Weak Approximation on Algebraic Varieties

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This survey paper consists of two parts. The first one is an introduction to the topic: definitions and first properties, classical examples and counterexamples. The second one is about cohomological methods : Brauer-Manin obstruction, descent theory of Colliot-Thélène and Sansuc, non-abelian descent theory.

In the whole article we let k be a number field with algebraic closure $\bar{k}$ and absolute Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. We denote by Ω_k the set of all places of k (including the archimedean places) and k_v stands for the completion of k at the place v. The ring of integers of k (resp. k_v for v finite) is denoted by $\mathcal{O}_k$ (resp. $\mathcal{O}_v$).

1. Classical results

1.1. Basic facts

Theorem 1.1.1 (Weak Approximation) *Let $\Sigma \subset \Omega_k$ be a finite set of places of k. Let $\alpha_v \in k_v$ for $v \in \Sigma$. Then there is an $\alpha \in k$ which is arbitrarily close to α_v for $v \in \Sigma$.*

For a complete proof, see [La], Theorem 1 p.35. This result is a refinement of the Chinese remainder theorem. One reformulation of it is as follows: the diagonal embedding $k \hookrightarrow \prod_{v \in \Omega_k} k_v$ is dense, the product being equipped with the product of the v-adic topologies.

We have the slight variant: $\mathbf{P}^1(k)$ is dense in $\prod_{v \in \Omega_k} \mathbf{P}^1(k_v)$ (here we have just replaced the affine line $\mathbf{A}^1_k$ by the projective line $\mathbf{P}^1_k$).

Definition 1.1.2 Let X/k be a geometrically integral algebraic variety. Then X satisfies *weak approximation* if given $\Sigma \subset \Omega_k$ a finite set of places and $M_v \in X(k_v)$ for $v \in \Sigma$, there exists a k-rational point $M \in X(k)$ which is arbitrarily close to M_v for $v \in \Sigma$.

Care must be taken if $\prod_{v \in \Omega_k} X(k_v)$ is empty; by convention, we will say that in this case X satisfies weak approximation even though $X(k)$ is empty. When $\prod_{v \in \Omega_k} X(k_v) \neq \emptyset$ but $X(k) = \emptyset$, one says that the *Hasse principle* fails[1].

[1] As Swinnerton-Dyer says, this corresponds to weak approximation failing dramatically.

We see that weak approximation is equivalent to the statement that $X(k)$ is dense in $\prod_{v\in\Omega_k} X(k_v)$ (equipped with the product of the v-adic topologies).

Remark 1.1.3 Let $\mathcal{X} \to \operatorname{Spec}\mathcal{O}_k$ be a flat model of X over $\operatorname{Spec}\mathcal{O}_k$; denote by $X(\mathbb{A}_k)$ the set of *adelic points* of X, that is: the restricted product of the sets $X(k_v)$ ($v \in \Omega_k$) with respect to the sets $\mathcal{X}(\mathcal{O}_v)$ (it is clearly independent on the choice of $\mathcal{X}$). If X is projective, then $X(\mathbb{A}_k) = \prod_{v\in\Omega_k} X(k_v)$ and weak approximation is equivalent to strong approximation, namely, $X(k)$ is dense in $X(\mathbb{A}_k)$ for the adelic topology.

Remark 1.1.4 Let X, X' be smooth. Assume that X is k-birational to X'. Then X satisfies weak approximation if and only if X' satisfies weak approximation: this an easy consequence of the implicit function theorem for k_v (a reference for this well known result is [Se92], p. 85).

We can therefore speak about weak approximation for a function field $k(X)$: this means that weak approximation holds for any smooth (projective) model of X (such a model exists by Hironaka's Theorem on resolution of singularities).

Example 1.1.5 It follows immediately from Theorem 1.1.1 that the affine line, the projective line, and more generally the affine space $\mathbf{A}_k^n$ and the projective space $\mathbf{P}_k^n$ satisfy weak approximation, as does any k-rational variety (see Remark 1.1.4), e.g. a smooth quadric with a k-point.

1.2. More examples

We begin with the most classical example of "local-global principle":

Theorem 1.2.1 *Let $Q \subset \mathbf{P}_k^n$ be a smooth projective quadric. Then Q satisfies weak approximation.*

Here, we do not assume that there is a k-rational point. This is the difficult part, proving the *Hasse principle*, that is: the existence of points everywhere locally implies the existence of a rational point. In the case of quadrics, this is the famous Hasse-Minkowski theorem (proven by Hasse around 1924). A detailed proof of this theorem for $k = \mathbf{Q}$ (the general case works just the same) can be found in Serre's book [Se70].

Here are some other results for complete intersections in $\mathbf{P}_k^n$:

Example 1.2.2 A smooth intersection of two quadrics $X \subset \mathbf{P}_k^n$ satisfies weak approximation if $n \geq 8$, or if $n \geq 4$ and there exists a pair of skew conjugate lines on X (Colliot-Thélène, Sansuc, Swinnerton-Dyer 1987 [CSS], Th. 10.1 and Prop. 5.2).

Example 1.2.3 Châtelet surfaces: let V be the affine surface $y^2 - az^2 = P(x)$, where $\deg P = 4$, $a \in k^* - k^{*2}$. If P is irreducible, then a smooth and projective model X of V satisfies weak approximation ([CSS], Th. 8.11).

Example 1.2.4 Let $X \subset \mathbf{P}^n_k$ a smooth cubic hypersurface, then weak approximation holds if $n \geq 16$ (Skinner 1997 [Ski]).

An interesting fact is that the proofs of the three previous results use different tools. The first statement is proved with the fibration method (see paragraph 1.3.), the second one with descent theory (see paragraph 2.2.). To deal with Example 1.2.4 one needs the Hardy-Littlewood circle method, which is especially efficient when the number of variables is substantially bigger than the degree. We shall not discuss further this analytic technique in these notes.

There are also results for linear algebraic groups.

Example 1.2.5 Let K/k be a cyclic field extension. Define the torus T by the equation with variables $x_1, ...x_r$: $N_{K/k}(x_1\omega_1 + ...x_r\omega_r) = 1$, where $(\omega_1, ..., \omega_r)$ is a basis of K/k. Then T satisfies weak approximation (this follows from [San], Cor. 3.5. (ii)). The Hasse principle for equations $N_{K/k}(x_1\omega_1 + ...x_r\omega_r) = a, a \in k^*$ goes back to Hasse (1924).

Example 1.2.6 If T is a k-torus, and $\dim T \leq 2$, then T satisfies weak approximation because T is k-rational (Voskresenskiĭ, [Vos], IV.9).

Example 1.2.7 If G is a semi-simple, simply connected linear k-group, then G satisfies weak approximation. This is due to Kneser ([K65], [K66]), Harder ([Ha]), and Platonov ([P69], [P70]).

We conclude this paragraph by two classical conjectures.

Conjecture 1.2.8 A smooth intersection of 2 quadrics in $\mathbf{P}^n$ for $n \geq 5$ satisfies weak approximation.

This is known when the variety has a rational point (Salberger and Skorobogatov 1991 [SaSk]; the case $n \geq 6$, and also $n = 5$ except a special situation were treated in [CSS], Th. 3.11). Thus the difficulty is now to prove the Hasse principle.

Conjecture 1.2.9 A smooth cubic hypersurface (of dimension at least 3) satisfies weak approximation.

Here the Hasse principle is known for *diagonal* hypersurfaces if we assume the finiteness of Tate-Shafarevich groups of elliptic curves (Swinnerton-Dyer [Sw01]).

We shall see later (paragraph 1.4.) that the similar conjectures for surfaces are false.

1.3. The fibration Method

The general idea of this method is quite natural: consider a pencil of varieties satisfying weak approximation over a base which satisfies also weak approximation. Does this imply that weak approximation holds for the total space of the fibration ? In general, the answer is no (even for examples for conic bundles over $\mathbf{P}^1_k$, see example 1.4.3 below) but with additional assumptions the result becomes true. Here is a useful statement in this direction:

Theorem 1.3.1 *Let $p : X \to B$ be a projective, flat surjective morphism with X smooth. Assume that*

1. *B is projective and satisfies weak approximation.*
2. *Almost all k-fibers of p satisfy weak approximation.*
3. *All fibers of p are geometrically integral.*

Then X satisfies weak approximation.

(Here almost all means on a Zariski-dense open subset; the hypothesis X smooth is not essential, but it makes the statement simpler; it is also possible to weaken the third assumption by replacing "geometrically integral" with "split" [2]).

There are refinements when B is the projective space : you can accept degenerate fibers on one hyperplane (using the strong approximation theorem for the affine space), see [Sk90].

The idea of this method goes back to the proof of Hasse-Minkowski Theorem (more precisely, the step consisting of going from four variables to five). The first subtle application of Theorem 1.3.1 appeared in [CSS] for intersection of two quadrics in $\mathbf{P}^n$: when $n \geq 8$ (here the authors used a fibration in Châtelet surfaces), and also when $n \geq 5$ when the intersection of two quadrics contains a pair of skew conjugate lines (the point is to go from $n = 4$ to $n \geq 5$ by induction). Another example of application consists of cubic hypersurfaces of dimension ≥ 4 with 3 conjugate singular points (Colliot-Thélène, Salberger 1989 [CSal]).

Sketch of proof of Theorem 1.3.1 : Start with a smooth k_v-point M_v for any $v \in \Omega_k$ on X and fix a finite set of places Σ. Project M_v to $P_v := p(M_v) \in B(k_v)$. Using weak approximation on B, we can approximate P_v by $P \in B(k)$ for $v \in \Sigma$. Consider the fibre $X_P := p^{-1}(P) \subset X$; Then X_P has a k_v-point M'_v close to M_v for $v \in \Sigma$ by the implicit function theorem. To apply weak approximation on X_P, it remains to check that $X_P(k_v) \neq \emptyset$ for each $v \notin \Sigma$; this is possible if Σ is sufficiently large by the Weil estimates :

[2] A k-variety is *split* if it contains a non-empty Zariski open subset which is geometrically integral. This notion was introduced by Skorobogatov in [Sk96].

here we use that all k-fibers are geometrically irreducible, which implies that the reduction mod. v of X_P also is for a sufficiently large v (independent of P). □

1.4. Some counterexamples

It has been known for a long time that for example elliptic curves do not satisfy weak approximation (the defect of weak approximation is described by Cassel's dual exact sequence [Cas]; see also Theorem 1.4.5 below). It is more difficult to find counterexamples to weak approximation among rational varieties (that is: varieties X such that $\overline{X} := X \times_k \bar{k}$ is $\bar{k}$-birational to the projective space). Here are some examples of this situation:

Example 1.4.1 Cubic surfaces do not satisfy the Hasse principle: the surface $5x^3 + 9y^3 + 10z^3 + 12w^3 = 0$ is a counterexample (Cassels and Guy 1966 [CG]). The existence of a rational point does not imply weak approximation; a counterexample is the surface defined in $P^3_{\mathbf{Q}}$ by the equation

$$t(x^2 + y^2) = (4z - 7t)(z^2 - 2t^2)$$

(Swinnerton-Dyer 1962 [Sw62]).

Example 1.4.2 In general a smooth intersection X of two quadrics in $\mathbf{P}^4_k$ does not satisfy the Hasse principle, and weak approximation does not hold even if $X(k) \neq \emptyset$. For example, the variety defined in $\mathbf{P}^4_{\mathbf{Q}}$ by the equations

$$x_0 x_1 - (x_2^2 - 5x_3^2) = 0$$

$$(x_0 + x_1)(x_0 + 2x_1) - (x_2^2 - 5x_4^2) = 0$$

does not satisfy the Hasse principle (Birch and Swinnerton-Dyer [BSD]) and the variety X:

$$x_0 x_1 - (x_2^2 + x_3^2) = 0$$

$$(4x_1 - 3x_0)(4x_0 - x_1) - (x_2^2 + x_4^2) = 0$$

is a counterexample to weak approximation with $X(\mathbf{Q}) \neq \emptyset$ ([CSS], 15.5).

Example 1.4.3 Let us explain how it is possible to construct counterexamples to weak approximation among Châtelet surfaces (which are special cases of conic bundles over $\mathbf{P}^1_k$). Consider the equation

$$X : y^2 + z^2 = f_1(x) f_2(x) \neq 0,$$

$\deg(f_1) = \deg(f_2) = 2$, $\gcd(f_1, f_2) = 1$ over the field $k = \mathbf{Q}$ of rational numbers. Set $K = \mathbf{Q}(\sqrt{-1})$, $K_v = K \otimes_{\mathbf{Q}} \mathbf{Q}_v$; then there exists a finite set $\Sigma_0 \subset \Omega_k$ such that if $v \notin \Sigma_0$ and $M_v \in X(\mathbf{Q}_v)$, then $f_1(M_v)$ is a norm of

$K_v/\mathbf{Q}_v$ (use a computation with valuations). If you find $v_0 \in \Sigma_0$ with the properties:

(i) there exists M_{v_0} such that $f_1(M_{v_0})$ is not a local norm,

(ii) for $v \neq v_0$ there exists M_v such that $f_1(M_v)$ is a local norm,

then there is no weak approximation thanks to global reciprocity law of class field theory, namely the exactness of the sequence

$$\mathbf{Q}^*/NK^* \to \bigoplus_{v \in \Omega_k} \mathbf{Q}_v^*/NK_v^* \to \mathbf{Z}/2$$

An explicit example of this situation is given by the equation

$$y^2 + z^2 = ((x-2)^2 - 3)((x+2)^2 + 3).$$

Here there is an obvious rational point $P = (0, 0, 1)$ such that $f_1(P) = 1$ is a global norm, hence this gives for any v a local point P_v such that $f_1(P_v)$ is a local norm. For $v = 2$ it is easy to construct a local point M_v such that $f_1(M_v)$ is not a local norm (Take $x = 2$ and use [Se70], p. 39).

It is even possible to obtain a counterexample to the Hasse principle, e.g. $y^2 + z^2 = (x^2 - 2)(3 - x^2)$ (Iskovskih, 1970). In this example, $f_1(M_v)$ is always a norm of $K_v/\mathbf{Q}_v$, except for $v = 2$, where it cannot be a norm, hence by the reciprocity law there is no rational point.

Example 1.4.4 The results of Example 1.2.5 cannot be extended to arbitrari tori. Let K/k be a biquadratic extension, then there are counterexamples to weak approximation like $T : N_{K/k}(x_1w_1 + \cdots + x_4w_4) = 1$, where $w_1, \ldots, w_4$ is a basis of K/k; this holds e.g. for $k = \mathbf{Q}$, $K = \mathbf{Q}(\sqrt{-1}, \sqrt{2})$, [San], (2.17) p. 237 and Th. 3.3.

All the previous counterexamples are related to reciprocity laws in global class field theory. In paragraph 2.2. we will describe a general framework for these, namely the Brauer-Manin obstruction.

We conclude this section with the following negative result ([Min]):

Theorem 1.4.5 (Minchev 1989) *Let X be a projective and smooth k-variety, assume that the geometric fundamental étale group $\pi_1(\overline{X})$ is not trivial, where $\overline{X} = X \otimes \bar{k}$. Suppose that $X(k) \neq \emptyset$, then X does not satisfy weak approximation.*

Proof (sketch of) : Enlarge the situation over $\operatorname{Spec} \mathcal{O}_{k,\Sigma_0}$ where Σ_0 is a finite set of places. By assumption, there is a nontrivial geometrically connected covering $Y \to X$, which for models gives $\mathcal{Y} \to \mathcal{X}$. Take an arbitrary $M \in X(k)$, then the fibre Y_M can be written as $Y = \operatorname{Spec} L$ where L is an étale algebra $L = k_1 \times \cdots \times k_r$; each k_i is unramified outside Σ_0, hence only finitely many k_i are possible (by Hermite's Theorem, cf. [La], Theorem 5 p.121). Find $v \notin \Sigma_0$ with v totally split for each k_i (such a v does exist by Cebotarev density Theorem, [La] Theorem 10 p. 169); find M_v such that the fibre of Y at M_v is not (this is possible because Y is geometrically connected, via a "geometric" Cebotarev-like Theorem as in [Ek], Lemma 1.2). Then M_v cannot be approximated by a rational point M (use Krasner's Lemma, [La], Proposition 3 p. 43). □

Here the obstruction to weak approximation cannot always be related to a reciprocity law as above. See paragraph 2.5. and [H00].

2. Cohomological Methods

Let X be a smooth and geometrically integral variety over k. From now on suppose that X is projective. We denote by $\overline{X(k)}$ the closure of $X(k)$ in $\prod_{v\in\Omega_k} X(k_v) = X(\mathbb{A}_k)$.

Here our aim is to: (i) explain the counterexamples to weak approximation; (ii) find 'intermediate' sets E between $\overline{X(k)}$ and $X(\mathbb{A}_k)$; (iii) in some cases, prove that $E = \overline{X(k)}$.

2.1. General setting

Let G/k be an algebraic group (usually linear, but not necessarily connected, e.g. G finite). If G is commutative, define the étale cohomology groups $H^i(X,G)$ ($i = 1,2$; the cohomological dimension of a non archimedean local field is two, making the higher cohomology groups uninteresting). In general, we have only the pointed set $H^1(X,G)$ (defined by Cech cocycles for the étale topology). If $X = \operatorname{Spec} k$, $H^1(X,G) = H^1(\Gamma, G(\bar{k}))$. If G is linear, then $H^1(X,G)$ corresponds to G-torsors (i.e. G-principal homogeneous spaces) over X up to isomorphism (cf. [Mil], III.4 and [Sk01], Chapter 2).

Take $f \in H^i(X,G)$, and define

$$X(\mathbb{A}_k)^f = \{(M_v) \in X(\mathbb{A}_k) : (f(M_v)) \in \operatorname{Im}[H^i(k,G) \to \prod_{v\in\Omega_k} H^i(k_v,G)]\}.$$

Obviously $X(k) \subset X(\mathbb{A}_k)^f$. We will see that in many cases $\overline{X(k)} \subset X(\mathbb{A}_k)^f$.

Example 2.1.1 Let $\operatorname{Br} X = H^2(X, \mathbf{G}_m)$ be the (cohomological) Brauer group of X; define the *Brauer-Manin* set of X

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \bigcap_{f\in \operatorname{Br} X} X(\mathbb{A}_k)^f.$$

Then $\overline{X(k)} \subset X(\mathbb{A}_k)^{\mathrm{Br}}$. Indeed X is projective and $\operatorname{Br}\mathcal{O}_v = 0$ for each finite place v ([Mil], IV.2.13), hence for each $\alpha \in \operatorname{Br} X$ there exists a finite set of places Σ_0 (the places of bad reduction for X or α) such that for any

$v \notin \Sigma$ and any $M_v \in X(k_v)$, we have $\alpha(M_v) = 0$. Let $(P_v) \in X(\mathbb{A}_k)$; if $P \in X(k)$ is sufficiently close to P_v for $v \in \Sigma_0$, then $\alpha(P) = \alpha(P_v)$ for any $v \in \Omega_k$, thus $\sum_{v \in \Omega_k} j_v(\alpha(P_v)) = 0$ because P is rational.

Manin ([Ma]) showed in 1970 that for a genus one curve with finite Tate-Shafarevich group, the condition $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies the existence of a rational point. The similar statement for abelian varieties is true and there is also an analogue about weak approximation ([Wa]).

Example 2.1.2 Let $f : Y \to X$ be a Galois, geometrically connected, nontrivial étale covering with group G. We can view f as an element of $H^1(X, G)$, where G is considered as a constant group scheme. Essentially the proof of Minchev's result (Theorem 1.4.5) consists of showing $\overline{X(k)} \subset X(\mathbb{A}_k)^f$ (this is the step which uses Hermite's Theorem), then to find $(M_v) \notin X(\mathbb{A}_k)^f$ thanks to a geometric Cebotarev Theorem.

Remark 2.1.3 If X is rational, then $\mathrm{Br}\, X/\mathrm{Br}\, k = H^1(k, \mathrm{Pic}\, \overline{X})$ is finite, where $\overline{X} = X \times_k \bar{k}$. Since for a constant element f of $\mathrm{Br}\, X$ (i.e. an element coming from $\mathrm{Br}\, k$) we obviously have $X(\mathbb{A}_k) = X(\mathbb{A}_k)^f$, we obtain that in this case $X(\mathbb{A}_k)^{\mathrm{Br}}$ is (at least in theory) 'computable'.

Theorem 2.1.4 (Harari, Skorobogatov) *Let X be a projective, smooth and geometrically integral k-variety. Let G be a linear k-group and $f \in H^1(X, G)$; then $\overline{X(k)} \subset X(\mathbb{A}_k)^f$ (and $X(\mathbb{A}_k)^f$ is "computable").*

The idea of the proof is to apply Borel-Serre finiteness Theorem ([Se94], III.4.6) instead of Hermite's Theorem. See [HS02] (Th. 4.7) or [Sk01] (5.3) for the details.

2.2. Abelian descent theory

This was developed by Colliot-Thélène and Sansuc [CSan], and recently completed by Skorobogatov [Sk99]. Recall that a *group of multiplicative type* S over k is a commutative linear k-group which is an extension of a finite group by a torus. The *module of characters* of S is the abelian group $\widehat{S} = \mathrm{Hom}(\overline{S}, \mathbf{G}_m)$, equipped with the action of the Galois group Γ, where $\overline{S} = S \times_k \bar{k}$. One of the main results of the theory consists of the following

Theorem 2.2.1 *Let X be a projective, smooth, and geometrically integral k-variety. Define*

$$X(\mathbb{A}_k)^{\mathrm{Br}_1} = \bigcap_{f \in \mathrm{Br}_1 X} X(\mathbb{A}_k)^f$$

where $\mathrm{Br}_1 X = \mathrm{Ker}\,(\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X})$. *Assume further that* $X(\mathbb{A}_k)^{\mathrm{Br}_1} \neq \emptyset$. *Then:*

1. *We have*

$$X(\mathbb{A}_k)^{\mathrm{Br}_1} = \bigcap_{\substack{f \in H^1(X,S) \\ S \text{ of multiplicative type}}} X(\mathbb{A}_k)^f.$$

2. *Assume further that* $\operatorname{Pic}\overline{X}$ *is of finite type, let* S_0 *be the group of multiplicative type with module of characters* $\operatorname{Pic}\overline{X}$*; then there exists a torsor* $f_0 : Y \to X$ *under* S_0 *(a* universal torsor*), such that*

$$X(\mathbb{A}_k)^{\mathrm{Br}_1} = X(\mathbb{A}_k)^{f_0}.$$

Intutitively, universal means "as nontrivial as possible"; in particular if there exists a universal torsor $f_0 : Y \to X$, then for any torsor $f : Z \to X$ under S_0 there exists a unique morphism of X-torsors $\varphi : Y \to Z$ such that $f_0 = f \circ \varphi$. See [Sk01], 2.3.3. for more details about the definition of universals torsors (this notion is due to Colliot-Thélène and Sansuc [CSan]).

Theorem 2.2.1 is difficult, see Skorobogatov's book [Sk01] for a complete account on the subject. One of the ideas is to recover the Brauer group of X (mod. $\operatorname{Br} k$) making cup-products $[Y] \cup a$, where $a \in H^1(k, \widehat{S_0})$ and $[Y]$ is the class of Y in $H^1(X, S_0)$. Another step (which is long to achieve) is to show that the condition $X(\mathbb{A}_k)^{\mathrm{Br}_1} \neq \emptyset$ implies the existence of a universal torsor.

Now assume that X is a rational variety, so $X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{Br}_1}$ (since $\operatorname{Br}\overline{X} = 0$). Assume $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$. Consider a universal torsor $f : Y \to X$. If $\sigma \in H^1(k, S_0)$, one can define the *twisted torsor* $f^\sigma : Y^\sigma \to X$ where

$$[Y^\sigma] = [Y] - \sigma \in H^1(X, S_0).$$

Then

$$X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^1(k,S_0)} f^\sigma(Y^\sigma(\mathbb{A}_k)).$$

The universal torsors are precisely the torsors $Y^\sigma, \sigma \in H^1(k, S_0)$. If you can prove that they satisfy weak approximation, then $\overline{X(k)} = X(\mathbb{A}_k)^f = X(\mathbb{A}_k)^{\mathrm{Br}}$, which means that the Brauer-Manin obstruction to weak approximation is the only one for X. In practice it is important to obtain *explicit* equations for the universal torsors (this is done in [CSan] Th. 2.3.1, see also [Sk01], 4.3.1.). Once the universal torsors are described by these equations, one can hope to prove (e.g. using fibration methods) that weak approximation holds for them because their Brauer group is trivial (that is: consists of constant elements), hence the Brauer-Manin obstruction vanishes for them. Here are some examples where this approach works completely:

Example 2.2.2 Consider a Châtelet surface: $y^2 - az^2 = P(x)$, $a \in k^*/k^{*2}$, $\deg P = 4$. Colliot-Thélène, Sansuc, Swinnerton-Dyer showed in [CSS] (Th. 8.11) that for a projective and smooth model X, the equality $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$ holds. Here weak approximation on universal torsors follows from the similar statement for intersection of two quadrics in $\mathbf{P}_k^n$ ($n \geq 4$) with a pair of skew conjugate lines (cf. Example 1.2.2).

If P is irreducible, then $\mathrm{Br}\, X/\mathrm{Br}\, k = 0$, so X satisfies weak approximation. It is worth noting that it seems impossible to deal with this special case without using descent, even though the Brauer-Manin obstruction already vanishes on X.

If P is reducible, we can have a counterexample to weak approximation, cf. Example 1.4.3. Here the obstruction is given by the Hilbert symbol $f = (a, f_1)$. This reinterpretes the reciprocity obstruction explained in Example 1.4.3 as a special case of the Brauer-Manin obstruction.

Example 2.2.3 Let X be a conic bundle surface over $\mathbf{P}^1$ with at most 5 degenerate fibres. Then $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$. Works by Salberger ([Sal]) and Colliot-Thélène ([Col]) covered at most 4 degenerate fibres via the descent method. Salberger and Skorobogatov ([SaSk]) treated the case of 5 bad fibres, using descent and K-theory. It is widely believed that the Brauer-Manin obstruction to weak approximation is the only one for a conic bundle over $\mathbf{P}^1$ with an arbitrary number of bad fibres. This was proved by Serre (unpublished) under Schinzel's hypothesis [3] in 1992. Another proof and several extensions of his result (in particular an unconditionnal zero-cycle version) can be found in [CSw].

We conclude this paragraph by the following general result about algebraic groups ([San]):

Theorem 2.2.4 (Sansuc, 1981) *Let G be a linear connected algebraic k-group and X a smooth compactification of G. Then the Brauer-Manin obstruction to weak approximation on X is the only one:*

$$\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}.$$

2.3. Open descent

In the last paragraph, we have considered descent over projective varieties. But the general results of the theory still hold over a geometrically integral variety U as soon as the only invertible functions on $\overline{U}$ are constant; this is often useful to obtain torsors described by nice equations. Descent over an open subset U of a projective variety X was introduced in 2000 by Colliot-Thélène and Skorobogatov. In particular they showed ([CSk], Prop. 1.1):

[3] Schinzel's hypothesis is a (rather wild) generalization of Dirichlet's Theorem on primes in an arithmetic progression, see [CSw].

Proposition 2.3.1 *Let X be a smooth, proper and geometrically integral k-variety. Let U be a non-empty Zariski open subset of X. Assume that* $\operatorname{Br} U/\operatorname{Br} k$ *is of finite index in* $\operatorname{Br} X/\operatorname{Br} k$.
Then $U(\mathbb{A}_k)^{\mathrm{Br}}$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$ for the adelic topology.

Note that elements of $\operatorname{Br} U$ do not necessarily belong to $\operatorname{Br} X$. This proposition is a consequence of the "formal lemma" ([H94], 2.6.1; see also next paragraph). With the help of Proposition 2.3.1, it is sometimes possible to prove that $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$ with a descent over a well chosen U instead of the whole X; this works for example for certain varieties fibred over the projective line ([CSk], Th. A and B).

Another application of the open descent is the following recent result ([HBS]); a new tool is to use the circle method to prove that universal torsors over U satisfy weak approximation.

Theorem 2.3.2 (Heath-Brown, Skorobogatov 2001) *Take $K/\mathbf{Q}$ a finite field extension. Consider the affine variety V, defined by an equation of norm type*

$$t^{a_0}(1-t)^{a_1} = N_{K/k}(x_1\omega_1 + \dots x_r\omega_r)$$

where $(\omega_1, \dots, \omega_r)$ is a basis of $K/\mathbf{Q}$, a_0, a_1 are two coprime integers, and $t, x_1, \dots, x_r$ are variables. Then the Brauer-Manin obstruction to weak approximation is the only one for a smooth and projective model X of V.

2.4. Back to fibration methods

If $p : X \to B$ is a fibration, we saw that if the base and the fibres satisfy weak approximation, under certain circumstances then X satisfies weak approximation.

Here we consider $p : X \to \mathbf{P^1}$, a projective, surjective morphism (and the generic fibre X_η is smooth). Assume also that X_η has a $\bar{k}(\eta)$-point (this technical condition is satisfied in most applications, e.g. if X_η is geometrically rationnally connected by a recent result of Graber, Harris and Starr [GHS]). A natural question is the following: If $\overline{X_P(k)} = X_P(\mathbb{A}_k)^{\mathrm{Br}}$ for almost all fibres X_P, $P \in \mathbf{P}^1(k)$, can you prove that $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$? The following result ([H94], [H97]) gives a partial answer to this question.

Theorem 2.4.1 *Under the notation and assumptions as above, we have* $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$ *if we assume that:*

1. $\operatorname{Pic} \overline{X_\eta}$ *is torsion-free, where $\overline{X_\eta} = X_\eta \times_K \overline{K}$, $K = k(\eta)$; e.g. X_η rational, or smooth complete intersection of dimension at least three.*

2. $\operatorname{Br} \overline{X_\eta}$ *is finite.*

3. *Either all fibres but one are geometrically integral, or X_η has a $k(\eta)$-point.*

Here again it is possible to replace "geometrically integral" by "split" in the third condition [4].

If we compare the proof of Theorem 2.4.1 with the proof of Theorem 1.3.1, there are two additional ingredients:

1. Show that the specialization map $\mathrm{Br}\, X_\eta/\mathrm{Br}\, K \to \mathrm{Br}\, X_P/\mathrm{Br}\, k$ is an isomorphism for many k-fibres X_P ('many' in the sense of Hilbert's irreducibility theorem). This is a consequence of assumptions 1. and 2. ([H94], 3.5.1. and [H97], 2.3.1.).

2. If $\alpha_1, \ldots, \alpha_r$ are elements of $\mathrm{Br}\, X_\eta$ which generate $\mathrm{Br}\, X_\eta/\mathrm{Br}\, k(\eta)$, choose an open subset $U \subset X$ such that $\alpha_i \in \mathrm{Br}\, U$. Then apply the following 'formal lemma' ([H94], 2.6.1.): Let $(M_v) \in X(\mathbb{A}_k)^{\mathrm{Br}}$, $M_v \in U$, and Σ_0 a finite set of places; then there exists $(P_v) \in X(\mathbb{A}_k)$, $P_v \in U$, and $\Sigma \supset \Sigma_0$ finite such that:

   (a) $P_v = M_v$ for $v \in \Sigma_0$;

   (b) $\sum_{v\in\Sigma} j_v(\alpha_i(P_v)) = 0$ for $1 \leq i \leq r$, where $j_v : \mathrm{Br}\, k_v \to \mathbf{Q}/\mathbf{Z}$ is the local invariant.

   This formal lemma is a consequence of

   Theorem 2.4.2 ([H94], 2.1.1.) *Let $\alpha \in \mathrm{Br}\, U$, suppose $\alpha \notin \mathrm{Br}\, X$. Then there exist infinitely many places v of k such that the image of the evaluation map $[U(k_v) \to \mathrm{Br}\, k_v, M_v \mapsto \alpha(M_v)]$ is not zero.*

Theorem 2.4.1 has several applications:

Example 2.4.3 We can recover Sansuc's result just knowing the case of a torus (which essentially goes back to [Vos]). Here we apply Theorem 2.4.1 in a situation when X_η has a $k(\eta)$-point, [H94], 5.3.1.

Example 2.4.4 If $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$ for any smooth projective cubic surface (this is a widely-believed conjecture), then by induction the same holds for hypersurfaces ([H94], 5.2.2.); therefore if $\dim X \geq 3$, then X satisfies weak approximation (The Brauer group of smooth hypersurfaces of dimension at least 3 is trivial).

It is also possible to cross open descent with the fibration method to obtain generalizations of Theorem 2.4.1 when at most 2 (or 3 in very special cases) fibres are degenerate. See [HS03].

[4] This refinement is especially useful if we have to deal with a non projective morphism because the "split" condition remains valid after compactification of the morphism. See [H97], Proof of Prop. 3.1.1.

2.5. Nonabelian descent

In the last few years it has become apparent that the Brauer-Manin obstruction can be refined if we consider non-abelian cohomology. In particular if G/k is a finite but not commutative k-group, it may happen that for $f \in H^1(X, G)$, $X(\mathbb{A}_k)^f \not\supset X(\mathbb{A}_k)^{\mathrm{Br}}$. The following result was the first unconditional counterexample to the Hasse principle not accounted for by the Brauer-Manin obstruction ([Sk99]).

Theorem 2.5.1 (Skorobogatov) *There exists a bielliptic surface X over $\mathbf{Q}$ such that $X(\mathbf{Q}) = \emptyset$, $X(\mathbb{A}_{\mathbf{Q}})^{\mathrm{Br}} \neq \emptyset$.*

Actually one can show ([HS02], 5.1) that $X(\mathbb{A}_{\mathbf{Q}})^f = \emptyset$ for some $f \in H^1(X, G)$, where G is a finite k-group satisfying $G(\overline{\mathbf{Q}}) = (\mathbf{Z}/4\mathbf{Z})^2 \rtimes \mathbf{Z}/2\mathbf{Z}$.

There are similar statements for weak approximation ([H00])), e.g. take X/k any bielliptic surface, $X(k) \neq \emptyset$, then $\overline{X(k)} \subsetneq X(\mathbb{A}_k)^{\mathrm{Br}}$.

Nevertheless the Brauer-Manin condition is quite strong, as shows the following result ([H02]; compare Th. 2.2.1 above):

Theorem 2.5.2 *Let X be a projective, smooth, and geometrically integral k-variety. Then:*

1. *If G/k is a linear connected k-group, $f \in H^1(X, G)$, then*
$$X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)^f.$$

2. *If G is any commutative k-group, $f \in H^2(X, G)$, then*
$$X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)^f.$$

Let us conclude with an open question : is the first part of this theorem still true for a (non-commutative) G which is an extension of a finite *abelian* group by a connected linear group (e.g. a torus) ? My guess is "no".

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