Descent on simply connected surfaces over algebraic number fields

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Introduction

Let X be a smooth projective surface over an algebraic number field k, and denote by $X(k)$ its set of k-rational points. Let $\overline{k}$ be an algebraic closure of k and $\overline{X} = X \times_k \overline{k}$. If $Pic(\overline{X})$ is finitely generated and torsion free, Colliot-Thélène and Sansuc [CT/S1,2] have established a theory of descent, which associates to X a finite set of auxiliary objects, *universal torseurs*. These are simpler arithmetically than X, in that there is no Brauer-Manin obstruction to the Hasse principle for the algebraic part of the Brauer group of a smooth compactification of such a universal torseur (see below for more details and definitions). The images of the rational points of the universal torseurs via structure morphisms to X give a finite partition of $X(k)$. This works well in a large number of cases for surfaces of geometric genus zero. For example, if X is a surface that becomes birational to the projective plane over $\overline{k}$, the theory has led to a much greater understanding of the arithmetic of X, and in some examples, an almost complete solution to most diophantine problems (see e.g. [CT/S/SwD] and [Sk2], §7).

On the other hand, we have little general understanding of the arithmetic of surfaces of positive geometric genus. In particular, we don't know very well the role played by the "transcendental" part of the Brauer group in the existence and distribution of rational points. Is the Brauer-Manin obstruction

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to the Hasse principle the only one for any geometrically simply connected surface? If a $K3$ surface X has a k-rational point, does it have infinitely many, and is $X(k)$ Zariski dense in X? For a surface X of general type, are almost all of the rational points in the images of the k-rational points of finitely many rational maps to X from abelian and rational varieties? If U denotes the complement of the images of these maps, is $U(L)$ finite for all number fields L (Bombieri-Lang conjecture)? See e.g. ([HiSi], Part F.5, Conjectures F.5.2.1 and F.5.2.2) for more precise statements of the conjectures. For $K3$ surfaces, see [BM] for conjectures on the growth of rational points of bounded height, and [BT1,2] for some results on the density of rational points.

The purpose of this paper is to establish a formal framework that we hope will lead to a better understanding of the arithmetic of a (geometrically) simply connected surface X of nonzero geometric genus. Our theory associates to X a conjecturally finite set of auxiliary objects that are not schemes, in general. Rather, they are *gerbes* which are bound by the second étale cohomology group of $\overline{X}$ with $\mathbf{Z}/n\mathbf{Z}(2)$-coefficients. The images of the rational points of these objects via the structure morphisms to X partition $X(k)$, and we expect, but cannot show at the moment, that they are "simpler" than X, in terms of the Brauer-Manin obstruction to the Hasse principle. We have also not made much progress in describing these gerbes in terms of explicit "equations," so that they may be computationally useful. Nonetheless, we hope that this note might inspire others to go further, and we believe that $K3$ surfaces of geometric Picard number 20 provide a promising class of varieties for computation. We develop the theory in some detail in this case below. For surfaces of general type, we hope that our theory might eventually yield some insight into the Bombieri-Lang conjecture, but this is probably a long way off.

The exposition here is a bit uneven, in that some proofs are given in quite a bit of detail, while others are only sketched or omitted. Complete details as well as more examples will be given elsewhere, but we hope what is written here will give the reader a good idea of how the theory should work.

The first author would like to thank J.-L. Colliot-Thélène for introducing him to the theory of descent on rational surfaces some years ago, and for several helpful discussions about this work. Since that time, there have been major advances in the theory of algebraic cycles (especially torsion 0-cycles

on surfaces of nonzero geometric genus), and this paper may be regarded as an attempt to apply these to make a more general theory. The idea that gerbes could be useful in this theory and are reasonable algebro-geometric objects was inspired by the work of Harari and Skorobogatov (see [HS] and [Sk2]). Harari [Ha] was also the first person to consider obstructions to the Hasse principle for the transcendental part of the Brauer group. We thank E. Peyre and N. Yui for helpful discussions. This paper was completed while the first author enjoyed the hospitality of Université de Paris-Sud, Institut Fourier (Grenoble) and Université Denis Diderot (Paris 7). The audience in the first author's course on this material in Grenoble listened patiently as he explained some rather raw material. Finally, we thank B. Poonen and Y. Tschinkel for organizing and giving us the opportunity to present this work at this excellent conference, and the American Institute of Mathematics for providing a very stimulating atmosphere.

1 Notation and Preliminaries

Let k be a field and X a smooth, projective, geometrically connected variety over k. We denote by $\overline{k}$ a separable closure of k, by G the absolute Galois group of k and $\overline{X} = X \times_k \overline{k}$. We will say that X is *geometrically simply connected* if $\pi_1^{alg}(\overline{X}) = \{1\}$. When we assume this condition, we really only need that the abelianized fundamental group of $\overline{X}$ is trivial. Let ℓ be a prime number different from the characteristic of k. We denote by $\mathbf{Z}/\ell^m\mathbf{Z}(r)$ the étale sheaf $\mathbf{Z}/\ell^m\mathbf{Z}$ Tate-twisted r-times.

The notation $Br(X)$ will be used for the cohomological Brauer group, $H^2_{et}(X, \mathbf{G}_m)$. The Hochschild-Serre spectral sequence:

$$E_2^{r,s} = H^r(k, H^s(\overline{X}, \mathbf{G}_m)) \Longrightarrow H^{r+s}(X, \mathbf{G}_m)$$

gives an exact sequence:

$$\textbf{(1.1)}\quad Br(k) \to \ker[Br(X) \to Br(\overline{X})] \xrightarrow{f} H^1(k, Pic(\overline{X})) \to H^3(k, \mathbf{G}_m).$$

The group $\ker[Br(X) \to Br(\overline{X})]$ is called the *algebraic part* of the Brauer group. If k is a number field, then $H^3(k, \mathbf{G}_m) = 0$, and so the map f is surjective. The image of $Br(X)$ in $Br(\overline{X})$ will be called the *transcendental part* of the Brauer group. Of course, this is a quotient of $Br(X)$.

For A an abelian group and ℓ a prime number, we denote by $A\{\ell\}$ the ℓ-primary part of A and $A[\ell^n]$ the subgroup of A of elements killed by ℓ^n. *The ℓ-Tate module* of A is $\varprojlim_n A[\ell^n]$ and will be denoted $T_\ell(A)$. If A is either a finitely generated $\mathbf{Z}_\ell$-module or an ℓ-primary torsion module of finite cotype, we denote by A^* the Pontryagin dual of A. We denote by Div the maximal divisible subgroup of an ℓ-primary torsion module of finite cotype.

If X is smooth and projective over an algebraically closed field, we denote by $NS(X)$ the group of divisors modulo algebraic equivalence. This is a finitely generated abelian group. If X is simply connected, then this group is equal to the Picard group, $Pic(X)$. By the *geometric Picard number* of X over an arbitrary field k, we shall mean the rank of the Néron-Severi group of $\overline{X}$. By the *geometric genus* of a smooth projective surface X over k, we mean the dimension of the k-vector space of global holomorphic 2-forms on X.

If k is an algebraic number field of finite degree over $\mathbf{Q}$, we denote places of k by v, the completion of k at such a place by k_v and $\mathbf{A}_k$ the ring of adeles of k. If S is a finite set of places of k, we let G_S denote the Galois group of a maximal extension of k that is unramified outside S. If X is a variety over k and we are speaking of the ℓ-adic étale cohomology of $\overline{X}$, then S will always contain the archimedean places, the places above ℓ and the places of bad reduction of X.

We shall denote by $CH^i(X)$ the group of codimension i-cycles modulo rational equivalence; when $i = \dim(\mathrm{X})$, we denote this group by $CH_0(X)$ and by $A_0(X)$ the subgroup of zero-cycles of degree zero.

It will be helpful for the reader to have some familiarity with the theory of descente of Colliot-Thélène/Sansuc (see [CT/S1,2] and [Sk2]), but in the next section, we will quickly review their theory.

2 Rapid review of the theory of Colliot-Thélène-Sansuc

The basic references for this section are [CT/S1,2] and [Sk2]. Let X a smooth projective surface such that $Pic(\overline{X})$ is torsion free, let S be the

Galois module whose group of characters is $Pic(\overline{X})$, and put $S_X = S \times_k X$. Recall the fundamental exact sequence of descente (see Théorème 1.5.1 of [CT/S2]):

$$0 \to H^1(k,S) \to H^1(X,S) \stackrel{\chi}{\to} H^1(\overline{X},S)^G \stackrel{\partial}{\to} H^2(k,S) \to H^2(X,S) \to \cdots .$$

As shown by Harari-Skorobogatov ([HS], Proposition in Appendix B and [Sk2], Proposition 2.3.1), this can be obtained from the exact sequence of terms of low degree for the Hochschild-Serre spectral sequence:

$$H^r(k, H^s(\overline{X}, S)) \Longrightarrow H^{r+s}(X, S).$$

We have that $H^1(\overline{X},S)^G \cong \mathrm{End}_G(Pic(\overline{X}))$. A *universal torseur* is an element of $H^1(X,S)$ whose image via χ is the identity map of $\mathrm{End}_G(Pic(\overline{X}))$. The image of the identity map in $H^2(k,S)$ via ∂ in the exact sequence above is called the *elementary obstruction.* It is proved in ([CT/S2], Théorème 2.1.2(a)) that the Picard group of a smooth compactification $\overline{\mathcal{T}^c}$ of a universal torseur is a permutation module. Hence, by the exact sequence (**1.1**) of §1, the algebraic part of the Brauer group is reduced to the image of the Brauer group of k. Thus there is no Brauer-Manin obstruction to the Hasse principle for the algebraic part of the Brauer group on $\overline{\mathcal{T}^c}$. Colliot-Thélène/Sansuc also derive explicit equations for the restriction of the universal torseurs to suitable open subsets of X ([CT/S2], Théorème 2.3.1 and [Sk2], §4.2, p. 71 and §4.3), and they interpret the Brauer-Manin obstruction to the Hasse principle in terms of the universal torseurs ([CT/S2], Théorème 3.5.1; see also [Sk2], §6.1).

If the geometric genus of X is zero, then we have that $Pic(\overline{X})/n \cong H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(1))$ and $S[n] = H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2))$. We in effect take $S[n]$ instead of S as the point of departure for our theory for surfaces of nonzero geometric genus, because in this case there is no group-scheme S such that $S[n] = H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2))$ for all n. *This is the fundamental reason why gerbes are required instead of torseurs.*

It is somewhat tedious but straightforward to verify that we can recover the theory of [CT/S1,2] for surfaces of geometric genus zero using our theory below, although for the applications in those papers our theory is much clumsier.

3 The Homological Algebra of Descent

In this section, we outline the homological algebra needed to set up the general theory of descent. Let

$$p: \ X \to \operatorname{Spec} k$$

be the structure morphism. If Y is another geometrically integral k-variety, let:

$$q: \ X \times_k Y \to X$$

be projection onto the first factor. Consider the Leray spectral sequence:

$$E_2^{r,s} = H^r(X, R^s q_* \mathbf{Z}/n\mathbf{Z}(2)) \Longrightarrow H^{r+s}(X \times_k Y, \mathbf{Z}/n\mathbf{Z}(2)).$$

Assume now that Y is proper over k. Then by the proper base change theorem, the stalk of $R^s q_* \mathbf{Z}/n\mathbf{Z}(2)$ at a geometric point $\overline{x} \in X$ is equal to the étale cohomology group

$$H^s(Y \times_X \overline{x}, \mathbf{Z}/n\mathbf{Z}(2)).$$

By the smooth and proper base change theorem, this sheaf is locally constant, represented by $H^s(\overline{Y}, \mathbf{Z}/n\mathbf{Z}(2))$.
Let

$$H^4(X \times_k Y, \mathbf{Z}/n\mathbf{Z}(2))^0 = \ker[H^4(X \times_k Y, \mathbf{Z}/n\mathbf{Z}(2)) \to H^0(X, R^4 q_* \mathbf{Z}/n\mathbf{Z}(2))].$$

Now assume that Y is a smooth, proper, geometrically connected and simply connected surface over k. Then from the spectral sequence, we get a map:

$$H^4(X \times_k Y, \mathbf{Z}/n\mathbf{Z}(2))^0 \to H^2(X, R^2 q_* \mathbf{Z}/n\mathbf{Z}(2)).$$

Consider the cycle map:

$$CH^2(X \times_k Y)/n \to H^4(X \times_k Y, \mathbf{Z}/n\mathbf{Z}(2));$$

Let $CH^2(X \times_k Y)^0$ denote the inverse image of $H^4(X \times_k Y, \mathbf{Z}/n\mathbf{Z}(2))^0$. We denote by $c_{2,2}$ the composite map

$$c_{2,2}: \ CH^2(X \times_k Y)^0/n \to H^4(X \times_k Y, \mathbf{Z}/n\mathbf{Z}(2))^0 \to H^2(X, R^2 q_* \mathbf{Z}/n\mathbf{Z}(2))$$

Now take $Y = X$ in this discussion. Then from the Künneth formula (see e.g. [Mi], Ch. Vi, Lemma 8.7), we get:

Lemma 1

$$H^2(\overline{X}, R^2q_*\mathbf{Z}/n\mathbf{Z}(2)) \cong \mathrm{End}(H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(1))).$$

Definition 1 *A universal n-gerbe is an element of $H^2(X, R^2q_*\mathbf{Z}/n\mathbf{Z}(2))$, which is in the subgroup generated by the image of $c_{2,2}$ and the image of $H^2(k, R^2q_*\mathbf{Z}/n\mathbf{Z}(2))$, and whose image in $H^2(\overline{X}, R^2q_*\mathbf{Z}/n\mathbf{Z}(2))$ via the natural map is the identity endomorphism. A universal ℓ-adic gerbe is an inverse system of universal ℓ^m-gerbes $\mathcal{G}_{\ell^m}$.*

For some basic notions about gerbes, see the Appendix to this paper, and for more details, see ([Mi], Ch. IV, §2 and [L/MB], §3). Note that since X and Y are geometrically simply connected, the kernel of the map:

$$H^2(X, R^2q_*\mathbf{Z}/n\mathbf{Z}(2)) \to H^2(\overline{X}, R^2q_*\mathbf{Z}/n\mathbf{Z}(2)),$$

is the image of $H^2(k, H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2)))$. Thus the set of universal n-gerbes is either empty or a principal homogeneous space under the image of $H^2(k, H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2)))$ in $H^2(X, R^2q_*\mathbf{Z}/n\mathbf{Z}(2))$.

Let $Ger(X, k, n)$ be the set of universal n-gerbes. If this is nonempty, then we have a pairing:

$$X(k) \times Ger(X, k, n) \to H^2(k, H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2)))$$

$$(P, \mathcal{G}) \mapsto \mathcal{G}_P,$$

where $\mathcal{G}_P$ denotes the pullback of $\mathcal{G}$ via the inclusion $P \to X$. We have that $\mathcal{G}_P = 0$ if and only if $P \in p_{\mathcal{G}}\mathcal{G}(k)$, where $p_{\mathcal{G}} : \mathcal{G} \to X$ is the structure morphism. Assume there is a universal n-gerbe, $\mathcal{G}$. For $\alpha \in H^2(k, H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2)))$, let $\mathcal{G}_\alpha$ be the universal n-gerbe whose class in $H^2(X, R^2q_*\mathbf{Z}/n\mathbf{Z}(2))$ is given by $[\mathcal{G}] - \alpha$. Then we have:

$$X(k) = \bigsqcup_{\alpha \in H^2(k, H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2)))} p_{\mathcal{G}_\alpha}\mathcal{G}_\alpha(k).$$

If k is a number field, then this disjoint union should be finite and bounded independently of n, and we will show this below for $K3$ surfaces of geometric Picard number 20.

3.1 The elementary obstruction

An obstruction to X having a rational point is it having a universal n-gerbe for all n. If X has a k-rational point, then the class of the diagonal in $CH^2(X \times_k X)$ can be modified by subtracting the class of $X \times P$, where $P \in X(k)$, to give a class in $H^2(X, R^2q_*\mathbf{Z}/n\mathbf{Z}(2))$ that gives the identity map in

$$H^2(\overline{X}, R^2q_*\mathbf{Z}/n\mathbf{Z}(2)) \cong \mathrm{End}(H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(1))).$$

The same can be done if X has a zero-cycle of degree one. But it is not true, in general, that X has a universal n-gerbe, and we call this *the elementary obstruction.*

Lemma 2 *Every section*

$$j:\ Br(X)[n] \to Br(k)[n]$$

to the natural map:

$$Br(k)[n] \to Br(X)[n]$$

determines a unique universal n-gerbe $\mathcal{G}$ with the property that the image of $\mathcal{G}$ under the map

$$H^2(X, R^2q_*\mathbf{Z}/n\mathbf{Z}(2)) \to H^2(k, R^2q_*\mathbf{Z}/n\mathbf{Z}(2))$$

is trivial. Conversely, a universal n-gerbe determines such a section.

Unfortunately, we have not been able as of yet to compute the elementary obstruction very effectively. We believe strongly that it should be described by the G-extension class of the exact sequence obtained from the Gersten-Qullen complex:

$$0 \to H^1(\overline{X}, \mathcal{K}_2) \to \bigoplus_{x \in \overline{X}^1} \overline{k}(x)^*/[K_2\overline{k}(X)/H^0(\overline{X}, \mathcal{K}_2)] \to \bigoplus_{x \in \overline{X}^2} \mathbf{Z} \to CH^2(\overline{X}) \to 0.$$

While it is expected that for k a number field and X geometrically simply connected over k, $CH^2(\overline{X}) = \mathbf{Z}$, this is not known for one single surface of positive geometric genus. Nonetheless, since $A_0(\overline{X})$ is uniquely divisible in this case (Roitman theorem [Ro]), we have:

$$Ext^2_G(CH^2(\overline{X}), H^1(\overline{X}, \mathcal{K}_2)) \cong Ext^2_G(\mathbf{Z}, H^1(\overline{X}, \mathcal{K}_2)),$$

and so the elementary obstruction should be an element of $H^2(k, H^1(\overline{X}, \mathcal{K}_2))$, which can be regarded as an element of $H^2(k, H^2(\overline{X}, \mathbf{Q}/\mathbf{Z}(2))$, since

$$H^1(\overline{X}, \mathcal{K}_2)_{tors} \cong H^2(\overline{X}, \mathbf{Q}/\mathbf{Z}(2))$$

, and the torsion free quotient of $H^1(\overline{X}, \mathcal{K}_2)$ is uniquely divisible (see [CT/R], Theorems 2.1 and 2.2 for basic results about the structure of $\mathcal{K}_2$-cohomology). Note that when X has geometric genus zero and is geometrically connected, it follows from ([CT/R], Theorem 2.12) that the map:

$$NS(\overline{X}) \otimes_{\mathbf{Z}} \overline{k}^* \to H^1(\overline{X}, \mathcal{K}_2)$$

is injective with uniquely divisible cokernel, and $S(k) \cong NS(\overline{X}) \otimes \overline{k}^*$. Thus

$$H^2(k, S) \cong H^2(k, H^1(\overline{X}, \mathcal{K}_2)),$$

and our proposal for the elementary obstruction would generalize that of Colliot-Thélène and Sansuc.

Now suppose the elementary obstruction is trivial everywhere locally. Then a diagram chase, using the fact that the map:

$$H^2(k, Br(\overline{X})(1)) \to \bigoplus_v H^2(k_v, Br(\overline{X})(1))$$

is injective ([J1], Theorem 3d)), would then show:

Proposition 1 *Assume the elementary obstruction lies in $H^2(k, H^2(\overline{X}, \mathbf{Q}/\mathbf{Z}(2)))$ as explained above, and that it vanishes on X_v for all v. Then the elementary obstruction for X lies in the group:*

$$\operatorname{Im} \mathrm{f} \cap \operatorname{Im} \partial,$$

where

$$H^2(k, Pic(\overline{X}) \otimes \mathbf{Q}/\mathbf{Z}(1)) \xrightarrow{f} \bigoplus_v H^2(k_v, Pic(\overline{X}) \otimes \mathbf{Q}/\mathbf{Z}(1))$$

is the localization map, and

$$\bigoplus_v H^1(k_v, Br(\overline{X})(1)) \xrightarrow{\partial} \bigoplus_v H^2(k_v, Pic(\overline{X}) \otimes \mathbf{Q}/\mathbf{Z}(1))$$

is the boundary map in the Galois cohomology sequence for the exact sequence of $Gal(\overline{k}_v/k_v)$-modules:

$$(**)\ 0 \to Pic(\overline{X}) \otimes \mathbf{Q}/\mathbf{Z}(1)) \to H^2(\overline{X}, \mathbf{Q}/\mathbf{Z}(2)) \to Br(\overline{X})(1) \to 0.$$

Remark 1 *It should be possible to reformulate the theory using the group $H^1_{et}(X, R^1q_*\mathcal{K}_2)$. We should have an injection:*

$$H^1_{et}(X, R^1q_*\mathcal{K}_2)/n \to H^2(X, R^2q_*\mathbf{Z}/n\mathbf{Z}(2)),$$

but we have been unable to prove this.

4 The Tate conjecture and the Brauer group

Let X be smooth and projective over a field k that is finitely generated over its prime subfield. Denote by (T_ℓ) the statement that the divisor class map:

$$Pic(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to H^2(\overline{X}, \mathbf{Q}_\ell(1))^G$$

is surjective (the Tate conjecture for divisors). Note that this has been proved for abelian varieties by Faltings [Fa], and together with the Kuga-Satake construction ([KS]; see also [De]), this implies that (T_ℓ) is true for any ℓ for $K3$ surfaces over a field of characteristic 0. Note also that the statement (T_ℓ) is trivial for varieties X for which $Br(\overline{X})$ is finite, which is the case for surfaces of geometric genus zero. This is why (T_ℓ) does not figure into the theory of Colliot-Thélène/Sansuc.

Proposition 2 *Let X be a geometrically simply connected surface over k as above, and assume statement (T_ℓ) is true. Then $Br(X)\{\ell\}/Br(k)\{\ell\}$ is a finite group. If (T_ℓ) is true for every ℓ and there exists a smooth specialization Y of X modulo a place of k with the same geometric Picard number as X and for which (T_ℓ) is true for some ℓ for Y, then the whole group $Br(X)/Br(k)$ is finite.*

Sketch of Proof: This follows easily from the following facts

(i) The simple connectivity of X and the Hochschild-Serre spectral sequence give an isomorphism:

$$H^2(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))/H^2(k, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1)) \cong H^2(\overline{X}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))^G.$$

(ii) If (T_ℓ) is true, then $Br(X)\{\ell\}/Br(k)\{\ell\}$ is the quotient of $H^2(\overline{X}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))^G$ by its maximal ℓ-divisible subgroup, which is given by the image of $NS(X) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell$.

(iii) The cohomology sequence of the Kummer sequence for the étale topology identifies the quotient

$$[H^2(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))/H^2(k, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))]/NS(X) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell$$

with $Br(X)\{\ell\}/Br(k)\{\ell\}$.

(iv) If (T_ℓ) is true for all ℓ and there is a smooth specialization Y of X modulo a place v of k with the same Picard number for which (T_ℓ) is true, then Tate has shown ([Ta], Theorem 5.2) that (T_ℓ) is true for every $\ell \neq \operatorname{char} F_v$, and the whole prime-to-char F_v-part of the Brauer group of Y is finite. Comparing the fixed modules of $Br(\overline{X})^G$ with $Br(\overline{Y})^{Gal(\overline{F}_v/F_v)}$ using the Kummer sequence and the smooth and proper base change theorem and the fact that the (co)-specialization map:

$$Br(\overline{X}) \to Br(\overline{Y})$$

is an isogeny on prime to char F_v-parts because the Picard numbers are the same, we get the finiteness of the prime to $p = \operatorname{char} F_v$-part of $Br(X)/Br(k)$. Since (T_p) is true, we get finiteness of $Br(X)\{p\}/Br(k)\{p\}$. This completes the proof of the proposition.

Proposition 3 *(Shioda-Inose [SI]) Let X be a $K3$ surface over $\mathbf{C}$ of Picard number 20. Then X is defined over a number field, and may be realized as a double cover of the Kummer surface associated to the abelian surface $E \times E'$, where E, E' are isogenous elliptic curves with complex multiplication.*

Theorem 1 *Let X be a $K3$ surface of geometric Picard number 20 over a field k that is finitely generated over the prime subfield. Then the group $Br(X)/Br(k)$ is finite.*

Proof: By the Shioda-Inose theorem, there are isogenous CM elliptic curves E and E' such that X is a double cover of the Kummer surface Y of $E \times E'$. Since the Picard numbers and second Betti numbers of Y and X are the same, the natural map:

$$Br(X) \to Br(Y)$$

is an isogeny, with kernel killed by 2. Since (T_2) is true for X, $Br(X)\{2\}/Br(k)\{2\}$ is finite, and it will then suffice to prove that the prime to 2 part of $Br(Y)/Br(k)$ is finite. Let K be the CM field of E and let $\wp$ be a prime ideal of k with residue field $\mathbf{F}_p$, for p an odd prime number. Then E and E' have good ordinary reductions $E_\wp$ and $E'_\wp$ modulo $\wp$. The Picard number of the Kummer surface $Y_\wp$ associated to $E_\wp \times E'_\wp$ is 20, and since this is ordinary, the Tate conjecture is known (see [Ny] or [NO]). By Proposition 1, we then get finiteness of $Br(Y)/Br(k)$. This completes the proof.

Remark 2 *Using results of Morrison [Mo], some of the results of this section can be extended to some $K3$-surfaces of geometric Picard number 19.*

5 Universal gerbes and the higher Abel-Jacobi mapping

Assume that X has a universal n-gerbe, $\mathcal{G}$. Then we can define a map:

$$\theta_{\mathcal{G}}:\ CH_0(X) \to H^2(k, H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2)))$$

by sending a rational point $P \in X(L)$ to $cor_{L/k}\mathcal{G}_P$, and extending by linearity. Here

$$cor_{L/k}\ : H^2(L, H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2))) \to H^2(k, H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2)))$$

is the corestriction map. If we restrict this map to $A_0(X)$, then it is independent of the choice of $\mathcal{G}$, since if we chose another one, say $\mathcal{G} - \alpha$, the α will cancel out after taking the difference between two cycles of the same degree.

Recall the higher Abel-Jacobi mapping [R]: the Hochschild-Serre spectral sequence

$$H^r(k, H^s(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2))) \Longrightarrow H^{r+s}(X, \mathbf{Z}/n\mathbf{Z}(2))$$

and the geometric simple connectivity of X allow us to define a map:

$$d_{2,n}:\ A_0(X) \to H^2(k, H^2(\overline{X}, \mathbf{Z}/n\mathbf{Z}(2))).$$

The proof of the following proposition is hypertechnical, and we will give it elsewhere. It uses the explicit description of $d_{2,n}$ in terms of certain 2-extensions obtained from the long exact sequence of cohomology of support in a codimension 2 cycle (see [J2]).

Proposition 4 *If X is geometrically simply connected, the map $\theta_{\mathcal{G}}$, restricted to $A_0(X)$, is the same (up to sign) as the higher Abel-Jacobi mapping $d_{2,n}$.*

Now take $n = \ell^m$, a power of a prime number ℓ. We can then consider the higher Abel-Jacobi mapping in continuous ℓ-adic étale cohomology (see [R] for this mapping with $\mathbf{Q}_\ell$-coefficients; the geometric simple connectivity of X allows us to define it with $\mathbf{Z}_\ell$-coefficients using the same arguments as used above for $\mathbf{Z}/n\mathbf{Z}$-coefficients):

$$d_{2,\ell}:\ A_0(X) \to H^2(k, H^2(\overline{X}, \mathbf{Z}_\ell(2))).$$

Proposition 5 *The image of $d_{2,\ell}$ is a finitely generated $\mathbf{Z}_\ell$-module (which is conjecturally torsion).*

Proof: This follows easily from the fact that for a suitable finite set of places S of k, we can factor $d_{2,\ell}$ through $H^2(G_S, H^2(\overline{X}, \mathbf{Z}_\ell(2)))$ (see §1 for notation), and this group is a finitely generated $\mathbf{Z}_\ell$-module.

Theorem 2 *If X is a $K3$ surface of geometric Picard number 20 over* $\mathbf{Q}$ *or K (the CM-field of the elliptic curves in the Shioda-Inose theorem), then the image of the higher ℓ-adic Abel-Jacobi mapping $d_{2,\ell}$ is finite for all ℓ and zero for almost all ℓ.*

Sketch of Proof: As in the proof of the last result, we are reduced to the case of a Kummer surface X associated to the product A of two isogenous CM elliptic curves. Now the map

$$Br(\overline{X}) \to Br(\overline{A})$$

is an isogeny, with kernel killed by 2, as one can see by comparing the Picard numbers (4 for A, 20 for X) with the second Betti numbers (6 for A, 22 for X). Thus it suffices to prove the result for A. In this case, the statement follows from work of Wiles [W] and Dee [D] on the finiteness of the Selmer group of the symmetric square of a CM elliptic curve. This completes the proof of the theorem.

6 Geometric interpretation of the Brauer-Manin obstruction

Let X be a geometrically integral variety over a number field k, and assume that for every place v of k, X has rational points in k_v. If $\mathcal{A} \in Br(X)$ and $P_v \in X(k_v)$, let $\mathcal{A}_{P_v}$ denote the pullback of $\mathcal{A}$ via the morphism:

$$P_v : \operatorname{Spec} k_v \to X$$

determined by the point P_v. Let $inv_v : Br(k_v) \to \mathbf{Q}/\mathbf{Z}$ denote the homomorphism giving the invariant of a central simple algebra, which is an isomorphism for v non-archimedean and an injection onto the subgroup of elements of order 2 in $\mathbf{Q}/\mathbf{Z}$ for v real. Let T be a subgroup of $Br(X)$ containing $Br(k)$, and let

$$X(\mathbf{A}_k)^T = \{(P_v) \in \prod_v X(k_v) : \forall \mathcal{A} \in T, \sum_v inv_v \mathcal{A}_{P_v} = 0\}.$$

The Brauer-Hasse-Noether theorem implies that:

$$X(k) \subseteq X(\mathbf{A}_k)^T.$$

If $T = Br(X)$, we will denote this set by $X(\mathbf{A}_k)^{Br}$.

We will say that X *is a counterexample to the Hasse principle* if $X(\mathbf{A}_k) \neq \emptyset$, but $X(k) = \emptyset$, that *there is no Brauer-Manin obstruction for T to the Hasse principle for X* if $X(\mathbf{A}_k)^T \neq \emptyset$, and that X *is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction* if $X(\mathbf{A}_k)^{Br} = \emptyset$.

Theorem 3 *Let X be a smooth projective geometrically simply connected surface over an algebraic number field k. Let ℓ be a prime number, and assume*

(i) $X(\mathbf{A}_k) \neq \emptyset$

(ii) There is no elementary obstruction to the existence of an ℓ-adic gerbe.

(iii) The Tate conjecture for divisors (T_ℓ) is true (see §4).

Then there is no Brauer-Manin obstruction for $Br(X)\{\ell\}$ if and only if there exists a universal ℓ-adic gerbe with rational points in every k_v.

Proof: This proof is similar in outline to the one in ([CT/S2], Théorème 3.5.1), except we replace Tate-Nakayama duality with Poitou-Tate duality. By Poincaré duality, we have a nondegenerate pairing:

$$H^2(\overline{X}, \mathbf{Z}_\ell(2)) \times H^2(\overline{X}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1)) \to H^4(\overline{X}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(3)) \cong \mathbf{Q}_\ell/\mathbf{Z}_\ell(1).$$

By Lemma 2, the vanishing of the elementary obstruction gives us a section of the natural map $Br(k) \to Br(X)$ and a universal gerbe, $\mathcal{G}$, that is trivial on that section.

Let S be a finite set of places of k including the archimedean places, the places above ℓ and the bad reduction places of X. By the Poitou-Tate global duality theorem, we have an exact sequence:

$$\cdots H^2(G_S, H^2(\overline{X}, \mathbf{Z}_\ell(2))) \to \prod_{v \in S} H^2(G_v, H^2(\overline{X}, \mathbf{Z}_\ell(2))) \xrightarrow{\rho} H^0(G_S, H^2(\overline{X}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1)))^* \to 0.$$

The map ρ is derived from the perfect pairings:

$$H^2(k_v, H^2(\overline{X}, \mathbf{Z}_\ell(2))) \times H^0(k_v, H^2(\overline{X}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))) \to H^2(k_v, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))$$

by taking the sum of the invariants. Consider the following diagram:

$$\begin{array}{ccccc} \prod_{v\in S} X(k_v) & \times & \prod_{v \in S} Br(X_v) & \to & Br(k_v) \\ \downarrow & & \uparrow & & \\ \prod_{v\in S} H^2(k_v, H^2(\overline{X}, \mathbf{Z}_\ell(2))) & \times & \prod_{v\in S} H^0(k_v, H^2(\overline{X}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))) & \to & Br(k_v). \end{array}$$

The map on the left is given by $\theta_{\mathcal{G}_v}$ (see §5), where $\mathcal{G}$ was fixed at the beginning of the proof here. The map on the right is the surjection

$$H^2(\overline{X}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))^{G_v} \overset{\cong}{\leftarrow} H^2(X_v, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))/H^2(k_v, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1)) \to Br(X_v)\{\ell\}/Br(k_v)\{\ell\}$$

explained in the proof of Proposition 1, followed by the map

$$Br(X_v)\{\ell\}/Br(k_v)\{\ell\} \to Br(X_v)\{\ell\}$$

that we get from the universal ℓ-adic gerbe, $\mathcal{G}_v$ and Lemma 2. This diagram is commutative.

Now suppose there is no Brauer-Manin obstruction for $Br(X)\{\ell\}$, and let (P_v) be a family of points such that

$$\sum_v inv_v \mathcal{A}_{P_v} = 0$$

for all $\mathcal{A} \in Br(X)\{\ell\}$. Since $Br(X)\{\ell\}/Br(k)\{\ell\}$ is finite, by enlarging S, if necessary, we may assume that there is a smooth proper model $\mathcal{X}_S$ of X over the ring of S-integers of k and a surjection $Br(\mathcal{X}_S)\{\ell\} \to Br(X)\{\ell\}/Br(k)\{\ell\}$. Then the Poitou-Tate exact sequence above and commutativity of the diagram show that the family $(\theta_{\mathcal{G}_v}(P_v)) \in \prod_{v\in S} H^2(G_v, H^2(\overline{X}, \mathbf{Z}_\ell(2)))$ comes from an element α of $H^2(G_S, H^2(\overline{X}, \mathbf{Z}_\ell(2)))$. Let $\mathcal{G}$ be the universal ℓ-adic gerbe that was fixed at the beginning of the proof and let α' be the image of α in $H^2(k, H^2(\overline{X}, \mathbf{Z}_\ell(2)))$ Then the universal gerbe $\mathcal{G}_{\alpha'}$ with class $\mathcal{G} - \alpha'$ has the property that $(\mathcal{G}_{\alpha'}, P_v) = 0$ for all v, so that $\mathcal{G}_{\alpha'}$ has points everywhere locally, as desired.

For the other direction, if there is a universal gerbe $\mathcal{G} \xrightarrow{q} X$ with points everywhere locally, then choosing a family $(P_v) \in \mathcal{G}_v(k_v)$, we have $\theta_{\mathcal{G}}(q(P_v)) = 0$, so this element pairs to zero with any $\mathcal{A} \in Br(X)$. Thus $(P_v) \in X(\mathbf{A}_k)^{Br(X)\{\ell\}}$. This completes the proof of the theorem.

Remark 3 *(i) Theorem 3 is a much weaker analogue of Proposition 3.3.2 and Théorème 3.5.1 of [CT/S2], which prove the result without assuming the existence of a universal ℓ-adic gerbe (universal torseur in their situation). We hope to be able to remove this assumption, but we have had trouble expressing the elementary obstruction (see §2.1 above) in terms of other computable cohomological invariants.*

(ii) We hope that this theorem may lead to other obstructions to the Hasse principle. For the theorem effectively allows one to replace the Brauer-Manin obstruction by the existence of an auxiliary algebro-geometric object that must have rational points everywhere locally. Using analogues of the homological algebra developed in §1 above, one can define other auxiliary objects associated to higher cohomology groups of varieties of higher dimension. For example, let X be a threefold in $\mathbf{P}^4$. Then since $Br(X)$ is the image of $Br(k)$, there is no Brauer-Manin obstruction to the Hasse principle. However, there is still the interesting cohomology group $H^3(\overline{X}, \mathbf{Z}/n\mathbf{Z}(3))$. One can define the notion of "universal 3-gerbe" associated to this group using similar formalism

as used above, and this can be used to formulate an obstruction to the Hasse principle, that there should be one such with points everywhere locally. This sounds rather abstract, but maybe it could be made more concrete in some cases.

7 Descent on curves

If X is a smooth projective curve over a number field, we can develop a similar theory using $H^1(\overline{X}, \mathbf{Z}/n(1))$. In this case, we are dealing with principal homogeneous spaces over X under the group-scheme $J[n]$, where J is the Jacobian of X, and these are parametrized by $J(k)/n$. As pointed out to us by Skorobogatov, these are related to the heterogeneous spaces of Coombes-Grant [CG]. The elementary obstruction is an element of $H^2(k, H^1(\overline{X}, \mathbf{Z}/n(1)))$ given by the class of the 2-extension:

$$0 \to H^1(\overline{X}, \mathbf{Z}/n(1)) \to \overline{k}(X)^*/n \to Div(\overline{X})/n \stackrel{deg}{\to} \mathbf{Z}/n \to 0$$

that is obtained by reducing the exact sequence:

$$0 \to \overline{k}(X)^*/\overline{k}^* \to Div(\overline{X}) \to Pic(\overline{X}) \to 0$$

modulo n, using the fact that $J(\overline{k})$ is divisible. If the genus of X is greater than one, the universal torseurs will be of higher genus than X, and seemingly more complicated. It should be the case that the Brauer-Manin obstruction to the Hasse principle is simpler on these spaces than on X, but we cannot prove this. See [Sk] §6.2 for more on descent on curves and abelian varieties.

8 Concluding Remarks

Our theory above suffers from two major shortcomings:

(i) We are not able to show that the Brauer-Manin obstruction on a universal n-gerbe is "simpler" than on X, and we cannot even say at the moment what "simpler" should mean. Even in the case of curves of genus at least two, we face a similar problem.

(ii) We have also not been able to describe the universal n-gerbes in terms of explicit local "equations." In this case, we have some idea of what form this might take, but have not been able to describe these explicitly in any concrete examples. Briefly, our idea is to replace a suitable Zariski open set U in the theory of Colliot-Thélène/Sansuc (with $Pic(\overline{U}) = 0$) with a quasi-finite étale morphism $U \to X$ which splits the n-torsion of the Brauer group of X. We can write a diagram similar to the one in ([CT/S2], 1.6.10), but we have not been able to find explicit "equations" to describe the universal n-gerbes. If the Galois group acts trivially on $H^2(\overline{X}, \mathbf{Z}/n(2))$, we can describe them as follows: the universal torseur associated to $S = \mathrm{Hom}(Pic(\overline{X}), \mathbf{G}_m)$ may be described by taking line bundles $\mathcal{L}_i$ that form a basis of $Pic(\overline{X})$, removing their zero sections and taking the product. Thus we need to describe the universal gerbe bound by $Br(\overline{X})[n](1)$. This may be done as in ([Mi], Ch. IV, §2) by taking a spanning set of elements of $Br(\overline{X})[n]$, describing them as gerbes as in the Appendix below, removing their zero sections and taking the product. It is a great challenge to be able to describe them in a more arithmetic situation, as is needed here.

Appendix

In this appendix, we briefly recall the definition of stacks and gerbes. We pick and choose material from the books of Milne ([Mi], Chapter IV, §2, p. 144-45) and Laumon/Moret-Bailly ([L/M-B], §§1-3). Let $\phi : F \to (\mathcal{C}/X)_E$ be a functor from a category to the underlying category of a site. For our purposes, this site will almost always be the big étale site. Given an object U of $(\mathcal{C}/X)_E$, we denote by $F(U)$ the category consisting of objects u of F such that $\phi(u) = U$ and morphisms f between such objects that cover the identity morphism of U. Given a covering $U_i \xrightarrow{g_i} U$ and an element of $F(U)$, we get via g_i^* elements of $F(U_i)$ which agree on $F(U_i \times U_j)$ and satisfy the cocycle condition on three-fold fibre products. If any family $F(U_i)$ satisfying these condition arises from such an element of $F(U)$, and if for any $u_1, u_2 \in F(U)$, the functor:

$$(V \xrightarrow{g} U) \mapsto Hom_{F(V)}(g^* u_1, g^* u_2)$$

is a sheaf, then ϕ is a *stack* (champ). It is a *gerbe* if it is a stack of groupoids, there is a covering U_i of U such that each $F(U_i)$ is nonempty, and if any two objects of $F(U)$ are locally isomorphic. A gerbe is bound by an abelian sheaf $\mathcal{F}$ if for any object U of $\mathcal{C}/X$ and any $u \in F(U)$, we have

$$\mathcal{F}(U) \cong Aut_{F(U)}(u).$$

A basic theorem of Giraud is that the set of gerbes bound by an abelian sheaf $\mathcal{F}$ up to equivalence is isomorphic to the second cohomology group $H^2(X_E, \mathcal{F})$.

In ([L/MB], Définition 3.1.5), a gerbe is defined to be a stack of groupoids $\mathcal{G}$ with a structure morphism $\mathcal{G} \xrightarrow{A} X$ such that both A and the diagonal morphism:

$$\mathcal{G} \times_{A,X,A} \mathcal{G}$$

are epimorphisms.

Any scheme X is a stack and a gerbe for the Zariski topology via the sheaf $Hom(-, X)$ it represents. Thus a stack for the étale topology may be vaguely (but incorrectly) regarded as a "scheme for the étale topology."

Description of the Brauer group in terms of gerbes

The group $H^2(X, \mathbf{G}_m)$ has a nice description in terms of the gerbes it classifies (up to equivalence). This may be done as follows (see [Mi], Ch. IV, §2, p. 145): let $\mathcal{A}$ be an Azumaya algebra on X. For U étale over X, let $F(U)$ be the set of pairs (E, α) where E is a locally free sheaf of $\mathcal{O}_U$-modules and $\alpha : \mathcal{A}(U) \to \mathrm{End}_U(E)$ is an isomorphism. Then descent theory shows that this is a stack, and the definition of an Azumaya algebra (see e.g. ibid. Ch. IV, §2, 2.1) shows that it is a gerbe. It is bound by $\mathbf{G}_m$ since the map

$$\mathbf{G}_m(U) \to Aut_U(E, \alpha)$$

that sends an element a of $\mathbf{G}_m(U)$ to multplication by a is an isomorphism.

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