UNIVERSAL TORSORS AND COX RINGS

by

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Abstract. — We study the equations of universal torsors on rational surfaces.

Contents

Introduction

The study of surfaces over nonclosed fields k leads naturally to certain auxiliary varieties, called *universal torsors*. The proofs of the Hasse principle and weak approximation for certain Del Pezzo surfaces required a very detailed knowledge of the projective geometry, in fact, explicit equations, for these torsors [**6**], [**8**], [**7**], [**20**], [**22**], [**23**]. More recently, Salberger proposed using universal torsors to count rational points of

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The first author was partially supported by the Sloan Foundation and by NSF Grants 0196187 and 0134259. The second author was partially supported by NSF Grant 0100277.

bounded height, obtaining the first sharp upper bounds on split Del Pezzo surfaces of degree 5 and asymptotics on split toric varieties over $\mathbb{Q}$ [**?**]. This approach was further developed in [**18**], [**19**], [**2**], [**13**].

Colliot-Thélène and Sansuc have given a general formalism for writing down equations for these torsors. We briefly sketch their method: Let X be a smooth projective variety and $\{D_j\}_{j\in J}$ a finite set of irreducible divisors on X such that $U := X \setminus \cup_{j\in J} D_j$ has trivial Picard group. In practice, one usually chooses generators of the effective cone of X, e.g., the lines on the Del Pezzo surface. Consider the resulting exact sequence:

$$0 \longrightarrow \bar{k}[U]^*/\bar{k}^* \longrightarrow \oplus_{j\in J}\mathbb{Z}D_j \longrightarrow \mathrm{Pic}(X_{\bar{k}}) \longrightarrow 0.$$

Applying $\mathrm{Hom}(-,\mathbb{G}_m)$, one obtains an exact sequence of tori

$$1 \longrightarrow T(X) \longrightarrow T \longrightarrow R \longrightarrow 1,$$

where the first term is the *Néron-Severi torus* of X. Suppose we have a collection of rational functions, invertible on U, which form a basis for the relations among the $\{D_j\}_{j\in J}$. These can be interpreted as a section $U \to R \times U$, and thus naturally induce a $T(X)$-torsor over U, which canonically extends to the universal torsor over X. In practice, this extension can be made explicit, yielding equations for the universal torsor.

However, when the cone generated by $\{D_j\}_{j\in J}$ is simplicial, there are *no* relations and this method gives little information. In this paper, we outline an alternative approach to the construction of universal torsors and illustrate it in specific examples where the effective cone of X is simplicial.

We will work with varieties X such that the Picard and the Néron-Severi groups of X coincide and such that the ring

$$\mathrm{Cox}(X) := \bigoplus_{L\in \mathrm{Pic}(X)} \Gamma(X, L),$$

is finitely generated. This ring admits a natural action of the Néron-Severi torus and the corresponding affine variety is a natural embedding of the universal torsor of X. The challenge is to actually compute $\mathrm{Cox}(X)$

in specific examples; Cox has shown that it is a polynomial ring precisely when X is toric [**9**].

Here is a roadmap of the paper: In Section 1 we introduce Cox rings and discuss their general properties. Finding generators for the Cox ring entails embedding the universal torsor into affine space, which yields embeddings of our original variety into toric quotients of this affine space. We have collected several useful facts about toric varieties in Section 2. Section 3 is devoted to a detailed analysis of the unique cubic surface S with an isolated singularity of type $\mathbf{E}_6$. We compute the (simplicial) effective cone of its minimal desingularization $\tilde{S}$, and produce 10 distinguished sections in $\mathrm{Cox}(\tilde{S})$. These satisfy a unique equation and we show the universal torsor naturally embeds in the corresponding hypersurface in $\mathbb{A}^{10}$. More precisely, we get a homomorphism from the coordinate ring of $\mathbb{A}^{10}$ to $\mathrm{Cox}(\tilde{S})$ and the main point is to prove its surjectivity. Here we use an embedding of $\tilde{S}$ into a simplicial toric threefold Y, a quotient of $\mathbb{A}^{10}$ under the action of the Néron-Severi torus so that $\mathrm{Cox}(Y)$ is the polynomial ring over the above 10 generators. The induced restriction map on the level of Picard groups is an isomorphism respecting the moving cones. We conclude surjectivity for each degree by finding an appropriate birational projective model of Y and using vanishing results on it. Finally, in Section 4 we write down equations for the universal torsors (the Cox rings) of a split and a nonsplit cubic surface with an isolated singularity of type $\mathbf{D}_4$.

Acknowledgments: The results of this paper have been reported at the American Institute of Mathematics conference "Rational and integral points on higher dimensional varieties". We benefited from the comments of the other participants, in particular, V. Batyrev and J.L. Colliot-Thélène. We also thank S. Keel for several helpful discussions about Cox rings and M. Thaddeus for advice about the geometric invariant theory of toric varieties.

1. Generalities on the Cox ring

For any finite subset Ξ of a real vector space, let $\mathrm{Cone}(\Xi)$ denote the closed cone generated by Ξ.

Let X be a normal projective variety of dimension n over an algebraically closed field k of characteristic zero. Let $\mathrm{A}_{n-1}(X)$ and $\mathrm{N}_{n-1}(X)$ denote Weil divisors on X up to linear and numerical equivalence, respectively. Let $\mathrm{A}_1(X)$ and $\mathrm{N}_1(X)$ denote the classes of curves up to equivalence. Let $\mathrm{NE}_{n-1}(X) \subset \mathrm{N}_{n-1}(X)_{\mathbb{R}}$ denote the cone of (pseudo)effective divisors, i.e., the smallest real closed cone containing all the effective divisors of X. Let $\mathrm{NE}_1(X) \subset \mathrm{N}_1(X)_{\mathbb{R}}$ denote the cone of effective curves and $\mathrm{NM}^1(X) \subset \mathrm{N}_{n-1}(X)_{\mathbb{R}}$ the cone of nef Cartier divisors, which is dual to the cone of effective Cartier divisors. By Kleiman's criterion, this is the smallest real closed cone containing all ample divisors of X.

Let $L_1, \ldots, L_r$ be invertible sheaves on X. For $\nu = (n_1, \ldots, n_r) \in \mathbb{N}^r$ write

$$L^\nu := L_1^{\otimes n_1} \otimes \ldots \otimes L_r^{\otimes n_r}.$$

Consider the ring

$$R(X, L_1, \ldots, L_r) := \bigoplus_{\nu \in \mathbb{N}^r} \Gamma(X, L^\nu),$$

which need not be finitely generated in general.

By definition, an invertible sheaf L on X is *semiample* if L^N is globally generated for some $N > 0$:

Proposition 1.1. — *([**14**], Lemma 2.8) If $L_1, \ldots, L_r$ are semiample then $R(X, L_1, \ldots, L_r)$ is finitely generated.*

Remark 1.2. — If the L_i are ample then, after replacing each L_i by a large multiple, $R(X, L_1, \ldots, L_r)$ is generated by

$$\Gamma(X, L_1) \otimes \ldots \otimes \Gamma(X, L_r).$$

However, this is not generally the case if the L_i are only semiample (despite the assertion in the second part of Lemma 2.8 of [**14**]). Indeed, let $X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a double cover and L_1 and L_2 be the pull-backs of the polarizations on the $\mathbb{P}^1$'s to X. For suitably large n_1 and n_2, $L_1^{n_1} \otimes L_2^{n_2}$

is very ample and its sections embed X. However,

$$\Gamma(X, L_1^{n_1}) \otimes \Gamma(X, L_2^{n_2}) \simeq \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n_1)) \otimes \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n_2)),$$

and any morphism induced by these sections factors through $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 1.3. — *Let $L_1, \ldots, L_r$ be a set of invertible sheaves on X such that L_j is generated by sections $s_{j,0}, ..., s_{j,d_j}$. Assume that the induced morphism $X \to \prod_j \mathbb{P}^{d_j}$ is birational into its image. Then the ring generated by the $s_{j,k}$'s has the same fraction field as $R(X, L_1, \ldots, L_r)$.*

Proof. — Both rings have fraction field $k(X)(t_1, ..., t_r)$, where t_j is a nonzero section of L_j. □

Definition 1.4. — [**14**] *Let X be a nonsingular projective variety so that* Pic(X) *is a free abelian group of rank r. The* Cox ring *for X is defined*

$$\mathrm{Cox}(X) := R(X, L_1, \ldots, L_r),$$

where $L_1, \ldots, L_r$ are lines bundles so that

1. *the L_i form a $\mathbb{Z}$-basis of* Pic(X)*;*
2. *the cone* Cone$(\{L_1, \ldots, L_r\})$ *contains* $\mathrm{NE}_{n-1}(X)$.

This ring is naturally graded by Pic(X)*: for $\nu \in$* Pic(X) *the ν-graded piece is denoted* $\mathrm{Cox}(X)_\nu$.

Proposition 1.5. — [**14**] *The ring* Cox(X) *does not depend on the choice of generators for* Pic(X).

Proof. — Consider two sets of generators $L_1, \ldots, L_r$ and $M_1, \ldots, M_r$. Since Cone$(\{L_i\})$ and Cone$(\{M_i\})$ contain all the effective divisors, the nonzero graded pieces of both $R(X, L_1, \ldots, L_r)$ and $R(X, M_1, \ldots, M_r)$ are indexed by the effective divisor classes in Pic(X). Choose isomorphisms

$$M_j \simeq L^{(a_{1j}, \ldots, a_{rj})}, \quad i = 1, \ldots, r, A = (a_{ij})$$

which naturally induce isomorphisms

$$\Gamma(M^\nu) \simeq \Gamma(L^{A\nu}), \quad A\nu = (a_{11}\nu_1 + \ldots + a_{1r}\nu_r, \ldots, a_{r1}\nu_1 + \ldots + a_{rr}\nu_r).$$

Thus we find $R(X, L_1, \ldots, L_r) \simeq R(X, M_1, \ldots, M_r)$. □

As $\mathrm{Cox}(X)$ is graded by $\mathrm{Pic}(X)$, a free abelian group of rank r, the torus

$$T(X) := \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{G}_m)$$

acts on $\mathrm{Cox}(X)$. Indeed, each $\nu \in \mathrm{Pic}(X)$ naturally yields a character χ_ν of $T(X)$, and the action is given by

$$t \cdot \xi = \chi_\nu(t)\xi, \quad \xi \in \mathrm{Cox}(X)_\nu, t \in T(X).$$

Thus the isomorphism constructed in Proposition 1.5 is *not* canonical: Two such isomorphisms differ by the action of an element of $T(X)$. It is precisely this ambiguity that makes descending the universal torsor to nonclosed fields an interesting question.

The following conjecture is a special case of 2.14 of [**14**]:

Conjecture 1.6 (Finiteness of Cox ring). — Let X be a log Fano variety. Then $\mathrm{Cox}(X)$ is finitely generated.

Remark 1.7. — Note that if $\mathrm{Cox}(X)$ is finitely generated it follows trivially that $\mathrm{NE}_{n-1}(X)$ is finitely generated. Moreover, the nef cone $\mathrm{NM}^1(X)$ is also finitely generated.

Indeed, the nef cone corresponds to one of the chambers in the group of characters of $T(X)$ governed by the stability conditions for points $v \in \mathrm{Spec}(\mathrm{Cox}(X))$. These chambers are bounded by finitely many hyperplanes (see Theorem 0.2.3 in [**10**] for more details).

It has been conjectured by Batyrev [**1**] that the pseudo-effective cone of a Fano variety is finitely generated. However, the finiteness of the Cox ring is not a formal consequence of the finiteness of the pseudo-effective cone.

Example 1.8. — Let $p_1, \ldots, p_9 \in H \subset \mathbb{P}^3$ be nine distinct coplanar points given as a complete intersection of two generic cubic curves in the hyperplane H, and let X be the blow-up of $\mathbb{P}^3$ at these points. Then $\mathrm{NE}^1(X)$ is finitely generated but $\mathrm{Cox}(X)$ is not. Indeed, X is an equivariant compactification of the additive group $\mathbb{G}_a^3$, acting by translation on the affine space $\mathbb{P}^3 - H$. The group action can be used to show that $\mathrm{NE}^1(X)$ is generated by the boundary components (see [**12**]). Similarly, one can show that the cone $\mathrm{NE}_1(X)$ is generated by classes of curves in the boundary components, e.g., the proper transform $\tilde{H} \subset X$ of H. It is well-known that $\mathrm{NE}_1(\tilde{H})$ is infinite [**16**] §1.23(4): The pencil of cubic plane curves with base locus $p_1, \dots, p_9$ induces an elliptic fibration,

$$\tilde{H} \to \mathbb{P}^1,$$

for which the nine exceptional curves of $\tilde{H} \to H$ are sections. Addition in the group law gives an infinite number of sections, which are also (-1)-curves and generators of $\mathrm{NE}_1(\tilde{H})$. These are also generators of $\mathrm{NE}_1(X)$, since the sections (other than the nine exceptional curves) intersect $\tilde{H}$ negatively. It follows that $\mathrm{NE}_1(X)$ and $\mathrm{NM}^1(X)$ are not finitely generated and hence $\mathrm{Cox}(X)$ is not finitely generated (see Remark 1.7).

Proposition 1.9. — *Let X be a nonsingular projective variety whose anticanonical divisor $-K_X$ is nef and big. Suppose that D is a nef divisor on X. Then $H^i(X, D) = 0$ for each $i > 0$ and D is semiample.*

Proof. — The first assertion is a consequence of Kawamata-Viehweg vanishing [**16**] §2.5. The second is a special case of the Kawamata Basepoint-freeness Theorem [**16**] §3.2. □

Proposition 1.9 largely determines the Hilbert function of the Cox ring:

Corollary 1.10. — *Retain the assumptions of Proposition 1.9. Then for nef classes ν we have*

$$\dim \mathrm{Cox}(X)_\nu = \chi(\mathcal{O}_X(\nu)).$$

Remark 1.11. — In practice, this will help us to find generators of $\mathrm{Cox}(X)$.

2. Generalities on toric varieties

We recall quotient constructions of toric varieties, following Brion-Procesi [**3**], Cox [**9**], and Thaddeus [**24**].

Let $T \simeq \mathbb{G}_m^r$ be a torus with character group $\mathfrak{X}^*(T)$. Suppose that T acts faithfully on the polynomial ring $k[x_1, \dots, x_{n+r}]$ by the formula

$$t(x_j) = \chi_j(t)x_j, \quad t \in T,$$

where $\{\chi_1, \dots, \chi_{n+r}\} \subset \mathfrak{X}^*(T)$. Define M as the kernel of the surjective morphism

$$\chi := (\chi_1, \dots, \chi_{n+r}) : \mathbb{Z}^{n+r} \to \mathfrak{X}^*(T).$$

We interpret M as the character group of the quotient torus $\mathbb{G}_m^{n+r}/T$. Set $N = \mathrm{Hom}(M, \mathbb{Z})$ so that dualizing gives

$$(\mathbb{Z}^{n+r})^* \to N \to 0.$$

Let $e_1, \dots, e_{n+r}$ and $e_1^*, \dots, e_{n+r}^*$ denote the coordinate vectors in $\mathbb{Z}^{n+r}$ and $(\mathbb{Z}^{n+r})^*$; let $\bar{e}_1^*, \dots, \bar{e}_{n+r}^* \in N$ denote the images of the e_i^* in N. Concretely, the $\bar{e}_i^*$ are the columns of the $n \times (n+r)$ matrix of dependence relations among the χ_j.

Consider a toric n-fold X associated with a fan having one-skeleton $\{\bar{e}_1^*, \dots, \bar{e}_{n+r}^*\}$. In particular, we assume that none of $\bar{e}_i^*$ is zero or a positive multiple of any of the others. The variety X is a categorical quotient of an invariant open subset $U \subset \mathbb{A}^{n+r}$ under the action of T described above (see [**9**] 2.1). Elements $\nu \in \mathfrak{X}^*(T)$ classify T-linearied invertible sheaves $\mathcal{L}_\nu$ on $\mathbb{A}^{n+r}$ and

$$\Gamma(\mathbb{A}^{n+r}, \mathcal{L}_\nu) \simeq k[x_1, \dots, x_{n+r}]_\nu.$$

We have $\mathrm{A}_{n-1}(X) \simeq \mathfrak{X}^*(T)$ and we can identify

$$\Gamma(\mathcal{O}_X(D)) \simeq k[x_1, \dots, x_n]_{\nu(D)},$$

where $\nu(D) \in \mathfrak{X}^*(T)$ is associated with the divisor class of D. The variables x_i are associated with the irreducible torus-invariant divisors D_i on X (see [**11**] §3.4), and the cone of effective divisors $\mathrm{NE}_{n-1}(X)$ is generated by $\{D_1, \dots, D_{n+r}\}$. Geometrically, the effective cone in $\mathfrak{X}^*(T)$ is the image of the standard simplicial cone generated by $e_1, \dots, e_{n+r}$ under the projection homomorphism $\chi : \mathbb{Z}^{n+r} \to \mathfrak{X}(T)$.

Recall that the *moving cone*

$$\mathrm{Mov}(X) \subset \mathrm{NE}_{n-1}(X)$$

is defined as the smallest closed subcone containing the effective divisors on X without fixed components.

Proposition 2.1. — *Retaining the notation and assumptions above,*

$$\mathrm{Mov}(X) = \bigcap_{i=1,\dots,n+r} \mathrm{Cone}(\chi_1, \dots, \chi_{i-1}, \chi_{i+1}, \dots, \chi_{n+r})$$

and has nonempty interior.

Proof. — The fixed components of $\Gamma(X, \mathcal{O}_X(D))$ are necessarily invariant under the torus action, hence are taken from $\{D_1, \ldots, D_{n+r}\}$. Moreover, D_i is fixed in each $\Gamma(X, \mathcal{O}_X(dD)), d > 0$ if and only if x_i divides each element of $k[x_1, \ldots, x_{n+r}]_{d\nu(D)}$. This is the case exactly when

$$\nu(D) \in \mathrm{Cone}(\chi_1, \ldots, \chi_{n+r}) - \mathrm{Cone}(\chi_1, \ldots, \chi_{i-1}, \chi_{i+1}, \ldots, \chi_{n+r}).$$

Suppose that the interior of the moving cone is empty. After permuting indices there are two possibilities: Either $\mathrm{Cone}(\chi_2, \ldots, \chi_{n+r})$ has no interior, or the cones $\mathrm{Cone}(\chi_2, \ldots, \chi_{n+r})$ and $\mathrm{Cone}(\chi_1, \chi_3, \ldots, \chi_{n+r})$ have nonempty interiors but meet in a cone with positive codimension. As the T-action is faithful, the χ_i span $\mathfrak{X}^*(T)$. In the first case, $\chi_2, \ldots, \chi_{n+r}$ span a codimension-one subspace of $\mathfrak{X}^*(T)$ that does not contain χ_1, so that each dependence relation

$$c_1\chi_1 + \ldots + c_{n+r}\chi_{n+r} = 0$$

has $c_1 = 0$. This translates into $\bar{e}_1^* = 0$, a contradiction. In the second case, $\chi_3, \ldots, \chi_{n+r}$ span a hyperplane, and χ_1 and χ_2 are on opposite sides of this hyperplane. Putting the dependence relations among the χ_i in row echelon form, we obtain a unique relation with nonzero first and second entries, and these two entries are both positive. This translates into the proportionality of $\bar{e}_1^*$ and $\bar{e}_2^*$. □

We now seek to characterize the projective toric n-folds X with one-skeleton $\{\bar{e}_1^*, \ldots, \bar{e}_{n+r}^*\}$. These are realized as Geometric Invariant Theory quotients $\mathbb{A}^{n+r}//T$ associated with the various linearizations of our T-action. We consider the graded ring

$$R := \sum_{d \geq 0} \Gamma(\mathbb{A}^{n+r}, \mathcal{L}_{d\nu}) = \sum_{d \geq 0} k[x_1, \ldots, x_{n+r}]_{d\nu}.$$

Proposition 2.2 (see [**24**] §2,3). — *Retain the notation above and set* $X := \mathrm{Proj}(R)$.

1. *X is projective over k if and only if 0 is not contained in the convex hull of $\{\chi_1, \ldots, \chi_{n+r}\}$.*
2. *X is toric of dimension n if ν is in the interior of*

$$\mathrm{Cone}(\chi_1, \ldots, \chi_{n+r}).$$

3. *In this case, the one-skeleton of X is contained in $\{\bar{e}_1^*, \ldots, \bar{e}_{n+r}^*\}$. Equality holds if ν is in the interior of the moving cone*

$$\bigcap_{i=1,\ldots,n+r} \mathrm{Cone}(\chi_1, \ldots, \chi_{i-1}, \chi_{i+1}, \ldots, \chi_{n+r}).$$

REMARK 2.3. — Our proof will show that X may still be of dimension n even when ν is contained in a facet of

$$\mathrm{Cone}(\chi_1, \ldots, \chi_{n+r}).$$

Similary, the one-skeleton of X may still be $\{\bar{e}_1^*, \ldots, \bar{e}_{n+r}^*\}$ even when ν is contained in a facet of

$$\bigcap_{i=1,\ldots,n+r} \mathrm{Cone}(\chi_1, \ldots, \chi_{i-1}, \chi_{i+1}, \ldots, \chi_{n+r}).$$

Proof. — The monomials which appear in R are in one-to-one correspondence to solutions of

$$a_1\chi_1 + \ldots + a_{n+r}\chi_{n+r} = d\nu, \quad a_i \in \mathbb{Z}_{\geq 0}.$$

In geometric terms, the monomials appearing in R coincide with the elements of $\mathbb{Z}^{n+r}$ in the cone

$$\chi^{-1}(\mathrm{Cone}(\nu)) \cap \mathrm{Cone}(e_1, \ldots, e_{n+r}).$$

By Gordan's Lemma in convex geometry, R is generated as a k-algebra by a finite set of monomials $x^{m_1}, \ldots, x^{m_s}$. The monomials appearing in the dth graded piece R_d coincide with elements of $\mathbb{Z}^{n+r}$ in the polytope

$$P_{d\nu} := \chi_{\mathbb{R}}^{-1}(d\nu) \cap \mathrm{Cone}(e_1, \ldots, e_{n+r}).$$

Note that $\chi^{-1}(d\nu)$ is a translate of M.

For the first part, recall that $\mathrm{Proj}(R)$ is projective over $\mathrm{Spec}(R_0)$, where R_0 is the degree-zero part. Now 0 is in the convex hull of $\{\chi_1, \ldots, \chi_{n+r}\}$ if and only if there are nonconstant elements of R of degree zero. Our hypothesis just says that $R_0 = k$ and thus is equivalent to the projectivity of X over k.

As for the second part, T acts on R by homotheties and thus acts trivially on $\mathrm{Proj}(R)$, so we have an induced action of $\mathbb{G}_m^{n+r}/T$ on $\mathrm{Proj}(R)$. We claim this action is faithful, so the quotient is toric of dimension n. Let $\mu_1, \ldots, \mu_n$ be generators for $M = \mathfrak{X}^*(\mathbb{G}_m^{n+r}/T)$. Choose $v \in \mathbb{Z}^{n+r}$ in the interior of $\mathrm{Cone}(e_1, \ldots, e_{n+r})$ so that $\chi_{\mathbb{R}}(\mathrm{Cone}(v)) = \mathrm{Cone}(\nu)$.

Replacing v by a suitably large integral multiple, we may assume each $v+\mu_i, i=1,\ldots,n$, is in $\mathrm{Cone}(e_1,\ldots,e_{n+r})$. If $\chi(v)=d\nu$ then R_d contains a set of generators for M, so the induced representation of $\mathbb{G}_m^{n+r}/T$ on R_d is faithful.

For the third part, we extract the fan classifying X from $P_{d\nu}$, following [**11**] §1.5 and 3.4: For each face Q of $P_{d\nu}$, consider the cone

$$\sigma_Q=\{v\in N_{\mathbb{R}}:\langle u,v\rangle\leq\langle u',v\rangle \text{ for all } u\in Q, u'\in P_{d\nu}\}.$$

This assignment is inclusion reversing, so the one-dimensional cones of the fan correspond to facets of $P_{d\nu}$. Moreover, each facet of $P_{d\nu}$ is induced by one of the facets of $\mathrm{Cone}(e_1,\ldots,e_{n+r})$. The corresponding one-dimensional cone in $N_{\mathbb{R}}$ is spanned by $\bar{e}_i^*$. It remains to verify that each facet of $\mathrm{Cone}(e_1,\ldots,e_{n+r})$ actually induces a facet of $P_{d\nu}$. The hypothesis that ν is in the moving cone means that $P_{d\nu}$ intersects each of the $\mathrm{Cone}(e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_{n+r})$. If ν is in the interior of the moving cone then the intersection of $P_{d\nu}$ with $\mathrm{Cone}(e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_{n+r})$ meets the relative interior of this cone, hence this cone induces a facet of $P_{d\nu}$. □

Proposition 2.2 yields the following nice consequence:

Proposition 2.4. — *Let X be a complete toric variety and ν a divisor class in the interior of* $\mathrm{Mov}(X)$. *Then there exists a projective toric variety Y_ν, with the same one-skeleton as X, and polarized by ν.*

For generic T-linearized invertible sheaves on $\mathbb{A}^{n+r}$, all semistable points are actually stable; hence Y_ν is a simplicial toric variety for generic ν (see [**3**] 1.2 and [**9**] 2.1). For the special values ν_0, contained in the walls of the chamber decomposition of [**24**], this fails to be the case. However, for each special ν_0, there exists a generic ν so that $\mathrm{Cone}(\nu)$ is very close to $\mathrm{Cone}(\nu_0)$ and there is a projective, torus-equivariant morphism $Y_\nu\to Y_{\nu_0}$ [**24**] 3.11. The polarization associated to ν_0 pulls back to Y_ν, so we obtain the following:

Proposition 2.5. — *Let X be a complete toric variety and ν_0 a divisor class in the moving cone of X. Then there exists a simplicial projective toric variety Y, with the same one-skeleton as X, so that ν_0 is semiample on Y.*

Of course, ν_0 is big when it is in the interior of the effective cone.

3. The $\mathbf{E}_6$ cubic surface

By definition, the $\mathbf{E}_6$ *cubic surface* is given by the homogeneous equation

$$S = \{(w, x, y, z) : xy^2 + yw^2 + z^3 = 0\} \subset \mathbb{P}^3. \tag{3.1}$$

We recall some elementary properties (see [**4**] for more details on singular cubic surfaces):

Proposition 3.1. —

1. *The surface S has a single singularity at the point $p := (0, 1, 0, 0)$, of type $\mathbf{E}_6$.*
2. *S is the unique cubic surface with this property, up to projectivity.*
3. *S contains a unique line, satisfying the equations $y = z = 0$.*

Any smooth cubic surface may be represented as the blow-up of $\mathbb{P}^2$ at six points in 'general position'. There is an analogous property of the $\mathbf{E}_6$ cubic surface:

Proposition 3.2. — *The $\mathbf{E}_6$ cubic surface S is the closure of the image of $\mathbb{P}^2$ under the linear series*

$$w = a^2c \quad x = -(ac^2 + b^3) \quad y = a^3 \quad z = a^2b,$$

where

$$\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = \langle a, b, c\rangle.$$

This map is the inverse of the projection of S from the double point p. The affine open subset

$$\mathbb{A}^2 := \{a \neq 0\} \subset \mathbb{P}^2$$

is mapped isomorphically onto $S - \ell$. In particular, $S \setminus \ell \simeq \mathbb{A}^2$, so the $\mathbf{E}_6$ cubic surface is a compactification of $\mathbb{A}^2$.

Remark 3.3. — Note that S is not an *equivariant* compactification of $\mathbb{G}_a^2$, so the general theory of [**5**] does not apply.

Indeed, if S were an equivariant compactification of $\mathbb{G}_a^2$ then the projection from p would be $\mathbb{G}_a^2$-equivariant (see [**12**]). Therefore, the map $\mathbb{P}^2 \dashrightarrow S$ given above has to be $\mathbb{G}_a^2$-equivariant. The only $\mathbb{G}_a^2$-action

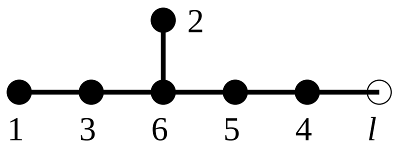

FIGURE 1. Dynkin diagram of $\mathbf{E}_6$

on $\mathbb{P}^2$ under which a line is invariant is the standard translation action [**12**]. However, the linear series above is not invariant under the standard translation action

$$b \mapsto b + \beta a \quad c \mapsto c + \gamma a.$$

We proceed to compute the effective cone of the minimal resolution $\phi_\ell : \tilde{S} \to S$. Let $\ell \subset \tilde{S}$ be the proper transform of the line mentioned in Proposition 3.1.

PROPOSITION 3.4. — *The Picard group* $\mathrm{Pic}(\tilde{S})$ *is a free abelian group of rank seven, generated by* ℓ *and the exceptional curves of* ϕ_ℓ*. For a suitable ordering* $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ *of the exceptional curves, the intersection pairing takes the form*

$$\begin{array}{c|ccccccc}
 & F_1 & F_2 & F_3 & \ell & F_4 & F_5 & F_6 \\
\hline
F_1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\
F_2 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\
F_3 & 1 & 0 & -2 & 0 & 0 & 0 & 1 \\
\ell & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
F_4 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
F_5 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
F_6 & 0 & 1 & 1 & 0 & 0 & 1 & -2
\end{array} . \tag{3.2}$$

PROPOSITION 3.5. — *The effective cone* $\mathrm{NE}(\tilde{S})$ *is simplicial and generated by* $\Phi := \{F_1, F_2, F_3, \ell, F_4, F_5, F_6\}$*. Each nef divisor is contained in* *the monoid generated by the divisors*

$$\begin{array}{rcl} A_1 & = & F_2+F_3+2\ell+2F_4+2F_5+2F_6 \\ A_2 & = & F_1+F_2+2F_3+3\ell+3F_4+3F_5+3F_6 \\ A_3 & = & F_1+2F_2+2F_3+4\ell+4F_4+4F_5+4F_6 \\ A_\ell & = & 2F_1+3F_2+4F_3+3\ell+4F_4+5F_5+6F_6 \\ A_4 & = & 2F_1+3F_2+4F_3+4\ell+4F_4+5F_5+6F_6 \\ A_5 & = & 2F_1+3F_2+4F_3+5\ell+5F_4+5F_5+6F_6 \\ A_6 & = & 2F_1+3F_2+4F_3+6\ell+6F_4+6F_5+6F_6 \end{array}$$

. Moreover A_ℓ is the anticanonical class $-K_{\tilde{S}}$ and its sections induce the resolution morphism $\phi_\ell : \tilde{S} \to S$.

Proof. — The intersection form in terms of $A := \{A_1, \ldots, A_6\}$ is:

(3.3)

	A_1	A_2	A_3	A_ℓ	A_4	A_5	A_6
A_1	0	1	1	2	2	2	2
A_2	1	1	2	3	3	3	3
A_3	1	2	2	4	4	4	4
A_ℓ	2	3	4	3	4	5	6
A_4	2	3	4	4	4	5	6
A_5	2	3	4	5	5	5	6
A_6	2	3	4	6	6	6	6 .

This is the inverse of the intersection matrix (3.2) written in terms of the basis Φ, so the A_i generate the dual to Cone(Φ). Observe that all the entries of matrix (3.3) are nonnegative and

$$\mathrm{Cone}(A) \subset \mathrm{Cone}(\Phi).$$

Suppose that D is an effective divisor on $\tilde{S}$. We write D as a sum of the fixed components contained in $\{F_1, \ldots, F_6, \ell\}$ and the parts moving relative to Φ:

$$D = M_\Phi + a_1F_1 + \ldots + a_6F_6 + a_\ell \ell, \quad a_1, \ldots, a_6, a_\ell \geq 0.$$

A priori, M_Φ may have fixed components, but they are not contained in Φ (however, see Lemma 3.6). It follows that M_Φ intersects each element of Φ nonnegatively, i.e., it is contained in Cone(A) and thus in Cone(Φ). We conclude that $D \in \mathrm{Cone}(\Phi)$. Since $A_1, ..., A_6, A_\ell$ generate $\mathrm{N}_1(\tilde{S})$ over

$\mathbb{Z}$ each nef divisor can be written as a nonnegative linear combination of these divisors.

To see that A_ℓ is the anticanonical divisor, we apply adjunction

$$K_{\tilde{S}}F_i = 0, i = 1, \dots, 6 \quad K_{\tilde{S}}\ell = -1.$$

Nondegeneracy of the intersection form implies $A_\ell = -K_{\tilde{S}}$. Since S has rational double points, the resolution map ϕ_ℓ is crepant, i.e., $\phi_\ell^* K_S = K_{\tilde{S}}$. Thus

$$\Gamma(A_\ell) = \Gamma(-K_{\tilde{S}}) = \Gamma(-\phi_\ell^* K_S) = \Gamma(\phi_\ell^* \mathcal{O}_S(+1))$$

so the sections of A_ℓ induce ϕ_ℓ. $\square$

Choose nonzero sections $\xi_1, \dots, \xi_\ell$ generating $\Gamma(F_1), \dots, \Gamma(\ell)$:

$$\Gamma(F_1) = \langle \xi_1 \rangle, \dots, \Gamma(F_6) = \langle \xi_6 \rangle, \Gamma(\ell) = \langle \xi_\ell \rangle.$$

These are canonical up to scalar multiplication. Each effective divisor

$$D = b_1F_1 + b_2F_2 + b_3F_3 + b_\ell \ell + b_4F_4 + b_5F_5 + b_6F_6$$

has a distinguished nonzero section

$$\xi^{(b_1,b_2,b_3,b_\ell,b_4,b_5,b_6)} := \xi_1^{b_1} \dots \xi_6^{b_6} \xi_\ell^{b_\ell}.$$

The distinguished section of A_j is denoted $\xi^{\alpha(j)}$. Note that we have an injective ring homomorphism

$$k[\xi_1, \dots, \xi_6, \xi_\ell] \rightarrow \mathrm{Cox}(\tilde{S}). \tag{3.4}$$

There is a partial order on the monoid of effective divisors of $\tilde{S}$: $D_1 \prec D_2$ if $D_2 - D_1$ is effective. The restriction of this order to the generators of the nef cone is illustrated in the diagram below:

$$\begin{array}{ccccc}
 & & A_6 & & \\
 & & | & & \\
 & & A_5 & & \\
 & & | & & \\
 & & A_4 & & \\
 & / & & \backslash & \\
A_\ell & & & & A_3 \\
 & \backslash & & / & \\
 & & A_2 & & \\
 & & | & & \\
 & & A_1 & &
\end{array}$$

Whenever $D_1 \prec D_2$ we have an inclusion

$$\Gamma(D_1) \hookrightarrow \Gamma(D_2)$$

which is natural up to scalar multiplication. Indeed, express

$$D_1 - D_2 = b_1 F_1 + b_2 F_2 + \ldots + b_6 F_6 + b_\ell \ell, \quad b_j \geq 0$$

so we have

$$\begin{array}{ccc} s_1 & \mapsto & \xi^{(b_1,b_2,b_3,b_\ell,b_4,b_5,b_6)} s_1 \\ \Gamma(D_1) & \hookrightarrow & \Gamma(D_2). \end{array}$$

The homomorphism (3.4) is not surjective, and we now look for generators of $\mathrm{Cox}(\tilde{S})$ beyond the ξ_j. Consider the subring

$$\mathrm{Cox}_a(\tilde{S}) = \bigoplus_{\nu \in \mathrm{NM}(\tilde{S})} \mathrm{Cox}(\tilde{S})_\nu$$

obtained by restricting to degrees corresponding to nef classes on $\tilde{S}$. The following lemma implies that any homogeneous element $\mathsf{s}_D \in \mathrm{Cox}(\tilde{S})$ can be written in the form

$$\mathsf{s}_D = m_D \xi_1^{b_1} \cdots \xi_6^{b_6} \xi_\ell^{b_\ell}$$

with nonnegative exponents and $m_D \in \mathrm{Cox}_a(\tilde{S})$.

Lemma 3.6. — *Let D be an effective divisor on $\tilde{S}$ with fixed part F_D and moving part M_D. Then F_D is supported in $\{F_1, \ldots, F_6, \ell\}$, and M_D is a linear combination of $A_1, \ldots, A_6, A_\ell$ with nonnegative coefficients.*

Proof. — Clearly M_D is nef, so the description of the nef divisors in Proposition 3.5 gives the expression in terms of the A_i. Proposition 1.9 shows M_D is semiample with vanishing higher cohomology; the last part of Proposition 3.5 gives the requisite positivity of the anticanonical class.

Let F be a fixed component of D not supported in $\{F_1, \ldots, F_6, \ell\}$. To arrive at a contradiction, we need to show that $h^0(M_D + F) > h^0(M_D)$. Since M_D has vanishing higher cohomology and

$$h^2(F + M_D) = h^0(K - F - M_D) = 0$$

it suffices to show that

$$\chi(F + M_D) > \chi(M_D).$$

By Riemann-Roch, it suffices to show that

$$F^2 + 2M_D F - K_{\tilde{S}} F > 0$$

or, equivalently,

$$F^2 + K_{\tilde{S}} F + 2M_D F - 2K_{\tilde{S}} F = 2g(F) - 2 + 2M_D F - 2K_{\tilde{S}} F > 0.$$

Since F is irreducible, $g(F) \geq 0$ and $M_D F \geq 0$ and $-K_{\tilde{S}} F > 1$, as M_D is nef and $-K_{\tilde{S}}$ is nonpositive only along the exceptional curves and has degree 1 only on the line ℓ (see Proposition 3.1). □

Corollary 1.10 gives the dimensions of the graded pieces of $\mathrm{Cox}_a(\tilde{S})$. We focus first on the generators of the nef cone, introducing sections $\tau_j \in \Gamma(A_j)$ as needed to achieve the prescribed dimensions:

$$\begin{aligned}
\Gamma(A_1) &= \left\langle \xi^{\alpha(1)}, \tau_1 \right\rangle \\
\Gamma(A_2) &= \left\langle \xi^{\alpha(2)}, \xi^{\alpha(2)-\alpha(1)}\tau_1, \tau_2 \right\rangle \\
\Gamma(A_\ell) &= \left\langle \xi^{\alpha(\ell)}, \xi^{\alpha(\ell)-\alpha(1)}\tau_1, \xi^{\alpha(\ell)-\alpha(2)}\tau_2, \tau_\ell \right\rangle
\end{aligned}$$

The sections of A_ℓ induce $\phi_\ell : \tilde{S} \to S \subset \mathbb{P}^3$ by Proposition 3.5, and can be identified with the coordinates w, x, y, z of Equation (3.1). Since $A_1 \prec A_2 \prec A_\ell$, we have

$$\Gamma(A_1) \hookrightarrow \Gamma(A_2) \hookrightarrow \Gamma(A_\ell).$$

We can identify $\Gamma(A_1) = \langle y, z \rangle$; these correspond to projecting S from the line $\ell = \{y = z = 0\}$ and induce a conic bundle structure

$$\phi_1 : \tilde{S} \to \mathbb{P}^1.$$

We have $\Gamma(A_2) = \langle x, y, z \rangle$; these correspond to projecting S from the singularity $p = \{w = y = z = 0\}$ and induce the blow-up realization

$$\phi_2 : \tilde{S} \to \mathbb{P}^2.$$

Therefore, we may choose τ_1, τ_2, and τ_ℓ so that

$$y = \xi^{\alpha(\ell)} \quad w = \xi^{\alpha(\ell)-\alpha(2)}\tau_2 \quad z = \xi^{\alpha(\ell)-\alpha(1)}\tau_1 \quad x = \tau_\ell.$$

We obtain the following induced sections for A_3, A_4, A_5, and A_6:

$$\begin{aligned}
\Gamma(A_3) &= \left\langle \xi^{\alpha(3)}, \xi^{\alpha(3)-\alpha(1)}\tau_1, \xi^{\alpha(3)-\alpha(2)}\tau_2, \xi^{\alpha(3)-2\alpha(1)}\tau_1^2 \right\rangle \\
\Gamma(A_4) &= \left\langle \xi^{\alpha(4)}, \xi^{\alpha(4)-\alpha(1)}\tau_1, \xi^{\alpha(4)-\alpha(2)}\tau_2, \xi^{\alpha(4)-\alpha(\ell)}\tau_\ell, \xi^{\alpha(4)-2\alpha(1)}\tau_1^2 \right\rangle \\
\Gamma(A_5) &= \left\langle \xi^{\alpha(5)}, \xi^{\alpha(5)-\alpha(1)}\tau_1, \xi^{\alpha(5)-\alpha(2)}\tau_2, \xi^{\alpha(5)-\alpha(\ell)}\tau_\ell, \xi^{\alpha(5)-2\alpha(1)}\tau_1^2, \right. \\
&\qquad \left. \xi^{\alpha(5)-\alpha(1)-\alpha(2)}\tau_1\tau_2 \right\rangle \\
\Gamma(A_6) &= \left\langle \xi^{\alpha(6)}, \xi^{\alpha(6)-\alpha(1)}\tau_1, \xi^{\alpha(6)-\alpha(2)}\tau_2, \xi^{\alpha(6)-\alpha(\ell)}\tau_\ell, \xi^{\alpha(6)-2\alpha(1)}\tau_1^2, \right. \\
&\qquad \left. \xi^{\alpha(6)-\alpha(1)-\alpha(2)}\tau_1\tau_2, \xi^{\alpha(6)-2\alpha(2)}\tau_2^2, \xi^{\alpha(6)-3\alpha(1)}\tau_1^3 \right\rangle
\end{aligned}$$

Equation (3.1) gives the relation

$$\tau_\ell\xi^{2\alpha(\ell)} + \tau_2^2\xi^{3\alpha(\ell)-2\alpha(2)} + \tau_1^3\xi^{3\alpha(\ell)-3\alpha(1)} = 0.$$

Dividing by a suitable monomial ξ^β, we obtain

$$\tau_\ell\xi_\ell^3\xi_4^2\xi_5 + \tau_2^2\xi_2 + \tau_1^3\xi_1^2\xi_3 = 0,$$

a dependence relation in $\Gamma(A_6)$. This is the only such relation: Any other relation, after multiplying through by ξ^β, yields a cubic form vanishing on $S \subset \mathbb{P}^3$, but equation (3.1) is the only such form. It follows that the sections given above for $A_1, \dots, A_5$ form bases for $\Gamma(A_1), \dots, \Gamma(A_5)$.

Since

$$A_3 \prec A_4 \prec A_5 \prec A_6 \prec 2A_\ell$$

we have

$$\begin{aligned}
\Gamma(A_3) &\hookrightarrow \Gamma(A_4) \hookrightarrow \Gamma(A_5) \hookrightarrow \Gamma(A_6) \\
&\hookrightarrow \Gamma(2A_\ell) = \left\langle w^2, wx, wy, x^2, xy, xz, y^2, yz, z^2 \right\rangle
\end{aligned}$$

and identifications

$$\begin{aligned}
\Gamma(A_3) &= \left\langle y^2, yz, wy, z^2 \right\rangle \\
\Gamma(A_4) &= \left\langle y^2, yz, wy, xy, z^2 \right\rangle \\
\Gamma(A_5) &= \left\langle y^2, yz, wy, xy, z^2, wz \right\rangle \\
\Gamma(A_6) &= \left\langle y^2, yz, wy, xy, z^2, wz, w^2 \right\rangle.
\end{aligned}$$

The sections of A_3 induce a morphism

$$\phi_3 : \tilde{S} \rightarrow \mathbb{P}^3$$

onto a quadric surface with a single ordinary double point. The sections of A_4 induce a morphism

$$\phi_4 : \tilde{S} \to \mathbb{P}^4$$

with image a quartic Del Pezzo surface with a rational double point of type $\mathbf{D}_5$. The sections of A_5 induce a morphism

$$\phi_5 : \tilde{S} \to \mathbb{P}^5$$

with image a quintic Del Pezzo surface with a rational double point of type $\mathbf{D}_4$. The sections of A_6 induce a morphisms

$$\phi_6 : \tilde{S} \to \mathbb{P}^6$$

with image a sextic Del Pezzo surface with two rational double points, of types $\mathbf{A}_1$ and $\mathbf{A}_2$.

We summarize this analysis in the following proposition

Proposition 3.7. — *Every section of $A_j, j = 1, 2, 3, \ell, 4, 5, 6,$ can be expressed as a polynomial in $\xi_1, \ldots, \xi_6, \xi_\ell, \tau_1, \tau_2, \tau_6$. The only dependence relation among these is*

$$\tau_\ell \xi_\ell^3 \xi_4^2 \xi_5 + \tau_2^2 \xi_2 + \tau_1^3 \xi_1^2 \xi_3 = 0$$

in $\Gamma(A_6)$. Each A_j is globally generated and induces a morphism

$$\phi_j : \tilde{S} \to \mathbb{P}^{\chi-1}, \quad \chi = \chi(\mathcal{O}_{\tilde{S}}(A_j)).$$

The remainder of this section is devoted to proving the following:

Theorem 3.8. — *The homomorphism*

$$\varrho \;:\; k[\xi_1, ..., \xi_6, \xi_\ell, \tau_1, \tau_2, \tau_\ell]/\langle \tau_\ell \xi_\ell^3 \xi_4^2 \xi_5 + \tau_2^2 \xi_2 + \tau_1^3 \xi_1^2 \xi_3 \rangle \to \mathrm{Cox}(\tilde{S})$$

is an isomorphism.

If ϱ were not injective, its kernel would have nontrivial elements in degree $\nu = dA_\ell$, for some d sufficiently large. These translate into homogeneous polynomials of degree d vanishing on $S \subset \mathbb{P}^3$. All such polynomials are multiples of the cubic form defining S, which itself is a multiple of the relation we already have.

It remains to show that ϱ is surjective. By Proposition 3.5, Lemma 3.6 and the analysis of the sections of the A_i, it suffices to prove:

PROPOSITION 3.9. — *ϱ is surjective in degrees corresponding to nef divisor classes of $\tilde{S}$.*

LEMMA 3.10. — *For any positive integers $c_1, c_2, c_3, c_\ell, c_4, c_5, c_6$, the image of*

$$\Gamma(A_1)^{\otimes c_1} \otimes \ldots \otimes \Gamma(A_\ell)^{\otimes c_\ell} \otimes \ldots \otimes \Gamma(A_6)^{\otimes c_6} \longrightarrow \Gamma(c_1 A_1 + \ldots + c_6 A_6)$$

is a linear series embedding $\tilde{S}$.

Proof. — Proposition 3.7 says that each A_j is globally generated, so if the image of

$$\Gamma(A_1) \otimes \ldots \otimes \Gamma(A_6) \longrightarrow \Gamma(A_1 + \ldots + A_6)$$

embed $\tilde{S}$ then the general result follows. We use the standard criterion: a linear series gives an embedding iff any length-two subscheme $\Sigma \subset \tilde{S}$ imposes two independent conditions on the linear series.

First, suppose the support of Σ is not contained in the exceptional locus of $\phi_\ell : \tilde{S} \to S$, i.e., the curves $F_1, F_2, F_3, F_4, F_5, F_6$. Then ϕ_ℓ maps Σ to a subscheme of length two, which imposes independent conditions on $\Gamma(A_\ell)$, and thus independent conditions on the linear series in question. Second, suppose that $\Sigma \subset F_j$ for some j (resp. $\Sigma \subset \ell$). Since $A_j \cdot F_j = 1$ (resp. $A_\ell \cdot \ell = 1$), ϕ_j maps F_j (resp. ℓ) isomorphically onto a line. It follows that Σ imposes independent conditions on $\Gamma(A_j)$. Third, suppose that Σ is reduced with support in F_i and F_j, but is not contained in either F_i or F_j. Consider the chain of rational curves containing F_i and F_j (see Figure 3.) There exists a curve F_k in this chain so that $\phi_k(F_i) \neq \phi_k(F_j)$, so Σ imposes independent conditions on $\Gamma(A_k)$. Fourth, suppose that Σ is nonreduced and supported in F_j but not contained in any F_i or ℓ. The morphism ϕ_j ramifies at points where F_j meets one of the other exceptional curves, and the kernel of the tangent morphism $d\phi_j$ consists of the tangent vectors to the curves contracted by ϕ_j. It follows that $\phi_j(\Sigma)$ has length two and imposes independent conditions on $\Gamma(A_j)$. □

The polynomial ring

$$k[\xi_1, \ldots, \xi_6, \xi_\ell, \tau_1, \tau_2, \tau_\ell]$$

is graded by the Néron-Severi group of $\tilde{S}$

$$\deg(\xi_j) = F_j, j = 1, \ldots, 6 \ \ \deg(\xi_\ell) = \ell \ \ \deg(\tau_j) = A_j, j = 1, 2, \ell.$$

This gives an action of the Néron-Severi torus $T(\tilde{S})$ on the corresponding affine space $\mathbb{A}^{10}$.

We consider the projective toric varieties that arise as quotients of $\mathbb{A}^{10}$ by $T(\tilde{S})$. As sketched in §2, these varieties have the one-skeleton

$$x_1 = (0,1,2), x_2 = (1,1,3), x_3 = (1,2,4), x_\ell = (2,3,3), x_4 = (2,3,4)$$

$$x_5 = (2,3,5), x_6 = (2,3,6), t_1 = (-1,0,0), t_2 = (0,-1,0), t_\ell = (0,0,-1)$$

where the x_j correspond to the ξ_j and the t_j correspond to the τ_j.

LEMMA 3.11. — *Let X be a toric threefold with one-skeleton $\{x_1, \ldots, t_\ell\}$ and divisor class-group $\mathfrak{X}^*(T(\tilde{S})) = \mathrm{N}_1(\tilde{S})$. Then*

$$\mathrm{Mov}(X) = \mathrm{Cone}(A_1, \ldots, A_6, A_\ell).$$

Proof. — Proposition 2.1 reduces this to computing the intersection of the cones generated by subsets of

$$\{F_1, \ldots, F_6, \ell, A_1, A_2, A_\ell\}$$

with nine elements. Since A_1, A_2, A_ℓ are effective combinations of the classes $F_1, \ldots, F_6$, and ℓ, it suffices to compute

$$\mathrm{Cone}(F_1, \ldots, \hat{\ell}, \ldots, A_\ell) \bigcap \left(\bigcap_{i=1,\ldots,6} \mathrm{Cone}(F_1, \ldots, \hat{F}_i, \ldots, A_\ell) \right).$$

This intersection obviously contains A_1, A_2, and A_ℓ, and it is a straightforward computation to show that it also contains A_3, A_4, A_5, A_6. For the reverse inclusion, suppose that D is contained in the intersection. Considering D as a divisor on $\tilde{S}$, we see that

$$D \cdot F_1, \ldots, D \cdot F_6, D \cdot \ell$$

are all nonnegative. Thus D is an effective sum of A_j by Proposition 3.5. □

Combining Lemmas 3.11 and 3.10 with Propositions 2.2 and 3.7, we obtain the following

PROPOSITION 3.12. — *Let ν be an ample divisor on $\tilde{S}$. Then there exists a projective toric variety Y_ν with one-skeleton $\{x_1, \ldots, t_\ell\}$ and polarization ν, and an embedding $\tilde{S} \hookrightarrow Y_\nu$ with the following properties:*

1. *the divisor class group of Y_ν is isomorphic to the divisor class group of $\tilde{S}$ so that the moving cone of Y_ν is identified with the nef cone of $\tilde{S}$;*
2. *the equation for $\tilde{S}$ in the Cox ring of Y_ν is*
$$\tau_\ell \xi_\ell^3 \xi_4^2 \xi_5 + \tau_2^2 \xi_2 + \tau_1^3 \xi_1^2 \xi_3 = 0$$
and $[\tilde{S}] = A_6$ in the divisor class group of Y_ν;
3. $\mathrm{Cox}(Y_\nu) = k[\xi_1, \ldots, \tau_\ell]$ *and is mapped isomorphically to the image of the homomorphism ϱ.*

For each toric variety Y_ν, we can consider the exact sequence of sheaves
$$0 \rightarrow I_{\tilde{S}} \rightarrow \mathcal{O}_{Y_\nu} \rightarrow \mathcal{O}_{\tilde{S}} \rightarrow 0,$$
where $I_{\tilde{S}} \simeq \mathcal{O}_{Y_\nu}(-A_6)$ is the ideal sheaf of $\tilde{S}$. Given an element θ in the divisor class group of Y_ν, we can twist to obtain
$$0 \rightarrow I_{\tilde{S}}(\theta) \rightarrow \mathcal{O}_{Y_\nu}(\theta) \rightarrow \mathcal{O}_{\tilde{S}}(\theta) \rightarrow 0.$$
We should make precise what we mean by the twist $\mathcal{F}(\theta)$ of a coherent sheaf $\mathcal{F}$ on Y_ν: Realize $\mathcal{F}$ as the sheafification of a graded module F over $\mathrm{Cox}(Y_\nu)$ (which exists by [**17**] Theorem 1.1, [**9**] Proposition 3.1), shift F by θ, and then resheafify the shifted module to obtain $\mathcal{F}(\theta)$. Twisting respects exact sequences [**9**] 3.1.

The anticanonical divisor of a toric variety is the sum of the invariant divisors [**11**] p. 89, so
$$-K_{Y_\nu} = F_1 + \ldots + F_6 + \ell + A_1 + A_2 + A_\ell = A_\ell + A_6$$
and we can rewrite our exact sequence as
$$0 \rightarrow \mathcal{O}_{Y_\nu}(K_{Y_\nu} + A_\ell + \theta) \rightarrow \mathcal{O}_{Y_\nu}(\theta) \rightarrow \mathcal{O}_{\tilde{S}}(\theta) \rightarrow 0.$$

Suppose that θ corresponds to a nef class on $\tilde{S}$; we shall prove that ϱ is surjective in degree θ, thus proving Proposition 3.9 and Theorem 3.8. Since
$$\Gamma(\mathcal{O}_{Y_\nu}(\theta)) \simeq k[\xi_1, \ldots, \xi_6, \xi_\ell, \tau_1, \tau_2, \tau_\ell]_\theta$$
it suffices to show that
$$H^1(\mathcal{O}_{Y_\nu}(K_{Y_\nu} + A_\ell + \theta)) = 0.$$
We apply Proposition 2.5, with $\nu_0 = A_\ell + \theta$, to get a *simplicial* toric variety Y_ν on which $A_\ell + \theta$ is nef. As A_ℓ is in the interior of the effective

cone of Y_ν, $A_\ell+\theta$ is also big. Note that Y_ν has finite-quotient singularities, which are log terminal [**16**] §5.2. The desired vanishing follows from Theorem 2.17 of [**15**]. Alternately, we could apply Theorem 0.1 of [**17**], which applies in arbitrary characteristic and obviates the need to pass to a simplicial model.

4. $\mathbf{D}_4$ cubic surface

The strategy of the previous section can applied to other surfaces as well. Here we illustrate it in the case of a cubic surface given by the homogeneous equation

$$S = \{(x_1, x_2, x_3, w) \,:\, w(x_1 + x_2 + x_3)^2 = x_1x_2x_3\} \subset \mathbb{P}^3.$$

We summarize its properties:

1. S has a single singularity at the point $(0, 0, 0, 1)$ of type $\mathbf{D}_4$.
2. S contains 6 lines with the equations

$$\begin{array}{ll} \ell'_1 := \{w = x_1 = 0\} & m'_1 := \{x_1 = x_2 + x_3 = 0\} \\ \ell'_2 := \{w = x_2 = 0\} & m'_2 := \{x_2 = x_1 + x_3 = 0\} \\ \ell'_3 := \{w = x_3 = 0\} & m'_3 := \{x_3 = x_1 + x_2 = 0\} \end{array}$$

3. S is given as a blow-up of $\mathbb{P}^2$ by the linear series

$$x_1 = u_1(u_1 + u_2 + u_3)^2,\ x_2 = u_2(u_1 + u_2 + u_3)^2,\ x_3 = u_3(u_1 + u_2 + u_3)^2,$$

$$w = u_1u_2u_3,$$

where $\langle u_1, u_2, u_3\rangle = \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

Remark 4.1. — There are two isomorphism classes of cubic surfaces with a $\mathbf{D}_4$ singularity [**4**] Lemma 4. The other class is

$$w(x_1 + x_2 + x_3)^2 = x_1x_2(-x_1 - x_2);$$

it is obtained from S by substituting

$$(w, x_1, x_2, x_3) \mapsto (t^{-2}w, x_1, x_2, tx_3 + (t - 1)x_1 + (t - 1)x_2)$$

and letting $t\to 0$ in the resulting equation.

Let $\beta : \tilde{S} \to S$ denote the minimal desingularization of S and

$$\ell_1, \ell_2, \ell_3, m_1, m_2, m_3$$

the strict transforms of the lines. The rational map $S \dashrightarrow \mathbb{P}^2$ induces a morphism $\tilde{S} \to \mathbb{P}^2$ and let L denote the pullback of the hyperplane class. Let E_0, E_1, E_2, E_3 be the exceptional divisors of β, ordered so that we have the following intersection matrix:

$$\begin{array}{c|ccccccc} & L & E_1 & E_2 & E_3 & m_1 & m_2 & m_3 \\ \hline L & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ E_2 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ E_3 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ m_1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ m_2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ m_3 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array} . \tag{4.1}$$

This is a rank seven unimodular matrix; since the Picard group of $\tilde{S}$ has rank seven, it is generated by $L, E_1, E_2, E_3, m_1, m_2, m_3$. In particular, we have

$$E_0 = L - (E_1 + E_2 + E_3 + m_1 + m_2 + m_3) \quad \text{and} \quad \ell_j = L - E_j - 2m_j.$$

The anticanonical class is given by

$$-K_{\tilde{S}} = 3L - (E_1 + E_2 + E_3) - 2(m_1 + m_2 + m_3) = \ell_1 + \ell_2 + \ell_3.$$

Proposition 4.2. — *The effective cone* $\mathrm{NE}(\tilde{S})$ *is generated by*

$$\Xi := \{E_0, E_1, E_2, E_3, m_j, \ell_j\}.$$

Proof. — Each effective divisor D can be expressed as a sum

$$D = M_\Xi + b_{E_0} E_0 + b_{E_1} E_1 + \ldots + b_{\ell_3} \ell_3,$$

with nonnegative coefficients, where M_Ξ intersects each of the elements in Ξ nonnegatively and thus is in the dual cone to $\mathrm{Cone}(\Xi)$. Direct computation shows that the dual to $\mathrm{Cone}(\Xi)$ has generators

$$L, L - E_i - m_i, 2L - E_i - 2m_i, 2L - E_i - E_j - 2m_i - 2m_j,$$

$$2L - E_i - E_j - m_i - 2m_j.$$

Each of these is contained in Cone(Ξ):

$$\begin{array}{rcl} L & = & \ell_i + E_i + 2m_i, \\ 2L - E_i - 2m_i & = & 2\ell_i + E_i + 2m_i, \\ 2L - E_i - E_j - m_i - 2m_j & = & \ell_i + \ell_j + m_i, \\ L - E_i - m_i & = & \ell_i + m_i, \\ 2L - E_i - E_j - 2m_i - 2m_j & = & \ell_i + \ell_j. \end{array}$$

It follows that M_Ξ and D are sums of elements in Ξ with nonnegative coefficients. $\square$

Each of the divisors m_i, ℓ_i and E_i has a distinguished nonzero section (up to a constant), denoted μ_i, λ_i and η_i, respectively. We have

$$\{\lambda_i \eta_i \mu_i^2, \eta_0\eta_1\eta_2\eta_3\mu_1\mu_2\mu_3\} \subset \Gamma(L),$$

and we may identify

$$u_i = \lambda_i\eta_i\mu_i^2 \quad \text{and} \quad u_1 + u_2 + u_3 = \eta_0\eta_1\eta_2\eta_3\mu_1\mu_2\mu_3$$

after suitably normalizing the μ_i, λ_i, and η_i. The dependence relation among the sections in $\Gamma(L)$ translates into

$$\lambda_1\eta_1\mu_1^2 + \lambda_2\eta_2\mu_2^2 + \lambda_3\eta_3\mu_3^2 = \eta_0\eta_1\eta_2\eta_3\mu_1\mu_2\mu_3. \tag{4.2}$$

An argument similar to the one given at the end of Section 3 proves that the natural homomorphism

$$k[\eta_0, ..., \eta_3, \mu_i, \lambda_i]/\langle \lambda_1\eta_1\mu_1^2 + \lambda_2\eta_2\mu_2^2 + \lambda_3\eta_3\mu_3^2 - \eta_0\eta_1\eta_2\eta_3\mu_1\mu_2\mu_3\rangle \to \mathrm{Cox}(\tilde{S})$$

is an isomorphism.

The cubic surface S admits an $\mathfrak{S}_3$-action on the coordinates x_1, x_2, x_3. In particular, it admits nonsplit forms over nonclosed ground fields. They can be expressed as follows: Let K/k be a cubic extension with Galois closure E/k. Fix a basis $\{\gamma, \gamma', \gamma''\}$ for K over k so that elements $Y \in K$ can be represented as

$$Y = y\gamma + y'\gamma' + y''\gamma''$$

with $y, y', y'' \in k$. Choose $\sigma \in \mathrm{Gal}(E/k)$ so that σ and σ^2 are coset representatives $\mathrm{Gal}(E/k)$ modulo $\mathrm{Gal}(E/K)$. Then

$$w \cdot \mathrm{Tr}_{K/k}(Y)^2 = \mathrm{N}_{K/k}(Y)$$

is isomorphic, over E, to S:

$$\begin{array}{rcl} x_1 &=& Y = y\gamma + y'\gamma' + y''\gamma'' \\ x_2 &=& \sigma(Y) = y\sigma(\gamma) + y'\sigma(\gamma') + y''\sigma(\gamma'') \\ x_3 &=& \sigma^2(Y) = y\sigma^2(\gamma) + y'\sigma^2(\gamma') + y''\sigma^2(\gamma'') \end{array}.$$

Assigning elements $U, V, W \in K$ to η_1, μ_1 and λ_1, respectively, the torsor equation (4.2) takes the form

$$\mathrm{Tr}_{K/k}(UV^2W) = \eta_0 \mathrm{N}_{K/k}(UV).$$

References

[1] V. V. Batyrev – "The cone of effective divisors of threefolds", *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)* (Providence, RI), Contemp. Math., vol. 131, Amer. Math. Soc., 1992, p. 337–352.

[2] R. de la Bretèche – "Nombre de points de hauteur bornée sur les surfaces de del Pezzo de degré 5", *Duke Math. J.* **113** (2002), no. 3, p. 421–464.

[3] M. Brion and C. Procesi – "Action d'un tore dans une variété projective", Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math., vol. 92, Birkhäuser Boston, Boston, MA, 1990, p. 509–539.

[4] J. W. Bruce and C. T. C. Wall – "On the classification of cubic surfaces", *J. London Math. Soc. (2)* **19** (1979), no. 2, p. 245–256.

[5] A. Chambert-Loir and Y. Tschinkel – "On the distribution of points of bounded height on equivariant compactifications of vector groups", *Invent. Math.* **148** (2002), no. 2, p. 421–452.

[6] J.-L. Colliot-Thélène and J.-J. Sansuc – "La descente sur les variétés rationnelles. II", *Duke Math. J.* **54** (1987), no. 2, p. 375–492.

[7] J.-L. Colliot-Thélène, J.-J. Sansuc and P. Swinnerton-Dyer – "Intersections of two quadrics and Châtelet surfaces. I", *J. Reine Angew. Math.* **373** (1987), p. 37–107.

[8] ——, "Intersections of two quadrics and Châtelet surfaces. II", *J. Reine Angew. Math.* **374** (1987), p. 72–168.

[9] D. A. Cox – "The homogeneous coordinate ring of a toric variety", *J. Algebraic Geom.* **4** (1995), no. 1, p. 17–50.

[10] I. V. Dolgachev and Y. Hu – "Variation of geometric invariant theory quotients", *Inst. Hautes Études Sci. Publ. Math.* (1998), no. 87, p. 5–56, With an appendix by Nicolas Ressayre.

[11] W. Fulton – *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.

[12] B. Hassett and Y. Tschinkel – "Geometry of equivariant compactifications of $\mathbf{G}_a^n$", *Internat. Math. Res. Notices* (1999), no. 22, p. 1211–1230.

[13] D. R. Heath-Brown – "The density of rational points on Cayley's cubic surface", to appear.

[14] Y. Hu and S. Keel – "Mori dream spaces and GIT", *Michigan Math. J.* **48** (2000), p. 331–348, Dedicated to William Fulton on the occasion of his 60th birthday.

[15] J. Kollár – "Singularities of pairs", Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, p. 221–287.

[16] J. Kollár and S. Mori – *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[17] M. Mustaţă – "Vanishing theorems on toric varieties", *Tohoku Math. J. (2)* **54** (2002), no. 3, p. 451–470.

[18] E. Peyre – "Terme principal de la fonction zêta des hauteurs et torseurs universels", *Astérisque* (1998), no. 251, p. 259–298, Nombre et répartition de points de hauteur bornée (Paris, 1996).

[19] ______, "Torseurs universels et méthode du cercle", Rational points on algebraic varieties, Progr. Math., vol. 199, Birkhäuser, Basel, 2001, p. 221–274.

[20] P. Salberger and A. N. Skorobogatov – "Weak approximation for surfaces defined by two quadratic forms", *Duke Math. J.* **63** (1991), no. 2, p. 517–536.

[21] P. Salberger – "Tamagawa measures on universal torsors and points of bounded height on Fano varieties", *Astérisque* (1998), no. 251, p. 91–258, Nombre et répartition de points de hauteur bornée (Paris, 1996).

[22] A. Skorobogatov – *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, Cambridge, 2001.

[23] A. N. Skorobogatov – "On a theorem of Enriques-Swinnerton-Dyer", *Ann. Fac. Sci. Toulouse Math. (6)* **2** (1993), no. 3, p. 429–440.

[24] M. Thaddeus – "Toric quotients and flips", Topology, geometry and field theory, World Sci. Publishing, River Edge, NJ, 1994, p. 193–213.