K3 surfaces over number fields with geometric Picard number one

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A long-standing question in the theory of rational points of algebraic surfaces is whether a K3 surface X over a number field K acquires a Zariski-dense set of L-rational points over some finite extension L/K. In this case, we say X has *potential density of rational points.* In case $X_{\mathbb{C}}$ has Picard rank greater than 1, Bogomolov and Tschinkel [2] have shown in many cases that X has potential density of rational points, using the existence of elliptic fibrations on X or large automorphism groups of X. By contrast, we do not know a single example of a K3 surface X/K with geometric Picard number 1 which can be shown to have potential density of rational points; nor is there an example which we can show *not* to have potential density of rational points. In fact, the situation is even worse; the moduli space of polarized K3 surfaces of a given degree contains a countable union of subvarieties, each parametrizing a family of K3 surfaces with geometric Picard number greater than 1. Since $\bar{\mathbb{Q}}$ is countable, it is not *a priori* obvious that these subvarieties don't cover the $\bar{\mathbb{Q}}$-points of the moduli space. In other words, it is a non-trivial fact that there exists a K3 surface over any number field with geometric Picard number 1!

In this note, we correct this slightly embarrassing situation by proving the following theorem:

Theorem 1. *Let d be an even positive integer. Then there exists a number field K and a polarized K3 surface X/K, of degree d, such that* $\operatorname{rank}\operatorname{Pic}(X_{\mathbb{C}}) = 1$.

The main idea is to use an argument of Serre on ℓ-adic groups to reduce the problem to proving the existence of K3 surfaces whose associated mod-n Galois representations have large image for some finite n; we then use Hilbert's irreducibility theorem and global Torelli for K3's to complete the proof.

Acknowledgment: This note is the result of a conversation between the author, Brendan Hassett, and A.J. de Jong, which took place at the American Institute of Mathematics during the workshop, "Rational and integral points on higher-dimensional varieties." It should also be pointed out that the main idea, in case $d = 4$, is implicit in the final remark of [3].

We begin by recalling some notations and basic facts regarding K3 surfaces. An element x of an abelian group L is called *primitive* if it is not contained in kL for any integer $k > 1$. Let X be a K3 surface over a number field K, and write $\bar{X}$ for $X \times_K \bar{K}$. The group $H^2(X_{\mathbb{C}}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{22}$; the cup product on $H^2(X_{\mathbb{C}}, \mathbb{Z})$ is a quadratic form with signature $(3, 19)$, which we denote

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$\langle , \rangle$. A *polarized K3 surface* is a pair $(X, \mathcal{L})$, where X/K is a K3 surface and $\mathcal{L}$ is an ample line bundle on X. If X is a polarized K3, we let x be the class of $\mathcal{L}$ in $H^2(X_{\mathbb{C}}, \mathbb{Z})$; then the positive even integer $\langle x, x\rangle$ is called the *degree* of X. We denote by L_X the orthogonal complement of x in $H^2(X_{\mathbb{C}}, \mathbb{Z})$. Denote by Γ the group of isometries of $H^2(X_{\mathbb{C}}, \mathbb{Z})$ which fix x and which lie in the identity component of $\mathrm{Aut}(H^2(X_{\mathbb{C}}, \mathbb{R}))$. So Γ is an arithmetic subgroup of $SO(2, 19)(\mathbb{Q})$.

For each prime ℓ we denote by G_ℓ the group of linear transformations α of $L_X \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ such that there exists $\chi(\alpha) \in \mathbb{Z}_\ell^*$ satisfying

$$\langle \alpha x, \alpha x\rangle = \chi(\alpha)\langle x, x\rangle$$

for all $x \in L_X \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. There is a natural inclusion

$$\iota : \Gamma \to G_\ell$$

and we denote by H_ℓ the closure, in the ℓ-adic topology, of $\iota(\Gamma)$.

When a polarized K3 surface X is defined over a number field K, the inclusion

$$L_X \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \subset H^2(\bar{X}, \mathbb{Z}_\ell)$$

induces a $\mathrm{Gal}(\bar{K}/K)$-module structure on $L_X \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$; we denote by

$$\rho_X : \mathrm{Gal}(\bar{K}/K) \to G_\ell$$

the resulting ℓ-adic Galois representation.

We begin by showing that the desired statement about $\mathrm{Pic}\, X_{\mathbb{C}}$ follows if the image of ρ_X is large enough.

Lemma 2. *Suppose $\rho_X(\mathrm{Gal}(\bar{K}/K))$ contains a finite-index subgroup of H_ℓ. Then* $\mathrm{rank}\, \mathrm{Pic}\, X_{\mathbb{C}} = 1$.

Proof. Suppose $\mathrm{rank}\, \mathrm{Pic}(X_{\mathbb{C}})$ is greater than 1; that is, there is divisor on $X_{\mathbb{C}}$ whose class is linearly independent from the class of the polarization. This divisor can be defined over some finite extension L/K. It follows that $\rho_X(\mathrm{Gal}(\bar{K}/L))$ is contained in the stabilizer of a line in $L_X \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. But this stabilizer does not contain a finite-index subgroup of H_ℓ. □

We also need a general lemma on linear ℓ-adic groups.

Lemma 3. *Let H be a closed subgroup of $\mathrm{GL}_m(\mathbb{Z}_\ell)$. Let $\Gamma_H(\ell^n)$ be the kernel of projection from H to $\mathrm{GL}_m(\mathbb{Z}/\ell^n\mathbb{Z})$. Then there exists an integer N such that no proper closed subgroup of H projects surjectively onto $H/\Gamma(\ell^N)$.*

Proof. Since H is a closed subgroup of $\mathrm{GL}_m(\mathbb{Z}_\ell)$, it is an analytic subgroup. In particular, there is a subspace $L \subset M_m(\mathbb{Q}_\ell)$ and a positive integer N such that, for all $n \geq N$, the group $\Gamma_H(\ell^n)$ is precisely the set of matrices $\exp(\lambda)$, where λ ranges over $\ell^n M_m(\mathbb{Z}_\ell) \cap L$. Thus, every element of $\Gamma_H(\ell^n)$ can be written as $\exp(\ell\lambda)$ for some $\lambda \in L$; in particular, for every $u \in \Gamma(\ell^n)$ there exists $v \in \Gamma_H(\ell^{n-1})$ with $v^\ell = u$. (See [4] for basic facts used here about ℓ-adic Lie groups.) We also require $N \geq 2$.

We now proceed as in [6, IV.3.4, Lemma 3], which proves the lemma in the case $H = \mathrm{SL}_2$. Suppose H_0 is a proper closed subgroup projecting surjectively onto $H/\Gamma_H(\ell^N)$. It suffices to prove that H_0 projects surjectively onto $H/\Gamma_H(\ell^n)$ for all $n > N$. We proceed by induction and assume H_0 projects surjectively onto $H/\Gamma_H(\ell^{n-1})$. We therefore need only show that, for all $x \in \Gamma_H(\ell^{n-1})$, there exists $h \in H_0$ with $h^{-1}x \in \Gamma_H(\ell^n)$. Since $n - 1 \geq N$, there exists $y \in \Gamma_H(\ell^{n-2})$ such that

$y^\ell = x$. We may write $y = 1 + \ell^{n-2}Y + \ell^{n-1}M_1$ for matrices $Y, M_1 \in M_m(\mathbb{Z}_\ell)$. By hypothesis, there exists $h' \in H_0$ such that $(h')^{-1}y \in \Gamma_H(\ell^{n-1})$. Then

$$h' = 1 + \ell^{n-2}Y + \ell^{n-1}M_2.$$

for some $M_2 \in \mathrm{GL}_m(\mathbb{Z}_\ell)$. So take

$$h = (h')^\ell = 1 + \ell^{n-1}Y + \ell^n M_2 + (1/2)(\ell)(\ell-1)\ell^{2n-3}Y^2 + \dots$$

which is congruent to x mod ℓ^n, since $n > N \geq 2$. $\square$

The purpose of Lemma 3 is to reduce the problem of showing that an ℓ-adic representation has large image to the corresponding problem for a mod ℓ^N representation. Below we show how to use Hilbert irreducibility to produce K3 surfaces X such that ρ_X has large image mod ℓ^N, where $N > 0$ is an integer to be specified at the end.

Write L_d for the rank-19 lattice $\langle -d\rangle \oplus H \oplus H \oplus E_8 \oplus E_8$. Then L_X is isomorphic to L_d for any polarized K3 of degree d.

By a *level* m structure on a polarized K3 we mean a choice of isometry

$$\phi : L_X/mL_X \cong L_d/mL_d.$$

We denote by $\Gamma(m)$ the kernel of the map $\Gamma \to \mathrm{GL}(L_d/mL_d)$. Choose a p large enough so that $\Gamma(p)$ is a torsion-free group. (It suffices to choose p larger than the order of any finite-order element of $GL(L_d)$.) If (X, ϕ) is a polarized K3 with level p structure, any automorphism $\alpha : X \to X$ preserving the polarization and ϕ must have finite order (because it preserves the polarization) and thus must act trivially on L_X (by the hypothesis on p). But then α is trivial by the Torelli theorem for K3's [5].

Let $\tilde{\mathcal{M}}/\mathbb{Q}$ be the moduli space of pairs (X, ϕ_p), where X is a polarized K3 surface of degree d and ϕ_p is a level p structure. We can construct this moduli space by GIT, as in the final remark of [1]. The fact that (X, ϕ_p) admits no nontrivial automorphisms implies that $\tilde{\mathcal{M}}$ is a *fine* moduli space. Now let $\tilde{\mathcal{M}}(\ell^N)$ be the space of pairs $(X, \phi_p, \phi_{\ell^N})$, where ϕ_{ℓ^N} is a level ℓ^N structure on X. Note that $\tilde{\mathcal{M}}$ and $\mathcal{M}\tilde{(\ell^N)}$ are not *a priori* connected.

Using again the Torelli theorem for K3 surfaces, we know that the analytic moduli space of polarized K3 surfaces of degree d is a quotient $\Gamma\backslash\Omega$, where Ω is a certain connected 19-dimensional domain of periods. (See [1, §3], noting that our Γ is an index-2 subgroup of Beauville's Γ_q.) It follows that $\Gamma(p)\backslash\Omega$ is a connected component of the analytification $\tilde{\mathcal{M}}^{an}$ of $\tilde{\mathcal{M}}$, and $\Gamma(p\ell^N)\backslash\Omega$ is a connected component of $\tilde{\mathcal{M}}(\ell^N)^{an}$. Denote by $\mathcal{M}$ and $\mathcal{M}(\ell^N)$ the connected components of $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}(\ell^N)$ corresponding to the quotients above; then, for some number field K, the map $\pi : \mathcal{M}(\ell^N) \to \mathcal{M}$ is a Galois cover of varieties over K with Galois group $\Gamma(p)/\Gamma(p\ell^N)$. Denote this finite group by $\bar{\Gamma}$.

Now let $p : \mathcal{M} \to \mathbb{P}^{19}$ be a generically finite map of degree n. Then the composition $p \circ \pi$ expresses the function field $K(\mathcal{M}(\ell^d))$ as a finite extension of $K(\mathbb{P}^{19})$. Let U be a Galois cover of $\mathbb{P}^{19}$ whose function field is the Galois closure of $K(\mathcal{M}(\ell^d))/K(\mathbb{P}^{19})$. Then the Galois group G of $K(U)/K(\mathbb{P}^{19})$ is naturally contained in the wreath product W of $\bar{\Gamma}$ with S_n. The group W fits in an exact sequence

$$1 \to \bar{\Gamma}^n \to W \to S_n \to 1$$

and the intersection of G with a Cartesian factor of $\bar{\Gamma}^n$ is the full group $\bar{\Gamma}$, since $\bar{\Gamma}$ is the Galois group of the cover π.

Now, by the Hilbert irreducibility theorem, there is a Zariski-dense subset of $\mathbb{P}^{19}(K)$ consisting of points x such that the Galois group of $(p \circ \pi)^{-1}(x)$ over x is the full group G. Let x be such a point, and let y be a $\bar{\mathbb{Q}}$-point of $\mathcal{M}$ lying over x. Then $Y \in \mathcal{M}(L)$ for some number field L, and the Galois group of $\pi^{-1}(y)$ over u is the full group $\bar{\Gamma}$. If X/L is the K3 surface corresponding to the point y, the map

$$\mathrm{Gal}(\bar{\mathbb{Q}}/L) \to GL_2(L_X \otimes_{\mathbb{Z}} (\mathbb{Z}/\ell^N\mathbb{Z}))$$

given by the Galois action on $H^2_{et}(X, \mathbb{Z}/\ell^N\mathbb{Z})$ has image $\bar{\Gamma}$. Now apply Lemma 3, taking H to be the closure in the ℓ-adic topology of the image of $\Gamma(p)$ in $GL_2(L_X \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$. We conclude that, having chosen N large enough, we can find a degree d polarized K3 surface X over a number field L such that the image of ρ_X contains H, which is a finite-index subgroup of H_ℓ. Now X has geometric Picard number 1 by Lemma 2.

Remark 4. Lemmas 2 and 3, in principle, should allow one to write down a K3 of any desired degree which has geometric Picard number 1. One would first compute suitable values of ℓ and N, as Lemma 3 guarantees we can. It remains to write down a K3 surface X such that the representation of Galois on $H^2_{et}(X, \mathbb{Z}/\ell^N Z)$ is as large as possible. In case $d = 4$, this computation is precisely the one suggested in the final remark of [3]. In order to make this computation more tractable, it might be a good idea to restrict to a family of quartic surfaces whose monodromy group Γ_0 is smaller than Γ, but which still doesn't have any stabilizers of points in L_X as finite-index subgroups.

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