Dirichlet $L$-functions

- **Characters modulo $n$:** $\chi : \mathbb{Z} \to \mathbb{C}$ such that
  - $\chi(a) = \chi(a')$ for $a = a' \mod n$,
  - $\chi(a) = 0$ iff $(a, n) \neq 1$,
  - $\chi(ab) = \chi(a)\chi(b)$.

  The **trivial** character $\chi^0$ modulo $n$ is given by $\chi^0(a) = 1$, for $(a, n) = 1$, and zero otherwise. A character modulo $n$ is **primitive**, if $n$ is the (smallest) period of $\chi$; this $n$ is called **conductor** of $\chi$. Primitive characters form a group, with $\chi^{-1} = \bar{\chi}$ (complex conjugation).

- **Gauss sums:** given a primitive $\chi$ with conductor $f = f_\chi$, define
  $$\tau(\chi) := \sum_{a=1}^f \chi(a)e^{\frac{2\pi ai}{f}}.$$  

  We have
  $$|\tau(\chi)| = \sqrt{f}.$$

- **Generalized Bernoulli numbers**:
  $$F_{\chi}(t, x) := \sum_{a=1}^f \chi(a)t^{e(a+x)t} = \sum_{n \geq 0} B_{n,\chi}(x) \frac{t^n}{n!}.$$  

  and $B_{n,\chi} := B_{n,\chi}(0)$. These satisfy:
  - $B_{n,\chi}(x) \in \mathbb{Q}(\chi)[x]$,
  - $B_{0,\chi} = \frac{1}{f} \sum_{a=1}^f \chi(a) = 0$, for $\chi \neq \chi^0$,
  - $B_{n,\chi}(x) = \sum_{k=0}^n \binom{n}{k} B_{k,\chi} x^{n-k}$,
  - $(-1)^n B_{n,\chi}(-x) = \chi(-1) B_{n,\chi}(x)$.

- **$L$-functions**:
  $$L(s, \chi) := \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$  

  We have
  - $L(s, \chi^0) = \zeta(s) \prod_{p|f}(1 - p^{-s})$, with a pole at $s = 1$,
  - $L(s, \chi)$ admit a meromorphic continuation to $\mathbb{C}$, and are holomorphic for $\chi \neq \chi^0$,
  - functional equation
    $$\left( \frac{f}{\pi} \right)^{s/2} \Gamma \left( \frac{s + \delta}{2} \right) L(s, \chi) = w_{\chi} \left( \frac{f}{\pi} \right)^{(1-s)/2} \Gamma \left( \frac{1 - s + \delta}{2} \right) L(1-s, \bar{\chi}),$$
where $\delta = \delta_\chi = 0$ if $\chi(-1) = 1$ and $\delta = 1$ otherwise, and

$$w_\chi := \tau(\chi)/\sqrt{f}\delta.$$

- Special values:

$$L(1-n, \chi) = -B_{n,\chi}/n$$

and, for $\chi_a = (\frac{a}{d})$,

$$L(1, \chi_a) = \begin{cases} \frac{1}{\sqrt{d}} \log(\epsilon h) & d > 0 \\ \frac{2h}{\sqrt{|d|}} & d < -4 \end{cases}$$

where $h \in \mathbb{N}$, the class number of $K = \mathbb{Q}(\sqrt{d})/\mathbb{Q}$, and $\epsilon$ is the fundamental solution of Pell’s equation $x^2 - dy^2 = \pm 1$.

**Proposition 1.** If $\chi \neq \chi^0$ then $L(1, \chi) \neq 0$.

**Proof.** By orthogonality of characters, we have

$$\sum \chi \chi(a) = \begin{cases} \phi(n) & a = 1 \mod n \\ 0 & \text{otherwise} \end{cases}$$

Here the sum is over all characters modulo $n$. We need to show that

$$\Pi(s) := \prod_{\chi} L(s, \chi)$$

has a pole as $s = 1$, i.e., the pole from $L(s, \chi^0)$ is not cancelled by (potential) zeroes at $s = 1$ of other $L(s, \chi)$. Assume this is not the case and consider $s \in \mathbb{R}$. Then $|\Pi(s)| < \infty$, for $s > 0$. On the other hand,

$$\log(\Pi(s)) = \sum_{p,k} \frac{1}{p^{ks}} \sum_{\chi} \chi(p^k) = \phi(n) \sum_{p^k = 1 \mod n} \frac{1}{p^{ks}}.$$ 

The right side is estimated from below by

$$\sum_p \frac{1}{p^{\phi(n)s}}$$

which diverges for $s \to 1/\phi(n) + 0$. \hfill $\square$

**Theorem 2.** For $(a, n) = 1$, there are infinitely many $p = a \mod n$.

**Proof.** For $\Re(s) > 1$, we have

$$\log(L(s, \chi)) = \sum_p \sum_n \frac{\chi(p)}{kp^{ks}} = \sum_p \frac{\chi(p)}{p^s} + R(s),$$
where $R(s)$ is absolutely convergent for $\Re(s) > 1/2$. Choosing $b$ such that $ab = 1 \mod p$ and using the orthogonality of characters, we find
\[
\sum_{\chi} \chi(b) \log(L(s, \chi)) = \phi(n) \sum_{p=a \mod n} \frac{1}{p^s} + R_1(s),
\]
where $R_1(s)$ is absolutely convergent for $\Re(s) > 1/2$. By the previous theorem, $L(1, \chi) \neq 0$ for $\chi \neq \chi^0$. Considering the limit $s \to 1 + 0$ we get a pole on the left side, from $L(s, \chi^0)$. The sum
\[
\sum_{p=a \mod n} \frac{1}{p^s}
\]
becomes unbounded, for $s \to 1 + 0$. \qed