Weak Approximation on Del Pezzo surfaces of degree 4

Sir Peter Swinnerton-Dyer

Let V be a Del Pezzo surface of degree 4 (that is, the smooth intersection of two quadrics in $\mathbf{P}^4$) defined over an algebraic number field k. Salberger and Skorobogatov [2] have shown that the only obstruction to weak approximation on V is the Brauer-Manin obstruction. More precisely:

Theorem 1 *Suppose that $V(k)$ is not empty. Let $\mathcal{A}$ be the subset of the adelic space $V(\mathbf{A})$ consisting of the points $\prod P_v$ such that*

$$\sum \mathrm{inv}_v(A(P_v)) = 0 \text{ in } \mathbf{Q}/\mathbf{Z}$$

for all A in the Brauer group $\mathrm{Br}(V)$*. Then the image of $V(k)$ is dense in $\mathcal{A}$.*

In this note I give a simpler proof of this theorem. What I actually prove is Theorem 2 below, which is equivalent to Theorem 1 because of Lemmas 5 and 7. Readers who are content with Theorem 2 need not trouble themselves with the Brauer-Manin conditions. Theorem 3, though it is a prerequisite for the proof of Theorem 2, is also of independent interest, since it adds an approximation property to Theorem 5.1 of [1] for pencils of conics. However Colliot-Thélène has pointed out to me that a similar approximation theorem is implicit in Theorem 6.2 of [1].

I am grateful to Jean-Louis Colliot-Thélène and Alexei Skorobogatov for a number of valuable comments.

Before beginning the proof (and even the statement) of Theorem 2, I need to describe the Legendre-Jacobi function L, which is a mild modification of a function (also called L) which was defined in rather crude form in [3] and more correctly in [4]. Indeed, one purpose of this paper is to give a fuller account of this function than has yet appeared. Let $F(U, V), G(U, V)$ be homogeneous coprime square-free polynomials in $k[U, V]$. Some of the more interesting results only hold when $\deg F$ is even, which nearly always holds in applications; this parity condition did not appear in [4], but it is already needed if we are to make use of the results of [1]. Let $\mathcal{B}$ be a finite set of places of k containing the infinite places, the primes dividing 2, those at which any coefficient of F or G is not integral, and any other primes $\mathfrak{p}$ at which FG does not remain separable when reduced mod $\mathfrak{p}$. Note that we do not assume that $\mathcal{B}$ contains a base for the ideal class group of k.

We shall always denote by $\mathfrak{o}$ the ring of integers of k. Let $\mathcal{N}^2 = \mathcal{N}^2(k)$ be the set of $\alpha \times \beta$ with α, β integral and coprime outside $\mathcal{B}$, and let $\mathcal{N}^1 = \mathcal{N}^1(k)$ be $k \cup \{\infty\}$. For $\alpha \times \beta$ in $\mathbf{A}^2(k)$ with α, β not both zero, we shall write $\lambda = \alpha/\beta$ with λ in $\mathcal{N}^1(k)$. Provided $F(\alpha,\beta)$ and $G(\alpha,\beta)$ are nonzero, we define the function

$$L(\mathcal{B}; F, G; \alpha, \beta) : \ \alpha \times \beta \mapsto \prod_{\mathfrak{p}} (F(\alpha,\beta), G(\alpha,\beta))_{\mathfrak{p}} \tag{1}$$

on $\mathcal{N}^2$, where the outer bracket on the right is the multiplicative Hilbert symbol and the product is taken over all primes $\mathfrak{p}$ of k outside $\mathcal{B}$ which divide $G(\alpha,\beta)$. By the definition of $\mathcal{B}$, $F(\alpha,\beta)$ is a unit at any such prime. Clearly we can restrict the product in (1) to those $\mathfrak{p}$ which divide $G(\alpha,\beta)$ to an odd power; thus we can also write it as $\prod \chi_{\mathfrak{p}}(F(\alpha,\beta))$ where $\chi_{\mathfrak{p}}$ is the quadratic character mod $\mathfrak{p}$ and the product is taken over all $\mathfrak{p}$ outside $\mathcal{B}$ which divide $G(\alpha,\beta)$ to an odd power. This relationship with the quadratic residue symbol underlies the proof of Lemma 1.

The function L does depend on $\mathcal{B}$, but the effect on the right hand side of (1) if we increase $\mathcal{B}$ is obvious. Although in the applications we can usually take $\deg F$ even, in the course of the proofs we need to consider functions (1) with $\deg F$ odd; and for this reason it is expedient to introduce

$$M(\mathcal{B}; F, G; \alpha, \beta) = L(\mathcal{B}; F, G; \alpha, \beta)(L(\mathcal{B}; U, V; \alpha, \beta))^{(\deg F)(\deg G)}.$$

Here of course $L(\mathcal{B}; U, V; \alpha, \beta) = \prod (\alpha, \beta)_{\mathfrak{p}}$ taken over all $\mathfrak{p}$ outside $\mathcal{B}$ which divide β.

Lemma 1 *The value of M is continuous in the topology induced on $\mathcal{N}^2$ by $\mathcal{B}$. For each v in $\mathcal{B}$ there is a function $m(v; F, G; \alpha, \beta)$ with values in $\{\pm 1\}$ which is continuous on $\mathcal{N}^2$ in the v-adic topology and is such that*

$$M(\mathcal{B}; F, G; \alpha, \beta) = \prod_{v \in \mathcal{B}} m(v; F, G; \alpha, \beta). \tag{2}$$

.

Proof If $\deg F$ is even, so that $M = L$, the neatest proof of the lemma is by means of the evaluation formula in [1], Lemma 7.2.4. The case when $\deg G$ is even then follows from (4), and (3) gives the general case. (The proof in [1] is for $k = \mathbf{Q}$, but there is not much difficulty in modifying it to cover all

k.) However, the proof which we shall give, using the ideas of [4], provides a more convenient method of evaluation.

For this proof we have to impose on $\mathcal{B}$ the additional condition that it contains all primes whose absolute norm does not exceed $\deg(FG)$. As the proof in [1] shows, this condition is not needed for the truth of Lemma 1 itself; but we use it in the proof of (8) below, and the latter is crucial to the subsequent argument. In any case, to classify all small enough primes as bad is quite usual. We repeatedly use the fact that $L(\mathcal{B}; F, G)$ and $M(\mathcal{B}; F, G)$ are multiplicative in both F and G; the effect of this is that we can reduce to the case when both F and G are irreducible in $\mathfrak{o}'[U, V]$, where $\mathfrak{o}'$ is the ring of elements of k integral outside $\mathcal{B}$. Introducing M and dropping the parity condition on $\deg F$ are not real generalizations since if we increase $\mathcal{B}$ so that the leading coefficient of F is a unit outside $\mathcal{B}$ then

$$M(\mathcal{B}; F, G) = L(\mathcal{B}; F, GV^{\deg G}) \tag{3}$$

by (5), and we can apply (4) to the right hand side.

It follows from the product formula for the Hilbert symbol that

$$L(\mathcal{B}; f, g; \alpha, \beta) L(\mathcal{B}; g, f; \alpha, \beta) = \prod_{v \in \mathcal{B}} (f(\alpha, \beta), g(\alpha, \beta))_v, \tag{4}$$

subject to conditions on $\mathcal{B}$ analogous to those stated before (1). The right hand side of (4) is the product of continuous terms each of which only depends on a single v in $\mathcal{B}$. This formula enables us to interchange F and G when we want to, and in particular to require that $\deg F \geq \deg G$ in the reduction process which follows. We also have

$$L(\mathcal{B}; f, g; \alpha, \beta) = L(\mathcal{B}; f - gh, g; \alpha, \beta) \tag{5}$$

for any homogeneous h in $k[U, V]$ with $\deg h = \deg f - \deg g$ provided the coefficients of h are integral outside $\mathcal{B}$, because corresponding terms in the two products are equal. Both (4) and (5) also hold for M.

We deal first with two special cases:

- G is a constant. Now $M(\mathcal{B}; F, G) = 1$ because all the prime factors of G must be in $\mathcal{B}$, so that the product in the definition of $L(\mathcal{B}; F, G)$ is empty.

- $G = V$. Choose H so that $F - GH = \gamma U^{\deg F}$ for some nonzero γ. Now $M(\mathcal{B}; F, G) = 1$ follows from the previous case and (5), since all the prime factors of γ must be in $\mathcal{B}$.

We now argue by induction on $\deg(FG)$. Since we can assume that F and G are irreducible, we need only consider the case when

$$\deg F \geq \deg G > 0, \quad G = \gamma U^{\deg G} + \dots, \quad F = \delta U^{\deg F} + \dots$$

for some nonzero γ, δ. Let $\mathcal{B}_1$ be obtained by adjoining to $\mathcal{B}$ those primes of k not in $\mathcal{B}$ at which γ is not a unit. By (5) we have

$$M(\mathcal{B}_1; F, G) = M(\mathcal{B}_1; F - \gamma^{-1}\delta G U^{\deg F - \deg G}, G). \tag{6}$$

By taking a factor V out of the middle argument on the right, and using (4), the second special case above and the induction hypothesis, we see that $M(\mathcal{B}_1; F, G)$ is continuous in the topology induced by $\mathcal{B}_1$ and is a product taken over all v in $\mathcal{B}_1$ of continuous terms each one of which depends on only one of the v. Hence the same is true of $M(\mathcal{B}; F, G)$, because this differs from $M(\mathcal{B}_1; F, G)$ by finitely many continuous factors, each of which depends only on one prime in $\mathcal{B}_1 \setminus \mathcal{B}$.

But $\mathcal{B}_1 \setminus \mathcal{B}$ only contains primes whose absolute norm is greater than $\deg(FG)$. Thus by an integral unimodular transformation from U, V to U, V_1 we can arrange that $G = \gamma_1 U^{\deg G} + \dots$ and $F = \delta_1 U^{\deg F} + \dots$ where γ_1 is a unit at each prime in $\mathcal{B}_1 \setminus \mathcal{B}$. Let $\mathcal{B}_2$ be obtained from $\mathcal{B}$ by adjoining all the primes at which γ_1 is not a unit; then $M(\mathcal{B}; F, G)$ has the same properties with respect to $\mathcal{B}_2$ that we have already shown that it has with respect to $\mathcal{B}_1$. Since $\mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{B}$, this implies that $M(\mathcal{B}; F, G)$ already has these properties with respect to $\mathcal{B}$. □

Of course there will be finitely many values of α/β for which the right hand side of (2) appears to be indeterminate; but by means of a preliminary linear transformation on U, V one can in fact ensure that the formula is meaningful except when $F(\alpha, \beta)$ or $G(\alpha, \beta)$ vanishes.

When $\deg F$ is even, the value of $L(\mathcal{B}; F, G; \alpha, \beta)$ is already determined by $\lambda = \alpha/\beta$ regardless of the values of α and β separately; here λ lies in $k \cup \{\infty\}$ with the roots of $F(\lambda, 1)$ and $G(\lambda, 1)$ deleted. We shall therefore also write this function as $L(\mathcal{B}; F, G; \lambda)$. But note that it is not necessarily a continuous function of λ; see the discussions in [3] and §9 of [1], and Lemma 4 below. Moreover if $\mathcal{B}$ does not contain a base for the ideal class group of k then not all elements of $k \cup \{\infty\}$ can be written in the form α/β with α, β integers coprime outside $\mathcal{B}$; so we have not yet defined $L(\mathcal{B}; F, G; \lambda)$ for all λ. To go further in the case when $\deg F$ is even, we modify the definition (1) so that it extends to all $\alpha \times \beta$ in $k \times k$ such that $F(\alpha, \beta)$ and $G(\alpha, \beta)$ are nonzero. For any such α, β and any $\mathfrak{p}$ not in $\mathcal{B}$, choose $\alpha_\mathfrak{p}, \beta_\mathfrak{p}$ integral at $\mathfrak{p}$, not both divisible by $\mathfrak{p}$ and such that $\alpha/\beta = \alpha_\mathfrak{p}/\beta_\mathfrak{p}$. Write

$$L(\mathcal{B}; F, G; \alpha, \beta) = \prod (F(\alpha_\mathfrak{p}, \beta_\mathfrak{p}), G(\alpha_\mathfrak{p}, \beta_\mathfrak{p}))_\mathfrak{p} \tag{7}$$

where the product is taken over all $\mathfrak{p}$ not in $\mathcal{B}$ such that $\mathfrak{p} | G(\alpha_\mathfrak{p}, \beta_\mathfrak{p})$. This is a finite product whose value does not depend on the choice of the $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$; indeed it only depends on $\lambda = \alpha/\beta$ and when α, β are integers coprime outside $\mathcal{B}$ it is the same as the function given by (1). Thus we can again write it as $L(\mathcal{B}; F, G; \lambda)$. This generalization is not really needed until we come to (11); but at that stage we need $\mathcal{B}$ to be independent of K. Its disadvantage is that L is no longer necessarily a continuous function of $\alpha \times \beta$; we investigate this situation in more detail after the proof of Lemma 3.

In discussing the continuity properties of L as a function of λ, we shall need the following lemma.

Lemma 2 *Let $\lambda_0 = \alpha_0/\beta_0$ with α_0, β_0 non-zero and integral outside $\mathcal{B}$; and let $\mathfrak{a}$ be an integral ideal in k not divisible by any prime in $\mathcal{B}$. Then we can find α, β in k, integral outside $\mathcal{B}$, with $(\alpha, \beta) = \mathfrak{a}(\alpha_0, \beta_0)$ and such that $\alpha \times \beta$ is arbitrarily close to $\alpha_0 \times \beta_0$ at each finite prime in $\mathcal{B}$, α/β is arbitrarily close to α_0/β_0 at each infinite place of k and α/α_0 and β/β_0 are positive at each real infinite place of k.*

Proof Let $\mathcal{S}$ be the set of primes which divide α_0 or β_0. We can write $\mathfrak{a} = (\gamma_1, \gamma_2)$ where γ_1 and γ_2 are units at every prime in $\mathcal{B}$ and both $\gamma_1/\mathfrak{a}$ and $\gamma_2/\mathfrak{a}$ are units at every prime in $\mathcal{S}$. Let δ in $\mathfrak{o}$, a unit outside $\mathcal{B}$, be such that $\alpha_0\delta$ and $\beta_0\delta$ are in $\mathfrak{o}$. Choose positive coprime integers a, b in $\mathbf{Z}$ which are close to 1 at every finite prime in $\mathcal{B}$ and units at all the primes which divide γ_1 or γ_2; and let M, N be large positive integers. By writing $\alpha_0\delta a^M/\gamma_1$ in terms of a base for $\mathfrak{o}/\mathbf{Z}$ and changing the coefficients by elements of $\mathbf{Q}$ which are small at each finite prime in $\mathcal{B} \cup \mathcal{S}$ and $O(a)$ at the infinite place of $\mathbf{Q}$, we can obtain an integer α_1 in $\mathfrak{o}$ which is prime to a and $\gamma_2/\mathfrak{a}$ and such that $\alpha_0\delta a^M/\alpha_1\gamma_1$ is close to 1 at each place in $\mathcal{B}$ and α_0, α_1 are divisible by the same power of $\mathfrak{p}$ for each $\mathfrak{p}$ in $\mathcal{S}$. Similarly we can obtain β_1 in $\mathfrak{o}$ which is prime to b and $\gamma_1/\mathfrak{a}$ and such that $\beta_0\delta b^N/\beta_1\gamma_2$ is close to 1 at each place of $\mathcal{B}$ and β_0, β_1 are divisible by the same power of $\mathfrak{p}$ for each $\mathfrak{p}$ in $\mathcal{S}$. We can further ensure that β_1 is prime to α_1 outside $\mathcal{B} \cup \mathcal{S}$. Now $\alpha = \alpha_1 b^N \gamma_1/\delta$ and $\beta = \beta_1 a^M \gamma_2/\delta$ satisfy all the requirements in the lemma. The only difficult thing to verify is that $(\alpha, \beta) = \mathfrak{a}(\alpha_0, \beta_0)$. So far as primes in $\mathcal{B}$ are concerned, the two sides agree; and

$$(\alpha, \beta) = (\alpha_1\gamma_1, \beta_1\gamma_2) = \mathfrak{a}(\alpha_1(\gamma_1/\mathfrak{a}), \beta_1(\gamma_2/\mathfrak{a})) = \mathfrak{a}(\alpha_1, \beta_1)$$

up to such primes. □

The proof of Lemma 1 constructs an evaluation formula all of whose terms come from the right hand side of (4) for various pairs f, g. For $\alpha \times \beta$ in $\mathcal{N}^2$, the formula can therefore be described by an equation of the form

$$m(v; F, G; \alpha, \beta) = \prod_j (\phi_j(\alpha, \beta), \psi_j(\alpha, \beta))_v. \tag{8}$$

Here the ϕ_j, ψ_j are homogeneous elements of $k[U, V]$ which depend only on F and G and not on v or $\mathcal{B}$, and which can be freely divided by squares. The decomposition (8) is not unique, and our next task is to display an invariant aspect of it.

Let $\theta = \gamma_1 U + \gamma_2 V$ be a linear form with γ_1, γ_2 coprime integers in k. By using $(\phi, \psi)_v = (\phi, \theta\psi)_v(\phi, \theta)_v$ and $(-\theta, \theta)_v = 1$, we can ensure that all the ϕ_j, ψ_j in (8) have even degree except that $\psi_0 = \theta$. Denote by Θ the group of elements of k^* which are not divisible to an odd power by any prime of k outside $\mathcal{B}$, and by $\Theta_0 \subset \Theta$ the subgroup consisting of those ξ which are quadratic residues mod $\mathfrak{p}$ for all $\mathfrak{p}$ outside $\mathcal{B}$; thus we are free to multiply ϕ_0 by any element of Θ_0. (Actually $\Theta_0 = k^{*2}$, but we shall not use this fact.)

Lemma 3 *Suppose that* $\deg F$ *is even. With the convention for the* ϕ_j, ψ_j *just adopted, we can take* ϕ_0 *to be in* Θ.

Proof Let γ in k^* be a unit outside $\mathcal{B}$, and apply (8) to the identity

$$L(\mathcal{B}; F, G; \gamma\alpha, \gamma\beta) = L(\mathcal{B}; F, G; \alpha, \beta),$$

where $\alpha \times \beta$ is in $\mathcal{N}^2$. On cancelling common factors, we obtain

$$\prod_{v \in \mathcal{B}} (\phi_0(\alpha, \beta), \gamma)_v = 1. \tag{9}$$

If we can choose $\alpha \times \beta$ in $\mathcal{N}^2$ so that $\phi_0(\alpha, \beta)$ is not in Θ, this gives a contradiction. For let δ prime to $\phi_0(\alpha, \beta)$ be such that $\prod(\phi_0(\alpha, \beta), \delta)_{\mathfrak{p}} = -1$ where the product is taken over all primes $\mathfrak{p}$ outside $\mathcal{B}$ at which $\phi_0(\alpha, \beta)$ is not a unit. Let $\mathcal{B}_1$ be obtained by adjoining to $\mathcal{B}$ all the primes at which δ is not a unit; then $\prod(\phi_0(\alpha,\beta),\delta)_v = -1$ by the Hilbert product formula, where the product is taken over all places v in $\mathcal{B}_1$. Recalling that ϕ_0 does not depend on $\mathcal{B}$ and writing $\mathcal{B}_1, \delta$ for $\mathcal{B}, \gamma$ in (9), we obtain a contradiction. It follows that $\phi_0(\alpha,\beta)$ lies in Θ for all α, β; this can only happen if $\phi_0(U,V)$ is itself in Θ modulo squares. $\square$

Let $\mathcal{S}$ be the set of primes $\mathfrak{p}$ outside $\mathcal{B}$ for which $\mathfrak{p}|F(\alpha_\mathfrak{p},\beta_\mathfrak{p})$ or $\mathfrak{p}|G(\alpha_\mathfrak{p},\beta_\mathfrak{p})$ in the notation of (7). We can write $\lambda = \alpha/\beta$ where (α,β) is not divisible by any prime in $\mathcal{S}$. Let $\mathfrak{a}$ be an integral ideal in the class of (α,β) not divisible by any prime in $\mathcal{S}$, and let γ be such that $(\gamma) = \mathfrak{a}/(\alpha,\beta)$; then $\lambda = \alpha\gamma/\beta\gamma$ and $(\alpha\gamma,\beta\gamma) = \mathfrak{a}$. If $\mathcal{B}_1$ is obtained from $\mathcal{B}$ by adjoining all the primes which divide $\mathfrak{a}$, then

$$L(\mathcal{B};F,G;\lambda) = L(\mathcal{B};F,G;\alpha\gamma,\beta\gamma) = L(\mathcal{B}_1;F,G;\alpha\gamma,\beta\gamma),$$

where the second equality holds because the two products involved are term by term the same. By (8) the right hand side is equal to

$$\begin{aligned}&\prod_{v\in\mathcal{B}_1}\prod_j(\phi_j(\alpha\gamma,\beta\gamma),\psi_j(\alpha\gamma,\beta\gamma))_v\\ &\qquad = \left\{\prod_{v\in\mathcal{B}_1}\prod_j(\phi_j(\alpha,\beta),\psi_j(\alpha,\beta))_v\right\}\prod_{v\in\mathcal{B}_1}(\phi_0(\alpha,\beta),\gamma)_v\end{aligned}$$

because of the parity properties above. If we further require that no prime which divides $\mathfrak{a}$ divides any of the $\phi_j(\alpha,\beta)$ or $\psi_j(\alpha,\beta)$, then each of the terms in curly brackets with v in $\mathcal{B}_1\backslash\mathcal{B}$ is trivial; so the outer product there reduces to a product over v in $\mathcal{B}$. By the Hilbert product formula the product outside the curly brackets can be replaced by a product over all v not in $\mathcal{B}_1$. In view of Lemma 3 we can reduce this to a product over those v outside $\mathcal{B}_1$ which divide (α,β). If $\chi_\mathfrak{p}$ is again the quadratic residue symbol mod $\mathfrak{p}$, we can write the result which we have just obtained in the form

$$L(\mathcal{B};F,G;\lambda) = \left\{\prod_{v\in\mathcal{B}}\prod_j(\phi_j(\alpha,\beta),\psi_j(\alpha,\beta))_v\right\}\prod\chi_\mathfrak{p}(\phi_0) \tag{10}$$

where the final product is taken over those $\mathfrak{p}$ outside $\mathcal{B}$ which divide (α,β) to an odd power.

Lemma 4 *Suppose that* $\deg F$ *is even and the conventions of Lemma* 3 *hold. Then* ϕ_0 *is uniquely determined by* F *and* G *as an element of* Θ/Θ_0*; and* ϕ_0 *is in* Θ_0 *if and only if* $L(\mathcal{B}; F, G; \lambda)$ *is continuous in* λ *in the topology induced by* $\mathcal{B}$.

Proof Suppose first that ϕ_0 is in Θ_0. Thus the final product in (10) is trivial. Now let $\lambda = \alpha/\beta$ and let λ' be close to λ in the topology induced by $\mathcal{B}$. Let γ in $\mathfrak{o}$ be such that $\lambda'\beta\gamma$ is integral. Applying (10) to the representations

$$\lambda = \alpha\gamma/\beta\gamma \quad \text{and} \quad \lambda' = \lambda'\beta\gamma/\beta\gamma$$

we deduce that $L(\mathcal{B}; F, G; \lambda) = L(\mathcal{B}; F, G; \lambda')$.

Conversely suppose that ϕ_0 is in Θ but not in Θ_0. Choose a prime $\mathfrak{p}$ outside $\mathcal{B}$ at which ϕ_0 is not a quadratic residue. As before, let $\lambda_0 = \alpha_0/\beta_0$, and let $\lambda = \alpha/\beta$ where α, β have the properties stated in Lemma 2 with $\mathfrak{a} = \mathfrak{p}$. Arguing as in the previous paragraph, but taking account of the final product in (10), we obtain

$$L(\mathcal{B}; F, G; \lambda) = L(\mathcal{B}; F, G; \lambda_0)\chi_{\mathfrak{p}}(\phi_0) = -L(\mathcal{B}; F, G; \lambda_0).$$

So $L(\mathcal{B}; F, G; \lambda)$ is not continuous at $\lambda = \lambda_0$ — which means that it is continuous nowhere.

Now suppose that $L(\mathcal{B}; F, G; \alpha, \beta)$ has two representations, say by the ϕ'_i, ψ'_i and the ϕ''_j, ψ''_j. Taking their quotient, we obtain

$$1 = \prod_{v \in \mathcal{B}} \left\{ (\phi'_0/\phi''_0, \theta(\alpha, \beta))_v \prod_{i>0} (\phi'_i(\alpha, \beta), \psi'_i(\alpha, \beta))_v \prod_{j>0} (\phi''_j(\alpha, \beta), \psi''_j(\alpha, \beta))_v \right\}.$$

This is a representation of a function of λ which is continuous; and it is of a kind to which we can apply the results of the previous two paragraphs. Hence ϕ'_0/ϕ''_0 is in Θ_0.

It remains only to show that ϕ_0 is independent of the choice of θ. Using a notation like that of the previous paragraph, there is a representation of 1 in which the terms with subscript 0 produce a quotient

$$\prod_{v \in \mathcal{B}} \{ (\phi'_0/\phi''_0, \theta')_v (\phi''_0, \theta'\theta'')_v \};$$

and since $\deg(\theta'\theta'')$ is even it follows as there that ϕ'_0/ϕ''_0 is in Θ_0. □

In practice, what we usually need to study is the subspace of $\mathcal{N}^2$ given by n conditions $L(\mathcal{B}; F_\nu, G_\nu; \alpha, \beta) = 1$, or the subspace of $\mathcal{N}^1$ given by the $L(\mathcal{B}; F_\nu, G_\nu; \lambda) = 1$, where the $\deg F_\nu$ are all even. Let Λ be the abelian group of order 2^n whose elements are the n-tuples each component of which is ± 1; then there is a natural identification, which we shall write τ, of each element of Λ with a partial product of the $L(\mathcal{B}; F_\nu, G_\nu)$. Thus each element of Λ can be interpreted as a condition, which we shall write as $\mathcal{L} = 1$. If ϕ_0 is as in Lemma 3, there is a homomorphism

$$\phi_0 \circ \tau : \Lambda \to \Theta/\Theta_0;$$

let Λ_0 denote its kernel. In view of Lemma 4, the conditions which are continuous in λ are just those which come from Λ_0. The following lemma corresponds to Harari's Formal Lemma (Theorem 3.2.1 of [1]); it shows that for most purposes we need only consider the conditions coming from the elements of Λ_0.

Lemma 5 *Suppose that* $\deg F$ *is even and all the conditions corresponding to* Λ_0 *hold at some given* λ_0. *Then there exists* λ *arbitrarily close to* λ_0 *such that all the conditions* $L(\mathcal{B}; F_\nu, G_\nu) = 1$ *hold at* λ.

Proof Let $\lambda_0 = \alpha_0/\beta_0$. For a suitably chosen $\mathfrak{a} = (\gamma)$ we show that we can take $\lambda = \alpha/\beta$, where $\alpha \times \beta$ is as in Lemma 2. For any c in Λ, write $\phi_{0c} = \phi_0 \circ \tau(c)$ for the corresponding element of Θ/Θ_0. If θ is as defined just before Lemma 3, the corresponding partial product $\mathcal{L}$ of the $L(\mathcal{B}; F_\nu, G_\nu; \lambda)$ is equal to

$$f_c(\lambda) \prod_{v \in \mathcal{B}} (\phi_{0c}, \theta(\alpha_0, \beta_0))_v \prod_{v \in \mathcal{B}} (\phi_{0c}, \gamma)_v$$

where f_c comes from the ϕ_j, ψ_j with $j > 0$ and is therefore continuous. The map $c \mapsto f_c(\lambda)$ is a homomorphism $\Lambda \to \{\pm 1\}$ for any fixed λ; moreover if two distinct c give rise to the same ϕ_{0c} their quotient comes from an element of Λ_0 and therefore the quotient of the corresponding f_c takes the value 1 at λ_0. In other words, if λ is close enough to λ_0 then $f_c(\lambda)$ only depends on the class of c in Λ/Λ_0. The map $c \mapsto \phi_{0c}$ is an embedding $\Lambda/\Lambda_0 \to \Theta/\Theta_0$, by Lemma 4. The homomorphism $\mathrm{Image}(\Lambda/\Lambda_0) \to \{\pm 1\}$ induced by $c \mapsto f_c(\lambda)$ can be extended to a homomorphism $\Theta/\Theta_0 \to \{\pm 1\}$ because Θ/Θ_0 is killed by 2; and any such homomorphism can be written in the form

$$\theta \to \prod_{v \in \mathcal{B}} (\theta, \gamma)_v$$

for a suitably chosen γ, because the Hilbert symbol induces a nonsingular form on Θ/Θ_0. But given any such γ we can construct $\lambda = \alpha/\beta$ having the properties listed in Lemma 2 with $\mathfrak{a} = (\gamma)$. □

We shall need analogues of these last results for positive 0-cycles, and this will require more notation. We continue to assume that $\deg F$ is even. Let K be the direct product of finitely many fields k_i each of finite degree over k, and let $\mathfrak{B}$ be the set of places of K lying over some place v in $\mathcal{B}$, and $\mathfrak{B}_i$ the corresponding set of places of k_i. (The place $\prod v_i$, where v_i is a place of k_i, lies over v if each v_i does so.) For λ in $\mathbf{P}^1(K)$ write $\lambda = \prod \lambda_i$ with λ_i in $\mathbf{P}^1(k_i)$; for each place w in k_i write $\lambda_i = \alpha_{iw}/\beta_{iw}$ where α_{iw}, β_{iw} are in k_i and integral at w and at least one of them is a unit at w. For any λ in K such that each $F(\lambda_i, 1)$ and $G(\lambda_i, 1)$ is nonzero, we define the function

$$L^*(\mathcal{B}; K; F, G; \lambda) : \ \lambda \mapsto \prod_{\mathfrak{P}_i} (F(\alpha_{iw}, \beta_{iw}), G(\alpha_{iw}, \beta_{iw}))_{\mathfrak{P}_i} \tag{11}$$

where w is the place associated with the prime $\mathfrak{P}_i$ in k_i and the product is taken over all i and all primes $\mathfrak{P}_i$ of k_i not lying in $\mathfrak{B}_i$ and such that $G(\alpha_{iw}, \beta_{iw})$ is divisible by $\mathfrak{P}_i$. As with (1), we can restrict the product to those $\mathfrak{P}_i$ which divide $G(\alpha_{iw}, \beta_{iw})$ to an odd power. Note that the functions ϕ_j, ψ_j in the evaluation formula (8) are the same for $k_i \supset k$ as they are for k. Now let $\mathfrak{a}$ be a positive 0-cycle on $\mathbf{P}^1$ defined over k and let $\mathfrak{a} = \cup \mathfrak{a}_i$ be its decomposition into irreducible components. Let λ_i be a point of $\mathfrak{a}_i$ and write $k_i = k(\lambda_i)$. If $K = \prod k_i$ and $\lambda = \prod \lambda_i$, write

$$L^*(\mathcal{B}; F, G; \mathfrak{a}) = L^*(\mathcal{B}; K; F, G; \lambda) = \prod\nolimits_i L(\mathfrak{B}_i; F, G; \lambda_i). \tag{12}$$

This is legitimate, because the right hand side does not depend on the choice of the λ_i. If $K = k$ this L^* is the same as the previous function L. Moreover $L^*(\mathfrak{a} \cup \mathfrak{b}) = L^*(\mathfrak{a}) L^*(\mathfrak{b})$. We can define a topology on the set of positive 0-cycles $\mathfrak{a}$ of given degree N by means of the isomorphism between that set and the points on the N-fold symmetric power of $\mathbf{P}^1$. With this topology, it is straightforward to extend to L^* the results already obtained for L.

The product in (11) is finite; so there is a finite set $\mathcal{S}$ of primes of k, disjoint from $\mathcal{B}$ and such that every $\mathfrak{P}_i$ which appears in this product lies above a prime in $\mathcal{S}$. For each i we can write $\lambda_i = \alpha_i/\beta_i$ with α_i, β_i integers in k_i. As in the argument which follows the proof of Lemma 3, let $(\alpha_i, \beta_i) = \mathfrak{a}_i$ and choose an integral ideal $\mathfrak{b}_i$ in k_i which is prime to $\mathfrak{a}_i$, in the same ideal class as $\mathfrak{a}_i$ and such that no prime of k_i which divides $\mathfrak{b}_i$ also divides $G(\alpha_i, \beta_i)$

or any $\phi_j(\alpha_i, \beta_i)$ or $\psi_j(\alpha_i, \beta_i)$ or lies above any prime in $\mathcal{S}$. Let γ_i be such that $(\gamma_i) = \mathfrak{b}_i/\mathfrak{a}_i$ and let $\mathcal{B}_1$ be obtained from $\mathcal{B}$ by adjoining all the primes of k which lie below any prime of k_i which divides $\mathfrak{b}_i$. For most purposes it costs us nothing to replace $\mathcal{B}$ by $\mathcal{B}_1$, and we then have

$$\lambda = \prod \lambda_i = \prod (\alpha_i\gamma_i/\beta_i\gamma_i) \text{ where } \alpha_i\gamma_i \times \beta_i\gamma_i \text{ is in } \mathcal{N}^2(k_i).$$

The following lemma is a trivial consequence of earlier results.

Lemma 6 *Suppose that* $\deg F$ *is even, and let* $\mathcal{L} = 1$ *be a continuous condition derived from the* L *and* $\mathcal{L}^* = 1$ *the corresponding condition derived from the* L^*. *For each* v *in* $\mathcal{B}$ *there is a function* $\ell^*(v; F, G; \mathfrak{a})$ *with values in* $\{\pm 1\}$ *which is a continuous function of* $\mathfrak{a}$ *in the* v*-adic topology and is such that*

$$\mathcal{L}^*(\mathcal{B}; F, G; \mathfrak{a}) = \prod_{v \in \mathcal{B}} \ell^*(v; F, G; \mathfrak{a}). \tag{13}$$

With these preliminaries out of the way, consider the Del Pezzo surface $V = Q_1 \cap Q_2$ where Q_1, Q_2 are quadrics in $\mathbf{P}^4$. Choose coordinates so that the given point of $V(k)$ is $(1, 0, 0, 0, 0)$ and the tangents to Q_1, Q_2 at this point are $X_1 = 0, X_2 = 0$ respectively. Thus the equations of Q_1 and Q_2 can be written

$$X_0X_1 + f_1(X_1, \dots, X_4) = 0, \quad X_0X_2 + f_2(X_1, \dots, X_4) = 0 \tag{14}$$

where f_1, f_2 are homogeneous quadratic. The variety (14) is birationally equivalent to the cubic surface $X_2 f_1 = X_1 f_2$, which is obtained by blowing up the given point of $V(k)$; and this cubic surface is birationally equivalent to the pencil of affine conics

$$V f_1(U, V, X_3, X_4) = U f_2(U, V, X_3, X_4), \tag{15}$$

which with some abuse of language can be parametrized by the points (U, V) of $\mathbf{P}^1$. Diagonalizing this equation and then making it homogeneous gives a pencil of projective conics of the form

$$Z_0^2 g_1(U, V) + Z_1^2 g_2(U, V)/g_1(U, V) + Z_3^2 g_5(U, V)/g_2(U, V) = 0,$$

where g_r is homogeneous of degree r. Writing

$$Z_0 = g_2 Y_0, \quad Z_1 = g_1 Y_1, \quad Z_2 = g_1 g_2 Y_2$$

and dividing by $g_1 g_2$ we obtain

$$g_2 Y_0^2 + Y_1^2 + g_1 g_5 Y_2^2 = 0. \tag{16}$$

We shall assume that the g_r are coprime in pairs in $k[U, V]$; if not, there is a further simplification of (16) and of the subsequent argument which is left to the reader. Every point of the line $X_1 = X_2 = 0$ on the cubic surface also lies on one of the conics of the pencil (15), so there are an infinity of conics which contain points defined over k.

It costs nothing to set the next part of the argument in a broader context. Denote by W the surface fibred by the pencil of conics

$$a_0(U,V)Y_0^2 + a_1(U,V)Y_1^2 + a_2(U,V)Y_2^2 = 0, \tag{17}$$

and call the pencil *reduced* if a_0, a_1, a_2 are homogeneous elements of $k[U, V]$ coprime in pairs and such that

$$\deg a_0 \equiv \deg a_1 \equiv \deg a_2 \bmod 2.$$

After a linear transformation on U, V if necessary, we can also assume that $a_0 a_1 a_2$ is not divisible by V. Clearly any pencil of conics can be put into reduced form. Suppose that (17) is reduced and everywhere locally soluble. Let $\lambda = (\alpha, \beta)$ be a point of $\mathbf{P}^1(k)$; whether (17) is soluble at $\alpha \times \beta$ depends only on λ and not on the choice of α, β. Similar statements hold for local solubility at a place v and for solubility in the adeles. Denote by $c(U, V)$ a monic irreducible factor of $a_0 a_1 a_2$ in $k[U, V]$. Let $\mathcal{B}$ be a finite set of places of k containing the infinite places, the primes dividing 2, those at which any coefficient of any c or a_r is not integral, and any other primes $\mathfrak{p}$ at which $a_0 a_1 a_2$ does not remain separable when reduced mod $\mathfrak{p}$. For convenience, we also assume that $\mathcal{B}$ contains a base for the ideal class group of k.

We need to work not on $\mathbf{P}^1$ but on the set $\mathbf{L}^1$ obtained from $\mathbf{P}^1$ by deleting the roots of $a_0 a_1 a_2$; thus we do not have to worry about the singular fibres. Denote by W_0 the Zariski open subset of W which is the inverse image of $\mathbf{L}^1$, and define V_0 similarly in terms of the representation (16) of V. Let $\lambda \in k \cup \{\infty\}$ be a point of $\mathbf{L}^1(k)$, and write $\lambda = \alpha/\beta$ where α, β are integers of k coprime outside $\mathcal{B}$; it will not matter which pair α, β we choose.

There is a non-empty set $\mathcal{N} \subset \mathbf{L}^1(k)$, open in the topology induced by $\mathcal{B}$, such that the conic (17) is locally soluble at every place of $\mathcal{B}$ if and only if λ lies in $\mathcal{N}$. Let $\mathfrak{p}$ be a prime of k not in $\mathcal{B}$ and consider the solubility of (17) in $k_\mathfrak{p}$ at the point λ. If none of the $a_r(\alpha, \beta)$ is divisible by $\mathfrak{p}$, then solubility of (17) in $k_\mathfrak{p}$ is trivial. Otherwise there is just one c such that $c(\alpha, \beta)$ is divisible by $\mathfrak{p}$; to fix ideas, suppose that this c divides a_2. Then the condition for solubility in $k_\mathfrak{p}$ is

$$(-a_0(\alpha, \beta)a_1(\alpha, \beta), c(\alpha, \beta))_\mathfrak{p} = 1. \tag{18}$$

Hence necessary conditions for the local solubility of (17) at λ for all $\mathfrak{p}$ outside $\mathcal{B}$ are the conditions like

$$L(\mathcal{B}; -a_0a_1, c; \lambda) = \prod(-a_0(\alpha, \beta)a_1(\alpha, \beta), c(\alpha, \beta))_\mathfrak{p} = 1$$

where the product is taken over all $\mathfrak{p}$ outside $\mathcal{B}$ which divide $c(\alpha, \beta)$. There is one of these conditions for each c, and for each of them the first argument in the Hilbert symbol has even degree. As in the discussion which follows the proof of Lemma 4, these generate a group of conditions naturally isomorphic to Λ. In the light of Lemma 4, we shall call a condition in this group *continuous* if it comes from Λ_0. Since increasing $\mathcal{B}$ does not alter the ϕ_0, it does not alter the set of continuous conditions.

Lemma 7 *Let W_0 be everywhere locally soluble. Then the continuous conditions derived from* (17) *are collectively equivalent to the Brauer-Manin conditions for the existence of points of W_0 defined over k. The continuous conditions similarly derived from the $L^*(\mathfrak{a})$ are collectively equivalent to the Brauer-Manin conditions for the existence of positive* 0*-cycles of degree N on W_0 defined over k.*

Proof The first assertion is proved for $k = \mathbf{Q}$ in [1], §8; as with Lemma 1, the proof there can be extended to our more general case. The second sentence follows trivially from the first in the light of (12). □

Theorem 2 *Let $\mathcal{B}$ be a finite set of places of k, satisfying the conditions for* (16) *analogous to those stated above for* (17).

(i) *For each v in $\mathcal{B}$ let A_v be a point of $V_0(k_v)$, and let λ_v be its image under the projection to $\mathbf{L}^1$. Suppose that all the conditions like*

$$\prod_{v \in \mathcal{B}} m(v; -a_0a_1, c; \lambda_v) = 1 \tag{19}$$

hold. Then there is a point of $V_0(k)$ as close as we like to each A_v.

(ii) *Let λ be a point of $\mathbf{L}^1(k)$ such that all the conditions like*

$$L(\mathcal{B}; -a_0a_1, c; \lambda) = 1 \tag{20}$$

hold. Then there is a point in $V_0(k)$ whose projection on $\mathbf{L}^1(k)$ is as close as we like to λ in the topology induced by $\mathcal{B}$.

Since we can find λ arbitrarily close to each λ_v, it follows from (2) that the two parts of the theorem are equivalent. In view of Lemma 5, the conclusion of (ii) still follows if we only require the continuous conditions to hold. By the first assertion of Lemma 7 and the fact that weak approximation holds for conics, Theorem 2(ii) is equivalent to Theorem 1.

To prove Theorem 2, we need to construct points in the image of $V_0(k)$ in $\mathbf{L}^1(k)$ which satisfy strong local conditions; to do this, we construct a sequence of positive 0-cycles of gradually decreasing degrees. If N is large enough, we can generate positive 0-cycles of degree N on V_0 satisfying local conditions by means of an argument which depends on the partial fraction formula (22); its use in this context was pioneered by Salberger. Of the various versions of the consequent algorithm, Lemma 9 seems the simplest, both in its proof and in the way in which it is used; in particular, it does not involve an auxiliary set of primes and its proof does not depend on a deep result of Waldschmidt.

We need a preliminary lemma about approximation.

Lemma 8 *Let L be an algebraic number field, $\mathfrak{B}$ a finite set of places of L and $\mathfrak{S}$ a finite set of primes of L not necessarily disjoint from $\mathfrak{B}$. Let $b > 1$ be in $\mathbf{Z}$ and such that no prime of L which divides b is in $\mathfrak{B}$. Let $M > 0$ be a rational integer and for each v in $\mathfrak{B}$ let ξ_v be in L_v. Then there exists ξ in L^* as close as we like to each ξ_v and such that $\xi = \alpha\gamma^M$, where (α) is the product of a first degree prime $\mathfrak{p}$ not in $\mathfrak{B} \cup \mathfrak{S}$ and primes in $\mathfrak{B}$, and $\gamma = \gamma_1/\gamma_2$ for coprime integers γ_1, γ_2 such that the prime factorization of γ_1 does not include any prime in $\mathfrak{B} \cup \mathfrak{S} \cup \{\mathfrak{p}\}$ and the only primes which divide γ_2 also divide b.*

Proof. By Dirichlet's theorem on primes in arithmetic progression, we can choose $\mathfrak{p}$ and α as in the statement of the lemma so that α is as close as we like to ξ_v for each finite v in $\mathfrak{B}$ and $\xi_v/\alpha > 0$ for each real v in $\mathfrak{B}$. For each infinite v in $\mathfrak{B}$ we choose γ_v in L_v so that $\gamma_v^M = \xi_v/\alpha$. Using weak approximation, choose γ' in L, a unit at every finite prime in $\mathfrak{B} \cup \mathfrak{S} \cup \{\mathfrak{p}\}$, so that γ' is arbitrarily close to 1 at every finite place in $\mathfrak{B}$ and arbitrarily close to γ_v at every infinite place v in $\mathfrak{B}$. By writing $\gamma' b^N$ for large enough N in terms of a base for $\mathfrak{o}_L/\mathbf{Z}$ and changing the coefficients by elements of $\mathbf{Q}$ which are small at each finite prime in $\mathcal{B} \cup \mathcal{S}$ and bounded at every infinite place in $\mathcal{B}$, we can obtain an integer γ_1 which is prime to $\mathcal{S} \cup \{\mathfrak{p}\}$ and to b and close to $\gamma' b^N$ at every place in $\mathcal{B}$. Now take $\gamma_2 = b^N$; then $\xi = \alpha\gamma^M$ satisfies all our requirements. □

For the statement and proof of the following lemma, we shall call a place of k *bad* if it lies in $\mathcal{B}$ or divides b; and we shall call a place in $\mathbf{Q}$ or in a field containing k *bad* if it lies below or above a bad place of k. For our purposes, the most important difference between places in $\mathcal{B}$ and primes dividing b is that the latter have no approximation conditions associated with them.

Lemma 9 *Let k be an algebraic number field and $P_1(X), \ldots, P_n(X)$ monic irreducible non-constant polynomials in $k[X]$; and let $N \geq \sum \deg(P_i)$ be a given integer. Let $\mathfrak{B}$ be a finite set of places of k which contains the infinite places, the primes which divide 2, the primes at which some coefficient of some P_i is not integral and any other primes $\mathfrak{p}$ at which $\prod P_i(X)$ does not remain separable when reduced* mod $\mathfrak{p}$. *Let b be as in Lemma 8. For each v in $\mathfrak{B}$ let U_v be a non-empty open set of separable monic polynomials of degree N in $k_v[X]$. Let $M > 0$ be a fixed rational integer. Then we can find an irreducible monic polynomial $G(X)$ in $k[X]$ of degree N which lies in each U_v and for which λ, the image of X in $K = k[X]/G(X)$, satisfies*

$$(P_i(\lambda)) = \mathfrak{P}_i \mathfrak{A}_i \mathfrak{C}_i^M \tag{21}$$

for each i, where the $\mathfrak{P}_i$ are distinct first degree primes in K not lying above any prime in $\mathfrak{B}$, the $\mathfrak{A}_i$ are products of bad primes in K and the $\mathfrak{C}_i$ are integral ideals in K. Moreover we can arrange that $\lambda = \alpha/\beta$ where α is integral and β is an integer all of whose prime factors are bad.

Proof We shall need to apply Lemma 8 repeatedly with the same value of M as in Lemma 9. We can assume, after adding a constant to X if necessary, that none of the $P_i(X)$ is a multiple of X. Write $R(X) = \prod P_i(X)$ and $R_i(X) = R(X)/P_i(X)$. Any polynomial $G(X)$ in $k[X]$ can be written in just one way in the form

$$G(X) = R(X)Q(X) + \sum R_i(X)\psi_i(X) \tag{22}$$

with deg ψ_i < deg P_i; for if λ_i is a zero of $P_i(X)$ this is just the classical partial fraction formula

$$\frac{G(X)}{\prod P_i(X)} = Q(X) + \sum \frac{\psi_i(X)}{P_i(X)}$$

with $\psi_i(\lambda_i) = G(\lambda_i)/R_i(\lambda_i)$. This property determines a unique $\psi_i(X)$ in $k[X]$ of degree less than $\deg P_i$. The same result holds over any k_v. If the coefficients of G are integral at v, for some v not in $\mathfrak{B}$, then so are those of Q and each ψ_i because R and the R_i are monic and $R_i(\lambda_i)$ is a unit outside $\mathfrak{B}$. For each v in $\mathfrak{B}$ let $G_v(X)$ be a polynomial of degree N lying in U_v, and write

$$G_v(X) = R(X)Q_v(X) + \sum R_i(X)\psi_{iv}(X)$$

with deg ψ_{iv} < deg P_i. We adjoin to $\mathfrak{B}$ a further finite place w at which b is a unit, and associate with it a monic irreducible polynomial $G_w(X)$ in $k_w[X]$ with degree N; the only purpose of G_w is to ensure that the $G(X)$ which we shall construct is irreducible over k. We build $G(X)$, close to $G_v(X)$ for every $v \in \mathfrak{B}$ including w, in the following manner.

For the first step let $k_i = k[X]/P_i(X)$ and for each $v \in \mathfrak{B}$ let ϕ_{iv} be the class of ψ_{iv} in $k_v[X]/P_i(X) = k_i \otimes_k k_v$. Take $\mathfrak{S}$ to consist of those primes in k at which the constant terms of the $P_i(X)$ are not all units. We apply Lemma 8 to each set of ϕ_{iv} in turn, replacing L by k_i and $\mathfrak{B}$ and $\mathfrak{S}$ by the sets of places of k_i which lie above $\mathfrak{B}$ and $\mathfrak{S}$ respectively; let ϕ_i be the element of k_i thus obtained, and let $\mathfrak{P}_i$ be the associated prime in k_i. Let $\psi_i'(X)$ be the unique polynomial in $k[X]$ with $\deg \psi_i' < \deg P_i$ whose class in k_i is ϕ_i. Clearly $\psi_i'(X)$ is arbitrarily close to each $\psi_{iv}(X)$, and its coefficients are integers outside $\mathfrak{B}$ because $\mathfrak{B}$ contains all the primes which ramify in k_i/k. Now choose positive c, T in $\mathbf{Z}$ so that c is a unit at all bad primes, divisible by all the primes outside $\mathfrak{B} \cup \{\mathfrak{P}_i\}$ which divide the numerator of any ϕ_i, and close to b^T at the real place and at all the primes below primes in $\mathfrak{B}$. Let $\psi_i(X) = (c/b^T)^M \psi_i'(X)$.

We now choose $Q(X)$ to be close to $Q_v(X)$ for each v in $\mathfrak{B}$, and to be such that each coefficient other than the leading coefficient (which is 1) is integral except perhaps at bad primes and is divisible by c. We can do this by an argument like, but very much simpler than, that in the proof of Lemma 8. This construction ensures that $G(X)$ is monic and arbitrarily close to each $G_v(X)$ including $G_w(X)$. The assumptions made about $G_w(X)$ ensure that $G(X)$ is irreducible in k_w and therefore in k. Moreover, the coefficients of

$Q(X)$ are integers except perhaps at bad primes; and since $G(X)$ is monic the denominator of any $P_i(\lambda)$ only contains bad primes. A consequence of the choice of $\mathfrak{S}$ is that every λ_i, and therefore every $Q(\lambda_i)$, is prime to c.

We have still to prove (21). Let $\mathfrak{p}_i$ be the prime in k below $\mathfrak{P}_i$. By computing the resultant of $P_i(X)$ and $G(X)$ in two different ways, we obtain

$$\mathrm{Norm}_{K/k} P_i(\lambda) = \pm \mathrm{Norm}_{k_i/k} G(\lambda_i) = \pm \mathrm{Norm}_{k_i/k}(\phi_i R_i(\lambda_i)) \qquad (23)$$

where λ_i is a zero of $P_i(X)$. By hypothesis $R_i(\lambda_i)$ is a unit at every place of $k(\lambda_i)$ which does not lie above a place in $\mathfrak{B}$; and we have arranged that the denominator of $\mathrm{Norm}_{k_i/k}\phi_i$ is only divisible by bad primes, and its numerator is the product of the first degree prime $\mathfrak{p}_i$, powers of primes in $\mathfrak{B}$ and Mth powers of norms of primes which come from the $\mathfrak{C}_i$ of Lemma 8. Also λ, and therefore $P_i(\lambda)$, is integral outside bad primes in K. None of these lie above $\mathfrak{p}_i$. Hence $P_i(\lambda)$ is an integer at each prime of K lying above $\mathfrak{p}_i$. It follows that the ideal $(P_i(\lambda))$ is divisible by just one prime of K above $\mathfrak{p}_i$, and that to the first power. It only remains to show that, apart from this prime and bad primes, what we have is an Mth power.

Let L be a splitting field for all the $P_i(X)$ and let $\mathfrak{P}$ be a prime in $L(\lambda)$ which divides the numerator of $P_i(\lambda)$. By (23) and the remarks on either side of it, $\mathfrak{P}$ must divide $\mathrm{Norm}_{k_i/k}(\phi_i)$ and therefore must divide c. Hence

$$\tilde{G}(X) = \tilde{R}(X)\tilde{Q}(X) \qquad (24)$$

where the tilde denotes reduction mod $\mathfrak{P}$ of the coefficients. But the construction of $Q(X)$ has ensured that the resultant of $Q(X)$ and $R(X)$, which is $\pm \prod_i \mathrm{Norm}_{k_i/k}(Q(\lambda_i))$, is prime to c; hence $\tilde{R}(X)$ and $\tilde{Q}(X)$ are coprime. Moreover $\tilde{R}(X)$ is a product of distinct linear factors over the residue field of L at $\mathfrak{P}$. It follows that (24) can be lifted to a factorization of $G(X)$ in the completion of $L(\lambda)$ at $\mathfrak{P}$; and the roots of $G(X)$ in this field consist of one near each root of each $P_i(X)$ together with roots which come (after a further field extension) from the lift of $\tilde{Q}(X)$. The latter are not close to any root of any $P_i(X)$.

I now claim that the power of $\mathfrak{P}$ which divides $P_i(\lambda)$ is $\mathfrak{P}^m$ where m is a multiple of M. For if λ is not close to a root of $P_i(X)$ then $m = 0$. On the other hand, (22) can be written

$$G(X) = R_i(X)\psi_i(X) + f_i(X)P_i(X)$$

where

$$f_i(X) = R_i(X)Q(X) + \sum_{j \neq i} \psi_j(X) R_j(X)/P_i(X).$$

By construction, if λ is close to a root of $P_i(X)$ then $f_i(\lambda)$ is a unit at $\mathfrak{P}$, as is $R_i(\lambda)$. If λ_i is that root of $P_i(X)$ which is close to λ, then the standard successive approximation process shows that $\lambda - \lambda_i$ has the same valuation as $\psi_i(\lambda_i) = \phi_i$; and by construction $\mathfrak{P}^m \| \phi_i$ where $M|m$. It follows that $\mathfrak{P}^m \| P_i(\lambda)$ with $M|m$, as claimed, in both cases.

Now let $\mathfrak{p}$ be a prime in k which divides c, and let $\mathfrak{q}$ be any prime of $k(\lambda)$ above $\mathfrak{p}$. The factors of $P_i(\lambda)$ coming from primes of $L(\lambda)$ above $\mathfrak{q}$ have the form

$$\prod_{\mathfrak{P}|\mathfrak{q}} \mathfrak{P}^{m(\mathfrak{P})} \text{ where each } m(\mathfrak{P}) \text{ is divisible by } M. \tag{25}$$

This is equal to the corestriction of $\mathfrak{q}^n$, where $\mathfrak{q}^n$ is the exact power of $\mathfrak{q}$ which divides $P_i(\lambda)$. But the extension $L(\lambda)/k(\lambda)$ is unramified at $\mathfrak{q}$, because it is only ramified at places above places in $\mathfrak{B}$. Hence each $m(\mathfrak{P})$ in (25) is equal to n, and so n is divisible by M. This holds for all primes in $k(\lambda)$ which divide c. □

We apply Lemma 9 to the surface W_0 fibred by the pencil (17), and we assume that $\mathcal{B}$ satisfies the conditions listed after (17). We state the theorem below in the form which corresponds to Theorem 2(ii), leaving it to any reader who wishes to do so to formulate a version which corresponds to Theorem 2(i). At the price of some extra complications, one can replace the condition on N in Theorem 3 below by the weaker condition that N is at least equal to the number of singular fibres of the pencil (17). For (16) this would enable us to take $N = 5$, since only the roots of g_5 give singular fibres. One could then dispense with one step in the proof below of Theorem 2(i).

Theorem 3 *With the notation above, let $N \geq \deg(a_0 a_1 a_2)$ be a fixed integer. Let $\mathfrak{a}$ be a positive 0-cycle of degree N on $\mathbf{L}^1$ defined over k and for each place v of k suppose that W_0 contains a positive 0-cycle $\mathfrak{b}_v$ of degree N defined over k_v; for v in $\mathcal{B}$ suppose further that $\mathfrak{b}_v$ is so chosen that its projection on $\mathbf{L}^1$ is $\mathfrak{a}$. If all the continuous conditions derived from the conditions (20) hold, then there is a positive 0-cycle of degree N on W_0 defined over k whose projection is arbitrarily near to $\mathfrak{a}$ in the topology induced by $\mathcal{B}$.*

Proof We must first show that for the purpose of proving this theorem we are allowed to increase $\mathcal{B}$. Suppose that $\mathcal{B}_0$ satisfies the conditions which were imposed on $\mathcal{B}$ after (17), and let $\mathfrak{p}$ be a prime of k not in $\mathcal{B}_0$. Suppose also that the hypotheses of the theorem hold for $\mathcal{B} = \mathcal{B}_0$ and $\mathfrak{a} = \mathfrak{a}_0$. Having chosen $\mathfrak{b}_\mathfrak{p}$ we can find a positive 0-cycle $\mathfrak{a}'$ on $\mathbf{L}^1$ of degree N and defined over k which is close at every v in $\mathcal{B}_0$ to $\mathfrak{a}$ and close at $\mathfrak{p}$ to the projection of $\mathfrak{b}_\mathfrak{p}$. Now

$$L^*(\mathcal{B}_0 \cup \{\mathfrak{p}\}; -a_0a_1, c; \mathfrak{a}') = L^*(\mathcal{B}_0; -a_0a_1, c; \mathfrak{a}');$$

for writing both sides as products by means of (11), if there is a factor on the right hand side which is not present on the left, that factor must come from $\mathfrak{p}$ and is therefore equal to 1. But a continuous condition for $\mathcal{B}_0$ holds at $\mathfrak{a}'$ if and only if it holds at $\mathfrak{a}$, which it does by hypothesis. Hence the continuous conditions for $\mathcal{B}_0 \cup \{\mathfrak{p}\}$ hold at $\mathfrak{a}'$. Now suppose that the theorem holds for $\mathcal{B}_0 \cup \{\mathfrak{p}\}$; then there is a positive 0-cycle $\mathfrak{b}$ of degree N on W_0 defined over k whose projection on $\mathbf{L}^1$ is close to $\mathfrak{a}'$ in the topology induced by $\mathcal{B}_0 \cup \{\mathfrak{p}\}$. The same projection is close to $\mathfrak{a}$ in the topology induced by $\mathcal{B}_0$. So the theorem also holds for $\mathcal{B}_0$.

Note that if $\mathfrak{a}$ is actually the projection of a positive 0-cycle of degree N in W_0, then the continuous conditions certainly hold in view of (12); thus imposing the hypothesis that they all hold costs us nothing. To simplify the notation, we assume henceforth that K is an algebraic number field; this will be true for the application in this paper because K will be constructed by means of Lemma 9. In view of the previous paragraph, we can assume that $\mathcal{B}$ is so large that it satisfies the conditions imposed on $\mathfrak{B}$ in the statement of Lemma 9 and it contains the additional place w which was adjoined to $\mathfrak{B}$ in the first paragraph of the proof of Lemma 9; and if b is as in Lemma 9 we also adjoin to $\mathcal{B}$ all the primes in k which divide b. By the analogue of Lemma 5, we can now choose $\mathfrak{a}''$ close to $\mathfrak{a}$ so that all the conditions like $L^*(\mathcal{B}; -a_0a_1, c; \mathfrak{a}'') = 1$ hold. As was remarked in the last paragraph before (14), we can now increase $\mathcal{B}$ so that if $\lambda_0 = \alpha_0/\beta_0$ is a point of $\mathbf{L}^1(K)$ in $\mathfrak{a}''$ then α_0, β_0 are coprime and integral except perhaps at primes of K above a prime in $\mathcal{B}$. Now apply Lemma 9 with $M = 2$, where we take the $c(X, 1)$, normalized to be monic, to be the $P_i(X)$ and each U_v to be a small neighbourhood of the monic polynomial whose roots determine $\mathfrak{a}''$. Let $G(X)$ be given by Lemma 9; let $\mathfrak{a}'$ be the associated 0-cycle on $\mathbf{L}^1(k)$ and λ a point of $\mathbf{L}^1(K)$ in $\mathfrak{a}'$. For each v in $\mathcal{B}$, the cycle $\mathfrak{a}'$ is close to $\mathfrak{a}''$ in the v-adic topology; so (17) at λ is soluble in K_w for each w above v, by continuity. But $\lambda = \alpha/\beta$ with α, β coprime except at primes of K above a prime of $\mathcal{B}$.

So

$$\prod_{\mathfrak{P}}(-a_0(\alpha,\beta)a_1(\alpha,\beta),c(\alpha,\beta))_{\mathfrak{P}} = L^*(\mathcal{B};-a_0a_1,c;\alpha,\beta) = 1,$$

where the product is taken over all primes $\mathfrak{P}$ not above a prime in $\mathcal{B}$ and such that $c(\alpha,\beta)$ is divisible to an odd power by $\mathfrak{P}$. Here the first equality holds by definition and the second one follows from the evaluation formula (8) by continuity. But if $c(X,1) = P_i(X)$ then the product on the left reduces to the single term for which $\mathfrak{P}$ is the prime of K above $\mathfrak{p}_i$ whose existence was proved by means of (23). Hence (17) at λ is locally soluble at this prime; and because these are the only primes not lying above a prime of $\mathcal{B}$ which divide any $c(\alpha,\beta)$ or any $a_r(\alpha,\beta)$ to an odd power, they are the only primes not lying above a prime of $\mathcal{B}$ at which local solubility might present any difficulty. Thus λ can be lifted to a point of that conic which is the fibre above λ, and the theorem now follows because weak approximation holds on conics. $\square$

We now turn to the proof of Theorem 2(i), and therefore revert from W_0 to V_0 which we identify with the pencil of conics (16). In principle, the idea of the proof is to construct a sequence of positive 0-cycles defined over k of decreasing degrees, each satisfying the appropriate continuous conditions, until we obtain a point P_0 in $V_0(k)$ satisfying the given local conditions; and indeed this is what we shall do in the last part of the proof. But it is not obvious how the local descriptions of successive elements of the sequence are related. So although the application of Theorem 3 to (16) shows that there is a positive 0-cycle of degree 8 satisfying any assigned local conditions, we do not yet know what local conditions to impose on it for P_0 to be close in the topology induced by $\mathcal{B}$ to the adelic point which is our target. To cope with this, we first run the process backwards. We need only consider the continuous conditions $\mathcal{L}^* = 1$ introduced in Lemma 6.

¿From now on, any $\mathfrak{b}^r$ or $\mathfrak{b}_v^r$ will be a positive 0-cycle on V_0, defined over k or k_v respectively, and $\mathfrak{a}^r$ or $\mathfrak{a}_v^r$ will be its projection on $\mathbf{L}^1$. For each v in $\mathcal{B}$ we choose two distinct hyperplanes H_v' and H_v'', each defined over k_v and passing through A_v. Choose H', a hyperplane defined over k and close to each H_v', and similarly for H''. The intersection $H' \cap H'' \cap V$ is a positive 0-cycle $\mathfrak{b}^1$ of degree 4 defined over k; and though $\mathfrak{b}^1$ may be irreducible over k it is reducible over k_v for each v in $\mathcal{B}$ because it has one point close to A_v. Thus we can write $\mathfrak{b}^1 = \mathfrak{b}_v^2 \cup \mathfrak{b}_v^3$ where $\mathfrak{b}_v^2, \mathfrak{b}_v^3$ are positive 0-cycles of degrees

$1,3$ respectively defined over k_v and $\mathfrak{b}_v^2$ is close to A_v. Hence

$$\begin{aligned}1 =&\mathcal{L}^*(\mathcal{B}; -a_0a_1, c; \mathfrak{a}^1) = \prod \ell^*(v; -a_0a_1, c; \mathfrak{a}_v^2 \cup \mathfrak{a}_v^3)\\ =&\prod \ell^*(v; -a_0a_1, c; \mathfrak{a}_v^2) \prod \ell^*(v; -a_0a_1, c; \mathfrak{a}_v^3)\end{aligned}$$

where the products are each taken over all v in $\mathcal{B}$. But the first product in the second line is 1, by continuity applied to (19); hence

$$\prod \ell^*(v; -a_0a_1, c; \mathfrak{a}_v^3) = 1. \tag{26}$$

Now let P_1 and P_2 be two points of $V_0(k)$; there are ∞^6 curves on V which are the intersection of V with a quadric and have double points at P_1 and P_2. For each v in $\mathcal{B}$, let C_v' and C_v'' be two such curves defined over k_v each of which also passes through the three points of $\mathfrak{b}_v^3$, and let Q_v', Q_v'' be quadrics defined over k_v which contain C_v', C_v'' respectively but neither of which contains the whole of V. Choose Q', a quadric defined over k, close to each Q_v' and touching V at P_1 and P_2, and similarly for Q''; since Q' is given by a single equation and the tangency conditions are linear in the coefficients, this is just a matter of weak approximation. The intersection

$$Q' \cap Q'' \cap V = 4\{P_1\} \cup 4\{P_2\} \cup \mathfrak{b}^4.$$

(This fails if Q' and Q'' have a common component; but we can ensure that this does not happen by requiring P_1, P_2 and $\mathfrak{b}^1$ to be in sufficiently general position. Similar remarks are needed at each stage of the proof.)

Much as before, $\mathfrak{b}^4 = \mathfrak{b}_v^5 \cup \mathfrak{b}_v^6$ over k_v for each v in $\mathcal{B}$, where each $\mathfrak{b}_v^5$ has degree 3 and is close to $\mathfrak{b}_v^3$, and each $\mathfrak{b}_v^6$ has degree 5; hence

$$\prod \ell^*(v; -a_0a_1, c; \mathfrak{a}_v^5) = 1$$

follows from (26) by continuity. But

$$\mathcal{L}(\mathcal{B}; -a_0a_1, c; \lambda_1) = \mathcal{L}(\mathcal{B}; -a_0a_1, c; \lambda_2) = 1$$

where λ_1, λ_2 are the projections of P_1, P_2 on $\mathbf{L}^1$; so

$$\prod \ell^*(v; -a_0a_1, c; \mathfrak{a}_v^6) = 1.$$

Now let P_3, P_4, P_5 be three further points of $V_0(k)$; then there are ∞^9 curves on V which are the intersection of V with a quadric and pass through

P_3, P_4, P_5. For each v in $\mathcal{B}$, let D'_v and D''_v be two such curves defined over k_v each of which also passes through the five points of $\mathfrak{b}^6_v$, and let R'_v, R''_v be quadrics defined over k_v which contain D'_v, D''_v respectively but neither of which contains the whole of V. Choose R', a quadric defined over k, close to each R'_v and passing through P_3, P_4, P_5, and similarly for R''. The intersection

$$R' \cap R'' \cap V = \{P_3\} \cup \{P_4\} \cup \{P_5\} \cup \mathfrak{b}^7,$$

where $\mathfrak{b}^7_v$ has degree 13. Much as before, $\mathfrak{b}^7 = \mathfrak{b}^8_v \cup \mathfrak{b}^9_v$ over k_v for each v in $\mathcal{B}$, where each $\mathfrak{b}^9_v$ is close to $\mathfrak{b}^6_v$, so that $\mathfrak{b}^8_v$ has degree 8 and

$$\prod \ell^*(v; -a_0a_1, c; \mathfrak{a}^8_v) = 1.$$

We now have the necessary map of how to go back. By Theorem 3, we can find a positive 0-cycle $\mathfrak{d}^8$ of degree 8 on V_0, defined over k and arbitrarily near to each $\mathfrak{b}^8_v$. With the same P_3, P_4, P_5 as before, there is a pencil of curves on V which are the intersections of V with a quadric and pass through P_3, P_4, P_5 and the points of $\mathfrak{d}^8$. Let $\mathfrak{d}^5$, of degree 5, be the residual intersection of the curves of this pencil; since the pencil contains a curve close to each D'_v and another close to each D''_v, it follows that $\mathfrak{d}^5$ is close to each $\mathfrak{b}^5_v$. (This time, the curves in the pencil do not all have a common component, because one of them is arbitrarily close to $R' \cap V$ and another to $R'' \cap V$.)

In the same way, we successively generate a 0-cycle $\mathfrak{d}^3$ on V_0 of degree 3 and arbitrarily close to each $\mathfrak{b}^3_v$, and then a point of $V_0(k)$ arbitrarily close to each A_v. This last is the point which we want. □

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