A distribution on $\mathbb{Z}_p$ is a map
$$\mu : X \to \mathbb{Q}_p$$
defined on compact open subsets $X \subset \mathbb{Q}_p$, so that for all $a + p^NZ_p \subset X$ one has
$$\mu(a + p^NZ_p) = \sum_{b=0}^{p-1} \mu(a + bp^n + p^{N+1})\mathbb{Z}_p.$$  

An example is $\mu_{\text{Haar}}$ given by
$$\mu_{\text{Haar}}(a + p^NZ_p) = p^{-N}.$$  

A $p$-adic measure on $X$ is a distribution such that, for some constant $C$,
$$|\mu(U)|_p \leq C$$
for all compact open $U \subset X$.

Let $\mu$ be a $p$-adic measure on $\mathbb{Z}_p$ and $f : \mathbb{Z}_p \to \mathbb{Q}_p$ a continuous function. For each $N$, choose $x_{a,N} \in \{a + p^NZ_p\}$ and put
$$S_N := \sum_{0 \leq a \leq p^N-1} f(x_{a,N})\mu(a + p^NZ_p).$$

Then there exists a limit
$$\lim_{N \to \infty} S_N = : \int_{\mathbb{Z}_p^1} f \, d\mu,$$
and it is independent of the choices made.

Note that there are problems with $\mu_{\text{Haar}}$, even the function $f(x) = x$ is not integrable! Other distributions are given by
$$\mu_{B,k}(a + p^NZ_p) := p^{N(k-1)}B_k\left(\frac{a}{p^N}\right),$$
where $k \in \mathbb{N}$ and $B_k(x)$ is the $k$-th Bernoulli polynomial.

To regularize these distributions, choose $\alpha \in \mathbb{Q} \cap \mathbb{Z}_p^*$ and put
$$\mu_{k,\alpha}(U) := \mu_{B,k}(U) - \alpha^{-k}\mu_{B,k}(\alpha U).$$

**Theorem 1.** The distributions $\mu_{k,\alpha}$ are measures. Moreover, for all compact open $U \subset \mathbb{Z}_p$ one has
$$\mu_{1,\alpha}(U) \leq 1.$$
Kummer congruences

Throughout, we assume that $p > 2$.

**Theorem 2** (Kummer, von Staudt/Clausen). For $k \in \mathbb{N}$ and $B_k = B_k(0)$ one has:

- if $p - 1 \nmid k$ then $\left| \frac{B_k}{k} \right|_p \leq 1$;
- if $p - 1 \nmid k$ and $k = k' \mod (p - 1)p^N$ then
  
  $$(1 - p^{k-1}) \frac{B_k}{k} = (1 - p^{k'-1}) \frac{B_{k'}}{k'} \mod p^{N+1}.$$ 

- if $p - 1 \mid k$ then $pB_k = -1 \mod p$.

**Proof.** Clearly $\left| B_1/1 \right|_p = \left| -1/2 \right|_p = 1$. Choose $\alpha \in \{2, \ldots, p - 1\}$ as a primitive root modulo $p$. A computation shows that

$$- \frac{B_k}{k} = (\alpha^{-k} - 1)^{-1}(1 - p^{k-1})^{-1} \int_{\mathbb{Z}_p^*} x^{k-1} \mu_{1, \alpha}.$$

Passing to $\left| \cdot \right|_p$, and envoking Theorem 1, gives the first statement. \(\square\)