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COUNTING POINTS ON VARIETIES USING UNIVERSAL TORSORS

Emmanuel Peyre

Abstract. Around 1989, Manin initiated a program toward the understanding of the asymptotic behaviour of the rational points of bounded height on Fano varieties. This program led to the search of new methods to estimate the number of points of bounded height on various classes of varieties. Methods based on harmonic analysis were very successfull for compactifications of homogeneous spaces. However, they do not apply to other types of varieties. Universal torsors which were introduced by Colliot-Thélène and Sansuc in connection with the Hasse principle and the weak approximation turned out to be a useful tool to attack other varieties. The aim of this short survey is to describe how it has been used in various simple examples.

1. Introduction

If the rational points of a variety V over a number field k are Zariski dense, it is natural to equip the variety V with a height H and to study asymptotically the set of points of bounded height on V. In **[FMT]** and **[BM]**, Batyrev, Franke, Manin and Tschinkel gave strong evidence supporting conjectures relating the asymptotic behaviour of the number of points of bounded height on open subsets of V to geometrical invariants of V. This work motivated the development of several methods to estimate the number of points of bounded height on new classes of varieties. One of the most successfull method was the use of harmonic analysis on adelic groups. For example, it was used by Batyrev and Tschinkel in [**BT1**], [**BT2**], and [**BT4**] to handle the case of projective toric varieties, by Strauch and Tschinkel in [**ST1**] and [**ST2**] for toric bundles over flag varieties, and by Chambert-Loir and Tschinkel in [**CLT1**], [**CLT2**], and [**CLT3**] for equivariant compactification of vector spaces. However this type of methods apply only to equivariant compactifications of homogeneous spaces. One may say that almost all other methods have one step in common, namely the lifting to universal torsors. Universal torsors have been introduced by Colliot-Thélène and Sansuc in [**CTS1**], [**CTS2**], [**CTS3**], and [**CTS4**] to study the Hasse principle and the weak approximation. The interest of universal torsors is that, from an arithmetical point of view, these torsors should be much simpler that the variety itself. As an example, universal torsors over smooth projective toric varities are open subsets of an affine space. When the Fano variety V is a smooth complete intersection of dimension bigger than three in the projective space, the universal torsor may be described as the cone above the variety. In that case, if the dimension of the variety is big enough, the conjectural formula of Manin may be deduced from the formula given by the classical circle method. This reduction, which is described in [**FMT**] may be seen as a particular case of the lifting to the universal torsor. Salberger in [**Sa**] was the first to use explicitly universal torsors in relation with points of bounded height. In particular, he was able to give a new proof of the theorem of Batyrev and Tschinkel for smooth projective split toric varieties over $\mathbf{Q}$. This lifting to the universal torsor was then used by de la Bretèche in [**Bre1**] to give a better estimate for the number of points of bounded height on toric varieties. The lifting to universal torsors was later used by Salberger and de la Bretèche (see [**Bre2**]) to prove the asymptotic formula for the plane blown up in four points over $\mathbf{Q}$. In a more general setting, the author described in [**Pe2**] and [**Pe3**] how the conjectural asymptotic formula lifts naturally to universal torsors.

The aim of this short survey is to present in a quite self-contained way the usefulness of universal torsors for counting points of bounded height. In section 2, we describe the heights used throughout this paper, in section 3 we recall the empiric formula for the number of points on Fano varieties. In section 4 we give a short list of cases for which this formula holds, in section 5 we describe the counter-example of Batyrev and Tschinkel. In section 6 we describe briefly both methods: harmonic analysis and universal torsors. Section 7 is devoted to the case of an hypersurface in $\mathbf{P}^n$. The next section contains the definition of universal torsors in general. In section 9 we describe Cox's construction of universal torsors for toric varieties and explain how Salberger used it and in section 10 we turn to the case of the plane blown-up in four points, in which case the universal torsor was described by Salberger and Skorobogatov. The last section contains a short description of the generalization of these lifting arguments to a larger class of varieties.

2. Heights on projective varieties

Definition 2.1. The classical height on the projective space over $\mathbf{Q}$ is defined as follows:

$$\begin{aligned} H_N : \mathbf{P}^N(\mathbf{Q}) &\to \mathbf{R}_{>0} \\ (x_0 : \ldots : x_N) &\mapsto \sup_{0 \leqslant i \leqslant N} |x_i|, \text{ if } \begin{cases} x_i \in \mathbf{Z}, \\ \gcd(x_i) = 1. \end{cases} \end{aligned}$$

If K is a number field, one generalizes this construction in the following way:

$$\begin{aligned} H_N : \mathbf{P}^N(K) &\to \mathbf{R}_{>0} \\ (x_0 : \ldots : x_N) &\mapsto \prod_{v \in \Omega_K} \sup_{0 \leqslant i \leqslant N} |x_i|_v \end{aligned}$$

where Ω_K is the set of places of K and for any $x \in K_v$, $|x|_v = |N_{K_v/\mathbf{Q}_p}(x)|_p$ if $v \mid p$. Any morphism of varieties $\phi : V \to \mathbf{P}^N_K$ induces an exponential height

$$\begin{aligned} H : V(K) &\to \mathbf{R}_{>0} \\ x &\mapsto H_N(\phi(x)). \end{aligned}$$

If $U \subset V$ is an open subset, then we would like to describe and understand the asymptotic behavior of the counting function

$$N_{U,H}(B) = \#\{x \in U(K) \mid H(x) \leqslant B\}.$$

Let us first give a few examples:

Example 2.2. If $V = \mathbf{P}^N(\mathbf{Q})$, $\phi = \mathrm{id}$, then an easy Moebius inversion formula gives that

$$N_{V,H}(B) \sim \frac{2^N}{\zeta_{\mathbf{Q}}(N+1)} B^{N+1}$$

as $B \to \infty$ (see figure 1). This result was later generalized by Schanuel in [**Sc**] to the projective space over any number field (see figure 2).

FIGURE 1. Projective space

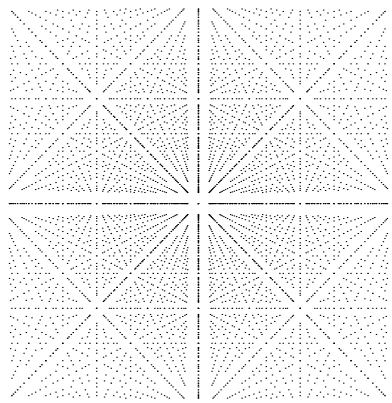
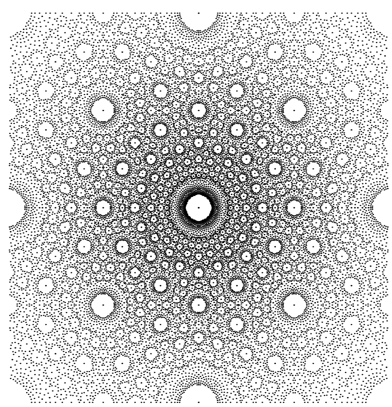


FIGURE 2. Projective line over $\mathbf{Q}(i)$



Example 2.3. If $V = V_1 \times V_2$, $H_i : V_i(K) \to \mathbf{R}_{>0}$, $U_i \subset V_i$ open subsets, then we have the height $H : V \to \mathbf{R}_{>0}$ defined by $H(x_1, x_2) = H_1(x_1)H_2(x_2)$ corresponding to Segre embedding. Assume that

$$N_{U_i,H_i}(B) = C_i B(\log B)^{t_i - 1} + O\left(B(\log B)^{t_i - 2}\right).$$

Then, by [**FMT**, proposition 2]

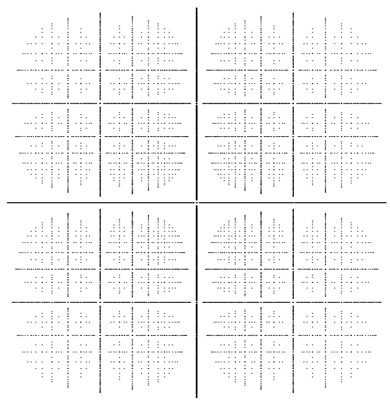
$$N_{U_1 \times U_2, H}(B) \sim \frac{(t_1-1)!(t_2-1)!}{(t_1+t_2-1)!} C_1 C_2 B(\log B)^{t_1+t_2-1}$$

as $B \to \infty$. For $\mathbf{P}^1 \times \mathbf{P}^1$, we get

$$N_{\mathbf{P}^1 \times \mathbf{P}^1, H}(B) \sim CB^2 \log B$$

(see figure 3).

FIGURE 3. Product of two projective lines



Example 2.4. Let $V \to \mathbf{P}^2(\mathbf{Q})$ be the blowing up of $\mathbf{P}^2$ at $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, and $P_3 = (0 : 0 : 1)$. Then V may be seen as a hypersurface in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ given by the equation $x_1 x_2 x_3 = y_1 y_2 y_3$. We put

$$H(P_1, P_2, P_3) = H_1(P_1) H_1(P_2) H_1(P_3)$$

which defines a height $H : V(\mathbf{Q}) \to \mathbf{R}_{>0}$.

On V, there are 6 exceptional lines $E_{i,j} : x_i = 0, y_j = 0$ for $i \neq j$. Let $U = V - \bigcup_{i \neq j} E_{ij}$. We have

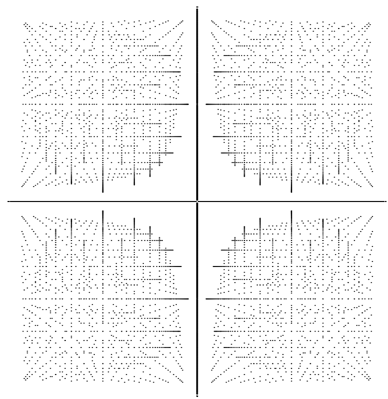
$$N_{(E_{i,j}),H}(B) \sim CB^2$$

and

$$N_{U,H}(B) \sim \frac{1}{6} \left(\prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right) B (\log B)^3$$

(see figure 4). We see that $N_{U,H}(B) = o(N_{(E_{i,j}),H}(B))$. Thus, in this case, the dominant term of the asymptotic behaviour of $N_{V,H}(B)$ is given by the number of points on the six lines. Therefore it can not reflect the geometry of the whole of V. One of the basic idea in the interpretation of the asymptotic behaviour of the number of points of bounded height is that one has to consider open subsets to be able to get a meaningful geometric interpretation.

FIGURE 4. The plane blown up



In all examples the author knows for which it was possible to give a precise estimate of the number of points of bounded height, the asymptotic behaviour is of the form

$$N_{U,H}(B) \sim CB^a(\log B)^{b-1}$$

with $C \geqslant 0$, $a \geqslant 0$ and $b \in \frac{1}{2}\mathbf{Z}$, $b \geqslant 1$. Thus one wishes to give a geometrical interpretation of a, b and C.

3. Manin's principle

We assume that V is a smooth, geometrically integral projective variety of dimension n over the number field K. We also assume that $\omega_V^{-1} = \Lambda^n T_V$ is very ample (in particular, V is a Fano variety). We look only at the height relative to this anticanonical divisor $\phi^*(\mathscr{O}_{\mathbf{P}^N}(1)) = \omega_V^{-1}$, and we assume that $V(K)$ is Zariski dense. The following question is a variant of the conjecture C′ in [**BM**]:

Question 3.1. *Does there exist a dense open subset $U \subset V$ and a constant $C > 0$ such that*

$$N_{U,H}(B) \sim CB(\log B)^{t-1}$$

as $B \to \infty$, where t is the rank of the Picard group of V. (Since V is Fano, $\operatorname{Pic} V$ *is a free* $\mathbf{Z}$*-module of finite rank.)*

In fact, it is even possible to give a conjectural interpretation of C, but to describe this conjectural constant, we first need to express the height in terms of metrics.

Notation 3.2. Let V be a geometrically integral smooth projective variety and H be the height corresponding to an embedding $\phi : V \to \mathbf{P}^N_K$. Let L be $\phi^*(\mathscr{O}_{\mathbf{P}^N}(1))$. We denote by $s_0, \dots, s_N$ the pull-backs in $\Gamma(V, L)$ of the sections $X_0, \dots, X_N$ of $\mathscr{O}_{\mathbf{P}^N}(1)$. We look at L as a line bundle over V and define for any place v of K a metric $\| \cdot \|_v : L(K_v) \to \mathbf{R}$ which is continuous for v-adic topology by the condition:

$$\forall x \in V(K_v), \quad \forall s \in \Gamma(V, L), \quad \|s(x)\|_v = \inf_{\substack{0 \leqslant i \leqslant N \\ s_i(x) \neq 0}} \left| \frac{s(x)}{s_i(x)} \right|_v .$$

Then the height H may be characterized by

$$\forall x \in V(K), \quad \forall s \in \Gamma(V, L), \quad s(x) \neq 0 \Rightarrow H(x) = \prod_{v \in \Omega_K} |s(x)|_v^{-1}.$$

From now on we assume that the above line bundle L is the anticanonical line bundle ω_V^{-1}. We now define a measure on the adelic space $V(\mathbf{A}_K)$ which coincides with the product $\prod_{v \in \Omega_K} V(K_v)$, since V is projective.

Definition 3.3. For any place v of K, we normalize the Haar measure $\mathrm{d}x_v$ on K_v by the conditions

- $\int_{\mathscr{O}_v} \mathrm{d}x_v = 1$ if v is finite,
- $\mathrm{d}x_v([0,1]) = 1$ if K_v is isomorphic to $\mathbf{R}$,
- $\mathrm{d}x_v = i\mathrm{d}z\mathrm{d}\overline{z} = 2\mathrm{d}x\mathrm{d}y$ if K_v is isomorphic to $\mathbf{C}$.

The measure $\boldsymbol{\omega}_v$ on $V(F_v)$ is defined locally by the formula

$$\boldsymbol{\omega}_v = \left\| \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n} \right\|_v \mathrm{d}x_{1,v} \dots \mathrm{d}x_{n,v}$$

if $(x_1, \dots, x_n)$ is a local system of coordinates on $V(K_v)$ in v-adic topology and where $\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}$ is seen as a section of ω_V^{-1}. The fact that these expressions glue together follow from the chosen normalization of the absolute value. Indeed the formula for a change of variables is given by

$$\mathrm{d}y_{1,v} \dots \mathrm{d}y_{n,v} = \left| \det \left(\frac{\partial y_i}{\partial x_j} \right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} \right| \mathrm{d}x_{1,v} \dots \mathrm{d}x_{n,v}$$

(see [**We**, §2.2.1]).

Remark 3.4. At any real place this construction is the classical formula to produce a measure on a differential variety from a continuous section of the canonical line bundle. At almost all finite places, using ideas of Tamagawa and Weil, one may prove the following proposition:

Proposition 3.5. *For allmost all finite* $\mathfrak{p}$ *in* Ω_K,

$$\boldsymbol{\omega}_{\mathfrak{p}}(V(K_{\mathfrak{p}})) = \frac{\# V(\mathbf{F}_{\mathfrak{p}})}{(\#\mathbf{F}_{\mathfrak{p}})^{\dim V}}$$

where $\mathbf{F}_{\mathfrak{p}}$ *is the residue field at* $\mathfrak{p}$.

In particular this implies that the product $\prod_{\mathfrak{p}} \boldsymbol{\omega}_{\mathfrak{p}}(V(K_{\mathfrak{p}}))$ diverges. Therefore we have to introduce convergence factors. These factors are suggested by the Grothendieck-Lefschetz formula.

Definition 3.6. We fix a finite set S of bad places containing all archimedean places and all places of bad reduction. Let $\overline{K}$ be an algebraic closure of K and put $\overline{V} = V \times_K \overline{K}$. Then one defines

$$\forall \mathfrak{p} \in \Omega_K - S, \quad L_{\mathfrak{p}}(s, \operatorname{Pic}(\overline{V})) = \frac{1}{\det(1 - \#\mathbf{F}_{\mathfrak{p}}^{-s} \mid \operatorname{Pic}(V_{\overline{\mathbf{F}}_{\mathfrak{p}}}) \otimes \mathbf{Q})}$$

and the global L-function is given by the Euler product

$$L_S(s, \operatorname{Pic}(\overline{V})) = \prod_{\mathfrak{p} \in \Omega_K - S} L_{\mathfrak{p}}(s, \operatorname{Pic}(\overline{V}))$$

which converges for $\operatorname{Re}(s) > 1$ and admits a meromorphic continuation to $\mathbf{C}$. We define the converging factors by

$$\lambda_v = \begin{cases} L_v(1, \operatorname{Pic}(\overline{V})) \text{ if } v \in \Omega_K - S, \\ 1 \text{ otherwise.} \end{cases}$$

The adelic measure on $V(\mathbf{A}_K)$ is then defined by the formula

$$\boldsymbol{\omega}_H = \lim_{s \to 1} (s-1)^t L_S(s, \operatorname{Pic}(\overline{V})) \, \frac{1}{\sqrt{d_K}^{\dim V}} \prod_{v \in \Omega_K} \lambda_v^{-1} \boldsymbol{\omega}_v,$$

where d_K is the absolute value of the discriminant of K.

Remarks 3.7. (i) The convergence of the product $\prod_{v \in \Omega_K} \lambda_v^{-1} \boldsymbol{\omega}_v$ follows from Lefschetz formula and Weil's conjecture about the absolute value of the eigenvalues of the Frobenius operator which was proven by Deligne [**Del**].

(ii) By definition, the measure $\boldsymbol{\omega}_H$ does not depend on S.

(iii) Note that $\sqrt{d_K}$ is the volume of $\mathbf{A}_K/K$ for the measure $\prod_{v \in \Omega_K} \mathrm{d}x_v$.

To define the conjectural constant it remains to multiply by two rational factors which are the object of the next definition.

Definition 3.8. Let $C^1_{\text{eff}}(V)$ be the cone in $\operatorname{Pic}(V) \otimes_{\mathbf{Z}} \mathbf{C}$ generated by the classes of the effective divisors and $C^1_{\text{eff}}(V)^\vee$ be the dual cone defined by

$$C^1_{\text{eff}}(V)^\vee = \{\, y \in \operatorname{Pic}(V) \otimes_{\mathbf{Z}} \mathbf{R}^\vee \mid \forall x \in C^1_{\text{eff}}(V),\ \langle x, y \rangle \geqslant 0 \,\}.$$

Then

$$\alpha(V) = \frac{1}{(t-1)!} \int_{C^1_{\text{eff}}(V)^\vee} e^{\langle \omega_V^{-1}, y \rangle} \mathrm{d}y$$

where the measure on $\operatorname{Pic}(V) \otimes_{\mathbf{Z}} \mathbf{R}^\vee$ is normalized so that the covolume of the dual lattice $\operatorname{Pic}(V)^\vee$ is one. We also consider the integer

$$\beta(V) = \# H^1(K, \operatorname{Pic}(\overline{V})).$$

Remarks 3.9. (i) The constant $\alpha(V)$ may also be defined as the volume of the domain

$$\{\, y \in C^1_{\text{eff}}(V)^\vee \mid \langle y, \omega_V^{-1} \rangle = 1 \,\}$$

for a suitable measure on the affine hyperplane $\langle y, \omega_V^{-1} \rangle = 1$ (see [**Pe1**, §2.2.5]). Therefore if there exists a finite family $(D_i)_{1 \leqslant i \leqslant r}$ of effective divisors on V such that

$$C^1_{\text{eff}}(V) = \sum_{i=1}^{r} \mathbf{R}_{\geqslant 0}[D_i]$$

then the constant $\alpha(V)$ is rational.

(ii) The constant $\beta(V)$ was introduced by Batyrev and Tschinkel in [**BT1**].

The conjectural constant is then defined as follows

Definition 3.10. We define

$$\theta_H(V) = \alpha(V)\beta(V)\boldsymbol{\omega}_H(\overline{V(K)}),$$

where $\overline{V(K)}$ denotes the closure of the rational points in the adelic space $V(\mathbf{A}_K)$.

We can now give a refined version of the question 3.1:

Empirical formula 3.11. *With notation as in question 3.1, there often exists a dense open subset $U_0 \subset V$ such that for any non-empty subset U of U_0, one has*

$$N_{U,H}(B) \sim \theta_H(V) B (\log B)^{t-1} \tag{F}$$

4. Results

The formula (F) is true in the following cases:

- $V = G/P$, G a reductive algebraic group over K, P a parabolic subgroup of G defined over K. It follows from the work of Langlands on Eisenstein series [**Lan**], (see Franke, Manin, Tschinkel [**FMT**] and [**Pe1**, §6]). We may take $U_0 = V$. In particular, it is true for any quadric.
- V is a smooth projective toric variety, that is an equivariant compactification of an algebraic torus. (see [**Pe1**, §8–11] for particular cases, Batyrev and Tschinkel [**BT1**], [**BT2**], and [**BT4**], Salberger [**Sa**], and de la Bretèche [**Bre1**]). One may take the open orbit as U_0. This case includes the plane blownup in 1, 2, or 3 points, and Hirzebruch surfaces.
- V is an equivariant compactification of an affine space for the action of the corresponding vector space (see Chambert-Loir, Tschinkel [**CLT1**], [**CLT2**], and [**CLT3**]).
- $V = \mathbf{P}^2_{\mathbf{Q}}$ blownup at $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$ (Salberger for an upper bound, de la Bretèche [**Bre2**]).

The formula (F) is compatible with:

- the circle method (In particular, it is true if $V \subset \mathbf{P}^n(\mathbf{Q})$ a hypersurface of degree d, smooth, if $n > 2^d(d-1)$ (see Birch [**Bir**]);
- the product of varieties (see [**FMT**], [**Pe1**, §4];
- numerical tests on computers for some diagonal cubic surfaces (see [**PT1**], [**PT2**]);
- lower bounds for some cubic surfaces (see Slater and Swinnerton-Dyer [**SSD**]). The problem of finding an optimal upper bound for cubic surfaces is still open.

All these examples support the empirical formula, however there is a counter-example which is the object of the next section.

5. The counter-example of Batyrev and Tschinkel

Take $V \subset \mathbf{P}^3 \times \mathbf{P}^3$ defined by $x_0y_0^3 + x_1y_1^3 + x_2y_2^3 + x_3y_3^3 = 0$. We have $\mathrm{Pic}(V) \simeq \mathrm{Pic}(\mathbf{P}^3 \times \mathbf{P}^3) = \mathbf{Z} \times \mathbf{Z}$ and $\omega_V^{-1} = \mathscr{O}_V(3,1)$. In particular, V is a Fano variety. We may use the height

$$H : V(\mathbf{Q}) \to \mathbf{R}_{>0}$$
$$(x_0 : \ldots : x_3), (y_0 : \ldots : y_3) \mapsto H_3(x)^3 H_3(y).$$

If (F) is true for V then there is some open subset U and a constant C such that

$$N_{U,H}(B) \sim CB \log B$$

as $B \to \infty$. There is a projection onto the first coordinate $\pi_1 : V \to \mathbf{P}^3$. If $(x_0 : \ldots : x_3) \in \mathbf{P}^3$, is such that $\prod_{i=0}^{3} x_i \neq 0$, $\pi_1^{-1}(x)$ is a smooth cubic surface; if $x_1/x_0, x_2/x_0, x_3/x_0$ are cubes, then $\operatorname{rk} \operatorname{Pic}(\pi_1^{-1}(x)) = 4$. If (F) is true for the fibre

$$N_{\pi^{-1}(x),H}(B) \sim C_x B (\log B)^3$$

as $B \to \infty$, but these fibers are Zariski dense, so the answer to the question 3.1 can not be positive for both V and the fibers. In fact Batyrev and Tschinkel prove the following more precise result:

***Theorem 5.1* (Batyrev and Tschinkel [BT3]).** *If K contains a cube root of unity, then for all $U \subset V$, $U \neq \emptyset$, (F) does not hold for U.*

6. Methods of counting

We now turn back to the methods used to prove the results given in section 4.

Harmonic analysis: Assume that there exists a dense open subset U of V which is of the form G/H where G is a reductive algebraic group, look at the height zeta function

$$\zeta_{U,H}(s) = \sum_{x \in U(K)} H(x)^{-s}$$

which converges when $\operatorname{Re} s \gg 0$.

The asymptotic behavior of $N_{U,H}(B)$ is given by the meromorphic properties of $\zeta_{U,H}(s)$. If $U = G$, one may use a Poisson formula. If $V = G/P$, $\zeta_{U,H}(s)$ is an Eisenstein series and we may apply the work of Langlands. In both cases the problem may be handled using harmonic analysis.

This type of method do not apply when the variety does not contain an homogeneous space. All other case appearing in the list of section 4 have one preliminary step in common: thay all use a lifting to the universal torsors:

Universal torsors: implicit in the case of a hypersurface in $\mathbf{P}^n(\mathbf{Q})$, it was made explicit by Salberger in [**Sa**] to give a new proof in the case of split toric varieties over $\mathbf{Q}$; it was then used by Salberger and de la Bretèche for the case of the plane blownup in 4 points. The end of this survey is devoted to the description of this preliminary step in those cases.

7. A basic example

In the case of a hypersurface of large dimension, the principle of Manin follows from the following deep theorem which is based upon the Hardy-Littlewood circle method.

***Theorem 7.1* (Birch [Bir]).** *Let $f \in \mathbf{Z}[x_0, \dots, x_N]$ be homogeneous of degree d, and let $W \subset \mathbf{A}^{N+1} - \{0\}$ be the cone defined by $f = 0$. Assume that:*

(i) *W is smooth,*
(ii) *$W(\mathbf{R}) \neq \emptyset$, and for all primes p, $W(\mathbf{Q}_p) \neq \emptyset$,*
(iii) *$N > 2^d(d-1)$.*

Let

$$M_W(B) = \#\Big\{x \in \mathbf{Z}^{N+1} - \{0\} \;\Big|\; f(x) = 0 \text{ and } \sup_{0 \leqslant i \leqslant N} |x_i| \leqslant B\Big\}.$$

Then there exists an explicit $C > 0$ and $\delta > 0$ such that

$$M_W(B) = CB^{N+1-d} + O(B^{N+1-d-\delta}).$$

Let $\pi : \mathbf{A}^{N+1} - \{0\} \to \mathbf{P}^N$, and let $V = \pi(W)$ be the corresponding projective hypersurface. Then $\omega_V^{-1} = \mathscr{O}_V(N+1-d)$, so we may take the height

$$H(x) = H_N(x)^{N+1-d}$$

where H_N was defined in section 2. Then

$$N_{V,H}(B) = \frac{1}{2}\#\left\{x \in \mathbf{Z}^{N+1} - \{0\} \;\middle|\; \begin{cases} f(x) = 0, \\ \sup |x_i|^{N+1-d} \leqslant B, \\ \gcd(x_i) = 1. \end{cases}\right\}.$$

Using Moebius inversion, we get

$$N_{V,H}(B) = \frac{1}{2}\sum_k \mu(k)\#\left\{x \in (k\mathbf{Z})^{N+1} - \{0\} \;\middle|\; \begin{cases} f(x) = 0, \\ \sup_i |x_i|^{N+1-d} \leqslant B \end{cases}\right\},$$

where $\mu : \mathbf{Z}_{>0} \to \{-1, 0, 1\}$ is the Moebius function. Then

$$\begin{aligned} N_{V,H}(B) &= \frac{1}{2}\sum_k \mu(k) M_W\Big(\frac{B^{1/(N+1-d)}}{k}\Big) \sim \frac{1}{2}C\sum_k \frac{\mu(k)}{k^{N+1-d}}B \\ &= \frac{1}{2}\frac{C}{\zeta(N+1-d)}B. \quad \square \end{aligned}$$

The idea behind the introduction of universal torsors is to generalize this simple descent argument to other varieties.

8. Universal torsors

Let V be a smooth, geometrically integral projective variety over K, where $\operatorname{char} K = 0$. Assume (for simplicity) that V is Fano, which means that ω_V^{-1} is ample. Thus, if $\overline{K}$ is an algebraic closure of K, and $\overline{V} = V \times_K \overline{K}$, then $\operatorname{Pic}(\overline{V})$ is a free abelian group of finite rank.

Assume $K = \overline{K}$ first. Let $L_1, \ldots, L_t$ be line bundles on V such that $[L_1], \ldots, [L_t]$ form a basis of $\operatorname{Pic}(V) = \operatorname{Pic}(\overline{V})$. Let $L_i^\times = L_i -$ zero section. Consider

$$\pi : L_1^\times \times_V L_2^\times \times_V \cdots \times_V L_t^\times \to V.$$

On the left we have an action of $\mathbf{G}_m^t$, this is 'the' *universal torsor* of V.

Proposition 8.1. *If $K = \overline{K}$, the universal torsor constructed above does not depend, up to isomorphism, on the chosen basis of the Picard group.*

Proof. Let $L_1', \ldots, L_t'$ be line bundles on V so that $[L_1'], \ldots, [L_t']$ form another basis of the Picard group of V. Let $M = (m_{i,j})$ in $\mathrm{GL}_n(\mathbf{Z})$ be the matrix such that

$$[L_i'] = \sum_{i=1}^{t} m_{j,i}[L_j].$$

In other words for each i in $\{1, \ldots, t\}$, we may fix an isomorphism

$$\psi_i : \bigotimes_{j=1}^{t} L_j^{\otimes m_{j,i}} \xrightarrow{\sim} L_i'.$$

But if $E_1, \ldots, E_m$ are one-dimensional vector spaces and $k_1, \ldots, k_m$ integers there is a canonical map

$$\begin{array}{rcl} \times_{i=1}^m (E_i - \{0\}) & \to & \bigotimes_{i=1}^m E_i^{\otimes k_i} \\ (y_1, \ldots, y_m) & \mapsto & \bigotimes_{i=1}^m y_i^{\otimes k_i} \end{array}$$

where for any vector space E of dimension one, and any non-zero y in E, $y^{\otimes -1}$ is the unique element of the dual $E^\vee$ of E such that $y^{\otimes -1}(y) = 1$. In that way, composing with ψ_i, we get maps

$$\rho_i : \mathop{\times}_{j=1}^{t} L_j^\times \to {L'}_i^\times.$$

This map is equivariant for the action of $\mathbf{G}_m^t$ in the following sense:

$$\forall (z_1, \ldots, z_t) \in \mathbf{G}_m^t(K), \forall y \in \mathop{\times}_{j=1}^{t} L_j^\times(K),\ \rho_i((z_1, \ldots, z_t).y) = \prod_{j=1}^{t} z_j^{m_{j,i}}.\rho_i(y).$$

Note that if ρ'_i is another map from $\times_{j=1}^t L_j^\times$ to $L'^\times_i$ with the same equivariance property, then there is a section $s \in \Gamma(V, \mathbf{G}_m)$ such that

$$\forall y \in \mathop{\times}_{j=1}^{t} L_j^\times(K), \quad \rho'_i(y) = s(\pi(y)).\rho_i(y).$$

But, since V is projective, $\Gamma(V, \mathbf{G}_m) = K^*$ and ρ_i is unique up to multiplication by a constant. The maps ρ_i yield a map

$$\rho : \mathop{\times}_{i=1}^{t} L_i^\times \to \mathop{\times}_{i=1}^{t} L'^\times_i.$$

The matrix M defines a morphism of algebraic groups

$$\begin{array}{rcl} \widetilde{M} : \mathbf{G}_m^t & \to & \mathbf{G}_m^t \\ (z_1, \dots, z_t) & \mapsto & (\prod_{j=1}^t z_j^{m_{j,i}})_{1 \leqslant i \leqslant t} \end{array}$$

and the map ρ is equivariant with respect to $\widetilde{M}$:

$$\forall z \in \mathbf{G}_m^t(K), \quad \forall y \in \mathop{\times}_{i=1}^{t} L_i^\times(K), \quad \rho(z.y) = \widetilde{M}(z).\rho(y).$$

Moreover if ρ' is another map with the same equivariance property, then there is $z \in \mathbf{G}_m^t(K)$ such that $\rho' = z.\rho$. Similarly we may define a map

$$\tau : \mathop{\times}_{i=1}^{t} L'^\times_i \to \mathop{\times}_{i=1}^{t} L_i^\times$$

which is equivariant with respect to $\widetilde{M}^{-1}$. Thus the composite map

$$\tau \circ \rho : \mathop{\times}_{i=1}^{t} L_i{}^\times \to \mathop{\times}_{i=1}^{t} L_i^\times$$

is equivariant with respect to the identity map and therefore coincides with the action of an element of $\mathbf{G}_m^t(K)$. Thus $\tau \circ \rho$ and $\rho \circ \tau$ are isomorphisms. □

For arbitrary fields, a universal torsor may be described as a K-structure on the above torsor. Let us define this notion more precisely:

Recall that there is a contravariant equivalence of categories between the category of algebraic tori, that is algebraic groups T such that $\overline{T}$ is isomorphic to $\mathbf{G}_{m,\overline{K}}^{\dim T}$, and the category of $\mathrm{Gal}(\overline{K}/K)$-lattices, that is $\mathrm{Gal}(\overline{K}/K)$-modules which are free abelian groups of finite rank. One functor is $T \mapsto X^*(T) = \mathrm{Hom}_{\mathrm{alg.gp.}}(\overline{T}, \mathbf{G}_m)$, and contrarily $M \mapsto \mathrm{Spec}(\overline{K}[M])^{\mathrm{Gal}(\overline{K}/K)}$.

Definition 8.2. Let the Neron-Severi torus, T_{NS}, be the torus corresponding to the $\mathrm{Gal}(\overline{K}/K)$-lattice $\mathrm{Pic}\,\overline{V}$.

If G is an algebraic group over K, then a *G-torsor* is a faithfully flat map $\pi : \mathcal{T} \to V$ with an action of G on $\mathcal{T}$ such that locally for the faithfully flat topology, $\mathcal{T} \times_V U \simeq G \times U$, where the isomorphism is compatible with the action of G. (In another language, these are principal homogeneous spaces.)

A T_{NS}-torsor $\mathcal{T} \to V$ is said to be *universal* if $\overline{\mathcal{T}} \to \overline{V}$ is isomorphic as a torsor to $L_1^* \times_V \cdots \times_V L_t^* \to V$.

Why are these universal torsors interesting? The following facts are due to Colliot-Thélène and Sansuc, who introduced the notion of universal torsor.

***Proposition 8.3* (Colliot-Thélène, Sansuc).** *With notations as above,*

- *For all $x \in V(K)$, there exists a unique (up to isomorphism) universal torsor $\pi : \mathcal{T} \to V$ such that $x \in \pi(\mathcal{T}(K))$.*
- *If K is a number field, there exist up to isomorphism only finitely many universal torsors $\pi : \mathcal{T} \to V$ such that $\mathcal{T}(K) \neq \emptyset$.*

This proposition gives us a nice decomposition of the set of rational points

$$V(K) = \bigsqcup_{1 \leqslant i \leqslant m} \pi_i(\mathcal{T}_i(K)).$$

Slogan 8.4. *From an arithmetical point of view, universal torsors should be much simpler than the variety V.*

A slogan does not need to be true, but we may justify this one with the following statement:

***Proposition 8.5* (Colliot-Thélène, Sansuc).** *If $\mathcal{T}^c$ is a smooth projective compactification of a universal torsor $\mathcal{T} \to V$, then $\mathcal{T}^c(\boldsymbol{A}_K)^{\mathrm{Br}} = \mathcal{T}^c(\boldsymbol{A}_K)$. In other words, there are no Brauer-Manin obstruction to the Hasse principle and the weak approximation.*

Example 8.6. Let $V \subset \mathbf{P}^N$ a hypersurface over $\mathbf{Q}$, $\dim V \geqslant 3$, $\deg V = d$, and $N+1-d > 0$, then the cone $W \subset \mathbf{A}^{N+1} - \{0\}$ above V is up to isomorphism the only universal torsor over V.

9. Toric varieties

The following construction is due to Cox. Let T be an algebraic torus and V be a smooth projective equivariant compactification of T. This means that there is an action of T on V, an open subset $U \subset V$, and an isomorphism from U to T compatible with the actions of T.

Denote by $\Sigma(1)$ the set of orbits of codimension 1 in $\overline{V}$. Then there is an exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules

$$0 \to X^*(T) \xrightarrow{\mathrm{div}} \mathbf{Z}^{\Sigma(1)} \xrightarrow{\rho} \mathrm{Pic}(\overline{V}) \to 0$$
$$e_\sigma \mapsto [D_\sigma]$$

where D_σ is the closure of the orbit σ in $\overline{V}$, which is an irreducible divisor of $\overline{V}$. Moreover we have

$$\omega_V^{-1} = \sum_{\sigma \in \Sigma(1)} [D_\sigma].$$

By duality we get an exact sequence of tori

$$1 \to T_{\mathrm{NS}} \to T_{\Sigma(1)} \xrightarrow{\pi} T \to 1.$$

But $T \subset V$ and we want to extend the map π to get a torsor over V. We do this in the following way: We consider the affine space

$$\mathbf{A}_{\Sigma(1)} = \mathrm{Spec}\big((\overline{K}[X_\sigma]_{\sigma \in \Sigma(1)})^{\mathrm{Gal}(\overline{K}/K)}\big)$$

and a closed subset $F \subset \mathbf{A}_{\Sigma(1)}$, defined over $\overline{K}$ as a union of affine subspaces,

$$\overline{F} = \bigcup_{\substack{I \subset \Sigma(1) \\ \bigcap_{\sigma \in I} D_\sigma = \emptyset}} \Big(\bigcap_{\sigma \in I} (X_\sigma = 0)\Big).$$

Note that $\overline{F}$ is stable under the action of the Galois group, so it is defined over K, so we take $\mathcal{T} = \mathbf{A}_{\Sigma(1)} - F$.

Claim 9.1. *For all $x \in T(K)$, the map*

$$T_{\Sigma(1)} \to T$$
$$t \mapsto \pi(t) \cdot x$$

extends to a map $\mathcal{T} \to V$ sending $1 \in T_{\Sigma(1)}$ to x.

Theorem 9.2 **(Colliot-Thélène, Sansuc, Salberger, Madore)**
The above construction gives a bijection between $T(K)/T_{\Sigma(1)}(K)$ and isomorphism classes of universal torsors over V.

We return now to the problem of counting points. We assume that $K = \mathbf{Q}$, that the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $X^*(T)$ and $\Sigma(1)$ are trivial, and that ω_V^{-1} is generated by global sections.

Then we consider

$$\mathscr{M} = \{m \in \mathbf{Z}^{\Sigma(1)} \mid \forall \sigma \in \Sigma(1),\, m_\sigma \geqslant 0 \text{ and } \rho(m) = \omega_V^{-1} \in \mathrm{Pic}\, V\}.$$

For all $m \in \mathscr{M}$, let $X^m \in \mathbf{Q}[X_\sigma]_{\sigma\in\Sigma(1)}$ be the corresponding monomial. We lift the height to the universal torsor by

$$\widetilde{H}((y_\sigma)_{\sigma\in\Sigma(1)}) = \sup_{m\in\mathscr{M}} |X^m((y_\sigma)_{\sigma\in\Sigma(1)})|.$$

***Theorem 9.3* (Salberger [Sa]).** *There exists a height H relative to ω_V^{-1} such that $N_{U,H}(B) = \widetilde{N}(B)/2^{\dim T_{\mathrm{NS}}}$, where $\widetilde{N}(B)$ is the number of $(y_\sigma)_{\sigma\in\Sigma(1)}$ in $\mathbf{Z}^{\Sigma(1)}$ such that*

$$\begin{cases} \widetilde{H}(y) \leqslant B, \\ \forall I \subset \Sigma(1), \bigcap_{\sigma\in I} D_\sigma = \emptyset \Rightarrow \gcd_{\sigma\in I}(y_\sigma) = 1. \end{cases}$$

To prove Manin's conjecture in that case one may then proceed as follows: By use of a Moebius inversion formula, reduce to give an estimate

$$\#\Big\{(y_\sigma)_{\sigma\in\Sigma(1)} \in (\mathbf{Z} - \{0\})^{\Sigma(1)} \;\Big|\; \widetilde{H}(y) \leqslant B\Big\}.$$

and prove that, when B goes to $+\infty$ this is equivalent to

$$\sim \mathrm{vol}\Big(\Big\{(y_\sigma)_{\sigma\in\Sigma(1)} \in \mathbf{R}^{\Sigma(1)} \;\Big|\; \widetilde{H}(y) \leqslant B\Big\}\Big) \sim CB(\log B)^{\operatorname{rk}\operatorname{Pic} V - 1},$$

which proves the Manin conjecture in this case.

10. The plane blownup in 4 Points

The construction is due to Salberger and Skorobogatov. We consider in this section the blowing up $\pi : V \to \mathbf{P}^2$ of $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$, and $P_4 = (1:1:1)$. The exceptional divisors on V are $E_{i,5} = \pi^{-1}(P_i)$ and $E_{i,j}$, the strict pullback of the line through P_k and P_l if $\{i,j,k,l\} = \{1,2,3,4\}$. Then $E_{i,j} \cap E_{k,l} = \emptyset$ if and only if $\{i,j\} \cap \{k,l\} \neq \emptyset$. Then we consider the Grassmannian variety $\mathrm{Gr}(2,5)$ of the planes in $\mathbf{Q}^5$; we may imbed it into $\mathbf{P}(\Lambda^2\mathbf{Q}^5)$. The cone above it, $W \subset \Lambda^2\mathbf{Q}^5$ is given by the Plücker relations:

$$\begin{cases} X_{1,2}X_{3,4} - X_{1,3}X_{2,4} + X_{1,4}X_{2,3} = 0, \\ X_{1,2}X_{3,5} - X_{1,3}X_{2,5} + X_{1,5}X_{2,3} = 0, \\ X_{1,2}X_{4,5} - X_{1,4}X_{2,5} + X_{1,5}X_{2,4} = 0, \\ X_{1,3}X_{4,5} - X_{1,4}X_{3,5} + X_{1,5}X_{3,4} = 0, \\ X_{2,3}X_{4,5} - X_{2,4}X_{3,5} + X_{2,5}X_{3,4} = 0. \end{cases}$$

Indeed the vector space $\Lambda^2\mathbf{Q}^5$ has dimension 10, with basis $e_i \wedge e_j$ if $i \neq j$, giving coordinates $X_{i,j}$. We consider the closed subset $F \subset W$ given by

$$F = \bigcup_{\{i,j\}\cap\{k,\ell\}\neq\emptyset} ((X_{i,j}=0)\cap(X_{k,\ell}=0))$$

and define $\mathcal{T} = W - F$. There is an action $\mathbf{G}_m^5 \subset GL_5(\mathbf{Q})$ on $\mathcal{T}$, and $\mathcal{T}/\mathbf{G}_m^5$ is isomorphic to V and $\mathcal{T} \to V$ is up to isomorphism the only universal torsor.

We put

$$\mathscr{M} = \left\{(m_{i,j}) \in \mathbf{Z}^{\{i,j\},i\neq j} \;\middle|\; \sum m_{i,j}E_{i,j} = \omega_V^{-1}\right\}$$

and for $m \in \mathscr{M}$, $X^m \in \mathbf{Q}[X_{i,j}]_{i\neq j}$. Then we may lift the height using

$$\widetilde{H}((y_{i,j})) = \sup_{m\in\mathscr{M}} |X^m(y)|.$$

***Proposition 10.1* (Salberger).** *There is a height H relative to ω_V^{-1} such that $N_{U,H}(B) = \widetilde{N}(B)/2^{\dim T_{\mathrm{NS}}}$ where $\widetilde{N}(B)$ is the number of $(y_{i,j})$ in $W(\mathbf{Z})$ such that*

$$\begin{cases} \widetilde{H}(y_{i,j}) \leqslant B, \\ \{i,j\}\cap\{k,l\} \neq \emptyset \Rightarrow \gcd(y_{i,j}, y_{k,l}) = 1. \end{cases}$$

This description, which was made by Salberger to get an upper bound on the number of points of bounded height, was the first step of the proof of the empirical formula (F) which was given by de la Bretèche in [**Bre2**].

11. Generalization

As was shown in [**Pe2**] and [**Pe3**], it is possible to generalize the lifting described in sections 7, 9, and 10 to a more general setting. The first remark which enables this generalization is the fact that universal torsors are equipped with a gauge form:

Proposition 11.1. *If V is a Fano variety over K, and $\mathcal{T}$ a universal torsor over V, then*

- *the canonical bundle $\omega_{\mathcal{T}}$ is trivial,*
- *$\Gamma(\mathcal{T}, \mathbf{G}_m) = K^*$.*

As usual, a gauge form yields a measure:

Definition 11.2. Up to a constant, there exists a unique non-vanishing section of the canonical line bundle. Let $\breve{\omega}_{\mathcal{T}}$ be such a section. This section defines for any place v of K a measure $\boldsymbol{\omega}_{\mathcal{T},v}$ on $\mathcal{T}(K_v)$ which is locally defined by

$$\boldsymbol{\omega}_{\mathcal{T},v} = \left|\left\langle \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_N}, \breve{\omega}_{\mathcal{T}} \right\rangle\right| \mathrm{d}x_{1,v} \ldots \mathrm{d}x_{N,v}$$

where $(x_1, \ldots, x_N)$ is an analytic local system of coordinates on $\mathcal{T}(K_v)$ for v-adic topology. We then define a canonical measure on $\prod_{v \in \Omega_K} \mathcal{T}(K_v)$ by

$$\boldsymbol{\omega}_{\mathcal{T}} = \frac{1}{\sqrt{d_K}^{\dim \mathcal{T}}} \prod_{v \in \Omega_K} \boldsymbol{\omega}_{\mathcal{T},v}.$$

Remarks 11.3. (i) If V is a hypersurface in $\mathbf{P}^n$, the measure $\boldsymbol{\omega}_{\mathcal{T},v}$ coincides with the classical Leray measure on $\mathcal{T}(K_v)$.

(ii) The measure $\boldsymbol{\omega}_{\mathcal{T}}$ does not depend on the choice of the section $\breve{\omega}_{\mathcal{T}}$.

(iii) The volume $\boldsymbol{\omega}_{\mathcal{T}}(\prod_{v \in \Omega_K} \mathcal{T}(K_v))$ is infinite, but if $S \subset \Omega_K$ is a finite set of places containing the archimedean ones and $\mathscr{T}$ a model of $\mathcal{T}$ over the ring of S-integers $\mathscr{O}_S$, then the product $\prod_{v \in \Omega_K - S} \boldsymbol{\omega}_{\mathcal{T},v}(\mathscr{T}(\mathscr{O}_v))$ converges.

Let $\mathcal{T}_1, \ldots, \mathcal{T}_r$ be torsors representing all isomorphism classes of universal torsors over V having a rational point. It is then possible to construct families of integrable functions $\Psi_{i,j,B} : \prod_{v \in \Omega_K} \mathcal{T}(K_v) \to \mathbf{R}_{>0}$ such that upper bounds for the difference

$$\left| \sum_{y \in \mathcal{T}_i(K)} \Psi_{i,j,B}(y) - \int_{\prod_{v \in \Omega_K} \mathcal{T}(K_v)} \Psi_{i,j,B}(y) \boldsymbol{\omega}_{\mathcal{T}} \right|$$

yield an upper bound for the difference

$$|N_{U,H}(B) - \theta_H(V) B (\log B)^{t-1}|.$$

The liftings described in the previous sections may be seen as particular case of this descent argument.

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