O}(-mK)|_{E_0}) \cong H^0(X_r, \mathcal{O}(D)|_{E_0}.$$

Therefore, it is enough to show that restrictions of the distinguished global sections of $\mathcal{O}(-mK)$ to E_0 generate the space $H^0(X_r, \mathcal{O}(-mK)|_{E_0})$. Our previous considerations have shown this for $m = 1$. The general case $m \geq 1$ follows now immediately from the fact that the homomorphism $H^0(X_r, \mathcal{O}(-K)) \to H^0(E_0, \mathcal{O}_{E_0}(1))$ is surjective and the space $H^0(E_0, \mathcal{O}_{E_0}(m))$ is spanned by tensor products of m elements from $H^0(E_0, \mathcal{O}_{E_0}(1))$. □

Corollary 3.3. *The semigroup $M_{\text{eff}}(X_r) \subset \text{Pic}(X_r)$ of classes of effective divisors on a Del Pezzo surfaces X_r $(2 \leq r \leq X_r)$ is generated by elements of degree 1. These elements are exactly the classes of exceptional curves if $r \leq 7$ and the classes of exceptional curves together with the anticanoncal class for $r = 8$.*

Proposition 3.4. *If D is an effective divisor of degree ≥ 2 on X_8, then the vector space $H^0(X_8, \mathcal{O}(D))$ is spanned by distinguished global sections of $\mathcal{O}(D)$ with supports only on exceptional curves.*

Proof. By 3.2 and 3.3, it is sufficient to check the statement for $D = -2K$ and for $D = -K + E$ for any exceptional curve. The latter case immediately follows from 3.2, because $D = -K + E$ is the pull back of the anticanonical sheaf on X_7 obtained by the contraction of E. In the case $D = -2K$, we obtain 120 distinguished global sections of $\mathcal{O}(D)$ from 120 pairs of exceptional curves E_i, E_j such that $(E_i, E_j) = 3$:

$$-2K = 6l_0 - 2l_1 - \ldots - 2l_8 = l_1 + (6l_0 - 3l_1 - 2l_2 \ldots - 2l_8).$$

□

Remark 3.5. Since $H^0(X_r, \mathcal{O}(E))$ is 1-dimensional for each exceptional curve $E \subset X_r$, we can choose a nonzero section $x_E \in H^0(X_r, \mathcal{O}(E))$ which is determined up to multiplication by a nonzero scalar. Therefore the affine algebraic variety $\mathbb{A}(X_r) := \text{Spec}Cox(X_r)$ is embedded into the affine space $\mathbb{A}^{N_r}$ on which the maximal torus $T_r \subset G(R_r)$ acts in a canonical way such that the space $\mathbb{A}^{N_r}$ can identified with the representation space $V(\varpi_r)$ of the algebraic group $G(R_r)$ (if $r \leq 7$).

In the case $r = 8$, all 240 exceptional curves on X_8 can be similarly identified with all non-zero weights of the adjoint representation of $G(E_8)$ in $V(\varpi_8)$. The space $V(\varpi_8)$ contains the weight-0 subspace of dimension 8, but the ring $Cox(X_r)$ has only 2-dimensional space of anticanonical sections. Thus, there is no any identification of degree-1 homogeneous component of $Cox(X_8)$ with the representation space $V(\varpi_8)$ of $G(E_8)$.

4. Quadratic relations in $Cox(X_r)$

Let us denote $P(X_r) := \text{Proj } Cox(X_r)$. If $4 \leq r \leq 7$, then $P(X_r)$ is canonically embedded into the projective space $\mathbb{P}^{N_r - 1}$ (N_r is the number of exceptional curves on X_r). For any exceptional curve $E \subset X$, we consider the open chart $U_E \subset \mathbb{P}^{N_r-1}$ defined by the condition $x_E \neq 0$. Thus, we obtain an open covering of $P(X_r)$ by N_r affine subsets $U_E \cap P(X_r)$. We denote by $A(X_r) \subset \mathbb{A}^{N_r}$ the affine cone over $P(X_r)$.

Proposition 4.1. *The ring $Cox(X_4)$ is isomorphic to the subring of all 3×3-minors of a generic 3×5-matrix. In particular, the projective variety $P(X_4)$ is isomorphic to the Grassmannian $Gr(3,5)$.*

Proof. To describe the multiplication in $R(X_5)$, one needs to choose a basis in $\mathrm{Pic}(X_5)$. We choose the basis $l_0, \dots, l_4$, as in Section 2. We choose a divisor in each basis class: l_0 being the preimage of the line at infinity w.r.t. a blow-down on $\mathbb{P}^2$, l_i the exceptional fibers of this blow-down. We identify the representatives of each Picard class with the subsheaves $\mathcal{O}(\sum c_i l_i)$ of the constant sheaf $\Bbbk(X_5)$. Then the multiplication in the ring is just the multiplication of the corresponding functions in the function field $\Bbbk(X_5)$ of X_5.

We choose the functions that represent x_E's in the following way: let $x : y : z$ be the homogeneous coordinates on $\mathbb{P}^2$ and let $(x_i : y_i : z_i)$, $i = 1, \dots, 4$, be the coordinates of the blown-up points. Consider the matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x/z \\ y_1 & y_2 & y_3 & y_4 & y/z \\ z_1 & z_2 & z_3 & z_4 & 1 \end{pmatrix}.$$

Let M_I for $I \subset [1,5], |I| = 3$, denote the maximal minor of M consisting of the columns with numbers in I, taken in the natural order. Then we set $x_{l_i} := M_{[1,4] \setminus \{i\}}$ for $i \in [1,4]$, $x_{l_0 - l_i - l_j} := M_{\{i,j,5\}}$ for $1 \leq i < j \leq 4$. All these functions lie in the corresponding $\mathcal{O}(D)$'s and are non-zero because the points are in general position.

It is known that the generators of the homogeneous coordinate ring of $G(3,5)$ are naturally identified with the maximal minors of a generic 3×5-matrix. We send these generic minors into the corresponding minors of the matrix above. This determines a homomorphism of the homogeneous coordinate ring of $G(3,5)$ to $Cox(X_5)$, as the equations of the Grassmann variety, being the relations between generic minors, hold for any particular minors. As $R(X_5)$ is generated by x_E's, it is surjective.

This homomorphism respects the Picard grading. Therefore this homomorphism respects the $\mathbb{Z}$-grading. As it is surjective, it induces a closed embedding of $P(X_5)$ into $G(3,5)$. As both varieties are irreducible of dimension 6 (because), it is an isomorphism of varieties, therefore they coincide as subvarieties in $\mathbb{P}^9$, therefore we have an isomorphism of rings. And we see that $Cox(X_5)$ is defined by quadratic relations, as the homogeneous coordinate ring of $G(3,5)$ is.

□

The article [G/P-I] describes a $\Bbbk$-basis for the homogeneous coordinate ring of G/P in the case, when P is a maximal parabolic subgroup containing a Borel subgroup B such that the fundamental weight ϖ corresponding to P is minuscule (see 2.2). It also shows that this ring is defined by quadratic relations.

A way to write explicitly the quadratic relations for the orbit of the highest weight vector for any representation of a semisimple Lie group is given in [Li]. A more geometric approach to these quadratic equations is contained in the proof of Theorem 1.1 in [L-T]:

Proposition 4.2. *The orbit G/P_ϖ of the highest weight vector in the projective space $\mathbb{P}V(\varpi)$ is the intersection of the second Veronese embedding of $\mathbb{P}V(\varpi)$ with the subrepresentation $V(2\varpi)$ of the symmetric square $S^2V(\varpi)$. Moreover, these quadratic relations generate the ideal of $G/P_\varpi \subset \mathbb{P}V(\varpi)$.*

We expect that the following statement is true:

Conjecture 4.3. *The ideal of relations between the degree-*1 *generators of* $Cox(X_r)$ *is generated by quadrics for all* $4 \leq r \leq 8$.

Proposition 4.4. *Let* X_{r-1} *the Del Pezzo surface obtained by the contraction of* E *on* X_r. *Then there exist an isomorphism*

$$U_E \cap P(X_r) \cong A(X_{r-1}).$$

Proof. The coordinate ring of the affine variety $U_E \cap P(X_r)$ consists of all fractions $f/x_E^{\deg D}$ such that $f \in H^0(X_r, \mathcal{O}(D))$. Let $\pi_E : X_r \to X_{r-1}$ be the contraction of E. Then any divisor class $[D] \in \mathrm{Pic}(X_r)$ can be uniquely represented as sum $[D'] + k[E]$ where $[D'] = \pi_E^*(\mathrm{Pic}(X_{r-1}))$.

Such a fraction is a meromorphic section over $D - (\deg D)E$ with possible poles at E, and two such fractions are equal exactly when they determine the same section (x_E is evidently a nonzerodivisor from the definition of the multiplication), so the affine coordinate ring is

$$\bigoplus_{\substack{D \in \mathrm{Pic}(X_r) \\ \deg D = 0}} H^0(X_r \setminus E, \mathcal{O}(D)) = \bigoplus_{D \in \mathrm{Pic}(X_r \setminus E)} H^0(X_{r-1} \setminus \pi_E(E), \pi_* \mathcal{O}(D)),$$

where $\pi : X_r \to X_{r-1}$ is the blow-down of E, because the condition $\deg D = 0$ determines a unique extension of each divisor from $X_r \setminus E$ to X_r and the blow-down is an isomorphism outside E. Now, in the last sum one needn't exclude $\pi_E(E)$ because it is a point on a normal surface X_{r-1}, so the last sum is just our ring $Cox(X_{r-1})$ for a Del Pezzo surface X_{r-1} with r one smaller, i.e. the affine coordinate ring of $A(X_{r-1})$. □

Corollary 4.5. *The singular locus of the algebraic varieties* $P(X_r)$ *and* $A(X_r)$ *has codimension* 7.

Proof. Since $A(X_3) \cong \mathbb{A}^6$, we obtain that $P(X_4)$ is a smooth variety covered by 10 affine charts which are isomorphic to $\mathbb{A}^6$. Using the isomorphism $P(X_4) \cong Gr(3,5)$ (see 4.1), we obtain that $A(X_4)$ has an isolated singularity at 0. Therefore, the singular locus of $P(X_5)$ consists of 16 isolated points. The singular locus of $P(X_6)$ is 1-dimensional and the singular locus of $P(X_7)$ is 2-dimensional. □

Definition 4.6. A divisor class D of a sum of two exceptional curves intersecting with multiplicity 1, or, equivalently, satisfying $(D,D) = 0,\ (D,-K) = 2$, is called *a ruling,* as the corresponding invertible sheaf determines a conic bundle $X_r \to \mathbb{P}^1$.

Lemma 5.3 of [F-M] says that the Weyl group acts transitively on rulings.

Let D be a sum of two exceptional curves which meet each other. Then it has several such decompositions which determine several monomials from $\Gamma(D)$. One can see that for $r \geq 4$ there are more such monomials than the dimension of this space, so they are linearly dependent and it determines quadratic relations between the generators of $Cox(X_r)$. If the curves coincide or do not intersect, then the first case in the proof of Proposition 3.2 shows that the decomposition is unique and there are no relations. Because of the Picard grading one has to look only at such homogeneous relations, living over one divisor.

The expectations are that the ring is defined by these relations, and for $r = 3, 4$ this follows from Lemma 3.1 and Proposition 4.1, but in general we have proved only a weaker result:

Theorem 4.7. *For $4 \le r \le 7$, the ring $Cox(X_r)$ is defined by quadratic relations up to radical.*

Proof. Let us look more closely at the relations for $r = 4, 5, 6$. A sum D of two intersecting exceptional curves can be written as $l_0 - l_1$ in a suitable basis, so the global sections of $\mathcal{O}(D)$ correspond to linear homogeneous polynomials on $\mathbb{P}^2$ that vanish at the point l_1, therefore $\dim \Gamma(D) = 2$, so there is a linear relation between any three monomials over D (but not between two, as they have different divisors). For $r \le 6$ the exceptional curves have only simple intersections, so if $E + F = E' + F'$ are two such decompositions, then $0 = (E, E + F) = (E, E' + F')$ requires $(E, E') = (F, F') = 0$ as the last two intersection numbers are nonnegative. We see that $D = l_0 - l_1 = l_i + (l_0 - l_1 - l_i), i \ge 2$ admits $r - 1$ such decompositions, so for $r = 4, 5, 6$ every monomial $x_E x_F, (E, F) \ne 0$ is equal to a polynomial in some $x_{E'}, (E', E) = 0$, i.e. *the affine coordinate ring of the quadratic variety in the affine chart $x_E \ne 0$ is generated by $\{x_F/x_E | (E, F) = 0\}$, i.e. by the variables corresponding to a blow-down of E.*

To show the italicized feature for $r = 7$ one has in addition to express the $x_{E'}/x_E$ with $(E, E') = 2$ in terms of $x_F/x_E, (E, F) \le 1$, because then the variables $x_F/x_E, (E, F) = 1$ can be reduced to those with intersection zero, as we've just shown.

But $E + E' = -K$ and a basis of sections over it was described in the proof of the proposition 3.2, namely, let E_1 be a point, E_2 and E_3 two lines through it, then $x_{E_i} x_{E_i'}$ is the basis (one can see from [Ma, 26.9] that $\forall E \exists! E' (E, E') = 2$, so the involution $\cdot'$ is well-defined). If E is another point (which it without loss of generality is), then writing $x_E x_{E'}$ in terms of this basis one obtains the desired result.

The quadratic relations determine another variety in the same projective space, which contains the torsor. We need to show that these varieties coincide, and it suffices to show it in every affine chart. Let $U = \{x_E \ne 0\}$ be this chart, X_{r-1} the blowdown of E. Let $R_q(X_{r-1})$ be the ring defined by quadratic relations for $X_{r-1}, R_q(X_d)_U$ the coordinate ring of the quadratic variety for r in our affine chart, $R(X_d)_U = R(X_{r-1})$ the coordinate ring of $P(X_d)$ in the chart = the homogeneous coordinate ring of $P(X_{r-1})$. Then it follows from the italicized feature that one has surjections $R_q(X_{r-1}) \to R_q(X_r)_U \to R(X_r)_U$, because the relations that we had for X_{r-1} hold for the lifts of the sections to X_r. Now by induction on r, the basis being $r = 3$ and no relations at all, we can assume that in a chart the map between the rings $R_q(X_{r-1}) \to R(X_{r-1})$ is the factorization modulo radical, so the varieties coincide in a chart, as we need. □

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