TRANSCENDENTAL BRAUER-MANIN OBSTRUCTION ON A PENCIL OF ELLIPTIC CURVES

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1. Introduction

Let $\mathrm{Br}(X)$ denote the cohomological Brauer group $H^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ of a scheme X. Let k be a number field and $\overline{k}$ be an algebraically closed extension of k. A class in the Brauer group of a projective smooth variety X over k is said to be *algebraic* if it belongs to the kernel of the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\overline{k}})$, *transcendental* otherwise; this property does not depend on the choice of $\overline{k}$. For any prime number ℓ, the ℓ-primary part of the Brauer group over $\mathbf{C}$ fits into an exact sequence

$$0 \longrightarrow (\mathbf{Q}_\ell/\mathbf{Z}_\ell)^{b_2-\rho} \longrightarrow \mathrm{Br}(X_{\mathbf{C}})\{\ell\} \longrightarrow H^3(X(\mathbf{C}), \mathbf{Z})\{\ell\} \longrightarrow 0,$$

where b_2 and ρ respectively denote the second Betti number and the Picard number of $X_{\mathbf{C}}$, and $M\{\ell\}$ denotes the ℓ-primary part of M. Although this sequence does prove the non-triviality of $\mathrm{Br}(X_{\mathbf{C}})$ in many cases, e.g. when X is a $K3$ surface, transcendental classes are in general difficult to exhibit.

Almost all known instances of Brauer-Manin obstruction are thus explained by algebraic classes, the only exceptions being Harari's examples [4] with conic bundles over $\mathbf{P}^2_{\mathbf{Q}}$. Besides, in the particular case of pencils of curves of genus 1, results on the Hasse principle have been obtained only under the assumption that the 2-primary part of the Brauer group be "vertical", and therefore algebraic (see [3], §4.7). The rôle of transcendental elements in the Brauer-Manin obstruction thus seems worthy of investigation. In this note we present an example of transcendental Brauer-Manin obstruction to weak approximation for an elliptic $K3$ surface over $\mathbf{Q}$, where "elliptic" means that it possesses a fibration in curves of genus 1, with a section, over $\mathbf{P}^1_{\mathbf{Q}}$. It should be noted that the class of order 2 which we will exhibit in $\mathrm{Br}(X_{\mathbf{C}})$ enjoys the property of being divisible (because $H^3(X(\mathbf{C}), \mathbf{Z}) = 0$ for a $K3$ surface), which was not the case in Harari's examples.

2. Preliminaries: 2-descent and the Brauer group of an elliptic curve

The subscript in $H^i_{\text{ét}}$ will be dropped, as we will only use étale cohomology. If G is an abelian group (resp. group scheme), ${}_nG$ will denote the n-torsion subgroup of G. Let k be a field of characteristic different from 2. The Hilbert symbol of a pair of elements $f, g \in k^\star$ will be denoted (f,g); it is the class of a quaternion algebra in ${}_2\mathrm{Br}(k)$. When X is a geometrically integral variety over k and L is an extension of k, $L(X)$ will denote the function field of X_L. The canonical morphism $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ is injective if in addition X is regular; this fact will be used without further mention. Let E be an elliptic curve over k whose 2-torsion is rational. Fix an isomorphism of k-group schemes $(\mathbf{Z}/2\mathbf{Z})^2 \xrightarrow{\sim} {}_2E$. The kernel of the evaluation map at the zero section $\mathrm{Br}(E) \to \mathrm{Br}(k)$ will be denoted $\mathrm{Br}^0(E)$.

Lemma 2.1. *The group* $\mathrm{Br}^0(E)$ *is canonically isomorphic to* $H^1(k, E)$.

Proof. Let us write the Leray spectral sequence for the structure morphism $f\colon E \to \mathrm{Spec}(k)$ and the étale sheaf $\mathbf{G}_{\mathrm{m}}$. Since $f_\star \mathbf{G}_{\mathrm{m}} = \mathbf{G}_{\mathrm{m}}$, $R^1 f_\star \mathbf{G}_{\mathrm{m}} = E \oplus \mathbf{Z}$ and $R^q f_\star \mathbf{G}_{\mathrm{m}} = 0$ for $q > 1$ by Tsen's theorem, we get an exact sequence

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(E) \longrightarrow H^1(k, E) \longrightarrow H^3(k, \mathbf{G}_{\mathrm{m}}) \longrightarrow H^3(E, \mathbf{G}_{\mathrm{m}}).$$

The zero section induces retractions of $\mathrm{Br}(k) \to \mathrm{Br}(E)$ and of $H^3(k, \mathbf{G}_{\mathrm{m}}) \to H^3(E, \mathbf{G}_{\mathrm{m}})$, hence the lemma. □

The Kummer sequence

$$0 \longrightarrow {}_2E \longrightarrow E \xrightarrow{z \mapsto 2z} E \longrightarrow 0,$$

together with the previous lemma and the chosen isomorphism $(\mathbf{Z}/2\mathbf{Z})^2 \xrightarrow{\sim} {}_2E$, yields the exact sequence

$$0 \longrightarrow E(k)/2E(k) \xrightarrow{\ \delta\ } (k^\star/k^{\star 2})^2 \xrightarrow{\ \gamma\ } {}_2\mathrm{Br}^0(E) \longrightarrow 0. \tag{1}$$

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We shall need explicit descriptions of the maps δ and γ. First choose distinct $p, q \in k^\star$ such that the Weierstrass equation

$$y^2 = x(x-p)(x-q) \tag{2}$$

defines E and the points $P = (p,0)$ and $Q = (q,0)$ are respectively sent to $(1,0)$ and $(0,1)$ via ${}_2E \xrightarrow{\sim} (\mathbf{Z}/2\mathbf{Z})^2$. It is well-known (see e.g. [9], p. 281) that $\delta(M) = (x(M)-q, x(M)-p)$ for $M \in E(k)$ if $M \notin {}_2E(k)$, that $\delta(P) = (p-q, p(p-q))$ and that $\delta(Q) = (q(q-p), q-p)$.

Proposition 2.2. *Let $f, g \in k^\star$. The classes of the quaternion algebras $(x-p, f)$ and $(x-q, g) \in \mathrm{Br}(k(E))$ actually belong to $\mathrm{Br}^0(E)$, and $\gamma(f,g) = (x-p, f) + (x-q, g)$.*

Proof. By symmetry, it is enough to prove that $\gamma(f,1) = (x-p, f)$ in $\mathrm{Br}(k(E))$. Choose a separable closure $\overline{k}$ of k and let G_k be its Galois group over k. Likewise, choose a separable closure $\overline{k(E)}$ of $\overline{k}(E)$ and let $G_{k(E)}$ be its Galois group over $k(E)$. It follows from the Hochschild-Serre spectral sequence, Tsen's theorem and Hilbert's theorem 90 that the inflation map $H^2(k, \overline{k}(E)^\star) \to \mathrm{Br}(k(E))$ is an isomorphism. Let $\rho\colon H^1(k,E) \to H^2(k, \overline{k}(E)^\star/\overline{k}^\star)$ denote the composition of the canonical isomorphism $H^1(k,E) \xrightarrow{\sim} H^1(k, \mathrm{Pic}(E_{\overline{k}}))$ and the boundary of the exact sequence

$$0 \longrightarrow \overline{k}(E)^\star/\overline{k}^\star \longrightarrow \mathrm{Div}(E_{\overline{k}}) \longrightarrow \mathrm{Pic}(E_{\overline{k}}) \longrightarrow 0.$$

As shown in the annexe of [2], the diagram

$$\begin{array}{ccccc}
\mathrm{Br}(k) & \longrightarrow & \mathrm{Br}(E) & \xrightarrow{\ \theta\ } & H^1(k,E) \\
\| & & \cap & & \\
\| & & \mathrm{Br}(k(E)) & & \downarrow{\scriptstyle -\rho} \\
\| & & \downarrow{\scriptstyle \wr} & & \\
\mathrm{Br}(k) & \longrightarrow & H^2(k, \overline{k}(E)^\star) & \longrightarrow & H^2(k, \overline{k}(E)^\star/\overline{k}^\star)
\end{array}$$

commutes, where θ denotes the map which stems from the Leray spectral sequence (see lemma 2.1). This enables us to carry out cocycle calculations for determining the image of $\gamma(f,1)$ in $H^2(k, \overline{k}(E)^\star/\overline{k}^\star)$. We shall use the standard cochain complexes. Let $\chi_f\colon G_k \to \mathbf{Z}$ be the map with image in $\{0,1\}$ whose composition with the projection $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ is the quadratic character associated with $f \in k^\star/k^{\star 2} = H^1(G_k, \mathbf{Z}/2\mathbf{Z})$. The image of $(f,1)$ in $H^1(k,E)$ is represented by the 1-cocycle $a\colon \sigma \mapsto \chi_f(\sigma)P$. If $M \in E(k)$, let $[M]$ denote the corresponding divisor on $E_{\overline{k}}$. The 1-cochain with values in $\mathrm{Div}(E_{\overline{k}})$ defined by $\sigma \mapsto \chi_f(\sigma)([P]-[0])$ is a lifting of a. Its differential $(\sigma,\tau) \mapsto (\chi_f(\sigma)+\chi_f(\tau)-\chi_f(\sigma\tau))([P]-[0])$ is, as expected, a 2-cocycle with values in $\overline{k}(E)^\star/\overline{k}^\star$, which we may rewrite as $(\sigma,\tau) \mapsto (x-p)^{\chi_f(\sigma)\chi_f(\tau)}$; it represents the image of $\gamma(f,1)$ in $H^2(k, \overline{k}(E)^\star/\overline{k}^\star)$. Since $x-p$ is invariant under G_k, the same formula defines a 2-cocycle on G_k with values in $\overline{k}(E)^\star$. We thus end up with a 2-cocycle

$$\begin{array}{rcl}
b\colon G_{k(E)} \times G_{k(E)} & \longrightarrow & \overline{k(E)}^\star \\
(\sigma,\tau) & \longmapsto & (x-p)^{\chi_f(\sigma)\chi_f(\tau)}
\end{array}$$

which represents the image of $\gamma(f,1)$ in $\mathrm{Br}(k(E))$, at least modulo $\mathrm{Br}(k)$, where χ_m now denotes the lifting with values in $\{0,1\}$ of the quadratic character on $k(E)$ associated with $m \in k(E)^\star$. (Note that k is separably closed in $k(E)$, so that G_k identifies with a quotient of $G_{k(E)}$.) Choose a square root s of $x-p$ in $\overline{k(E)}$. Dividing b by the differential of the 1-cochain $\sigma \mapsto s^{\chi_f(\sigma)}$ gives the 2-cocycle $(\sigma,\tau) \mapsto (-1)^{\chi_{x-p}(\sigma)\chi_f(\tau)}$, which does represent the image of the cup-product $(x-p) \cup f$ by the composite map $H^1(k(E), \mathbf{Z}/2\mathbf{Z})^{\otimes 2} \to H^2(k(E), \mathbf{Z}/2\mathbf{Z}) \to \mathrm{Br}(k(E))$.

We have now proved that $\gamma(f,1) = (x-p, f)$ in $\mathrm{Br}(k(E))/\mathrm{Br}(k)$, but the equality holds in $\mathrm{Br}(k(E))$ since $(x-p, f) = (y^2/(x-p)^3, f)$ evaluates to 0 at the zero section. ☐

3. An actual example

The reader is referred to [4] for the definitions of weak approximation, Brauer-Manin obstruction, residue maps and unramified Brauer group.

Let Ω denote the set of places of $\mathbf{Q}$. Define the polynomials $p, q \in \mathbf{Q}[t]$ by $p(t) = 3(t-1)^3(t+3)$ and $q(t) = p(-t)$. It will be useful to notice that $p(t) - q(t) = 48t$. Let E be the elliptic curve over $\mathbf{Q}(t)$ defined by (2). Denote by $\mathscr{E}$ its minimal proper regular model over $\mathbf{P}^1_{\mathbf{Q}}$ (see [8]); it is a smooth surface over $\mathbf{Q}$ endowed with a proper flat morphism $f\colon \mathscr{E} \to \mathbf{P}^1_{\mathbf{Q}}$ whose generic fibre is isomorphic to E. A geometric fibre of f is either smooth or is a union of rational curves whose intersection numbers may be computed with Tate's algorithm [10].

One finds the following reduction types, in Kodaira's notation [5]: I_2 above $t=0$, $t=3$ and $t=-3$; I_6 above $t=1$, $t=-1$ and $t=\infty$; the other fibres are smooth. Recall that a fibre of type I_n has n irreducible components $(C_i)_{1\leqslant i\leqslant n}$, with $(C_i.C_{i+1})=1$, $(C_1.C_n)=1$ and $(C_i.C_j)=0$ if $|j-i|>1$. Put

$$A=\gamma(6t(t+1),6t(t-1))=(x-p,6t(t+1))+(x-q,6t(t-1))\in \mathrm{Br}(E).$$

Proposition 3.1. *The class $A\in\mathrm{Br}(E)$ belongs to the subgroup $\mathrm{Br}(\mathscr{E})$.*

Proof. Let v be a discrete rank 1 valuation on $\mathbf{Q}(\mathscr{E})$ whose restriction to $\mathbf{Q}$ is trivial, and κ be its residue field. We shall prove that A has trivial residue at v. Let us choose a uniformiser π of v and put $\tilde z = z\pi^{-v(z)}$ for $z\in\mathbf{Q}(\mathscr{E})^\star$. It will be convenient to denote by $V\colon \mathbf{Q}(\mathscr{E})^\star\to\mathbf{Z}\times\kappa^\star$ the group homomorphism $z\mapsto(v(z),[\tilde z])$, where $[u]$ denotes the class in κ of $u\in\mathbf{Q}(\mathscr{E})$ if $v(u)=0$. For $f,g\in\mathbf{Q}(\mathscr{E})^\star$, the residue of the quaternion algebra (f,g) at v is given by the tame symbol formula

$$\partial_v(f,g)=(-1)^{v(f)v(g)}\left[\frac{f^{v(g)}}{g^{v(f)}}\right]=(-1)^{v(f)v(g)}\left[\tilde f\right]^{v(g)}\left[\tilde g\right]^{v(f)}\in\kappa^\star/\kappa^{\star 2}.$$

Note that it only depends on $V(f)$ and $V(g)$. Furthermore, if $V(f)$ is a double, i.e. if $v(f)$ is even and $\tilde f$ is a square modulo π, then $\partial_v(f,g)=1$. These remarks will be used implicitly throughout the proof.

Lemma 3.2. *The class $(-p,6t(t+1))+(-q,6t(t-1))\in\mathrm{Br}(\mathbf{Q}(t))$ is unramified over $\mathbf{P}^1_{\mathbf{Q}}$.*

Proof. The residue at a closed point of $\mathbf{P}^1_{\mathbf{Q}}$ other than $t=\alpha$ for $\alpha\in\{-3,-1,0,1,3,\infty\}$ is obviously trivial. It is straightforward to check that the remaining residues are also trivial. □

Let us now turn to showing that $\partial_v(A)=1$. As A is invariant under $t\mapsto -t$, we may assume $v(p)\leqslant v(q)$. If $v(x)<v(p)$, then $V(x-p)=V(x-q)=V(x)$, from which we deduce thanks to (2) that $V(x-p)$ and $V(x-q)$ are doubles. If $v(x)>v(q)$, then $V(x-p)=V(-p)$ and $V(x-q)=V(-q)$, hence the result by lemma 3.2. From now on, we may and will therefore assume $v(p)\leqslant v(x)\leqslant v(q)$.

To begin with, suppose $v(p)<v(q)$. In this case, either $v(t-3)>0$ or $v(t+1)>0$. If $v(x)=v(q)$, then $V(x-p)=V(-p)$, hence $\partial_v(A)=\partial_v(-q(x-q),6t(t-1))$ by lemma 3.2; but with a look at (2), one finds that both $v(-q(x-q))$ and $v(6t(t-1))$ are even. Suppose now $v(x)<v(q)$. It follows from (2) that $V(x-p)$ is a double, hence $\partial_v(A)=\partial_v(x-q,6t(t-1))=\partial_v(x,6t(t-1))$. If $v(x)$ is even or if $[6t(t-1)]$ is a square in κ, which happens if $v(t-3)>0$, we get $\partial_v(A)=1$. If on the other hand $v(t+1)>0$ and $v(x)$ is odd, then $[6t(t-1)]=12$, which (2) shows to be a square in κ.

We are now left with the case $v(p)=v(q)=v(x)$. If $v(t)=0$, then $v(t-3)=v(t-1)=v(t+1)=v(t+3)=0$, so $v(6t(t+1))=v(6t(t-1))=0$ and it suffices to prove that $v(x-p)$ and $v(x-q)$ are even, which follows from (2) and the equality $v(p)=v(x)=v(q)=v(p-q)=0$. If $v(t)<0$, then $V(6t(t+1))=V(6t(t-1))$, so that $\partial_v(A)=\partial_v(x,6t(t+1))$, which is trivial since both $v(x)=v(p)=4v(t)$ and $v(6t(t+1))$ are even. Suppose finally that $v(t)>0$. If $v(x-p)<v(t)$, then $V(x-p)=V(x-q)$ since $v(p-q)=v(t)$, and $\partial_v(A)=\partial_v(x-p,(t+1)(t-1))=\partial_v(x-p,-1)$; if $v(x-p)=0$, the residue is obviously trivial, and if $v(x-p)>0$, which means that $[\tilde x]=[\tilde p]=-9$, (2) shows that -1 is a square in κ. We therefore assume $v(x-p)\geqslant v(t)$, which still leads to $[\tilde x]=[\tilde p]=-9$. As $v(p-q)=v(t)$, at least one of $v(x-p)$ and $v(x-q)$ is equal to $v(t)$. In either case, (2) implies that $v(x-p)+v(t)$ is even, so $(-9)^{v(t)}(-1)^{v(x-p)}$ is a square, hence $\partial_v(A)=\partial_v(x,6t(t-1))+\partial_v(x-p,(t+1)(t-1))$ is trivial. □

We shall now prove the following.

Theorem 3.3. *The class $A\in\mathrm{Br}(\mathscr{E})$ is transcendental and yields a Brauer-Manin obstruction to weak approximation on the projective smooth surface $\mathscr{E}$ over $\mathbf{Q}$.*

Proof. Let us first deal with the second part of the assertion. A glance at equation (2) shows that $\mathscr{E}$ has a $\mathbf{Q}_2$-point M_2 with coordinates $x=1$ and $t=2$. (Indeed, this equation defines an affine surface over $\mathbf{Q}$ endowed with a morphism to $\mathbf{P}^1_{\mathbf{Q}}$ whose smooth locus identifies with an open subset of $\mathscr{E}$.) Using the formula given in [7], Ch. XIV, §4, one easily checks that $A(M_2)$ is non-trivial. Now choose $N\in\mathscr{E}(\mathbf{Q})$ in the image of the zero section and let $M_v\in\mathscr{E}(\mathbf{Q}_v)$ be equal to N for any $v\in\Omega\setminus\{2\}$. This defines an adelic point $(M_v)_{v\in\Omega}$. The class $A(M)\in\mathrm{Br}(\mathbf{Q})$ is trivial since $A\in\mathrm{Br}^0(E)$; consequently, the evaluation of A at $(M_v)_{v\in\Omega}$ is non-trivial, which is an obstruction to weak approximation.

It remains to be shown that A is transcendental. The exact sequence (1) reduces this to the computation of $E(\mathbf{C}(t))/2E(\mathbf{C}(t))$.

Lemma 3.4. *The surface $\mathscr{E}$ is a $K3$ surface.*

Proof. The topological Euler-Poincaré characteristic $e(\mathscr{E}_{\mathbf{C}})$ of $\mathscr{E}_{\mathbf{C}}$ can be expressed in terms of that of the fibres and that of the base ([1], p. 97, prop. 11.4), which leads to $e(\mathscr{E}_{\mathbf{C}}) = 24$. Let $\chi(\mathscr{O}_{\mathscr{E}})$ denote the Euler-Poincaré characteristic of the coherent sheaf $\mathscr{O}_{\mathscr{E}}$. The canonical bundle $\mathscr{K}_{\mathscr{E}}$ of $\mathscr{E}$ is simply $f^\star\mathscr{O}(\chi(\mathscr{O}_{\mathscr{E}})-2)$ (see [1], p. 162, cor. 12.3); in particular it has self-intersection 0, hence $\chi(\mathscr{O}_{\mathscr{E}}) = 2$ by Noether's formula. We have now proved the triviality of $\mathscr{K}_{\mathscr{E}}$. That $H^1(\mathscr{E}, \mathscr{O}_{\mathscr{E}}) = 0$ follows from $\chi(\mathscr{O}_{\mathscr{E}}) = 2$ and Serre duality. □

Lemma 3.5. *The elliptic curve E has Mordell-Weil rank* 0 *over* $\mathbf{C}(t)$.

Proof. Let $\rho(\mathscr{E}_{\mathbf{C}})$ be the Picard number of $\mathscr{E}_{\mathbf{C}}$ and R be the subgroup of the Néron-Severi group $\mathrm{NS}(\mathscr{E}_{\mathbf{C}})$ spanned by the zero section and the irreducible components of the fibres. As follows from the output of Tate's algorithm, R has rank 20. On the other hand, $\rho(\mathscr{E}_{\mathbf{C}}) \leqslant 20$ since $\mathscr{E}$ is a $K3$ surface. The Shioda-Tate formula

$$\rho(\mathscr{E}_{\mathbf{C}}) = \mathrm{rank}(E(\mathbf{C}(t))) + \mathrm{rank}(R)$$

thus yields the result. □

This lemma shows that the $\mathbf{F}_2$-vector space $E(\mathbf{C}(t))/2E(\mathbf{C}(t))$ has dimension 2. Now the classes $\delta(P) = (t, t(t-1)(t+3))$ and $\delta(Q) = (t(t+1)(t-3), t)$ are independent over $\mathbf{F}_2$, hence span the whole kernel of γ. On the other hand $(t(t+1), t(t-1))$ is evidently not a combination of $\delta(P)$ and $\delta(Q)$, so that A has non-zero image in $\mathrm{Br}(\mathbf{C}(\mathscr{E}))$ and is therefore transcendental. □

Remark 3.6. It is actually true that $A(M) = 0$ for *all* $M \in \mathscr{E}(\mathbf{Q})$. This is a consequence of the global reciprocity law and the fact that A vanishes on $\mathscr{E}(\mathbf{Q}_v)$ for all $v \in \Omega \setminus \{2\}$, which can be checked by a tedious computation.

Remark 3.7. It is possible to determine ${}_2\mathrm{Br}(\mathscr{E})$ completely if one is willing to compute explicit equations for $\mathscr{E}$. This involves blowing up the singular surface given by equation (2) a sufficient number of times. Alternately, one may observe that all fibres have type I_n (in other words, $\mathscr{E} \to \mathbf{P}^1_{\mathbf{Q}}$ is semi-stable), and then use the equations given by Néron in this case in [6], §III. Anyhow one finds that ${}_2\mathrm{Br}(\mathscr{E})$ is spanned by A modulo ${}_2\mathrm{Br}(\mathbf{Q})$ after writing out all possible residues of a general class $\gamma(f, g)$. On the other hand, the 2-torsion of the Brauer group of a complex $K3$ surface with Picard number 20 has rank 2 over $\mathbf{F}_2$, so ${}_2\mathrm{Br}(\mathscr{E}_{\mathbf{C}})$ is strictly larger than ${}_2\mathrm{Br}(\mathscr{E})/{}_2\mathrm{Br}(\mathbf{Q})$. It turns out that ${}_2\mathrm{Br}(\mathscr{E}_{\mathbf{C}})$ is spanned by A and the class of the quaternion algebra (x, t), which unexpectedly belongs to $\mathrm{Br}(\mathbf{Q}(\mathscr{E}))$ and only gets unramified after extension of scalars to $\mathbf{Q}(\sqrt{-1}, \sqrt{3})$.

Remark 3.8. In the semi-stable case, a computer program was written to carry out the calculations alluded to in the previous paragraph, as they often get quite lengthy. Its source code is available on request.

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