Metric fields

A metric on a field $K$ is a map $\varphi : K \to \mathbb{R}$ such that
- $\varphi(x) > 0$, $x \neq 0$, $\varphi(0) = 0$;
- $\varphi(x + y) \leq \varphi(x) + \varphi(y)$;
- $\varphi(xy) = \varphi(x)\varphi(y)$.

**Theorem 1** (Ostrovski). For $K = \mathbb{Q}$ the only possibilities are:
- $\varphi(x) = |x|^\alpha$, with $0 < \alpha \leq 1$;
- $\varphi(x) = \rho^n p(x)$, with $0 < \rho < 1$.

Hensel’s lemma

**Theorem 2.** Let $f(x) \in \mathbb{Z}_p[x]$ be a polynomial and $\alpha_1 \in \mathbb{Z}/p$ such that
- $f(\alpha_1) = 0 \mod p$ and
- $f'(\alpha_1) \neq 0 \mod p$.
Then there exists an $\alpha \in \mathbb{Z}_p$ such that $\alpha = \alpha_1 \mod p$ and $f(\alpha) = 0$.

**Proof.** Build inductively a sequence of $\alpha_n \in \mathbb{Z}_p$ such that $\alpha_{n+1} \equiv \alpha_n \mod p$ and $f(\alpha_n) = 0 \mod p^n$. The procedure is analogous to Newton’s method.

First step: $\alpha_2 = \alpha_1 + b_1p$. Then
$$f(\alpha_2) = f(\alpha_1) + f'(\alpha_1)b_1p + O(p^2).$$

We have $f(\alpha_1) = px$. Then $b_1 = -x/f'(\alpha_1)$ and
$$\alpha_2 = -\left(\frac{f(\alpha_1)}{p}\right) \cdot \frac{1}{f'(\alpha_1)} \cdot p + \alpha_1 = -\frac{f(\alpha_1)}{f'(\alpha_1)} + \alpha_1.$$

**Application:** for $p \nmid m$ we have $\zeta_m \in \mathbb{Q}_p$ iff $m \mid (p - 1)$.

Functions

Continuity: $f(x_n) \to f(x)$ for $x_n \to x$. Polynomial functions, rational functions, ... What about power series: $f(x) := \sum_n a_n x^n \in \mathbb{Q}_p[[x]]$?

Convergence radius $r(f) := (\limsup |a_n|^{1/n})^{-1}$.

**Proposition 3.** A power series $f(x) = \sum_n a_n x^n$ diverges for $|x|_p > r(f)$ and converges to a continuous function for $|x|_p < r(f)$.
Example 4.

- If $f \in \mathbb{Z}_p[[x]]$ then $r(f) \geq 1$.
- The series
  
  $$e^x = \sum_{n \geq 0} x^n / n!$$
  
  converges for $|x|_p \leq p^{-1/(p-1)}$.
- The series
  
  $$\log_p (1 + x) := \sum_{n \geq 1} (-1)^{n-1} x^n / n$$
  
  converges for $|x|_p < 1$.
- The binomial power series
  
  $$(1 + x)^a = \sum_n a(a-1) \cdots (a-n+1) x^n / n! \in \mathbb{Z}_p[[x]].$$

- For $a = 1/2$ and $x = 7/9$ the binomial power series converges
to $4/3$ in $\mathbb{R}$ and to $-4/3$ in $\mathbb{Q}_7$. 